Publ. RIMS, Kyoto Univ.
16 (1980), 499–511

On the Uniqueness of Dyer-Lashof Operations on the Bott Periodicity Spaces

By

Masatsugu NAGATA*

§1. Introduction

Bott periodicity ([5], [8]):

$$(1.1)\quad BO\times\mathbf{Z}\simeq\Omega(U/O),\quad U/O\simeq\Omega(Sp/U),\quad Sp/U\simeq\Omega Sp,\quad Sp\simeq\Omega BSp,$$
$$BSp\times\mathbf{Z}\simeq\Omega(U/Sp),\quad U/Sp\simeq\Omega(O/U),\quad O/U\simeq\Omega O,\quad O\simeq\Omega BO$$

and

$$(1.2)\qquad BU\times\mathbf{Z}\simeq\Omega U,\quad U\simeq\Omega BU$$

shows that each of these spaces is an infinite loop space.

Dyer-Lashof operations Q^r $(r\geq 0)$ are defined on the mod p homology (p prime) of any infinite loop space, depending on its infinite loop structure. If $p=2$, they are natural homology operations of degree r such that $Q^n(x_n)=x_n^2$, $Q^r(x_n)=0$ if $r<n$, which satisfy Cartan formula, Adem relations, Nishida relations, etc. Details are given in Section 3.

Dyer-Lashof operations on the mod 2 homology of the spaces in (1.1) and (1.2) have been determined by S. O. Kochman [10] and S. Priddy [14], where the infinite loop structure of each of these spaces is the one determined by Bott periodicity.

The purpose of this paper is to compute Dyer-Lashof operations on the mod 2 homology of O, O/U and U/Sp, with the infinite loop structure determined by Bott periodicity, by a slightly different method from that of Kochman [10], and to show that the action is determined only by the H-space structure for the infinite loop spaces SO and SO/U (Theorem (6.9)).

The plan of the paper is as follows: The Hopf algebra structure of the

Received September 8, 1978.
* Research Institute for Mathematical Sciences, Kyoto University.

mod 2 homology of the spaces in (1.1) and (1.2) is summarized in Section 2. In Section 3, we list the main properties of Dyer-Lashof operations. The computations are done in Section 4, and the results are summarized in Section 5. In Section 6, we consider some conditions which determine the operations on the homology of these spaces, half of these depending on the result of J. F. Adams and S. Priddy [1] on the uniqueness of the infinite loop structure, and the other half depending on the computations in Section 4.

The author would like to thank Professor H. Toda, Dr. G. Nishida and Dr. A. Kono for their many valuable suggestions and encouragement. He would also like to thank Professor J. P. May for informing him of S. O. Kochman's work [10].

§2. The Hopf Algebra Strtcture of the Mod 2 Homology

Notation. In this paper, $BO\times\mathbf{Z}$, U/O, etc. will mean the space with the infinite loop structure determined by Bott periodicity.

All homology will be with $\mathbf{Z}_2$-coefficients ($\mathbf{Z}_2$ denotes $\mathbf{Z}/2\mathbf{Z}$) and the elements will always be indexed by their degree.

$\mathbf{Z}_2[a, b,\dots]$ and $\wedge_2(a, b,\dots)$ will denote the polynomial algebra and the exterior algebra, respectively, over $\mathbf{Z}_2$ with generators $a, b,\dots$.

$\mathbf{Z}_2\{a, b,\dots\}$ will denote the free module over $\mathbf{Z}_2$ generated by $a, b,\dots$.

For any Hopf algebra A, we shall denote the coproduct by ψ and the submodule of primitive elements by $P(A)$.

It is known that ([8], [7])

(2.1) $$H_*(BO\times\mathbf{Z})=\mathbf{Z}_2[z_1, z_2, z_3\dots]\otimes\mathbf{Z}_2[[-1], [1]]/([1][-1]=1),$$
$$\psi(z_n)=\sum z_i\otimes z_{n-i}\quad (z_0=1),$$
$$H_*(U/O)=\mathbf{Z}_2[v_1, v_3, v_5,\dots],\quad P(H_*(U/O))=\mathbf{Z}_2\{(v_{2n-1})^{2^s}\},$$
$$H_*(Sp/U)=\mathbf{Z}_2[x_2, x_6, x_{10},\dots],\quad P(H_*(Sp/U))=\mathbf{Z}_2\{(x_{4n-2})^{2^s}\},$$
$$H_*(Sp)=\wedge_2(a_3, a_7, a_{11},\dots),\quad P(H_*(Sp))=\mathbf{Z}_2\{a_{4n-1}\},$$
$$H_*(BSp\times Z)=\mathbf{Z}_2[w_4, w_8, w_{12},\dots]\otimes\mathbf{Z}_2[[-1], [1]]/([1][-1]=1),$$
$$\psi(w_{4n})=\sum w_{4i}\otimes w_{4n-4i}\quad (w_0=1),$$
$$H_*(U/Sp)=\wedge_2(b_1, b_5, b_9,\dots),\quad P(H_*(U/Sp))=\mathbf{Z}_2\{b_{4n-3}\},$$
$$H_*(O/U)=\wedge_2(d_2, d_4, d_6,\dots)\otimes\mathbf{Z}_2[[-1]]/([-1]^2=1),$$

