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On the Homotopy Types of the Groups of Equivariant Diffeomorphisms

By

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§0. Introduction

The purpose of this paper is to study the homotopy type of the group of the equivariant diffeomorphisms of a closed connected smooth G-manifold M, when G is a compact Lie group and the orbit space M/G is homeomorphic to a unit interval $[0, 1]$.

Let $\mathrm{Diff}_G^\infty(M)_0$ denote the group of equivariant C^∞ diffeomorphisms of the G-manifold M which are G-isotopic to the identity, endowed with C^∞ topology. If M/G is homeomorphic to $[0, 1]$, then M has two or three orbit types G/H, G/K_0 and G/K_1. We can choose the isotropy subgroups H, K_0, K_1 satisfying $H\subset K_0\cap K_1$. Moreover the G-manifold structure of M is determined by an element η of a factor group $N(H)/H$, where $N(H)$ is the normalizer of H in G (see §1). Let $\Omega(N(H)/H; (N(H)\cap N(K_0))/H, (N(H)\cap N(\eta K_1\eta^{-1}))/H)_0$ denote the connected component of the identity of the space of paths a: $[0, 1]\to N(H)/H$ satisfying $a(0)\in(N(H)\cap N(K_0))/H$ and $a(1)\in(N(H)\cap N(\eta K_1\eta^{-1}))/H$.

Theorem. $\mathrm{Diff}_G^\infty(M)_0$ *has the same homotopy type as the path space* $\Omega(N(H)/H; (N(H)\cap N(K_0))/H, (N(H)\cap N(\eta K_1\eta^{-1}))/H)_0$.

The paper is organized as follows. In Section 1, we study the G-manifold structure of M and give a differentiable structure of M/G such that the functional structure of M/G is induced from that of M. This differentiable structure is important to study the structure of $\mathrm{Diff}_G^\infty(M)_0$. In Section 2, we define a group homomorphism P: $\mathrm{Diff}_G^\infty(M)_0\to\mathrm{Diff}^\infty[0, 1]_0$ and prove that P is a continuous homomorphism between topological groups. In Section 3, we

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prove that there exists a global continuous section of P and Ker P is a deformation retract of $\mathrm{Diff}_G^\infty(M)_0$. In Section 4, we study the group structure of Ker P. In Section 5 and Section 6, we prove our Theorem.

§1. G-Manifold Structure of M and the Functional Structure of M/G

In this paper we assume that all manifolds and all actions are differentiable of class C^∞.

In this section we study the G-manifold structure of M. First we see that it is sufficient for us to consider $\eta=1$ (see Lemma 1.1). Next we give a differentiable structure of M/G such that the functional structure of M/G is induced from that of M (see Lemma 1.2).

Let M be a closed connected smooth G-manifold such that M/G is homeomorphic to $[0, 1]$. We denote this homeomorphism by f. Let $\pi: M\to M/G$ be the natural projection. Put $M_0=(f\circ\pi)^{-1}([0, 1/2])$ and $M_1=(f\circ\pi)^{-1}([1/2, 1])$. Let x_i be a point of M with $f(\pi(x_i))=i$ for $i=0, 1$. Then M_i is a G-invariant closed tubular neighborhood of the orbit $G(x_i)$ (c.f. G. Bredon [3, Chapter VI, §6]). Moreover M is equivariantly diffeomorphic to a union of the G-manifolds M_0 and M_1 such that their boundaries are identified under a G-diffeomorphism $\eta': \partial M_0\to\partial M_1$. Let V_i be a normal vector space of $G(x_i)$ at x_i and K_i be the isotropy subgroup of x_i for $i=0, 1$. Then V_i is a representation space of K_i. From the differentiable slice theorem, M_i is equivariantly diffeomorphic to a smooth G-bundle $G\times_{K_i}D(V_i)$ which is associated to the principal K_i bundle $\pi_i: G\to G/K_i$, where $D(V_i)$ is a unit disc in V_i.

Let H be a principal isotropy subgroup of the G-manifold M. We can assume that H is a subgroup of $K_0\cap K_1$. Let $e_i\in S(V_i)$ be a point such that the isotropy subgroup of e_i is H for $i=0, 1$, where $S(V_i)$ is a unit sphere in V_i. There exists a G-diffeomorphism $h_i: G/H\to G\times_{K_i}S(V_i)$ given by $h_i(gH)=[g, e_i]$, $i=0, 1$. Then we have a G-diffeomorphism

$$\eta'': G/H \xrightarrow{h_0} G\times_{K_0}S(V_0)=\partial M_0 \xrightarrow{\eta'} \partial M_1=G\times_{K_1}S(V_1)\xrightarrow{h_1^{-1}} G/H.$$

Since any G-map $G/H\to G/H$ is given by a right translation of an element of $N(H)/H$, η'' defines an element $\eta\in N(H)/H$.

Put $x_i'=\eta\cdot x_i$. Then the isotropy subgroup K_1' of x_1' is $\eta K_1\eta^{-1}$. Let V_1' be a normal vector space of the orbit $G(x_1')=G(x_i)$ at x_1'. Put $e_1'=(d\eta)_{x_1}(e_1)\in S(V_1')$. There exists a G-diffeomorphism $u: G\times_{K_1}D(V_1)\to G\times_{K_1'}D(V_1')$ given by $u([g, v])=[g\eta^{-1}, \eta\cdot v]$. Then $(u\circ\eta')([g, e_0])=u([g\eta, e_1])=[g, e_1']$ for $[g, v] \in G\times_{K_0}S(V_0)$. Therefore M is equivariantly diffeomorphic to a union of the G-bundles $G\times_{K_0}D(V_0)$ and $G\times_{K_1'}D(V_1')$ such that their boundaries are identified under a G-diffeomorphism $u\circ\eta'$. Now we have

Lemma 1.1. *Let M be a closed connected smooth G-manifold such that M/G is homeomorphic to $[0, 1]$. Then M has two or three orbit types G/H, G/K_0 and G/K_1 with $H\subset K_0\cap K_1$, and there exist representation spaces V_i, $i=0, 1$, of K_i such that M is equivariantly diffeomorphic to a union of G-bundles $G\times_{K_0}D(V_0)$ and $G\times_{K_1}D(V_1)$ with their boundaries identified under a G-diffeomorphism η: $G\times_{K_0}S(V_0)\to G\times_{K_1}S(V_1)$. Moreover we may assume that $\eta([g, e_0])=[g, e_1]$, where e_i is a point of $S(V_i)$ such that the isotropy subgroup of e_i is H for $i=0, 1$.*

Hereafter we shall assume that M is a G-manifold as in Lemma 1.1. Let ξ: $[0, 1]\to R$ be a smooth function such that

$$\begin{aligned}&\xi(r)=r^2 \quad \text{for} \quad 0\leq r\leq 1/2,\\ &\xi'(r)>0 \quad \text{for} \quad 0<r\leq 1 \quad \text{and}\\ &\xi(r)=r-1/2 \quad \text{for} \quad 7/8<r\leq 1.\end{aligned}$$

Let θ: $M=G\times_{K_0}D(V_0)\cup_\eta G\times_{K_1}D(V_1)\to[0, 1]$ be a map given by

$$\begin{aligned}&\theta([g, v])=\xi(\|v\|) \qquad \text{for} \quad [g, v]\in G\times_{K_0}D(V_0),\\ &\theta([g, v])=1-\xi(\|v\|) \qquad \text{for} \quad [g, v]\in G\times_{K_1}D(V_1).\end{aligned}$$

Since θ is a G-map, there exists a map ϕ: $M/G\to[0, 1]$ such that $\phi\circ\pi=\theta$. It is easy to see that ϕ is a homeomorphism. We give a differentiable structure of M/G by ϕ.

Lemma 1.2. *In the above situation, we have*

(1) *θ is a smooth map,*

(2) *there exists a G-diffeomorphism α: $\theta^{-1}((0, 1))\to G/H\times(0, 1)$ such that $\theta\circ\alpha^{-1}$ is the projection on the second factor, and*

(3) *f: $M/G\to R$ is smooth if and only if $f\circ\pi$: $M\to R$ is smooth.*

Proof. (1) Let α_i: $G\times_{K_i}(D(V_i)-0)\to G/H\times(0, 1]$ be a map given by $\alpha_i([g, re_i])=(gH, r)$ for $g\in G$ and $r\in(0, 1]$ $(i=0, 1)$. Then it is easy to see that α_i is a G-diffeomorphism. Since $\alpha_1\circ\eta=\alpha_0$ on $G\times_{K_0}S(V_0)$, the composition β: $\theta^{-1}((0, 1))=G\times_{K_0}(D(V_0)-0)\cup_\eta G\times_{K_1}(D(V_1)-0)\xrightarrow{\alpha_0\cup\alpha_1} G/H\times(0, 1]\cup_{1_{G/H}\times 1} G/H\times(0, 1]=G/H\times(0, 2)$ is a G-diffeomorphism. Note that

$$(\theta\circ\beta^{-1})(gH, r)=\begin{cases}\xi(r) & \text{for } 0<r\leq 1,\\ 1-\xi(2-r) & \text{for } 1\leq r\leq 2.\end{cases}$$

Thus $\theta\circ\beta^{-1}$ is a smooth map, and θ is a smooth map on $\theta^{-1}((0, 1))$. From the definition, θ is a smooth map on $\theta^{-1}(r)$ for $r\neq 1/2$. Therefore θ is a smooth map.

