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Some Simple C^*-Algebras Constructed As Crossed Products with Discrete Outer Automorphism Groups[1)]

By

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Abstract

An analogue for C^*-algebras is given of the theorem of von Neumann (Theorem VIII of [24]) that the crossed product of a commutative von Neumann algebra by a discrete group acting freely and ergodically is a factor. The method of proof also works for certain noncommutative C^*-algebras, and so in these cases one obtains also an analogue of Kallman's noncommutative generalization of von Neumann's theorem (3.3 of [18]).

§1. Introduction

It seems to be a reasonable conjecture that the C^*-algebra crossed product, suitably defined, of a simple C^*-algebra with unit by a group of outer automorphisms should be simple. For a finite group (and an arbitrary simple ring with unit) this was proved by Azumaya in [2]. It follows from Azumaya's work that for any group the algebraic crossed product is simple (see also 1.1 of [16]).

If the group is not amenable, then the largest pre-C^*-algebra norm on the algebraic crossed product does not in general yield a simple C^*-algebra (see 5.2 of [25]), but in [25] Zeller-Meier introduced a smaller norm, determining what he called the reduced crossed product, and showed that if the automorphisms satisfy a condition which in general is rather more than being outer (and if the C^*-algebra is separable and the group of automorphisms countable) then the reduced crossed product is simple.

Recently, Cuntz in [6], and Bratteli in [3] gave examples of simple crossed products of simple C^*-algebras by groups of outer automorphisms not satisfying

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Zeller-Meier's stronger condition. These examples, which involve commutative groups, were subsumed by work of Olesen and Pedersen in [19] (based on work of Connes and Takesaki in [4]) which showed that the C^*-algebra crossed product of a simple C^*-algebra by a commutative group of outer automorphisms is simple. (Here, if there is no unit, outer means not determined by a multiplier.)

In the present paper, no restriction is placed on the group, but for technical reasons the C^*-algebra is assumed to be approximately finite-dimensional (that is, every finite subset is assumed to be arbitrarily close to a subset of a finite-dimensional sub-C^*-algebra). Then it is shown that if the C^*-algebra is simple and the automorphisms outer then the reduced crossed product is simple (3.2).

The proof is based on an analysis of outer automorphisms, which establishes (in 2.11) an approximate version of a property possessed exactly by the automorphisms considered by Cuntz in [6] (which are shifts). This is enough for Cuntz's proof of simplicity to be carried through essentially unchanged in this general setting (see 3.3, 3.4).

A similar argument shows that the reduced crossed product of a commutative C^*-algebra by a group of automorphisms none of which (except the identity) is equal to the identity on any nonzero ideal, and such that no nontrivial closed ideal is invariant under all the automorphisms, is simple. Under the stronger condition that no points in the spectrum are fixed by any of the automorphisms this was proved by Effros and Hahn in [9] and by Zeller-Meier in [25] (assuming countability conditions). In the case of a commutative group this also was proved by Olesen and Pedersen in [19].

With a suitable notion of properly outer automorphism (see 2.1), similar arguments (now incorporating earlier work of the author in [11], [12] and [10]) also show that the reduced crossed product of any separable approximately finite-dimensional C^*-algebra or of any postliminary C^*-algebra by a group of properly outer automorphisms which leaves no nontrivial closed two-sided ideal invariant is simple (3.2, 3.7).

§2. Properly Outer Automorphisms of C^*-Algebras

2.1. Definition. Let A be a C^*-algebra and let α be an automorphism of A. We shall say that α is properly outer if for every nonzero α-invariant closed two-sided ideal I of A and every unitary multiplier u of I,

$$\|\alpha | I\text{-Ad } u | I\| = 2 .$$

2.2. Remark. Let A be a C^*-algebra and let α be an automorphism of A. If α is not properly outer then, by the Kadison-Ringrose theorem (7 of [17]), on some nonzero invariant closed two-sided ideal I of A, $\alpha=(\mathrm{Ad}\, u)(\exp \delta)$ where δ is a derivation of I and u is a unitary multiplier of I. By [20], the converse holds if A is separable. (For approximately finite-dimensional separable C^*-algebras this was established in [11].) The converse of course also holds if A has an essential closed two-sided ideal all of whose derivations are determined by multipliers (by [10], [21] and [22] this is true if A is postliminary, a von Neumann algebra, or simple). In case A has such an ideal, that α is properly outer just means that on no nonzero closed two-sided ideal is α determined by a multiplier.