$$\psi(d_{2n})=\sum d_{2i}\otimes d_{2n-2i} \quad (d_0=1),$$

$$H_*(O)=\wedge_2(u_1, u_2, u_3,\dots)\otimes \mathbf{Z}_2[[-1]]/([-1]^2=1),$$
$$\psi(u_n)=\sum u_i\otimes u_{n-i} \quad (u_0=1)$$

and

$$H_*(BU\times \mathbf{Z})=\mathbf{Z}_2[\beta_2, \beta_4, \beta_6,\dots]\otimes \mathbf{Z}_2[[-1], [1]]/([1][-1]=1),$$
$$\psi(\beta_{2n})=\sum \beta_{2i}\otimes \beta_{2n-2i} \quad (\beta_0=1),$$

$$H_*(U)=\wedge_2(\alpha_1, \alpha_3, \alpha_5,\dots), \quad P(H_*(U))=\mathbf{Z}_2\{\alpha_{2n-1}\}.$$

The generators are taken as follows:

(2.2) $z_n\in H_*(BO\times\{0\})$ (resp. $\beta_{2n}\in H_*(BU\times\{0\})$, $w_{4n}\in H_*(BSp\times\{0\})$)

is the dual of the n-th power of the first Stiefel-Whitney class in $H^1(BO)$ (resp. the first Chern class in $H^2(BU)$, the first symplectic Pontrjagin class in $H^4(BSp)$).

$u_n\in H_*(SO)$ is defined by $u_n=\phi_*(e_n)$, where $e_n\in H_n(RP_\infty)$ is the unique generator and $\phi\colon RP_\infty\to SO$ is the map defined in [7].

$d_{2n}\in H_*(SO/U)$ is defined by $d_{2n}=p_*(u_{2n})$, where $p\colon SO\to SO/U$ is the canonical map ([8]).

$v_{2n-1}, x_{4n-2}, a_{4n-1}, b_{4n-3}, \alpha_{2n-1}$ are the unique non-zero primitive elements in their degree. Then Proposition 4.21 of Milnor-Moore [12] shows that

$$0\longrightarrow P(\xi A)\longrightarrow P(A)\longrightarrow Q(A)$$

is exact, where $Q(A)$ is the module of indecomposables and ξA is the sub Hopf algebra generated by the squares of elements, and hence, in the polynomial algebras $H_*(U/O)$ and $H_*(Sp/U)$ the primitives are $\{\xi^s$ (above generator)$\}$, and in the exterior algebras $H_*(Sp)$, $H_*(U/Sp)$ and $H_*(U)$ the primitives are {above generator}.

Note that the H-space structure of the spaces is determined by the de-looping in (1.1) and (1.2).

Notation (2.3) For any infinite loop space X with $\pi_0(X)=\mathbf{Z}$ or $\mathbf{Z}_2$ (resp. $\pi_0(X)=0$ and $\pi_1(X)=\mathbf{Z}$ or $\mathbf{Z}_2$), let X_0 be its connected component of the base point (resp. $\tilde{X}$ be its universal 1-connected covering space), with the infinite loop structure naturally induced by that of X.

$SO\subset O$, $BO\subset BO\times\mathbf{Z}$, $\mathrm{Spin}=\widetilde{SO}$, etc. will be considered to be with such infinite loop structure.

§3. Mod 2 Dyer-Lashof Operations

Dyer-Lashof operations and their properties are defined and studied in [3], [6], [9], [13] and [11]. The main properties of mod 2 Dyer-Lashof operations are as follows:

(3.1) Let X be an infinite loop space.

Dyer-Lashof operation

$$Q^r\colon H_*(X) \longrightarrow H_{*+r}(X) \qquad (r \geq 0)$$

is a natural homomorphism of degree r
such that

(i) $Q^0(1)=1$ and $Q^r(1)=0$ if $r>0$, where 1 is the unit element of the ring $H_*(X)$;

(ii) $Q^r(x_n)=0$ if $r<n$ and $Q^n(x_n)=x_n^2$;

(iii) $\sigma Q^r = Q^r \sigma$, where $\sigma\colon \tilde{H}_*(\Omega X) \to H_{*+1}(X)$ is the homology suspension (the infinite loop structure of ΩX is canonically induced by that of X);

(iv) $\chi Q^r = Q^r \chi$, where $\chi\colon H_*(X) \to H_*(X)$ is the conjugation (induced from the reversing of the loop coordinate);

(v) (Cartan formula) $Q^r(xy) = \sum_j Q^j(x) Q^{r-j}(y)$

and $\qquad \psi Q^r(x) = \sum_j \sum Q^j(x') \otimes Q^{r-j}(x'')$, where $\psi(x) = \sum x' \otimes x''$;

(vi) (Adem relations) If $a>2b$, then

$$Q^a Q^b = \sum_i \binom{i-b-1}{2i-a} Q^{a+b-i} Q^i;$$

(vii) (Nishida relations) If $t \geq k$, then

$$Sq_*^k Q^t = \sum_i \binom{t-k}{k-2i} Q^{t-k+i} Sq_*^i, \quad \text{where} \quad Sq_*^k\colon H_*(X) \longrightarrow H_{*-k}(X)$$

is the dual of the Steenrod squaring operation Sq^k $(k \geq 0)$, and $\binom{m}{n}$ denotes the binomial coefficient (see (5.2)).