(2) Let $\bar{\theta}$: $(0, 2)\to(0, 1)$ be a smooth map given by

$$\bar{\theta}(r)=\begin{cases}\xi(r) & \text{for } 0<r\leq 1,\\ 1-\xi(2-r) & \text{for } 1\leq r<2.\end{cases}$$

Since $\bar{\theta}'(r)>0$ for $0<r<2$, $\bar{\theta}$ is a diffeomorphism. Let α: $\theta^{-1}((0, 1))\to G/H\times(0, 1)$ be a G-diffeomorphism given by $\alpha=(1, \bar{\theta})\circ\beta$. Then $(\theta\circ\alpha^{-1})(gH, r)=(\theta\circ\beta^{-1})(gH, \bar{\theta}^{-1}(r))=r$, and $\theta\circ\alpha^{-1}$ is the projection on the second factor.

(3) Let f: $M/G\to R$ be a function such that $f\circ\pi$: $M\to R$ is smooth. We shall prove that $f\circ\phi^{-1}$: $[0, 1]\to R$ is smooth. Since

$$(f\circ\pi\circ\alpha^{-1})(gH, r)=(f\circ\phi^{-1}\circ\theta\circ\alpha^{-1})(gH, r)=(f\circ\phi^{-1})(r) \quad \text{for} \quad 0<r<1,$$

$f\circ\phi^{-1}$ is smooth on $(0, 1)$. Let i_0: $D_{1/2}(V_0)=\{v\in D(V_0);\ \|v\|\leq 1/2\}\to G\times_{K_0} D(V_0)$ be an inclusion given by $i_0(v)=[1, v]$. Note that $(\theta\circ i_0)(v)=\|v\|^2$ for $v\in D_{1/2}(V_0)$. By Corollary 5.3 of G. Bredon [3, Chapter VI, § 5], $f\circ\phi^{-1}$ is smooth if and only if $(f\circ\phi^{-1})\circ(\theta\circ i_0)$ is smooth. Since $(f\circ\phi^{-1})\circ(\theta\circ i_0)=f\circ\pi\circ i_0$, which is smooth, then $f\circ\phi^{-1}$ is smooth on $[0, 1/4]$. Similarly we can prove that $f\circ\phi^{-1}$ is smooth on $[3/4, 1]$. Since $(f\circ\phi^{-1})(r)=(f\circ\phi^{-1}\circ\theta\circ\alpha^{-1})(1H, r)=(f\circ\pi\circ\alpha^{-1})(1H, r)$ for $0<r<1, f\circ\phi^{-1}$ is smooth on $(0, 1)$. This completes the proof of Lemma 1.2.

Remark 1.3. Lemma 1.2 is essentially proved by G. Bredon [3, Chapter VI, § 5], and (3) implies that the functional structure of M/G is induced from that of M.

§ 2. On the Group Homomorphism *P*

In this section we shall define a group homomorphism P: $\mathrm{Diff}^{\infty}_{G}(M)_0\to \mathrm{Diff}^{\infty}[0, 1]$, and we shall prove P is continuous.

We shall identify the orbit space M/G with $[0, 1]$ by the homeomorphism ϕ in § 1, therefore the projection π: $M\to M/G$ is identified with the smooth map θ: $M\to[0, 1]$. Let h: $M\to M$ be a G-diffeomorphism of M which is G-isotopic to the identity 1_M, and let f: $[0, 1]\to[0, 1]$ be the orbit map of h. Since $f\circ\pi=\pi\circ h$ is a smooth map, f is a smooth map by Lemma 1.2 (3). Similarly the inverse map f^{-1} of f is smooth, and f is a diffeomorphism. Then there exists an abstract group homomorphism P: $\mathrm{Diff}_G^\infty(M)_0 \to \mathrm{Diff}^\infty[0, 1]$ which is given by $P(h)=f$, where $\mathrm{Diff}^\infty[0, 1]$ is the group of C^∞ diffeomorphisms of $[0, 1]$, endowed with C^∞ topology.

Proposition 2.1. P: $\mathrm{Diff}_G^\infty(M)_0 \to \mathrm{Diff}^\infty[0, 1]$ *is a continuous homomorphism of topological groups.*

Let $C^\infty(M_1, M_2)$ denote the set of all smooth maps from a compact smooth manifold M_1 to a smooth manifold M_2, endowed with C^∞ topology. Before the proof of Proposition 2.1, we begin with some lemmas.

Lemma 2.2. *Let M_i be a compact smooth manifold and N_i be a smooth manifold for $i=1, 2$. Then we have:*

(1) *Let ϕ: $N_1 \to N_2$ be a smooth map, and let ϕ_*: $C^\infty(M_1, N_1) \to C^\infty(M_1, N_2)$ be a map which is given by $\phi_*(f)=\phi\circ f$. Then ϕ_* is continuous.*

(2) *Let ϕ: $M_1 \to M_2$ be a smooth map, and let ϕ^*: $C^\infty(M_2, N_1) \to C^\infty(M_1, N_1)$ be a map which is given by $\phi^*(f)=f\circ\phi$. Then ϕ^* is continuous.*

(3) *Let ϕ: $M_1 \to N_2$ be a smooth map and let $\phi_\sharp$: $C^\infty(M_1, N_1) \to C^\infty(M_1, N_1 \times N_2)$ be a map which is given by $\phi_\sharp(f)=(f, \phi)$. Then $\phi_\sharp$ is continuous.*

(4) *Let ϕ: $M_2 \to N_2$ be a smooth map and let $\phi_!$: $C^\infty(M_1, N_1) \to C^\infty(M_1 \times M_2, N_1 \times N_2)$ be a map given by $\phi_!(f)=f\times\phi$. Then $\phi_!$ is continuous.*

(5) *Let κ: $C^\infty(M_1, N_1) \times C^\infty(M_1, N_2) \to C^\infty(M_1, N_1 \times N_2)$ be a map given by $\kappa(f, g)(x)=(f(x), g(x))$ for $x \in M_1$. Then κ is continuous.*

(6) *Let L be a smooth manifold. Let* comp: $C^\infty(M_1, N_1) \times C^\infty(N_1, L) \to C^\infty(M_1, L)$ *be a map given by* $\mathrm{comp}(f, g)=g\circ f$. *Then* comp *is continuous.*

Proof. (1) and (2) are proved by R. Abraham [2, Theorems 11.2 and 11.3]. It is easy to see (3), (4) and (5). From J. Cerf [4, Chapter I, §4, Proposition 5], (6) follows.

Lemma 2.3. *Let X be a topological space. Let M be a compact smooth manifold and N be a smooth manifold. Choose an open covering $\{U_i\}$ of M such that each closure $\overline{U}_i$ of U_i is a regular submanifold of M which is contained in a coordinate neighborhood of M. Then a map Ψ: $X \to C^\infty(M, N)$ is continuous if and only if each composition Ψ_i: $X \xrightarrow{\Psi} C^\infty(M, N) \xrightarrow{j_i^*} C^\infty(\overline{U}_i, N)$ is continuous for each i, where j_i: $\overline{U}_i \hookrightarrow M$ is an inclusion.*

Proof. From Lemma 2.2 (2), if Ψ is continuous, then Ψ_i is continuous for each i. We can choose $\{U_i\}$ as a coordinate covering of M. Let $\{V_\lambda\}$ be a coordinate covering of N. Let $f\in C^\infty(M, N)$ and $K\subset U_i$ be a compact subset such that $f(K)\subset V_\lambda$ for some λ. $N^r(f, U_i, V_\lambda, K, \varepsilon)$ $(r=0, 1, 2,\dots, 0<\varepsilon\leq\infty)$ denotes the set of C^r maps $g: M\to N$ such that $g(K)\subset V_\lambda$ and $\|D^k f(x)-D^k g(x)\|<\varepsilon$ for any $x\in K$, $k=0, 1, 2,\dots, r$. Then the C^∞ topology on $C^\infty(M, N)$ is generated by these sets $N^r(f, U_i, V_\lambda, K, \varepsilon)$ (see M. Hirsch [6, Chapter 2, § 1]).

Let $x\in X$ and let $f=\Psi(x)$. For any open neighborhood W of f, there exist above sets $N_k=N^{r_k}(f, U_{i_k}, V_{\lambda_k}, K_k, \varepsilon_k)$, $k=1, 2,\dots, n$, such that $\cap_{k=1}^n N_k\subset W$. Note that $j_{i_k}^*: C^\infty(M, N)\to C^\infty(\overline{U}_{i_k}, N)$ is an open map and $(j_{i_k}^*)^{-1}(j_{i_k}^*(N_k))=N_k$. Since Ψ_{i_k} is continuous, $\Psi^{-1}(N_k)=\Psi_{i_k}^{-1}(j_{i_k}^*(N_k))$ is an open neighborhood of x in X, for each k. Then $\cap_{k=1}^n \Psi^{-1}(N_k)$ is an open neighborhood of x in X. Since $\Psi(\cap_{k=1}^n \Psi^{-1}(N_k))\subset\cap_{k=1}^n N_k\subset W$, Ψ is continuous at x. This completes the proof of Lemma 2.3.

Remark. Lemma 2.2 and Lemma 2.3 hold in the cases of manifolds with corners.