2.3. Theorem (*cf.* 1.2.1 *of* [5]). *Let A be an approximately finite-dimensional C^*-algebra which is either separable or simple, and let α be a properly outer automorphism of A. Then for every nonzero projection $e\in A$ and every $\varepsilon>0$, there exists a nonzero projection $f\in eAe$ such that $\|f\alpha(f)\|\leq\varepsilon$.*

2.4. Lemma. *Let A be C^*-algebra and let α be an automorphism of A. Then*

$$\|\alpha-1\| = \sup_{-1\leq x\leq 1}\|(\alpha-1)(x)\| .$$

Proof. If the right side is equal to 2, so also is the left side. If the right side is strictly less than 2, inspection of the argument of Kadison and Ringrose in [17] shows that in the atomic representation, α is determined by a unitary operator u with spectrum contained in the open right halfplane, so that for a suitable $\lambda\in \boldsymbol{C}$, $\|u-\lambda\|<1$. Then

$$\|\alpha-1\| = \|\mathrm{Ad}\, u-1\| = \|\mathrm{ad}\, u\|\leq 2\|u-\lambda\|<2 .$$

2.5. Corollary. *Let A be a C^*-algebra and let α be an automorphism of A. Then*

$$\|\alpha-1\| = 2 \sup_{0\leq y\leq 1}\|(\alpha-1)(y)\| .$$

Proof. By the Kaplansky density theorem, it is sufficient to prove the conclusion with the bidual A^{**} of A in place of A and with α^{**} in place of α. In particular, we may suppose that A has a unit. If $-1\leq x\leq 1$ is as given by 2.4 with $\|(\alpha-1)(x)\|$ close to $\|\alpha-1\|$, then $y=(x+1)/2$ satisfies $0\leq y\leq 1$ and $2(\alpha-1)(y)=(\alpha-1)(x)$, so also $2\|(\alpha-1)(y)\|$ is close to $\|\alpha-1\|$.

2.6. Lemma. *Let A be an approximately finite-dimensional C^*-algebra and let α be an automorphism of A such that $\|\alpha-1\|=2$. Then for every $\varepsilon>0$ there exists a nonzero projection $f\in A$ such that $\|f\alpha(f)\|<\varepsilon$.*

Proof. By 2.5 there exists $y\in A$ with $0\leq y\leq 1$ such that $\|\alpha(y)-y\|$ is arbitrarily close to 1. Since we may suppose that y is contained in a finite-dimensional sub-C^*-algebra, we may suppose that y is a projection. Then by 1.6 of [14] there exists a finite-dimensional sub-C^*-algebra B of A containing both y and also a projection z arbitrarily close to $\alpha(y)$. Thus, inside B, $\|y-z\|$ is arbitrarily close to 1; this implies that either a nonzero subprojection f of y in B is almost orthogonal to z, in which case $\|f\alpha(f)\|$ is small, or else a nonzero subprojection of z in B is almost orthogonal to y. Since z and $\alpha(y)$ are close they are unitarily equivalent by a unitary close to 1 (1.8 of [14]), so in the latter case there exists a nonzero subprojection, say $\alpha(f)$, of $\alpha(y)$ such that $\alpha(f)$ is almost orthogonal to y — then $f\leq y$ and so $\|f\alpha(f)\|$ is small.

2.7. Lemma (*cf.* 1.2.4 *of* [5]). *Let e and f be projections in a C^*-algebra A such that $\|ef\|<1$. Then $e\vee f\in A$, where $e\vee f$ is the supremum of e and f in A^{**}, the bidual of A.*

(*a*) *Let $0<\lambda\leq 1$, and assume that for any $e',f'\in A$ with $0\leq e'\leq e$, $0\leq f'\leq f$ and $\|e'\|=\|f'\|=1$, one has $\|e'f\|\geq\lambda$, $\|ef'\|\geq\lambda$. Then*

$$|e\vee f-(e+f)|\geq\lambda(e\vee f)\,.$$

(*b*) *If the assumption in* (*a*) *holds for some $0<\lambda\leq 1$, then the partial isometry u of the polar decomposition of $e\vee f-(e+f)$ belongs to A and satisfies*

$$u=u^*,\ u^2=e\vee f,\ ueu^*=f,\ ufu^*=e\,.$$

Proof. $e\vee f\in A$ follows from $\|e\vee f-(e-f)^2\|=\|efe\|<1$ (see bottom of page 390 of [5]).