§4. Computations $H_*(O)$, $H_*(O/U)$ and $H_*(U/Sp)$

Theorem (4.1) *Let B be an "allowable AR-Hopf algebra" (i.e., Hopf algebra over $\mathbf{Z}_2$ with Dyer-Lashof operations and dual Steenrod operations* *which satisfies* (3.1) (*except* (iii)), *see* [11]) *such that*

$$B=\Lambda_2(u_1, u_2, u_3,\dots) \quad \textit{as algebra},$$
$$\psi(u_n)=\sum u_i\otimes u_{n-i} \quad (u_0=1) \quad \textit{as coalgebra}$$

and $Sq^r_*(u_n)=\binom{n-r}{r}u_{n-r}+$*decomposables* ($r<n$, $n\geq 1$) *as for dual Steenrod operations.*

If $Q^2(u_1)\neq 0$, *then* $Q^r(u_n)=\sum_{a+b+c=r+n}\binom{r-a-b-1}{n-a}u_a u_b u_c$ ($r\geq 0$, $n\geq 1$) *with notation* (5.2).

If $Q^2(u_1)=0$, *then* $Q^r(u_n)=0$ $\quad$ ($r\geq 0$, $n\geq 1$).

Lemma (4.2) *If* $Q^2(u_1)\neq 0$, *then* $Q^{t+1}(u_t)\not\equiv 0$ mod *decomposables in* B ($t\geq 1$).

Proof: Nishida relations (3.1 (vii)) show that: For $n\geq 2r$,

$$Sq^{2r}_*Q^{n+1}(u_n)=\sum_i\binom{n+1-2r}{2r-2i}Q^{n+1-2r+i}Sq^i_*(u_n)=Q^{n+1-r}Sq^r_*(u_n),$$

which follows from (3.1 (ii)) and the fact that

$$n+1-2r+i>n-i \quad \text{if and only if} \quad 2i\geq 2r.$$

(Note that $\deg Sq^i_*(u_n)=n-i$ and that B is an exterior algebra.)

Hence the coefficient of u_{2n+1} in $\binom{2n+1-2r}{2r}Q^{n+1}(u_n)$ is equal to that of $u_{2n+1-2r}$ in $\binom{n-r}{r}Q^{n+1-r}(u_{n-r})$ if $n\geq 2r$. Since $\binom{2n+1-2r}{2r}\equiv\binom{n-r}{r}$ mod 2 and $\binom{n-r}{r}=0$ if $r<n<2r$, this means that:

(*) If $\binom{n-r}{r}\equiv 1$ mod 2, $n>r$, then

$Q^{n+1}(u_n)\not\equiv 0$ mod *decomp.* if and only if $Q^{n-r+1}(u_{n-r})\not\equiv 0$ mod *decomp.*

As $\psi(u_n)=\sum u_i\otimes u_{n-i}$, the only non-zero primitive element of degree 3 is $u_3+u_1u_2$. Hence $Q^2(u_1)\neq 0$ implies $Q^2(u_1)\not\equiv 0$ mod *decomposables.*

Using "if"-part of (*), we get $Q^{2^s+1}(u_{2^s})\not\equiv 0$ mod *decomposables*, and if we consider the 2-primary expansion of t, we get $Q^{t+1}(u_t)\not\equiv 0$ mod *decomposables* for all $t\geq 1$, using "only if"-part of (*).

Lemma (4.2)′ *If* $Q^2(u_1)=0$, *then* $Q^{t+1}(u_t)\equiv 0$ mod *decomposables in* B ($t\geq 1$).

Proof: Same as the proof of (4.2).

Lemma (4.3) *If* $Q^2(u_1)\neq 0$, *then* $Q^r(u_n)=\binom{r-1}{n}u_{r+n}+$ *decomposables in* B $(n\geq 1)$.

Proof: Put $Q^r(u_n)=\varepsilon_{r,n}u_{r+n}+$ *decomposables*, $\varepsilon_{r,n}=0$ or 1.

Nishida relations show that: For $k\leq r$,

$$Sq_*^k Q^r(u_n)=\binom{r-k}{k}Q^{r-k}(u_n)+\sum_{i>0}\binom{r-k}{k-2i}Q^{r-k+i}Sq_*^i(u_n)\,.$$

We shall prove by induction on n that the numbers $\{\varepsilon_{r,n}\}$ are uniquely determined by this relation and the condition (4.2).