Let $C_e^\infty([-1/2, 1/2], R)$ denote the set of all smooth functions $f: [-1/2, 1/2]\to R$ satisfying $f(-x)=f(x)$, endowed with C^∞ topology. Let T: $C_e^\infty([-1/2, 1/2], R)\to C^\infty([0, 1/4], R)$ denote a map defined by $T(f)(x)=f(\sqrt{x})$. Then we have

Lemma 2.4. *The above map T is well defined and continuous.*

Proof. Put $f=T(\hat{f})$ for each $\hat{f}\in C_e^\infty([-1/2, 1/2], R)$. Since $\hat{f}$ is a C^∞ even function, we have the Taylor expansion

$$\hat{f}(x)=\hat{f}(0)+(\hat{f}''(0)/2)x^2+\cdots+(\hat{f}^{(2n-2)}(0)/(2n-2)!)x^{2n-2} +\left(\int_0^1((1-t)^{2n-1}/(2n-1)!)\hat{f}^{(2n)}(tx)dt\right)x^{2n}$$

for $-1/2\leq x\leq 1/2$, $n=1, 2,\dots$. Thus we have

$$f(x)=\hat{f}(0)+(\hat{f}''(0)/2)x+\cdots+(\hat{f}^{(2n-2)}(0)/(2n-2)!)x^{n-1} +\left(\int_0^1((1-t)^{2n-1}/(2n-1)!)\hat{f}^{(2n)}(t\sqrt{x})dt\right)x^n$$

for $0\leq x\leq 1/4$. By the composite mapping formula, we can compute the n-th derivative

$$D^n(\hat{f}^{(2n)}(t\sqrt{x})x^n) =\sum_{p=0}^n\sum_{q=0}^p\sum_{\substack{i_1+\cdots+i_q=p\\ i_1>0,\dots,i_q>0}} B(p, i_1,\dots, i_q)\hat{f}^{(2n+q)}(t\sqrt{x})x^{q/2}t^q,$$

where $B(p, i_1,\dots, i_q)$ is a real number. Put $f_i=T(\hat{f}_i)$ for $\hat{f}_i\in C_e^\infty([-1/2, 1/2], R)$

$(i=1, 2)$. Then there exists a positive number A_n such that

$$\sup_{0\leq x\leq 1/4} |D^n f_1(x) - D^n f_2(x)| \\ \leq A_n \cdot \max_{0\leq q\leq 3n} (\sup_{-1/2\leq x\leq 1/2} |D^q \hat{f}_1(x) - D^q \hat{f}_2(x)|)$$

for each $n=1, 2,\dots$. Note that

$$\sup_{0\leq x\leq 1/4} |f_1(x) - f_2(x)| = \sup_{-1/2\leq x\leq 1/2} |\hat{f}_1(x) - \hat{f}_2(x)|.$$

Therefore T is a continuous map, and this completes the proof of Lemma 2.4.

Proof of Proposition 2.1. Let J denote a closed interval $[0, 1/4]$, $[1/5, 4/5]$ or $[3/4, 1]$. By Lemma 2.3, it is sufficient to show that the composition P_J: $\mathrm{Diff}^\infty_G(M)_0 \xrightarrow{P} \mathrm{Diff}^\infty[0, 1] \xrightarrow{j^*} C^\infty(J, [0, 1])$ is continuous, where $j: J \hookrightarrow [0, 1]$ is an inclusion map.

We shall first consider the case $J=[0, 1/4]$. Let $\iota: [-1/2, 1/2] \to [0, 1/4]$ be a map given by $\iota(x)=x^2$. Let $\hat{\iota}: [-1/2, 1/2] \to G \times_{K_0} D(V_0) \hookrightarrow M$ be a map given by $\hat{\iota}(r)=[1, re_0]$, where e_0 is a point of $S(V_0)$ as in §1. Then $\pi\circ\hat{\iota}=\iota$. Let $\hat{P}_J$ denote the composition $\mathrm{Diff}^\infty_G(M)_0 \xrightarrow{\hat{\iota}^*} C^\infty([-1/2, 1/2], M) \xrightarrow{\pi_*} C^\infty([-1/2, 1/2], [0, 1])$. Then $\hat{P}_J(h)=\pi\circ h\circ\hat{\iota}=P(h)\circ\iota=\iota^*P(h)$ for $h\in \mathrm{Diff}^\infty_G(M)_0$, and the image of $\hat{P}_J$ is contained in $C^\infty_e([-1/2, 1/2], R)$. Note that $P_J=T\circ\hat{P}_J$. Combining Lemma 2.2 and Lemma 2.4, P_J is continuous.

Next consider the case $J=[1/5, 4/5]$. By Lemma 1.2, there is a G-diffeomorphism $\alpha: \pi^{-1}([1/5, 4/5]) \to G/H \times [1/5, 4/5]$. Let $i: \pi^{-1}([1/5, 4/5]) \hookrightarrow M$ be the inclusion map and let $k: [1/5, 4/5] \to G/H \times [1/5, 4/5]$ be a map given by $k(r)=(1H, r)$. Then P_J is the composition

$$\mathrm{Diff}^\infty_G(M)_0 \xrightarrow{(i\circ\alpha^{-1}\circ k)^*} C^\infty([1/5, 4/5], M) \xrightarrow{\pi_*} C^\infty([1/5, 4/5], [0, 1])$$

which is continuous by Lemma 2.2.

We can prove that P_J is continuous in the case $J=[3/4, 1]$ similarly as in the case $J=[0, 1/4]$, and this completes the proof of Proposition 2.1.

§3. A Continuous Global Section of P

In Section 2 we have proved that P: $\mathrm{Diff}^\infty_G(M)_0 \to \mathrm{Diff}^\infty[0, 1]$ is continuous. Thus the image of P is contained in the connected component $\mathrm{Diff}^\infty[0, 1]_0$ of the identity. In this section we shall construct a continuous global section of P: $\mathrm{Diff}^\infty_G(M)_0 \to \mathrm{Diff}^\infty[0, 1]_0$.

Let f be an element of $\mathrm{Diff}^\infty[0, 1]_0$. We shall define a map $\Psi(f): M \to M$ as follows: $\Psi(f)$ is defined on $\pi^{-1}((0, 1))$ by the composition $\pi^{-1}((0, 1)) \xrightarrow{\alpha}$

$G/H\times(0,1)\xrightarrow{(1,f)} G/H\times(0,1)\xrightarrow{\alpha^{-1}}\pi^{-1}((0,1))$, and $\Psi(f)=1$ on $\pi^{-1}(0)\cup\pi^{-1}(1)$.

Proposition 3.1. *$\Psi(f)$ is a G-diffeomorphism of M.*

In order to prove Proposition 3.1, we need the following lemma and notations.

Lemma 3.2. *Let Ψ_1: $\mathrm{Diff}^\infty[0,1]_0\to\mathrm{Diff}^\infty(D^n)$ be a map defined by*

$$\Psi_1(f)(v)=\begin{cases}(\sqrt{f(\|v\|^2)}/\|v\|)v & \text{for } v\neq 0,\\ 0 & \text{for } v=0,\end{cases}$$

where D^n denotes an n-dimensional unit disc. Then Ψ_1 is well defined and continuous.

Notations 3.3. *For $i=0, 1$, we shall use the following notations*

π_i: $G\to G/K_i$ *the natural projection,*

U_i *an open disc neighborhood of $1K_i$ in G/K_i,*

σ_i: $U_i\to G$ *a smooth local cross section of π_i,*

$\sigma_{i,a}$: $aU_i\to G$ $(a\in G)$ *a smooth local cross section of π_i defined by*

$$\sigma_{i,a}(x)=a\cdot\sigma_i(a^{-1}x).$$

Put $M_i=G\times_{K_i}D(V_i)$ and $M_i(r)=G\times_{K_i}D_r(V_i)$, where $D_r(V_i)$ is a closed r-disc in V_i $(0<r\leq 1)$.

p_i: $M_i\to G/K_i$, $p_{i,r}$: $M_i(r)\to G/K_i$ *the bundle projections,*

$\phi_{i,a}$: $p_i^{-1}(aU_i)\to U_i\times D(V_i)$ $(a\in G)$ *a chart of p_i over aU_i defined by*

$$\phi_{i,a}([g,v])=(a^{-1}\pi_i(g),\ ((\sigma_{i,a}\circ\pi_i)(g))^{-1}g\cdot v),$$

π_2: $G\to G/H$ *the natural projection,*

U_2 *an open disc neighborhood of $1H$ in G/H,*

σ_2: $U_2\to G$ *a smooth local cross section of π_2.*

Proof of Proposition 3.1. Put $h=\Psi(f)$. We shall first prove that h is smooth on $\pi^{-1}(0)$. Since $f(0)=0$, there exists a real number ε such that $0<\varepsilon\leq 1/2$ and $f(\varepsilon^2)\leq 1/4$. Then $h(\pi^{-1}([0,\varepsilon^2]))\subset\pi^{-1}([0,1/4])$, and $h(M_0(\varepsilon))\subset M_0(1/2)$. For $[g, re_0]\in G\times_{K_0}D_\varepsilon(V_0-0)$ $(0<r\leq\varepsilon)$, $h([g, re_0])=(\alpha^{-1}\circ(1,f)\circ\alpha)([g, re_0])=(\alpha^{-1}\circ(1,f))(gH, r^2)=\alpha^{-1}(gH, f(r^2))=[g, \sqrt{f(r^2)}e_0]$. Then, for $[g,v]\in G\times_{K_0}D_\varepsilon(V_0-0)$, $h([g,v])=[g, \sqrt{f(\|v\|^2)}/\|v\|\ v]=[g, \Psi_1(f)(v)]$. Since $h([g,0])=[g,0]$, $h([g,v])=[g,\Psi_1(f)(v)]$ for any $[g,v]\in M_0(\varepsilon)$. Then the composition

$$\begin{aligned}\tilde{h}: U_0\times D_\varepsilon(V_0) &\xrightarrow{(\phi_{0,a})^{-1}} p_{0,\varepsilon}^{-1}(aU_0)\\ &\xrightarrow{h} p_{0,1/2}^{-1}(aU_0)\\ &\xrightarrow{\phi_{0,a}} U_0\times D_{1/2}(V_0)\end{aligned}$$

is given by $\tilde{h}(x, v)=(x, \Psi_1(f)(v))$ for $a \in G$. Since $\Psi_1(f)$ is a smooth map by Lemma 3.2, h is smooth on $\pi^{-1}(0)$. Similarly we can prove that h is smooth on $\pi^{-1}(1)$. Since h is smooth on $\pi^{-1}((0, 1))$ by the definition, h is a smooth map. Since $h^{-1}=\Psi(f^{-1})$, h^{-1} is also a smooth map. Thus h is a G-diffeomorphism of M, and this completes the proof of Proposition 3.1.