(a) By the Kaplansky density theorem, the hypothesis remains true with A^{**} in place of A. Hence by 1.2.4 (a) of [5], $|e\vee f-(e+f)|\geq\lambda(e\vee f)$.

(b) $u\in A$ because $e\vee f\in A$ and (by (a)) $e\vee f-(e+f)$ is invertible in $(e\vee f)A(e\vee f)$. The other properties of u are just 1.2.4 (b) of [5].

2.8. Lemma. *Let A be a C^*-algebra, let α be a properly outer automorphism of A and let B be a nonzero hereditary sub-C^*-algebra of A invariant for α. Assume either that A is separable and approximately finite-dimensional, or that B has a nonzero closed two-sided ideal all of whose derivations are determined by multipliers* (*this holds if A has an essential closed two-sided ideal which is* *post-liminary, simple, or an ideal of a von Neumann algebra — cf.* 2.2). *Then* $\alpha|B$ *is a properly outer automorphism of* B.

Proof. Suppose that $\alpha|B$ is not properly outer. Then by 2.2, passing to a closed two-sided ideal of B, we may suppose that $\alpha|B=(Adu)(\exp\delta)$ where δ is a derivation of B and u is a unitary multiplier of B.

If A is separable and approximately finite-dimensional, then by (ii)$'\Rightarrow$ (i)$'\varepsilon$ ($\varepsilon<2$) of 4.2 of [12], α is not properly outer — a contradiction.

If, when B is replaced by a nonzero closed two-sided ideal of B, δ is determined by a multiplier of B, so that $\alpha|B=\mathrm{Ad}\,v|B$ where v is a unitary multiplier of B, then consider the closed two-sided ideal I of A generated by B. I is α-invariant, and by 4.4 of [13], the automorphism $(\alpha|I)^{**}$ of I^{**} is inner, determined by a unique unitary $w\in I^{**}$ such that $we=ew=v$ where e is the unit of $B^{**}\subset I^{**}$. By 3.1 of [1] (modified for automorphisms), w is a multiplier of I (the argument: if $a\in A$ and $b\in B$ then $wab=waw^*wb=\alpha(a)vb\in I$). This shows that $\alpha|I$ is not properly outer — a contradiction.

2.9. *Proof of* 2.3. Let e be a nonzero projection in A and set

$$\inf_{0\neq f\leq e, f^2=f}||f\alpha(f)|| = \lambda\,.$$

We must show that $\lambda=0$. As in 1.2.1 of [5] let us assume that $\lambda>0$ and deduce a contradiction.

First, note that

$$\lambda = \inf_{0\leq k\leq e, ||k||=1}||k\alpha(k)||\,.$$

Indeed, if $0\leq k\leq e$ and $||k||=1$ then, by assumption, for any $\gamma>0$ there is a nonzero projection $f\leq k+\gamma$. Then in particular, $f\leq e+\gamma$, and so if γ is small f may be changed slightly so that $f\leq e$ (one has $(1-e)f(1-e)\leq\gamma(1-e)$, whence $||f-efe||\leq 2\gamma^{1/2}$, so if γ is small, by 1.6 of [14] a function of efe is a projection $\leq e$ and close to f). If $f\leq k+\gamma$ with γ small, then $||f\alpha(f)||$ can be at most only slightly larger than $||k\alpha(k)||$.

If $\lambda=1$, then $\alpha|eAe$ is the identity. By 2.8 this contradicts the hypothesis that α is properly outer.