Let $n\geq 1$, and assume that $\varepsilon_{r,n}$ are known for $m<n$. (If $n=1$, $Sq_*^i(u_1)=0$ for $i>0$ and the induction hypothesis is not necessary.)

Then the relation implies

$$(**)\qquad \binom{r+n-k}{k}\varepsilon_{r,n}-\binom{r-k}{k}\varepsilon_{r-k,n}=known\qquad (k\leq r)\,.$$

Put $k=2^{s-1}$, $r=2^s t$ $(t,\ s\geq 1)$. Then $\binom{r-k}{k}=\binom{2^s t-2^{s-1}}{2^{s-1}}\equiv 1 \bmod 2$. Hence $\varepsilon_{2^s t-2^{s-1},n}=known$ if $\varepsilon_{2^s t,n}=known$.

If we consider the 2-primary expansion of r, we get:

$$\varepsilon_{r,n}=known \text{ for } 1\leq r\leq 2^s \text{ if } \varepsilon_{2^s,n}=known.$$

Thus it suffices to know $\varepsilon_{2^s,n}$.

If $s=0$, then $Q^1(u_n)=0$ (since B is an exterior algebra) and $\varepsilon_{1,n}=0$. Now assume inductively that $\varepsilon_{2^{s-1},n}$ is known.

In case $n<2^{s-1}$, put $k=2^{s-1}$, $r=2^s$ in (**). Then $\binom{r+n-k}{k}=\binom{2^{s-1}+n}{2^{s-1}}\equiv 1$ and $\varepsilon_{r-k,n}=\varepsilon_{2^{s-1},n}=known$, and hence $\varepsilon_{2^s,n}$ is known.

In case $2^{s-1}\leq n<2^s-1$, put $k=n+1$, $r=2^s$ in (**). Then $\binom{r+n-k}{k}=\binom{2^s-1}{n+1}\equiv 1$ and $r-k=2^s-n-1<2^{s-1}$, and hence $\varepsilon_{2^s,n}$ is known.

In case $n=2^s-1$, then $\varepsilon_{2^s,2^s-1}=1$ by (4.2).

In case $n\geq 2^s$, then $Q^{2^s}(u_n)=0$ since B is an exterior algebra, and hence $\varepsilon_{2^s,n}=0$.

Induction is now complete and it suffices for our purpose to show that $\left\{\varepsilon_{r,n}=\binom{r-1}{n}\right\}$ satisfies Nishida relations, i.e., that

$$\binom{r-1}{n}\binom{r+n-k}{k}+\sum_{\substack{n-i\geq i\\ r-k+i-1\geq n-i}}\binom{r-k}{k-2i}\binom{n-i}{i}\binom{r-k+i-1}{n-i}\equiv 0 \bmod 2$$

for $k\leq r$, or equivalently, that

$$(***)\qquad \begin{bmatrix} k+m-1 \\ n \end{bmatrix}\begin{bmatrix} m+n \\ k \end{bmatrix}+\sum_i \begin{bmatrix} m \\ k-2i \end{bmatrix}\begin{bmatrix} n-i \\ i \end{bmatrix}\begin{bmatrix} m+i-1 \\ n-i \end{bmatrix}\equiv 0 \bmod 2$$

for $k,\ m\geq 0,\ n\geq 1$,

where $\begin{bmatrix} a \\ b \end{bmatrix}=\begin{cases} \binom{a}{b} & \text{if } a\geq 0 \\ 0 & \text{if } a<0. \end{cases}$

(***) can be proved easily by an induction on m and n, using the identity:

$$\begin{bmatrix} a \\ b \end{bmatrix}=\begin{bmatrix} a-1 \\ b \end{bmatrix}+\begin{bmatrix} a-1 \\ b-1 \end{bmatrix} \qquad \text{for} \quad (a,b)\neq(0,0).$$

Thus we have proved (4.3).

Lemma (4.3)′ *If* $Q^2(u_1)=0$, *then* $Q^r(u_n)\equiv 0$ mod *decomposables in* B $(n\geq 1)$.

Proof: Same as the proof of (4.3).

Now we shall prove (4.1).

Cartan formula and (3.1 (i)) imply

$$\psi Q^r(u_n)=Q^r(u_n)\otimes 1+1\otimes Q^r(u_n)+\sum_{0<i<n}\sum_j Q^j(u_i)\otimes Q^{r-j}(u_{n-i})\,.$$

Hence if we assume inductively that $Q^r(u_m)$ are known for $m<n$ and $r\geq 0$, then $Q^r(u_n)$ is known mod primitives.

But since B is an exterior algebra, Proposition 4.21 of Milnor-Moore [12] shows that

$$P(B)\longrightarrow Q(B)\cong \mathbf{Z}_2\{u_1,u_2,\dots\}$$

is injective. Hence non-zero primitives are indecomposables, and $Q^r(u_n)$ is known, by (4.3).