In order to prove Lemma 3.2, we need the following assertion.

Assertion 3.4. *Let* Φ: $\mathrm{Diff}^\infty[0, 1]_0 \to C^\infty([0, 1], R)$ *be a map given by*

$$\Phi(f)(x)=\begin{cases} \sqrt{f(x)/x} & \text{for } x\neq 0\,, \\ \sqrt{f'(0)} & \text{for } x=0\,. \end{cases}$$

Then Φ *is well defined and continuous.*

Proof. For $f \in \mathrm{Diff}^\infty[0, 1]_0$, we have the Taylor expansion

$$f(x)=f'(0)x+x^2\int_0^1(1-t)f''(tx)dt \qquad \text{for} \quad 0\le x\le 1.$$

Put $F(x)=f'(0)+x\int_0^1(1-t)f''(tx)dt$ for $0\le x\le 1$. Then $\Phi(f)=\sqrt{F}$. Note that $F(x)>0$ for $0\le x\le 1$. It is easy to see that Φ is continuous.

Proof of Lemma 3.2. Let N: $D^n\to[0, 1]$ be a map given by $N(v)=\|v\|^2$. Let i: $D^n \hookrightarrow R^n$ be the inclusion and let μ: $R\times R^n\to R^n$ be the scalar multiplication. Since $\Psi_1(f)(v)=\Phi(f)(\|v\|^2)v$, $\Psi_1(f)$ is a smooth map by Assertion 3.4. Since $\Psi_1(f^{-1})=\Psi_1(f)^{-1}$, $\Psi_1(f)^{-1}$ is also a smooth map. Thus $\Psi_1(f)$ is a diffeomorphism of D^n. Note that Ψ_1 is the composition $\mathrm{Diff}^\infty[0, 1]_0 \xrightarrow{\Phi} C^\infty([0, 1], R) \xrightarrow{N^*} C^\infty(D^n, R) \xrightarrow{i_\#} C^\infty(D^n, R\times R^n) \xrightarrow{\mu_*} C^\infty(D^n, R^n)$. Combining Assertion 3.4 and Lemma 2.2, Ψ_1 is continuous. This completes the proof of Lemma 3.2.

Proposition 3.5. Ψ: $\mathrm{Diff}^\infty[0, 1]_0 \to \mathrm{Diff}^\infty_G(M)$ *is continuous.*

Proof. Let $B_i\subset U_i$ be a closed disc neighborhood of $1K_i$ in G/K_i for $i=0, 1$. Let $B_2\subset U_2$ be a closed disc neighborhood of $1H$ in G/H. We can choose $\{\mathrm{int}\,(p_{0,\varepsilon}^{-1}(aB_0)), \mathrm{int}\,(p_{1,\varepsilon}^{-1}(aB_1)), \mathrm{int}\,(\alpha^{-1}(aB_2\times[\varepsilon/2, 1-\varepsilon/2]))\,;\ a\in G\}$ as an open covering of M for $0<\varepsilon<1/2$. Put $W=\{f\in \mathrm{Diff}^\infty[0, 1]_0\,; f([0, \varepsilon^2])\subset [0, 1/4), f([1-\varepsilon^2, 1])\subset(3/4, 1]\}$. Then W is an open neighborhood of the identity in $\mathrm{Diff}^\infty[0, 1]_0$. Since Ψ is a homomorphism as an abstract group, it is enough to show that Ψ is continuous on W. Let C denote one of the sets $p_{0,\varepsilon}^{-1}(aB_0)$, $p_{1,\varepsilon}^{-1}(aB_1)$ or $\alpha^{-1}(aB_2\times[\varepsilon/2, 1-\varepsilon/2])$ for $a\in G$. If we can prove that the composition

$$\Psi_C\colon W \xrightarrow{\Psi} \mathrm{Diff}^\infty_G(M)_0 \xrightarrow{i^*} C^\infty(C, M)$$

is continuous for each C, then Ψ is continuous on W by Lemma 2.3, where $i\colon C \hookrightarrow M$ is an inclusion map.

First consider in the case $C = p_{0,\varepsilon}^{-1}(aB_0)$. $\Psi(f)(C)$ is contained in $p_{0,1/2}^{-1}(aU_0)$ for each $f \in W$. Note that $\Psi(f)([g, v]) = [g, \Psi_1(f)(v)]$ for $[g, v] \in C$ and $(\phi_{0,a} \circ \Psi(f) \circ \phi_{0,a}^{-1})(x, v) = (x, \Psi_1(f)(v))$ for $(x, v) \in B_0 \times D_\varepsilon(V_0)$. Thus Ψ_C is given by the composition

$$\begin{aligned} W &\xrightarrow{\Psi_1} C^\infty(D_\varepsilon(V_0), D(V_0)) \\ &\xrightarrow{j_!} C^\infty(B_0 \times D_\varepsilon(V_0), U_0 \times D(V_0)) \\ &\xrightarrow{\phi_{0,a}^*} C^\infty(C, U_0 \times D(V_0)) \\ &\xrightarrow{(k \circ \phi_{0,a}^{-1})_*} C^\infty(C, M), \end{aligned}$$

where $j\colon B_0 \hookrightarrow U_0$ and $k\colon p_0^{-1}(aU_0) \hookrightarrow M$ are inclusions. Combining Lemma 3.2 and Lemma 2.2, Ψ_C is continuous.

Now consider the case $C = \alpha^{-1}(B_0 \times [\varepsilon/2, 1-\varepsilon/2])$. Ψ_C is given by the composition

$$\begin{aligned} W &\xrightarrow{\iota^*} C^\infty([\varepsilon/2, 1-\varepsilon/2], (0, 1)) \\ &\xrightarrow{j_!} C^\infty(B_0 \times [\varepsilon/2, 1-\varepsilon/2], G/H \times (0, 1)) \\ &\xrightarrow{\alpha^*} C^\infty(C, G/H \times (0, 1)) \\ &\xrightarrow{(k \circ \alpha^{-1})_*} C^\infty(C, M), \end{aligned}$$

where $\iota\colon [\varepsilon/2, 1-\varepsilon/2] \hookrightarrow [0, 1]$, $j\colon B_0 \hookrightarrow G/H$ and $k\colon \pi^{-1}((0, 1)) \hookrightarrow M$ are inclusion maps. By Lemma 2.2, Ψ_C is continuous.

We can prove that Ψ_C is continuous in the case $C = p_{1,\varepsilon}^{-1}(aB_1)$ similarly as in the case $C = p_{0,\varepsilon}^{-1}(aB_0)$, and this completes the proof of Proposition 3.5.

By Proposition 3.5, $P\colon \mathrm{Diff}^\infty_G(M)_0 \to \mathrm{Diff}^\infty[0, 1]_0$ is a globally trivial fibration. Then we have

Corollary 3.6. $\mathrm{Diff}^\infty_G(M)_0$ *is homeomorphic to* $\mathrm{Diff}^\infty[0, 1]_0 \times \mathrm{Ker}\, P$.

§4. On the Group Ker P

In this section we shall define a group homomorphism $L\colon \mathrm{Ker}\, P \to Q$, where Q is a subgroup of $C^\infty([0, 1], N(H)/H)$, and we shall prove that L is a group monomorphism between topological groups (see Lemma 4.5 and Proposition 4.6).

Let h be an element of Ker P. Let $\hat{h}$ be the composition

$$G/H\times(0, 1) \xrightarrow{\alpha^{-1}} \pi^{-1}((0, 1)) \xrightarrow{h} \pi^{-1}((0, 1)) \xrightarrow{\alpha} G/H\times(0, 1).$$

Then $\hat{h}$ is a level preserving G-diffeomorphism. Let $a: (0, 1)\to N(H)/H$ be a smooth map satisfying $h(gH, r)=(ga(r), r)$ for $(gH, r)\in G/H\times(0, 1)$.

Proposition 4.1. *With the above notations, there exists a smooth map* $\bar{a}: [0, 1]\to N(H)/H$ *such that*

(1) $\bar{a}=a$ *on* $(0, 1)$,

(2) $\bar{a}(i)\in(N(H)\cap N(K_i))/H$ *for* $i=0, 1$.

To prove Proposition 4.1, we need the following lemmas.

Lemma 4.2. *Let G be a compact Lie group. Let K and N be closed subgroups of G. Let $\pi: G\to G/K$ be the natural projection. Then there exists a smooth local section σ of π, which is defined on an open neighborhood U of $1K$, such that $\sigma(1K)=1$ and $\sigma(x)\in N$ for $x\in\pi(N)\cap U$.*

Proof. Let $\pi_1: N\to N/(N\cap K)$ be a natural projection. Let $i: N\hookrightarrow G$ be the inclusion and let $I: N/(N\cap K)\to G/K$ be a map satisfying $\pi\circ i=I\circ\pi_1$. Since $I(N/(N\cap K))=\pi(N)$ is an orbit of the natural action $N\times G/K\to G/K$, I is an imbedding. Let U be a disc neighborhood around $\pi(1)$ in G/K and let U_1 be a disc neighborhood around $\pi_1(1)$ in $N/(N\cap K)$. Since I is an imbedding, we can assume $I(U_1)=U\cap I(N/(N\cap K))=U\cap\pi(N)$. Let $\sigma_1: U_1\to N$ be a smooth local section of π_1 satisfying $\sigma_1(\pi(1))=1$. Then $\sigma_1\circ I^{-1}$ is a smooth section defined on $I(U_1)$. We can extend $\sigma_1\circ I^{-1}$ to a smooth local section defined on U. Then $\sigma(\pi(1))=1$ and $\sigma(U\cap\pi(N))\subset N$. This completes the proof of Lemma 4.2.