If $0<\lambda<1$, then choose $\varepsilon>0$ such that $\varepsilon(\lambda+1)<\lambda$ and $\lambda+\varepsilon<1$, and choose a nonzero projection $f\leq e$ such that $||f\alpha(f)||<\lambda+\varepsilon$. Since $\lambda=\inf_{0\leq k\leq e,||k||=1}||k\alpha(k)||$, the hypotheses of 2.7 (a) are satisfied by f and $\alpha(f)$, so $f\vee\alpha(f)\in A$, and, with g denoting $f\vee\alpha(f)$,

$$\lambda g\leq |g-(f+\alpha(f))|\leq(\lambda+\varepsilon)g\,.$$

Moreover, the partial isometry u of the polar decomposition of $g-(f+\alpha(f))$ belongs to A, by 2.7 (b), and satisfies:

$$u=u^*,\ u^2=g,\ ufu^*=\alpha(f),\ u\alpha(f)u^*=f\,;$$
$$||\lambda u-(g-(f+\alpha(f)))||=||\lambda g-|g-(f+\alpha(f))|\,||\leq\varepsilon\,.$$

The automorphism $\alpha'=(\mathrm{Ad}\,u)\alpha\,|\,fAf$ of fAf is properly outer by 2.8, so by 2.6 (applied to fAf and α') there exists $0\neq k\leq f$ such that $||k||=1$ and $||k\alpha'(k)||\leq\varepsilon$. Then

$$||ku\alpha(k)||=||ku\alpha(k)u^*||\leq\varepsilon\;;$$

hence

$$||k(g-(f+\alpha(f)))\alpha(k)||\leq\lambda\varepsilon+\varepsilon\,.$$

But $kg\alpha(k)-kf\alpha(k)-k\alpha(f)\alpha(k)=-k\alpha(k)$ since $0\leq k\leq f$, and so $||k\alpha(k)||\leq\lambda\varepsilon+\varepsilon<\lambda$. This contradicts the relation $\lambda=\inf_{0\leq k\leq e,\,||k||=1}||k\alpha(k)||$ established in the second paragraph of this proof.

2.10. Remark. It follows from 2.3 that if A and α are as in 2.3 then $||(\alpha-1)\,|\,eAe||\geq 1$ for each nonzero projection $e\in A$.

Thus (restricting to the separable case) one obtains a proof of (iii)$'\varepsilon\Rightarrow$(i) of 4.2 of [12] for all $\varepsilon<1$; the proof in [12] is valid only for small ε (small enough that if $||\alpha(e)-e||<\varepsilon$ then by 1.8 of [14] $\alpha(e)$ is equivalent to e by a unitary suitably close to 1). Note that in the proof of 2.8 above we did use (iii)$'\varepsilon\Rightarrow$(i) of 4.2 of [12], as this is needed to prove (ii)$'\Rightarrow$(i) of that theorem, but we used it only for small ε (inspection of 4.2 of [12] shows that this is sufficient to prove (ii)$'\Rightarrow$(i)).

This result is optimal; for (iii)$'\varepsilon\Rightarrow$(i) of 4.2 of [12] to hold, ε must be <1, since it may happen that $e\alpha(e)=0$. On the other hand, for (iii)$\varepsilon\Rightarrow$(i) of that theorem to hold, ε need only be <2, since, for $\varepsilon<2$, by the Kadison-Ringrose theorem (7 of [17]) we have (iii)$\varepsilon\Rightarrow$(ii)$'$.

An alternative proof of $||(\alpha-1)\,|\,eAe||\geq 1$ for A separable and approximately finite-dimensional and α properly outer is as follows. By 4.2 of [11] (which again uses (iii)$'\varepsilon\Rightarrow$(i) of 4.2 of [12] for small ε), if α is properly outer then for every nonzero closed two-sided ideal I of A there exists an irreducible representation π of A such that $\pi(I)\neq 0$ and the representations π and $\pi\alpha$ are inequivalent. With I the closed two-sided ideal generated by the nonzero projection e, we get inequivalent representations $\pi\,|\,eAe$ and $\pi\alpha\,|\,eAe$ with

$\pi|eAe$ irreducible and $\pi\alpha|eAe$ either irreducible or 0. By 3 of [15] the direct sum of the weak closures of eAe in these two representations is the weak closure of eAe in the direct sum representation. In particular the unit of the weak closure of $\pi(eAe)$, say p, belongs to the weak closure of eAe in the direct sum representation, whence by the Kaplansky density theorem there exists $h\in eAe$, $0\leq h\leq 1$, with $\pi(\alpha(h))\oplus\pi(h)$ strongly close to p, so that $\pi(h)$ is strongly close to p and $\pi(\alpha(h))$ to 0. Then $\pi((\alpha-1)(h))$ is strongly close to p, which implies that $\|(\alpha-1)(h)\|$ is arbitrarily close to 1.