Hence it suffices to prove that

$$Q^r(u_n)=\sum_{a+b+c=r+n}\binom{r-a-b-1}{n-a}u_a u_b u_c \quad (r\geq 0,\ n\geq 1)\,,$$
$$Q^0(1)=1,\quad Q^r(1)=0 \qquad \text{if } r>0$$

satisfies Cartan formula and the condition (4.3).

Since $\binom{-n-1}{n}=(-1)^n\binom{2n}{n}\equiv 0$ if $n\geq 1$ and $\binom{-n-1}{-r}=0$ if $r>0$, (4.3) is satisfied. Cartan formula is also satisfied, by virtue of Theorem 25.3 of Adem [2]:

$$\sum_\nu \binom{a-\nu}{\nu}\binom{b+\nu}{c-\nu}\equiv\binom{a+b+1}{c} \bmod 2.$$

Thus the proof of (4.1) is completed.

The following results (4.4), (4.5) and (4.6) are due to S. O. Kochman [10] (Theorems 52, 61, 68, 84 and 88).

Corollary (4.4) *In* $H_*(O)$, $Q^r(u_n)=\sum_{a+b+c=r+n}\binom{r-a-b-1}{n-a}u_a u_b u_c$ $(r\geq 0, n\geq 1)$ *and* $Q^r([-1])=u_r$ $(r\geq 0)$.

Proof: Since $[-1]^2=1$, $Q^r([-1])$ is in the homology of the zero component, namely, in $H_*(SO)=\wedge_2(u_1, u_2, u_3,\dots)$. (See the construction of Dyer-Lashof operations in [9].) Hence applying Cartan formula to

$$\psi([-1])=[-1]\otimes[-1],$$

$Q^r([-1])$ is inductively determined, and we get $Q^r([-1])=u_r$, since $Q^1([-1])\neq 0$ because $\sigma([-1])=z_1\in H_*(BO)$ (see the arguments below).

As for $Q^r(u_n)$, we shall apply (4.1) to

$$B=H_*(SO)=\wedge_2(u_1, u_2, u_3,\dots).$$

Since $u_n=\phi_*(e_n)$, $e_n\in H_*(RP_\infty)$ ((2.2)),

$$Sq_*^r(u_n)=\binom{n-r}{r}u_{n-r}(r<n, n\geq 1).$$

Hence it suffices to prove that $Q^2(u_1)\neq 0$ in $H_*(SO)$.

Consider the homology Serre spectral sequence of the fibering

$$SO\simeq\Omega BSO\longrightarrow *\longrightarrow BSO.$$

Let $y_2\in H_2(BSO)$ be the unique generator. Then clearly $y_2=\sigma(u_1)$, and $\sigma Q^2(u_1)=Q^2(\sigma(u_1))=Q^2(y_2)=y_2^2\in H_4(BSO)$ by (3.1 (iii)).

Hence it suffices to prove that $y_2^2\neq 0$ in $H_4(BSO)$, or equivalently (by duality), that there is a non-primitive element in $H^4(BSO)$. Indeed,

$$H^*(BSO)=\mathbf{Z}_2[w_2, w_3, w_4,\dots]$$

where w_i is the Stiefel-Whitney class and

$$\psi(w_4)=w_4\otimes 1+w_2\otimes w_2+1\otimes w_4.\quad\text{(See [8].)}$$

Corollary (4.5) *In* $H_*(O/U)$, $Q^{2r}(d_{2n})=\sum_{a+b+c=r+n}\binom{r-a-b-1}{n-a}d_{2a}d_{2b}d_{2c}$ $(r\geq 0, n\geq 1)$ *and* $Q^{2r}([-1])=d_{2r}$ $(r\geq 0)$.

Proof: Since $d_{2n}=p_*(u_{2n})$ ((2.2)),

$$Sq_*^{2r}(d_{2n})=\binom{n-r}{r}d_{2n-2r}\qquad(r<n, n\geq 1).$$

(Note that $\binom{2n-2r}{2r}\equiv\binom{n-r}{r}$ mod 2.)

Hence again (4.1) (which still holds if all the degrees are doubled) is applied, and it suffices to prove that $Q^4(d_2)\neq 0$.

Consider the fibering

$$SO/U\simeq\Omega\mathrm{Spin}\longrightarrow *\longrightarrow \mathrm{Spin}=\widetilde{SO}.$$

Then $\sigma(d_2)=u'_3\in H_3(\mathrm{Spin})$, which maps to $u_3+\textit{decomposables}\in H_3(SO)$. Since $Q^4(u_3)=u_7+\textit{decomposables}\in H_*(SO)$ by (4.4), $Q^4(u'_3)\neq 0$ and so $Q^4(d_2)\neq 0$.

Theorem (4.6) *In* $H_*(U/Sp)$, $Q^{4r}(b_{4n-3})=\binom{r-1}{n-1}b_{4r+4n-3}$ $(r\geq n\geq 1)$, $Q^t(b_{4n-3})=0$ $(t\not\equiv 0 \bmod 4)$.