Lemma 4.3. *Let G be a compact connected Lie group. Let V be a representation of G such that G acts transitively and effectively on a unit sphere $S(V)$ of V. Let H be the isotropy subgroup of a point of $S(V)$. Then we have the following list:*

G	$SO(n)$ $(n\geq3)$	$SU(n)$ $(n\geq2)$	$U(n)$ $(n\geq1)$	$Sp(n)$ $(n\geq1)$	$Sp(n)\times_{Z_2}S^3$ $(n\geq1)$
H	$SO(n-1)$	$SU(n-1)$	$U(n-1)$	$Sp(n-1)$	H_1
$N(H)/H$	Z_2	$U(1)$	$U(1)$	$Sp(1)$	Z_2

$Sp(n)\times_{Z_2}S^1$ $(n\geq 1)$	G_2	Spin(7)	Spin(9)
H_2	$SU(3)$	G_2	Spin(7)
S^1	Z_2	Z_2	Z_2

where $H_1=\{[(q, A), q^{-1}]\in Sp(n)\times_{Z_2}S^3;\ (q, A)\in Sp(1)\times Sp(n-1)\subset Sp(n)\}$ *and* $H_2=\{[(z, A), z^{-1}]\in Sp(n)\times_{Z_2}S^1;\ (z, A)\in S^1\times Sp(n-1)\subset Sp(n)\}$.

Proof. It is known that G and H are the above Lie groups (c.f. W. C. Hsiang and W. Y. Hsiang [7, § 1]). We can determine the Lie group $N(H)/H$ by an immediate calculation except for $G=G_2$, Spin (7), Spin (9). For the cases $G=G_2$, Spin (7), Spin (9), we can determine $N(H)/H$ by using I. Yokota's definitions of these Lie groups in [9, Chapter 5].

Lemma 4.4. (1) *Let* $F: [-1, 1]\to R$ *be a smooth function such that* $F(0)=0$. *Put* $f(x)=F(x)/x$ *for* $x\neq 0$ *and* $f(x)=F'(0)$ *for* $x=0$. *Then* $f: [-1, 1]\to R$ *is a well defined smooth function.*

(2) *Put* $C_0^\infty([-1, 1], R)=\{F\in C^\infty([-1, 1], R);\ F(0)=0\}$, *endowed with* C^∞ *topology. Let* $\Phi: C_0^\infty([-1, 1], R)\to C^\infty([-1, 1], R)$ *be a map given by* $\Phi(F)(x)=f(x)$. *Then* Φ *is continuous.*

Proof. For $F\in C_0^\infty([-1, 1], R)$, we have $\Phi(F)(x)=f(x)=F'(0)+x\int_0^1(1-t)F''(tx)dt$. Then the n-th derivative $f^{(n)}(x)=x\int_0^1(1-t)t^nF^{(n+2)}(tx)dt+n\int_0^1(1-t)t^{n-1}F^{(n+1)}(tx)dt$. Thus there exists a positive number A such that $\|\Phi(F)\|_n\leq A\|F\|_{n+2}$, and Lemma 4.4 follows.

Proof of Proposition 4.1. Let ε $(0<\varepsilon\leq 1/2)$ be a real number. Let W_i and U_i be open neighborhoods of $1K_i$, satisfying $\overline{W}_i\subset U_i$, for $i=0, 1$. Put $O=\{h\in \mathrm{Ker}\ P;\ h(p_{i,\varepsilon}^{-1}(\overline{W}_i))\subset p_{i,\varepsilon}^{-1}(U_i)$ for $i=0, 1\}$. Then O is an open neighborhood of the identity in Ker P. By Corollary 3.6, Ker P is connected, and O generates the topological group Ker P. Thus we can assume $h\in O$.

Let $\tilde{h}$ be the composition

$$W_0\times D_\varepsilon(V_0)\xrightarrow{(\phi_{0,1})^{-1}}p_{0,\varepsilon}^{-1}(W_0)\xrightarrow{h}p_{0,\varepsilon}^{-1}(U_0)\xrightarrow{\phi_{0,1}}U_0\times D_\varepsilon(V_0).$$

Let $\rho_1: U_0\times D_\varepsilon(V_0)\to U_0$ and $\rho_2: U_0\times D_\varepsilon(V_0)\to D_\varepsilon(V_0)$ be projections on the first factor and the second factor, respectively. Let $i: [-\varepsilon, \varepsilon]\to W_0\times D_\varepsilon(V_0)$ be an imbedding given by $i(r)=(1K_0, re_0)$. Then the compositions $\tilde{h}_1=\rho_1\circ\tilde{h}\circ i: [-\varepsilon, \varepsilon]\to U_0$ and $\tilde{h}_2=\rho_2\circ\tilde{h}\circ i: [-\varepsilon, \varepsilon]\to D_\varepsilon(V_0)$ are smooth maps. Let $\bar{\pi}_0: G/H\to G/K_0$ be the natural projection. Note that

$$\begin{aligned}(\alpha\circ h\circ\phi_{0,1}^{-1})(1K_0, re_0) &= (\alpha\circ h)([1, re_0]) \\ &= (\hat{h}\circ\alpha)([1, re_0]) \\ &= \hat{h}(1H, r^2) \\ &= (a(r^2), r^2) \qquad \text{for} \quad |r|\le\varepsilon, r\ne 0.\end{aligned}$$

Then

$$\begin{aligned}\tilde{h}(1K_0, re_0) &= (\phi_{0,1}\circ\alpha^{-1})(a(r^2), r^2) \\ &= (\bar{\pi}_0(a(r^2)), \quad (\sigma_0\circ\bar{\pi}_0)(a(r^2))^{-1}\cdot a(r^2)\cdot re_0),\end{aligned}$$

and

$$\begin{aligned}&\tilde{h}_1(r) = \bar{\pi}_0(a(r^2)), \\ &\tilde{h}_2(r) = (\sigma_0\circ\bar{\pi}_0)(a(r^2))^{-1}\cdot a(r^2)\cdot re_0,\end{aligned}$$

for $|r|\le\varepsilon$, $r\ne 0$.

Here we can assume that $\sigma_0(1K_0)=1$ and $\sigma_0(\pi_0(N(H))\cap U_0)\subset N(H)$ by Lemma 4.2. Let $b\colon[-\varepsilon, \varepsilon]\to G$ be a smooth map given by $b(r)=\sigma_0(\tilde{h}_1(r))$. Then $b(r)=\sigma_0(\bar{\pi}_0(a(r^2)))\in\sigma_0(\pi_0(N(H))\cap U_0)$, and $b(r)\in N(H)$ for $r\ne 0$. Since b is a smooth map, $b(r)\in N(H)$ for $r=0$. For $[1, 0]\in\pi^{-1}(0)$, we have $h([1, 0]) =(h\circ\phi_{0,1}^{-1})(1K_0, 0)=(h\circ\phi_{0,1}^{-1})(i(0))=\phi_{0,1}^{-1}(\tilde{h}_1(0), 0)=[b(0), 0]$. Note that p_0 is a G-diffeomorphism on the zero section of p_0 and $h(\pi^{-1}(0))=\pi^{-1}(0)$. Then the composition $p_0\circ h\circ p_0^{-1}\colon G/K_0\to G/K_0$ is a G-diffeomorphism, and $(p_0\circ h\circ p_0^{-1})(1K_0)=(p_0\circ h)([1, 0])=p_0([b(0), 0])=b(0)K_0$. Thus $b(0)\in N(K_0)$, and $b(0)\in N(H)\cap N(K_0)$.

Put $J=[-\varepsilon, 0)\cup(0, \varepsilon]$. Let $c\colon J\to N(H)/H$ be a smooth map given by $c(r)=b(r)^{-1}\cdot a(r^2)$. Since $\bar{\pi}_0(c(r))=\bar{\pi}_0(\sigma_0(\bar{\pi}_0(a(r^2)))^{-1}\cdot a(r^2))=1K_0$, then $c(r)\in K_0/H$. Thus $c(r)\in N(H, K_0)/H$ for $r\in J$. Since Ker P is connected, the maps a, b and c are homotopic to the constant maps. Note that the identity component $(N(H, K_0)/H)^0$ of $N(H, K_0)/H$ is contained in $(N(H, K_0)\cap K_0^0)\cdot H/H$, and there exists an isomorphism $(N(H, K_0)\cap K_0^0)\cdot H/H\simeq(N(H, K_0)\cap K_0^0)/(H\cap K_0^0)$ as a Lie group, where K_0^0 is the identity component of K_0. Then there exists a smooth map $\hat{c}\colon J\xrightarrow{c}(N(H, K_0)/H)^0\hookrightarrow(N(H, K_0)\cap K_0^0)\cdot H/H\simeq(N(H, K_0)\cap K_0^0)/(H\cap K_0^0)\hookrightarrow N(H\cap K_0^0, K_0^0)/(H\cap K_0^0)$. Now we shall prove that $\hat{c}$ can be extended to a smooth map on $[-\varepsilon, \varepsilon]$, and so is c.