2.11. Corollary. *Let A and α be as in 2.9, let A_1 be a finite-dimensional sub-C^*-algebra of A, let e be a nonzero projection in A_1, and let $\varepsilon>0$. Then there exists a nonzero projection $f\in eA_1'$ such that $\|f\alpha(f)\|<\varepsilon$. (Here A_1' denotes the relative commutant of A_1 in A.)*

Proof. Clearly it is enough to consider the case that A_1 is simple and that e is the unit of A_1. We shall assume in addition that A_1 is isomorphic to the algebra M_2 of 2×2 matrices over $\boldsymbol{C}$—the proof for the case that A_1 is isomorphic to M_n for any other $n=3,4,\cdots$ is similar.

Suppose, then, that $e=e_1+e_2$ with e_1 and e_2 projections in A_1 such that for some $u\in A_1$,

$$u=u^*,\ u^2=e,\ ue_1u^*=e_2,\ ue_2u^*=e_1\,,$$

and A_1 is generated by e_1, e_2 and u.

By 2.9, there exists a nonzero projection $f_1\in e_1Ae_1$ such that $\|f_1\alpha(f_1)\|\leq \varepsilon/4$. Set $uf_1u^*=f_2$. Then also $uf_2u^*=f_1$.

By 2.9, we may replace f_1 by a smaller projection (the transform by u of a suitable subprojection of f_2) so that also $\|f_2\alpha(f_2)\|\leq\varepsilon/4$.

Now consider the automorphism α' of A defined by $\alpha'(a)=v\alpha(a)v^*$, $a\in A$, where $v=u+(1-e)$. By 2.9 we may replace f_1 by a smaller projection so that $\|f_1\alpha'(f_1)\|\leq\varepsilon/4$, and again so that also $\|f_2\alpha'(f_2)\|\leq\varepsilon/4$.

Since

$$u^*f_1\alpha'(f_1)u=f_2\alpha(f_1) \text{ and } u^*f_2\alpha'(f_2)u=f_1\alpha(f_2)\,,$$

we have $\|f_1\alpha(f_2)\|\leq\varepsilon/4$ and $\|f_2\alpha(f_1)\|\leq\varepsilon/4$. Hence, $\|(f_1+f_2)\alpha(f_1+f_2)\|\leq\varepsilon$. Since $f_1+f_2\in eA_1'$, we may take f_1+f_2 for f.

2.12. Remark. Since by 1.2.1 of [5] the conclusion of 2.9 holds for a properly outer automorphism α of a von Neumann algebra A (the hypothesis that A is countably decomposable is unnecessary), also the conclusion of 2.11

holds in this case. In this way one sees that the constant 1/35,000 in 2.1 of [23] can be replaced by 1, at least if the subfactor of type I involved is finite. Presumably the best possible constant would be 2.

2.13. Remark. Let A be a postliminary C^*-algebra and let α be a properly outer automorphism of A. Then the following much sharper property than that of 2.11 holds. For every nonzero closed two-sided ideal I of A there exists a nonzero closed two-sided ideal $J \subset I$ such that $J\alpha(J)=0$.