Proof: Consider the *cohomology* Serre spectral sequence of the fibering

$$U/Sp\simeq\Omega(SO/U)\longrightarrow *\longrightarrow SO/U.$$

It is known that

$$H^*(SO/U)=\mathbf{Z}_2[g_2, g_6, g_{10},\dots]$$

where g_{4n-2} is the dual of $d_{4n-2}+\textit{decomposables}\in H_*(SO/U)$ (see [8]).

Then the Borel transgression theorem shows that

$$H^*(\Omega(SO/U))=\Delta(\sigma^*(g_2), \sigma^*(g_6), \sigma^*(g_{10}),\dots),$$

where $\sigma^*\colon H^*(SO/U)\to H^{*-1}(\Omega(SO/U))$ is the cohomology suspension.

Hence by duality,

$$\sigma(b_{4n-3})\neq 0.$$

Since $\sigma(b_{4n-3})\in H_*(SO/U)$ is primitive, it is indecomposable in $H_*(SO/U)$ (see the proof of (4.1)), that is:

$$\sigma(b_{4n-3})=d_{4n-2}+\textit{decomposables},$$

and (4.6) follows from (4.5), since $P(H_*(U/Sp))=\mathbf{Z}_2\{b_{4n-3}\}$.

§5. The Results

We list here the values of Dyer-Lashof operations for the Bott periodicity spaces, which are due to S. Priddy [14] and S. O. Kochman [10].

(5.1) In $H_*(BO\times\mathbf{Z})$, $Q^n(z_n)=z_n^2$,

$$Q^r(z_n)=\sum_{\Sigma\nu\lambda_\nu=r+n}[\lambda_1,\dots,\lambda_{r+n}]^{(r)}z_1^{\lambda_1}\cdots z_{r+n}^{\lambda_{r+n}}\text{ (with notation (5.2) below)}$$

$$\equiv\binom{r-1}{n}z_{r+n}+\textit{decomposables}\ (r>n\geq 1),\ Q^r([1])=z_r[1]^2$$

$$Q^r([-1])=\sum_{\Sigma\nu\lambda_\nu=r}(\lambda_1,\dots,\lambda_r)z_1^{\lambda_1}\cdots z_r^{\lambda_r}[-1]^2\quad(r\geq 0).$$

In $H_*(U/O)$, $Q^r(v_n)=\binom{r-1}{n-1}v_{r+n}$ $(r\geq n\geq 1)$ with notation $v_{2n}=v_n^2$.

In $H_*(Sp/U)$, $Q^{2r}(x_{2n})=\binom{r-1}{n-1}x_{2r+2n}$ $(r\geq n\geq 1)$ with notation $x_{4n}=x_{2n}^2$.

In $H_*(Sp)$, $Q^{4r}(a_{4r-1})=\binom{r-1}{n-1}a_{4r+4n-1}$ $(r\geq n\geq 1)$.

In $H_*(BSp\times \mathbf{Z})$, $Q^{4n}(w_{4n})=w_{4n}^2$,

$$Q^{4r}(w_{4n})=\sum_{\Sigma\nu\lambda_\nu=r+n}[\lambda_1,\dots,\lambda_{r+n}]^{(r)}w_4^{\lambda_1}\cdots w_{4r+4n}^{\lambda_{r+n}}$$

$$\equiv\binom{r-1}{n}w_{4r+4n}+\textit{decomposables}\ (r>n\geq 1),\ Q^{4r}([1])=w_{4r}[1]^2,$$

$$Q^{4r}([-1])=\sum_{\Sigma\nu\lambda_\nu=r}(\lambda_1,\dots,\lambda_r)w_4^{\lambda_1}\cdots w_{4r}^{\lambda_r}[-1]^2\quad(r\geq 0).$$

In $H_*(U/Sp)$, $Q^{4r}(b_{4n-3})=\binom{r-1}{n-1}b_{4r+4n-3}$ $(r\geq n\geq 1)$.

In $H_*(O/U)$, $Q^{2r}(d_{2n})=\sum_{a+b+c=r+n}\binom{r-a-b-1}{n-a}d_{2a}d_{2b}d_{2c}$

$$\equiv\binom{r-1}{n}d_{2r+2n}+\textit{decomposables}\ (r\geq n\geq 1),\ Q^{2r}([-1])=d_{2r}\quad(r\geq 0).$$

In $H_*(O)$, $Q^r(u_n)=\sum_{a+b+c=r+n}\binom{r-a-b-1}{n-a}u_au_bu_c$

$$\equiv\binom{r-1}{n}u_{r+n}+\textit{decomposables}\ (r\geq n\geq 1),\ Q^r([-1])=u_r\quad(r\geq 0).$$

In $H_*(BU\times \mathbf{Z})$, $Q^{2n}(\beta_{2n})=\beta_{2n}^2$,

$$Q^{2r}(\beta_{2n})=\sum_{\Sigma\nu\lambda_\nu=r+n}[\lambda_1,\dots,\lambda_{r+n}]^{(r)}\beta_2^{\lambda_1}\cdots\beta_{2r+2n}^{\lambda_{r+n}}$$

$$\equiv\binom{r-1}{n}\beta_{2r+2n}+\textit{decomposables}\ (r>n\geq 1),\ Q^{2r}([1])=\beta_{2r}[1]^2,$$

$$Q^{2r}([-1])=\sum_{\Sigma\nu\lambda_\nu=r}(\lambda_1,\dots,\lambda_r)\beta_2^{\lambda_1}\cdots\beta_{2r}^{\lambda_r}[-1]^2\quad(r\geq 0).$$

In $H_*(U)$, $Q^{2r}(\alpha_{2n-1})=\binom{r-1}{n-1}\alpha_{2r+2n-1}$ $(r\geq n\geq 1)$.