Note that K_0 acts transitively on the unit sphere $S(V_0)$ of V_0. If $\dim S(V_0)=0$, then $K_0/H=Z_2$ and $N(H, K_0)/H=Z_2$. In this case $\hat{c}$ is a trivial map, and it is clear that $\hat{c}$ can be extended to a smooth map on $[-\varepsilon, \varepsilon]$. Now we assume $\dim S(V_0)>0$. Since $S(V_0)$ is connected, K_0^0 acts transitively on $S(V_0)$ and $K_0^0/(K_0^0\cap H)$ is diffeomorphic to $S(V_0)$. Put $D=\bigcap_{g\in K_0^0} g(K_0^0\cap H)g^{-1}$ which is the kernel of the action $K_0^0 \times S(V_0) \to S(V_0)$. Put $\bar{K}_0 = K_0^0/D$ and $\bar{H} = (H \cap K_0^0)/D$. Then $\bar{K}_0$ acts transitively and effectively on $S(V_0)$ and $\bar{K}_0/\bar{H}$ is diffeomorphic to $S(V_0)$. Put $\bar{N}_0 = N(\bar{H}, \bar{K}_0)/\bar{H}$ which is isomorphic to $N(H \cap K_0^0, K_0^0)/(H \cap K_0^0)$ as a Lie group. The pair $(\bar{K}_0, \bar{N}_0)$ is one of pairs $(G, N(H)/H)$ in the list of Lemma 4.3. Now we shall prove that $\hat{c}$ can be extended to a smooth map on $[-\varepsilon, \varepsilon]$. If $\bar{N}_0 = Z_2$, this is clear since $\hat{c}$ is a trivial map.

Consider the case $\bar{K}_0 = SU(n)$ $(n \geq 1)$ and $\bar{N}_0 = U(1)$. In this case V_0 is an n-dimensional complex vector space and $\bar{N}_0 = U(1)$ acts on V_0 as a scalar multiplication. We can regard C^n as a $2n$-dimensional real vector space R^{2n} and $\bar{N}_0$ as $SO(2)$. Then there exist smooth functions $c_i: J \to R$, $i = 1, 2$, such that

$$\hat{c}(r) = \begin{bmatrix} c_1(r) & -c_2(r) \\ c_2(r) & c_1(r) \end{bmatrix} \in SO(2) \qquad \text{for} \quad r \in J.$$

Note that $\tilde{h}_2: [-\varepsilon, \varepsilon] \to D_\varepsilon(V_0)$ is a smooth map and $\tilde{h}_2(r) = c(r) \cdot re_0 = \hat{c}(r) \cdot re_0$ for $r \neq 0$. In this case $e_0 = (1, 0, \dots, 0) \in S^{2n-1}$ and $\tilde{h}_2(r) = (c_1(r)r, c_2(r)r, 0, \dots, 0)$ for $r \in J$. Put $c_i(0) = \lim_{r \to 0} c_i(r)$ for $i = 1, 2$. From Lemma 4.4, $c_i: [-\varepsilon, \varepsilon] \to R$, $i = 1, 2$, are smooth functions and $\hat{c}$ can be extended to a smooth map on $[-\varepsilon, \varepsilon]$.

Now consider the case $\bar{K}_0 = Sp(n)$ $(n \geq 1)$ and $\bar{N} = Sp(1)$. In this case V_0 is an n-dimensional quaternionic vector space H^n and $\bar{N}_0 = Sp(1)$ acts on V_0 as a scalar multiplication. We can regard H^n as R^{4n} and $Sp(1)$ as a subgroup of $SO(4)$ naturally. By the similar way as in the case $K_0 = SU(n)$, there exist smooth functions $c_i: J \to R$, $i = 1, 2, 3, 4$, such that $h_2(r) = (c_1(r)r, c_2(r)r, c_3(r)r, c_4(r)r, 0, \dots, 0)$ for $r \in J$, and we can extend $\hat{c}$ to a smooth map on $[-\varepsilon, \varepsilon]$.

The proofs of the other cases are similar to those of the above cases. Thus we can extend c to a smooth map on $[-\varepsilon, \varepsilon]$. Since $c(r) \in N(H, K_0)/H$ for $r \neq 0$, we see $c(0) \in N(H, K_0)/H$. Put $\bar{a}(0) = b(0) \cdot c(0)$. Since $b(0) \in N(H) \cap N(K_0)$ and $c(0) \in N(H, K_0)/H$, we have $\bar{a}(0) \in (N(H) \cap N(K_0))/H$. Let $\hat{a}$: $[-1/2, 1/2] \to N(H)/H$ be a map given by $\hat{a}(r) = \bar{a}(r^2)$. Since $\hat{a}(r) = b(r) \cdot c(r)$ for $-\varepsilon \leq r \leq \varepsilon$, $\hat{a}$ is a smooth map. Since $\hat{a}$ is an even map and $\bar{a}(r) = \hat{a}(\sqrt{r})$ for $0 \leq r \leq 1/4$, $\bar{a}$ is a smooth map on $[0, 1/4]$ by Lemma 2.4. Thus we can extend the map a to a smooth map $\bar{a}$ on $[0, 1)$ satisfying $\bar{a}(0) \in (N(H) \cap N(K_0))/H$. Similarly we can extend a to a smooth map $\bar{a}$ on $[0, 1]$ satisfying $\bar{a}(1) \in (N(H) \cap N(K_1))/H$. This completes the proof of Proposition 4.1.

Let Q denote the set of smooth maps $f: [0, 1] \to N(H)/H$ satisfying $f(i) \in (N(H) \cap N(K_i))/H$ for $i = 0, 1$, endowed with C^∞ topology. Using Proposition

4.1, we define a map L: Ker $P \to Q$ by $L(h) = \bar{a}^{-1}$.

Lemma 4.5. *L: Ker $P \to Q$ is a group monomorphism.*

Proof. Let $h_i \in \text{Ker}\, P$ for $i=1, 2$. For $0<r<1$ and $g \in G$, we have

$$\begin{aligned}(g \cdot L(h_2 \circ h_1)(r)^{-1}, r) &= (\alpha \circ h_2 \circ h_1 \circ \alpha^{-1})(gH, r) \\ &= ((\alpha \circ h_2 \circ \alpha^{-1}) \circ (\alpha \circ h_1 \circ \alpha^{-1}))(gH, r) \\ &= (\alpha \circ h_2 \circ \alpha^{-1})(g \cdot L(h_1)(r)^{-1}, r) \\ &= (g \cdot L(h_1)(r)^{-1} \cdot L(h_2)(r)^{-1}, r).\end{aligned}$$

Thus $L(h_2 \circ h_1) = L(h_2) \cdot L(h_1)$ on $(0, 1)$. Since $L(h_1)$, $L(h_2)$ and $L(h_1 \circ h_2)$ are smooth maps on $[0, 1]$, $L(h_2 \circ h_1) = L(h_2) \cdot L(h_1)$ on $[0, 1]$. Thus L is a group homomorphism. Suppose $L(h) = 1$ for $h \in \text{Ker}\, P$. Then $(h \circ \alpha^{-1})(gH, r) = \alpha^{-1}(gH, r)$ for $g \in G$ and $0<r<1$, and $h = 1$ on $\pi^{-1}((0, 1))$. Thus $h = 1$ on M, and L is a monomorphism.

Proposition 4.6. *L is a continuous map.*

Proof. We shall use the notations in the proof of Proposition 4.1. Since L is a group homomorphism, it is sufficient to show that L: $O \to Q$ is continuous. Let I denote a closed interval $[0, \varepsilon^2]$, $[\varepsilon^2/2, 1-\varepsilon^2/2]$ or $[1-\varepsilon^2, 1]$. By Lemma 2.3, it is sufficient to prove that L_I: $O \xrightarrow{L} Q \xrightarrow{j^*} C^\infty(I, N(H)/H)$ is continuous, where j: $I \hookrightarrow [0, 1]$ is an inclusion map.

First we shall consider the case $I = [0, \varepsilon^2]$. Let L_1 be the composition

$$\begin{aligned}O &\xrightarrow{(k \circ \phi_{0,1}^{-1} \circ i)^*} C^\infty([-\varepsilon, \varepsilon], p_{0,\varepsilon}^{-1}(U_0)) \\ &\xrightarrow{(\sigma_0 \circ \rho_1 \circ \phi_{0,1})_*} C^\infty([-\varepsilon, \varepsilon], G),\end{aligned}$$

where k: $p_{0,\varepsilon}^{-1}(\overline{W}_0) \hookrightarrow M$ is an inclusion map. Then L_1 is continuous by Lemma 2.2. Note that $L_1(h) = b$.

Let L_2: $O \to C^\infty([-\varepsilon, \varepsilon], (N(H, K_0)/H)^0)$ be a map given by $L_2(h) = c$. We shall prove that L_2 is continuous. This is trivial in the case $N(H, K_0)/H = Z_2$. Consider the case $\overline{K}_0 = SU(n)$ $(n \geq 2)$. In this case $V_0 = C^n = R^{2n}$ and $\overline{N}_0 = U(1) = SO(2)$. Put $C_0^\infty([-\varepsilon, \varepsilon], V_0) = \{F \in C^\infty([-\varepsilon, \varepsilon], V_0);\ F(0) = 0\}$, endowed with C^∞ topology. Let Φ: $C_0^\infty([-\varepsilon, \varepsilon], V_0) \to C^\infty([-\varepsilon, \varepsilon], R^2)$ be a map defined by $\Phi(F) = (\Phi(F^1), \Phi(F^2))$, where $F = (F^1, \ldots, F^{2n})$ and $\Phi(F^i)$ is a map defined in Lemma 4.4. Then Φ is continuous by Lemma 4.4. Let m: $R^2 \to M_2(R)$ denote a smooth map defined by

$$m(x, y) = \begin{bmatrix} x & -y \\ y & x \end{bmatrix},$$

where $M_2(R)$ denote the set of all 2×2 matrices over R. Let L_2' denote the composition

$$O \xrightarrow{(k\circ\phi_{0,1}^{-1}\circ i)^*} C^\infty([-\varepsilon, \varepsilon], p_{0,1}^{-1}(U_0)) \xrightarrow{(p_2\circ\phi_{0,1})_*} C^\infty([-\varepsilon, \varepsilon], D_\varepsilon(V_0)).$$

From Lemma 2.2, L_2' is continuous. Note that $L_2'(h)=\tilde{h}_2$ and $L_2'(O)$ is contained in $C_0^\infty([-\varepsilon, \varepsilon], V_0)$. Let $\hat{L}_2$ denote the composition

$$O \xrightarrow{L_2'} C_0^\infty([-\varepsilon, \varepsilon], V_0) \xrightarrow{\Phi} C^\infty([-\varepsilon, \varepsilon], R^2) \xrightarrow{m_*} C^\infty([-\varepsilon, \varepsilon], M_2(R)).$$

Then $\hat{L}_2(h)=\hat{c}$ and $\hat{L}_2$ is continuous. This implies that L_2 is continuous by using Lemma 2.2. Similarly we can see that L_2 is continuous in the other cases.