To see this, note that if I and every closed two-sided ideal of I are α-invariant then by the proof of 2 of [10], α is determined by a multiplier on some nonzero closed two-sided ideal of I. If, on the other hand, $\alpha(I) \neq I$ for some closed two-sided ideal I of A, then, with J_0 a closed two-sided ideal of I such that $J_0^\wedge$ is Hausdorff and dense in I (4.4.5 of [7]), there exists $t_0 \in J_0^\wedge$ such that $\alpha(t_0) \neq t_0$ (otherwise, by continuity $\alpha(t)=t$ for all $t \in I^\wedge$, whence $\alpha(I)=I$). Now consider two cases. If $\alpha(t_0) \in J_0^\wedge$, then since $J_0^\wedge$ is Hausdorff there exist disjoint open neighbourhoods of t_0 and $\alpha(t_0)$, and hence by continuity of α on $A^\wedge$, an open neighbourhood V of t_0 such that $V \cap \alpha(V)=\emptyset$. If $\alpha(t_0) \notin J_0^\wedge$, and if, moreover, there does not exist $t_1 \in J_0^\wedge$ such that $\alpha(t_1) \neq t_1$ and $\alpha(t_1) \in J_0^\wedge$, then $J_0^\wedge$ is the union of two disjoint subsets—$\{t \in J_0^\wedge \mid \alpha(t)=t\}$ and $\{t \in J_0^\wedge \mid \alpha(t) \notin J_0^\wedge\}$. Since α is continuous on $A^\wedge$ and $A^\wedge \backslash J_0^\wedge$ is closed, both these subsets are relatively closed in $J_0^\wedge$. Since they are complementary they are also relatively open, and hence open. Then with V the second subset, $V \cap \alpha(V)=\emptyset$. In either case we may take J to be the ideal with $J^\wedge = V$.

§3. Reduced Crossed Products by Properly Outer Automorphism Groups

3.1. Definition. Let A be a C^*-algebra and let α be a representation of the group G by automorphisms of A. By the crossed product $C^*(A, \alpha)$ we shall mean the C^*-algebra generated by the universal covariant representation of (A, α). In other words, $C^*(A, \alpha)$ is generated by A and the image of a unitary representation u of G such that $u_g a u_g^* = \alpha_g(a)$ for $a \in A$ and $g \in G$, and every such C^*-algebra is a quotient of $C^*(A, \alpha)$.

By considering the covariant representation $\bar{\pi}$ on $H_\pi \otimes l^2(G)$, where π is a faithful representation of A, defined by $\bar{\pi}(a) = (g \mapsto \pi\alpha_g^{-1}(a)) \in l^\infty(G, B(H_\pi))$, and $\bar{\pi}(g) = 1 \otimes \lambda(g)$ where λ is the left regular representation of G, one sees that there is a projection of norm one P from $C^*(A, \alpha)$ to A such that $P(u_g)=0$ for all $g \neq 1$ (the identity of G). Clearly this property determines P uniquely.

By the reduced crossed product $C_r^*(A,\alpha)$ (see 4.6 of [25]) we shall mean the quotient of $C^*(A,\alpha)$ by the closed two-sided ideal $I_P=\{x\in C^*(A,\alpha)\mid P(xu_g)=0$ for all $g\in G\}$ (this is the largest closed two-sided ideal in the kernel of P). Since $P(I_P)=0$, P induces a projection of norm one from $C_r^*(A,\alpha)=C^*(A,\alpha)/I_P$ to A, which we shall denote by P also.

Let J be a closed two-sided ideal of $C^*(A,\alpha)$ such that $J\cap A=0$. Since $P(J)\subset A$, clearly $P(J)\subset J$ is equivalent to $P(J)=0$. On the other hand, since $Ju_g=J$ for $g\in G$, $P(J)=0$ implies and hence is equivalent to $J\subset I_P$. It follows that $C_r^*(A,\alpha)$ is characterized among quotients of $C^*(A,\alpha)$ by the existence of the projection of norm one P from $C_r^*(A,\alpha)$ to A such that $P(u_g)=0$ for $g\neq 1$ and such that P is not zero on any nonzero closed two-sided ideal of $C_r^*(A,\alpha)$.