And the other Q^t (generator in (2.2)) are all zero.

Notation (5.2) $\binom{-m}{n}$ is the coefficient of X^n in the power series $(1+X)^{-m}$,

$$(\lambda_1,\dots,\lambda_r)=\binom{\lambda_1}{\lambda_1}\binom{\lambda_1+\lambda_2}{\lambda_2}\binom{\lambda_1+\lambda_2+\lambda_3}{\lambda_3}\cdots\binom{\lambda_1+\cdots+\lambda_r}{\lambda_r}$$

and

$$[\lambda_1,\dots,\lambda_{r+n}]^{(r)}=\sum_{\substack{0\le u\le n\\ 0\le j\le r-n}}\binom{r-j-n+u-1}{u}(\lambda_1,\dots,\lambda_{n-u-1},\lambda_{n-u}-1,$$

$$\lambda_{n-u+1},\dots,\lambda_{r-j+u-1},\lambda_{r-j+u}-1,\lambda_{r-j+u+1},\dots,\lambda_{r+n}),$$

with priviso: if $j=r-n$ and $u=0$, replace by $(\lambda_1,\dots,\lambda_n-2,\dots,\lambda_{r+n})$; if $u=n$, replace by $(\lambda_1,\dots,\lambda_{r-j+n}-1,\dots,\lambda_{r+n})$.

§6. Some Conditions for the Uniqueness of the Dyer-Lashof Action

In this section, X always denotes an infinite loop space, BX denotes its first delooping space, $\simeq$ expresses a homotopy equivalence of topological spaces and $\underset{H}{\simeq}$ expresses an H-equivalence of H-spaces.

R denotes the mod 2 Dyer-Lashof algebra ([11]).

J. F. Adams and S. Priddy have proved in [1] the following:

(6.1) *If $X_{(2)}\simeq BSO_{(2)}$, then $X_{(2)}\simeq BSO_{(2)}$ as infinite loop spaces.*

If $X_{(2)}\simeq BO_{(2)}$ and $x_1^2\neq 0$ for the generator $x_1\in H_1(X)$, then $X_{(2)}\simeq BO_{(2)}$ as infinite loop spaces.

(Here $X_{(2)}$ denotes the localization at 2 of X.)

As a corollary,

(6.2) *If $X\simeq BO$ and $x_1^2\neq 0$ for the generator $x_1\in H_1(X)$, then R-action on $H_*(X)$ is the one in* (5.1).

(6.3) *If $X\simeq \widetilde{U/O}$, then R-action on $H_*(X)$ is the one in* (5.1).

Proof: $X\simeq\widetilde{U/O}$ means $\Omega X\underset{H}{\simeq}\Omega(\widetilde{U/O})\underset{H}{\simeq}BO$, and a homology transgression theorem shows that $H_*(X)$ is generated by σ-image of the polynomial generators of $H_*(BO)$. Thus R-action on $H_*(X)$ is determined by (3.1 (iii)).

(6.4) *If $X\simeq U/O$ and $H_*(X)\cong H_*(U/O)$ (or, in particular, $X\underset{H}{\simeq}U/O$), then R-action on $H_*(X)$ is the one in* (5.1).

Proof: Since $H_*(U/O)$ is a polynomial algebra, R-action on $H_*(X)$ is determined by that on $H_*(\tilde{X})$ and Cartan formula $(Q^r(v_1))^2=Q^{2r}(v_1^2)$.

(6.5) *If $X\simeq Sp/U$, then R-action on $H_*(X)$ is the one in* (5.1).

Proof: Since $\Omega X\underset{H}{\simeq}U/O$, this follows from (6.4), as in the proof of (6.3).

(6.6) *If $X\simeq Sp$, then R-action on $H_*(X)$ is the one in* (5.1).

Now suppose that $X \underset{H}{\simeq} SO$. Then $H^*(X) \cong H^*(SO)$ as a Hopf algebra over the Steenrod algebra. Hence we can apply (4.1) to $H_*(X)$, and we get:

(6.7) *If* $X \underset{H}{\simeq} SO$ *and* $Q^2(u_1) \neq 0$ *for the generator* $u_1 \in H_1(X)$, *then R-action on* $H_*(X)$ *is the one in* (4.4).