Let $\mu\colon G\times G/H\to G/H$ be a map defined by the left translation and let $\iota\colon (N(H, K_0)/H)^0 \hookrightarrow G/H$ be an inclusion map. Then the composition

$$\hat{L}\colon O \xrightarrow{(L_1, \iota_*\circ L_2)} C^\infty([-\varepsilon, \varepsilon], G)\times C^\infty([-\varepsilon, \varepsilon], G/H) \xrightarrow{\kappa} C^\infty([-\varepsilon, \varepsilon], G\times G/H) \xrightarrow{\mu_*} C^\infty([-\varepsilon, \varepsilon], G/H)$$

is continuous by Lemma 2.2, where κ is defined by $\kappa(f_1, f_2)(r)=(f_1(r), f_2(r))$. Note that $\hat{L}(h)=b\cdot c=\hat{a}$ and $\hat{L}(O)$ is contained in $C_e^\infty([-\varepsilon, \varepsilon], N(H)/H)$. Here $C_e^\infty([-\varepsilon, \varepsilon], N(H)/H)$ denotes the set of all smooth even maps $f\colon [-\varepsilon, \varepsilon] \to N(H)/H$, endowed with C^∞ topology. Let $T\colon C_e^\infty([-\varepsilon, \varepsilon], N(H)/H)\to C^\infty([0, \varepsilon^2], N(H)/H)$ be a map defined by $T(f)(r)=f(\sqrt{r})$. By the same argument as in the proof in Lemma 2.4, we can prove that T is continuous. Thus $L_I=T\circ L$ is continuous.

Now consider the case $I=[\varepsilon^2/2, 1-\varepsilon^2/2]$. L_I is given by the composition

$$O \xrightarrow{k^*} C^\infty(\pi^{-1}(I), \pi^{-1}(I)) \xrightarrow{(\alpha^{-1}\circ\iota)^*} C^\infty(I, \pi^{-1}(I)) \xrightarrow{(q_2\circ\alpha)_*} C^\infty(I, G/H),$$

where $k\colon \pi^{-1}(I)\hookrightarrow M$ is an inclusion, $\iota\colon I\to G/H\times I$ is a map given by $\iota(r)=(1H, r)$ and $q_2\colon G/H\times I\to G/H$ is the projection on the first factor. Thus L_I is continuous. We can see that L_I is continuous in the case $I=[1-\varepsilon^2, 1]$ similarly as in the case $I=[0, \varepsilon^2]$, and this completes the proof of Proposition 4.6.

§5. Subgroups of the Topological Groups Q and Ker P

In this section we shall consider subgroups Q_1 and S of the topological groups Q and $\mathrm{Ker}\,P$, respectively, such that $L(S)=Q_1$, and we shall prove that the inclusions $Q_1 \hookrightarrow Q_0$ and $S \hookrightarrow \mathrm{Ker}\,P$ are homotopy equivalences, where Q_0 is the identity component of Q.

Put $Q_1=\{a \in Q_0;\ a(r)=a(0) \text{ for } 0 \leq r \leq 1/4 \text{ and } a(r)=a(1) \text{ for } 3/4 \leq r \leq 1\}$. Then Q_1 is a topological subgroup of Q_0. Let $i\colon Q_1 \hookrightarrow Q_0$ be an inclusion.

Lemma 5.1. *$i\colon Q_1 \hookrightarrow Q_0$ is a homotopy equivalence.*

Proof. Let $\sigma\colon [0, 1] \to [0, 1]$ be a smooth map such that

$$\sigma(r)=0 \quad \text{for} \quad 0 \leq r \leq 1/4,$$
$$\sigma(r)=1 \quad \text{for} \quad 3/4 \leq r \leq 1.$$

Let $\mu_t\colon [0, 1] \to [0, 1]$ $(0 \leq t \leq 1)$ be a smooth homotopy given by $\mu_t(r)=t\sigma(r)+(1-t)r$. Since $(a \circ \mu_t)(i) \in (N(H) \cap N(K_i))/H$ for $i=0, 1$, $a \circ \mu_t$ is an element of Q. Define $q\colon Q_0 \times [0, 1] \to Q$ by $q(a, t)=a \circ \mu_t$. Let $\mu\colon [0, 1] \to C^\infty([0, 1], [0, 1])$ be a map given by $\mu(t)=\mu_t$. Then it is easy to see that μ is continuous. Note that q is given by the composition

$$Q_0 \times [0, 1] \xrightarrow{(1,\mu)} Q_0 \times C^\infty([0, 1], [0, 1])$$
$$\xrightarrow{\mathrm{comp}} C^\infty([0, 1], N(H)/H),$$

where comp is given by $\mathrm{comp}(a, f)=a \circ f$. By Lemma 2.2 (6), q is continuous. Then $q(Q_0 \times [0, 1])$ is contained in Q_0. Let $q_t\colon Q_0 \to Q_0$ be a map given by $q_t(a)=q(a, t)$. Since $\mu_1=\sigma$, $q_1(Q_0)$ is contained in Q_1. Thus q is a homotopy between $q_0=1_{Q_0}$ and $q_1=i \circ q_1$. Note that $q_t(Q_1)$ is contained in Q_1 for any t. Then $q\colon Q_1 \times [0, 1] \to Q_1$ is a homotopy between 1_{Q_1} and $q_1 \circ i$. Therefore Lemma 5.1 follows.

Put $S=L^{-1}(Q_1) \subset \mathrm{Ker}\,P$. Let $\iota\colon S \hookrightarrow \mathrm{Ker}\,P$ be an inclusion.

Lemma 5.2. *$\iota\colon S \hookrightarrow \mathrm{Ker}\,P$ is a homotopy equivalence.*

Proof. Put $a=L(h^{-1})$ for $h \in \mathrm{Ker}\,P$. Let $h_t\colon M \to M$ $(0 \leq t \leq 1)$ be a map as follows: h_t is given on $\pi^{-1}((0, 1))$ by the composition $\pi^{-1}((0, 1)) \xrightarrow{\alpha} G/H \times (0, 1) \xrightarrow{\hat{h}_t} G/H \times (0, 1) \xrightarrow{\alpha^{-1}} \pi^{-1}((0, 1))$, where $\hat{h}_t$ is defined by $\hat{h}_t(gH, r)=(g \cdot q_t(a)(r), r)$. $h_t(gK_i)=ga(i) \cdot K_i$ $(i=0, 1)$ for $g \in G$. Here we need the following

Assertion 5.3. *h_t is a smooth map for any t.*

Proof. By the definition, h_t is smooth on $\pi^{-1}((0, 1))$. We shall prove that h_t is smooth on $\pi^{-1}(0)$. Let a_0 be an element of G such that $a_0H=a(0)$ and $a_0\in N(H)\cap N(K_0)$. For $[g, 0]\in p_{0,1/2}^{-1}(1K_0)$, $(p_{0,1/2}\circ h)([g, 0])=\pi_0(ga_0)$ $=\pi_0(a_0)\in a_0U_0$. Then there exists a neighborhood W_0 of $1K_0$ in G/K_0 such that $(p_{0,1/2}\circ h)(p_{0,1/2}^{-1}(\overline{W}_0))$ is contained in a_0U_0. For $[g, re_0]\in p_{0,1/2}^{-1}(\overline{W}_0)$ and $0\leq t\leq 1$,

$$\begin{aligned}(p_{0,1/2}\circ h_t)([g, re_0])&=\bar{\pi}_0(gq_t(a)(r^2))\\ &=\bar{\pi}_0(ga((1-t)r^2))\\ &=(p_{0,1/2}\circ h)([g, \sqrt{1-t}re_0])\end{aligned}$$

which is contained in $(p_{0,1/2}\circ h)(p_{0,1/2}^{-1}(\overline{W}_0))\subset a_0U_0$. Then $h_t(p_{0,1/2}^{-1}(g\overline{W}_0))$ is contained in $p_{0,1/2}^{-1}(ga_0U_0)$ for $g\in G$ and $0\leq t\leq 1$.

Let $\tilde{h}$: $W_0\times D_{1/2}(V_0)\to U_0\times D_{1/2}(V_0)$ be a map given by $\tilde{h}=\phi_{0,ga_0}\circ h\circ\phi_{0,g}^{-1}$ for $g\in G$. Let ρ_1: $U_0\times D_{1/2}(V_0)\to U_0$ and ρ_2: $U_0\times D_{1/2}(V_0)\to D_{1/2}(V_0)$ be the projections on the first factor and the second factor respectively. Put $g'=ga_0$ and put $\tilde{h}^i=\rho_i\circ\tilde{h}$ for $i=0, 1$. Then $\tilde{h}^i$ is a smooth map and

$$\tilde{h}^1(x, rke_0)=g'^{-1}g\sigma_0(x)k\cdot\bar{\pi}_0(a(r^2)),$$
$$\tilde{h}^2(x, rke_0)=\sigma_{0,g'}(g\sigma_0(x)k\cdot\bar{\pi}_0(a(r^2))^{-1}g\sigma_0(x)ka(r^2)\cdot re_0$$

for $x\in W_0$ and $k\in K_0$, where $\bar{\pi}_0$: $G/H\to G/K_0$ is the natural projection. Put $\tilde{h}_t^i=\rho_i\circ\phi_{0,g'}\circ h_t\circ\phi_{0,g}^{-1}$ for $i=0, 1$. Then

$$\tilde{h}_t^1(x, rke_0)=g'^{-1}g\sigma_0(x)k\cdot\bar{\pi}_0(a(\mu_t(r^2))),$$
$$\tilde{h}_t^2(x, rke_0)=\sigma_{0,g'}(g\sigma_0(x)k\cdot\bar{\pi}_0(a(\mu_t(r^2)))^{-1}g\sigma_0(x)ka(\mu_t(r^2))\cdot re_0$$

for $x\in W_0$ and $k\in K_0$.