3.2. Theorem. *Let A be an approximately finite-dimensional C^*-algebra which is either separable or simple, and let α be a representation of the group G as automorphisms of A such that no closed two-sided ideal other than* 0 *or A is invariant under all α_g, $g\in G$, and such that for each $g\in G$ with $g\neq 1$, α_g is properly outer. Then the reduced crossed product $C_r^*(A,\alpha)$ is simple.*

3.3. Lemma. *Let A be an approximately finite-dimensional C^*-algebra which is either separable or simple, and let a be an element of A, S a finite subset of A, T a finite set of properly outer automorphisms of A, and $\varepsilon>0$. Then there exists $x\in A$ with $\|x\|=1$ such that*:

$$\|xax\|\geq\|a\|-\varepsilon\ ;$$
$$\|xb-bx\|\leq\varepsilon,\ b\in S\ ;$$
$$\|x\alpha(x)\|\leq\varepsilon,\ \alpha\in T\,.$$

Proof. We may suppose that a and S are contained in a finite-dimensional sub-C^*-algebra A_1 of A. Denote by e a minimal central projection in A_1 such that $\|ae\|=\|a\|$. Then by 2.11 there exists a nonzero projection $f_1\in eA_1'$ such that $\|f_1\alpha_1(f_1)\|\leq\varepsilon$ where α_1 is the first element of T. Again by 2.11, with f_1A_1 in place of A_1 and f_1 in place of e, there exists a nonzero projection $f_2\in f_1(f_1A_1)'\subset eA_1'$ such that $\|f_2\alpha_2(f_2)\|\leq\varepsilon$ where α_2 is the second element of T. Since $f_2\leq f_1$, also $\|f_2\alpha_1(f_2)\|\leq\varepsilon$. Continuing in this way, one obtains a nonzero projection f in eA_1' such that $\|f\alpha(f)\|\leq\varepsilon$ for all $\alpha\in T$. Then $\|faf\|=\|ae\|=\|a\|$, and $fb-bf=0$ for all $b\in S$, so we may take f for x.

3.4. *Proof of* 3.2. By 3.1, to show that $C_r^*(A,\alpha)$ is simple it is sufficient to show that $P(J)=0$ for any proper closed two-sided ideal of $C_r^*(A,\alpha)$. If J

is a proper closed two-sided ideal of $C_r^*(A,\alpha)$ then $J\cap A=0$, since $J\cap A$ is an α-invariant closed two-sided ideal of A. Since $P(C_r^*(A,\alpha))=A$, to prove that $P(J)=0$ it is then sufficient to prove that $P(J)\subset J$, that is, that P induces a projection of norm one onto A in the quotient of $C_r^*(A,\alpha)$ by J.

In other words, given a pre-C^*-algebra seminorm $||\cdot||$ on the subalgebra of $C_r^*(A,\alpha)$ generated by A and $\{u_g\,|\,g\in G\}$, which extends the norm of A, we must show that for each $a\in A$, each finite subset $T\subset G\setminus\{1\}$ and each family $(a_g)_{g\in T}$ in A,

$$||a||\leq||a+\textstyle\sum_{g\in T}a_g u_g||\,.$$

Fix $\varepsilon>0$. Then by 3.3 with $S=\{a_g\,|\,g\in T\}$ and $T=\{\alpha_g\,|\,g\in T\}$, there exists $x\in A$ with $||x||=1$ such that:

$$\begin{aligned}&||xax||\geq||a||-\varepsilon\;;\\&||xb-bx||\leq\varepsilon,\ b\in S\;;\\&||x\alpha_g(x)||\leq\varepsilon,\ \text{that is }\ ||xu_gx||\leq\varepsilon,\ g\in T\,.\end{aligned}$$

Then

$$\begin{aligned}||a||&\leq||xax||+\varepsilon\\&\leq||x(a+\textstyle\sum_{g\in T}a_gu_g)x||+||x(\sum_{g\in T}a_gu_g)x||+\varepsilon\,.\end{aligned}$$

Since

$$||x(\textstyle\sum_{g\in T}a_gu_g)x||\leq\sum_{g\in T}(||xa_g-a_gx||+||a_g||\;||xu_gx||)\,,$$

we obtain

$$||a||\leq||a+\textstyle\sum_{g\in T}a_gu_g||+\varepsilon(1+\sum_{g\in T}(1+||a_g||))\,.$$

Since $\varepsilon>0$ is arbitrary the desired inequality follows.