(6.7)′ *If* $X \underset{H}{\simeq} SO$ *and* $Q^2(u_1)=0$, *then R-action on* $H_*(X)$ *is trivial.*

Similarly,

(6.8) *If* $X \underset{H}{\simeq} SO/U$ *and* $Q^4(d_2) \neq 0$ *for the generator* $d_2 \in H_2(X)$, *then R-action on* $H_*(X)$ *is the one in* (4.5).

(6.8)′ *If* $X \underset{H}{\simeq} SO/U$ *and* $Q^4(d_4)=0$, *then R-action on* $H_*(X)$ *is trivial.*

For $X \simeq SO/U$, $H_*(\Omega^2 X) \cong H_*(BSp \times \mathbf{Z})$ as an algebra. Then Nishida relation

$$Sq_*^4 Q^8(w_4) = Q^4(w_4) = w_4^2 \neq 0$$

implies that $Q^8(w_4)$ is a non-zero primitive element in $H_{12}(\Omega^2 X)$, and hence is indecomposable.

Then as in the proof of (6.3), a homology transgression theorem shows that $Q^8(b_5)$ is also non-zero and primitive in $H_*(\Omega X)$, where $b_5 \in H_5(\Omega X)$ is the generator.

Since $H^*(\Omega X) \cong H^*(U/Sp)$ is an exterior algebra on generators of odd degrees, Borel's big transgression theorem ([4] Th. 13.1) shows that $H^*(X)$ is a polynomial algebra generated by the transgression of the generators of $H^*(\Omega X)$. Then as in the proof of (4.6), $\sigma(b_{4n-3}) \neq 0$ by duality, and $Q^8(d_6)$ is non-zero in $H_*(X)$. Namely (6.8)′ does not occur. Similar arguments show that (6.7)′ does not occur.

Thus we have obtained our main theorem:

Theorem (6.9) *Dyer-Lashof action on* $H_*(X)$, *with* $X \underset{H}{\simeq} SO$, SO/U *or* $X \simeq Sp$, Sp/U, $\widetilde{U/O}$, *is independent of the infinite loop structure of the space.*

Remark (6.10) The condition $X \underset{H}{\simeq} SO$, SO/U is a little too restrictive. Indeed, one can show that if $X \simeq SO/U$, then $H_*(X) \cong H_*(SO/U)$ as a Hopf algebra. Since $H_*(\Omega X) \cong H_*(U/Sp)$ as a Hopf algebra over the Steenrod algebra, the action of Sq_*^{2k} on d_{4n-2} is determined (mod decomposables). Thus one can replace the condition of (6.9) to be

$$X \simeq SO/U \quad \text{and} \quad Sq_*^2(d_{2^{r+1}}) = d_{2^{r+1}-2} \quad (r \geq 2)$$

or

$$X \simeq SO \quad \text{and} \quad Sq_*^1(u_{2r}) = u_{2r-1} \ (r \geq 2).$$

References

[1] Adams, J. F. and Priddy, S., Uniqueness of *BSO*, *Math. Proc. Camb. Phil. Soc.*, **80** (1976), 475–509.

[2] Adem, J., The relations on Steenrod powers of cohomology classes, *Algebraic Geometry and Topology* (A symposium in honor of S. Lefschetz, 1957).

[3] Araki, S. and Kudo, T., Topology of H_n-spaces and H-squaring operations, *Mem. Fac. Sci. Kyushu Univ., Ser. A.* **10** (1956), 85–120.

[4] Borel, A., Sur la cohomologie des espaces fibrés principaux et des espaces homogènes de groupes de Lie compacts, *Ann. of Math.*, **57** (1953), 115–207.

[5] Bott, R., The stable homotopy of the classical groups, *Ann. of Math.*, **70** (1959), 313–337.

[6] Browder, W., Homology operations and loop spaces, *Ill. J. Math.*, **4** (1960), 347–357.

[7] Cartan, H., *Séminaire Henri Cartan de l'Ecole Norm. Sup.*, 1959–60, exposé **17**.

[8] Dyer, E. and Lashof, R., A topological proof of the Bott periodicity theorems, *Annali di Mat. Pure Appl.*, **54** (1961), 231–254.

[9] Dyer, E. and Lashof, R., Homology of iterated loop spaces, *Amer. J. Math.*, **84** (1962), 35–88.

[10] Kochman, S. O., Homology of the classical groups over the Dyer-Lashof algebra, *Trans. Amer. Math. Soc.*, **185** (1973), 83–136.

[11] May, J. P., Homology operations on infinite loop spaces, *Proc. Symp. in Pure Math.*, **22** (1971), 171–186.

[12] Milnor, J. W. and Moore, J. C., On the structure of Hopf algebras, *Ann. of Math.*, **81** (1965), 211–264.

[13] Nishida, G., Cohomology operations in iterated loop spaces, *Proc. Japan Acad.*, **44** (1968), 104–109.

[14] Priddy, S., Dyer-Lashof operations for the classifying spaces of certain matrix groups, *Quart. J. Math. Oxford* (3), **26** (1975), 179–193.