3.5. Remark. In 4.20 of [25], Zeller-Meier proved that $C_r^*(A,\alpha)$ is simple if A is any separable C^*-algebra, G is a countable group, and α is a representation of G by automorphisms of A such that A has no nontrivial α-invariant closed two-sided ideal, provided that α satisfies a considerably stronger condition than in 3.2 — for each $g\neq 1$ there should be no factor representation of A which is quasi-invariant for α_g. For a commutative separable C^*-algebra this was proved by Effros and Hahn in 5.16 of [9]. For some rather special nonseparable commutative C^*-algebras and uncountable (commutative) groups this was proved by Douglas in [8] (also some earlier results of this kind are referred to in [8]).

With their strong form of the condition that the automorphisms should be properly outer, Zeller-Meier and Effros and Hahn showed that, if A does have nontrivial α-invariant closed two-sided ideals, then these ideals are in bijective correspondence with the closed two-sided ideals of $C_r^*(A, \alpha)$. This need not hold under the present assumption just that α_g is properly outer, $g \neq 1$. For example, take the group $\boldsymbol{Z}^2$ acting by translation on the C^*-algebra A of functions on $\boldsymbol{Z}^2$ generated by the characteristic functions of half-open intervals $[a, b[$ in $\boldsymbol{Z}^2$ with the lexicographical order. A has only one nontrivial invariant closed two-sided ideal, but the reduced crossed product has many closed two-sided ideals — it has a quotient with spectrum the circle.

3.6. Remark. In the case that the group G is commutative, 3.2 follows from a theorem of Olesen and Pedersen (6.5 of [19]) adapting a different result of Connes to C^*-algebras. A consequence of 6.5 of [19] is that if G is commutative and A is an arbitrary C^*-algebra with no nontrivial α-invariant closed two-sided ideals, and if for each $g \neq 1$ and each nonzero α-invariant hereditary sub-C^*-algebra B of A, $\|(\alpha_g - 1)|B\| = 2$, then $C^*(A, \alpha)$ is simple. This last condition is satisfied by a properly outer automorphism of a C^*-algebra which is either postliminary, simple, a von Neumann algebra, or separable and approximately finite-dimensional (see 2.8).

3.7. Remark. 3.3 and therefore also 3.2 hold for a postliminary C^*-algebra.

To show 3.3 when A is a postliminary C^*-algebra, choose a nonzero closed two-sided ideal I of A such that $\|a+t\| > \|a\| - \varepsilon$ for all $t \in I^\wedge$ (3.3.2 of [7]), and choose a nonzero closed two-sided ideal $J \subset I$ such that $J\alpha(J) = 0$, $\alpha \in T$ (2.13). Denote by e the unit of J^{**} in A^{**}. We have:

$$\|eae\| > \|a\| - \varepsilon ;$$
$$eb - be = 0, \; b \in S ;$$
$$e\alpha(e) = 0, \; \alpha \in T .$$

In fact e itself could be used in the proof of 3.2, but it is also possible to find x in A with the properties stipulated in 3.3. By the Kaplansky density theorem we may approximate e ultraweakly in J^{**} by $x \in J$. Of course for all such x, $exe = x$, and so $x\alpha(x) = 0$, $\alpha \in T$. For x sufficiently close to e ultraweakly, $\|xax\| > \|a\| - \varepsilon$, because the set of x with $\|xax\| \leq \|a\| - \varepsilon$ is ultraweakly closed. Since $xb - bx$ converges ultraweakly to 0 in J^{**}, that is, weakly to 0 in the Banach space J, taking convex combinations we may choose x so that

$xb-bx$ is close in norm to 0. Then x satisfies the conditions of 3.3.

In particular, as a consequence of 3.2 for a commutative C^*-algebra, one obtains an alternative (and easier) proof of the result of Cuntz in [6] that the C^*-algebra $\mathcal{O}_n$ generated by n isometries whose range projections have sum 1 is simple ($n=2, 3, \cdots$). Indeed, the tensor product of $\mathcal{O}_n$ and the elementary C^*-algebra M_∞ is just the crossed product of the commutative C^*-algebra generated by the projections in $L^2(\boldsymbol{R})$ corresponding to bounded intervals with n-adic rational endpoints by the group of transformations of $\boldsymbol{R}$ of the form $x\mapsto ax+b$ where a is a power of n and b is an n-adic rational. It should be noted that the hypothesis of 3.2 is satisfied here even though the hypothesis of Effros and Hahn in [6] or Zeller-Meier in [25] is not.

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