Since $\sigma(r^2)=0$ for $r\leq 1/2$, $\mu(r^2, t)=(1-t)r^2$ for $0\leq r\leq 1/2$. Then $\tilde{h}_t^1(x, v)$ $=\tilde{h}^1(x, \sqrt{1-t}v)$ for $0\leq t\leq 1$ and $\tilde{h}_t^2(x, v)=1/\sqrt{1-t}\,\tilde{h}^2(x, \sqrt{1-t}v)$ for $0\leq t<1$. Thus $\tilde{h}_t^1$ $(0\leq t\leq 1)$ and $\tilde{h}_t^2$ $(0\leq t<1)$ are smooth maps.

By the Taylor formula (c.f. J. Dieudonné [5, Chapter VIII, (8, 14, 3)]), we have

$$\tilde{h}^2(x, v)=\tilde{h}^2(x, 0)+\left(\int_0^1(D\tilde{h}^2)(x, \zeta v)d\zeta\right)v,$$

where $D\tilde{h}^2$ is the derivative of $\tilde{h}^2$. Since $\tilde{h}^2(x, 0)=0$,

$$\tilde{h}_t^2(x, v)=\left(\int_0^1(D\tilde{h}^2)(x, \sqrt{1-t}\zeta v)d\zeta\right)v \qquad \text{for} \quad 0\leq t<1.$$

Then $\tilde{h}_1^2(x, v)=\lim_{t\to 1}\tilde{h}_t^2(x, v)=(D\tilde{h}^2)(x, 0)v$, and $\tilde{h}_1^2$ is a smooth map. Therefore h_t is smooth on $\pi^{-1}(0)$ for any $0\leq t\leq 1$. Similarly we can prove that h_t is smooth on $\pi^{-1}(1)$, and Assertion 5.3 follows.

Proof of Lemma 5.2 *continued.* Let $\bar{q}$: $\operatorname{Ker} P\times[0, 1]\to\operatorname{Ker} P$ be a map defined by $\bar{q}(h, t)=h_t$. By Assertion 5.3, h_t and h_t^{-1} are smooth maps, and $\bar{q}$ is a well defined map. Next we shall prove that $\bar{q}$ is continuous. Let W_i be a neighborhood of $1K_i$ in G/K_i satisfying $\overline{W}_i\subset U_i$ for $i=0, 1$. Put $O=\{h\in\operatorname{Ker} P;$ $h(p_{i,1/2}^{-1}(\overline{W}_i))\subset p_{i,1/2}^{-1}(U_i)$ for $i=0, 1\}$. Then O is an open neighborhood of 1_M in $\operatorname{Ker} P$. For $h\in O, g\in G$ and $0\leq t\leq 1$, $h_t(p_{i,1/2}^{-1}(g\overline{W}_i))$ is contained in $p_{i,1/2}^{-1}(gU_i)$ $(i=0, 1)$. Let W_2 be an open neighborhood of $1H$ in G/H satisfying $\overline{W}_2\subset U_2$. Let C be one of the sets $\{p_{i,1/2}^{-1}(g\overline{W}_i)$ $(i=0, 1, g\in G)$, $\alpha^{-1}(g\overline{W}_2\times[1/5, 4/5])$ $(g\in G)\}$. By Lemma 2.3, it is sufficient to show that the composition $\bar{q}_C$: $O\times[0, 1]\xrightarrow{\bar{q}}\operatorname{Ker} P\xrightarrow{j_C^*}C^\infty(C, M)$ is continuous for any C, where j_C: $C\hookrightarrow M$ is an inclusion map.

First consider the case $C=p_{0,1/2}^{-1}(g\overline{W}_0)$. Let ν_1: $C^\infty(\overline{W}_0\times D_{1/2}(V_0), U_0)\times[0, 1]\to C^\infty(\overline{W}_0\times D_{1/2}(V_0), U_0)$ be a map given by $\nu_1(f, t)(x, v)=f(x, \sqrt{1-t}\,v)$. Let ν_2: $C^\infty(\overline{W}_0\times D_{1/2}(V_0), D_{1/2}(V_0))\times[0, 1]\to C^\infty(\overline{W}_0\times D_{1/2}(V_0), D_{1/2}(V_0))$ be a map given by $\nu_2(f, t)(x, v)=\left(\int_0^1(Df)(x, \sqrt{1-t}\,\zeta v)d\zeta\right)(v)$. It is easy to see that ν_1 and ν_2 are continuous. Note that $\bar{q}_C$ is the composition

$$
\begin{aligned}
O\times[0, 1] &\xrightarrow{(j_C^*,1)} C^\infty(C, p_{0,1/2}^{-1}(gU_0))\times[0, 1]\\
&\xrightarrow{((\phi_{0,g})_*\circ(\phi_{0,g})^*,1)} C^\infty(\overline{W}_0\times D_{1/2}(V_0), U_0\times D_{1/2}(V_0))\times[0, 1]\\
&\xrightarrow{((p_1)_*,(p_2)_*,1)} C^\infty(\overline{W}_0\times D_{1/2}(V_0), U_0)\\
&\qquad\qquad\times C^\infty(\overline{W}_0\times D_{1/2}(V_0), D_{1/2}(V_0))\times[0, 1]\\
&\xrightarrow{\nu} C^\infty(\overline{W}_0\times D_{1/2}(V_0), U_0)\times C^\infty(\overline{W}_0\times D_{1/2}(V_0), D_{1/2}(V_0))\\
&\xrightarrow{\kappa} C^\infty(\overline{W}_0\times D_{1/2}(V_0), U_0\times D_{1/2}(V_0))\\
&\xrightarrow{(\phi_{0,g}^{-1})_*\circ(\phi_{0,g}^{-1})^*} C^\infty(C, p_{0,1/2}^{-1}(gU_0))\hookrightarrow C^\infty(C, M).
\end{aligned}
$$

Here ν is given by $\nu(f_1, f_2, t)=(\nu_1(f_1, t), \nu_2(f_2, t))$ and κ is the map defined in Lemma 2.2 (5). Then $\bar{q}_C$ is continuous by Lemma 2.2.

Next consider the case $C=\alpha^{-1}(g\overline{W}_2\times[1/5, 4/5])$. Let m: $N(H)/H\times G/H\to G/H$ be a map defined by $m(nH, gH)=(gn)H$ and p_2: $G/H\times[1/5, 4/5]\to[0, 1]$ be a map given by $p_2(gH, r)=r$. Let δ: $Q_0\to Q_0$ be a map given by $\delta(a)=a^{-1}$. Then the map $\bar{q}_C$ is the composition

$$O\times[0,1]\xrightarrow{(L,1)}Q_0\times[0,1]\xrightarrow{\delta\circ q}Q_0\xrightarrow{p_2^*}C^\infty(G/H\times[1/5,4/5],N(H)/H)$$
$$\xrightarrow{(1_{G/H\times[1/5,4/5]})!}C^\infty(G/H\times[1/5,4/5],$$
$$N(H)/H\times G/H\times[1/5,4/5])$$
$$\xrightarrow{m_*}C^\infty(G/H\times[1/5,4/5],G/H\times[1/5,4/5])$$
$$\xrightarrow{(\alpha\circ j_C)^*\circ(\alpha^{-1})_*}C^\infty(C,\alpha^{-1}(G/H\times[1/5,4/5]))\hookrightarrow C^\infty(C,M),$$

which is continuous because L and q are continuous.

Similarly as in the case $C=p_{0,1/2}^{-1}(g\overline{W}_0)$, we can see that $\bar{q}_C$ is continuous in the case $C=p_{1,1/2}^{-1}(g\overline{W}_1)$. Thus $\bar{q}$ is continuous. Since $q_1(Q_0)\subset Q_1$, $\bar{q}_1(\operatorname{Ker}P)\subset S$. Therefore $\bar{q}$ is a homotopy between $\bar{q}_0=1_{\operatorname{Ker}P}$ and $\bar{q}_1=\iota\circ\bar{q}_1$. Since $q(Q_1\times[0,1])\subset Q_1$, $\bar{q}(S\times[0,1])\subset S$. Then $\bar{q}\colon S\times[0,1]\to S$ is a homotopy between 1_S and $\bar{q}_1\circ\iota$. Thus ι is a homotopy equivalence, and this completes the proof of Lemma 5.2.

§6. Proof of Theorem

In this section, we shall see that $L\colon S\to Q_1$ is an isomorphism between topological groups, and we shall prove our Theorem.

Proposition 6.1. *$L\colon S\to Q_1$ is an isomorphism between topological groups.*

Before the proof of Proposition 6.1, we begin with some lemmas. For any topological subgroup K of G, K^0 denotes the identity component of K.

Lemma 6.2. *For any $a\in N(K_0)^0\cap N(H)$, there exist $a'\in N(H^0)\cap K_0^0$ and $n\in\operatorname{Cent}(K_0^0)$ such that $a=n\cdot a'$, where $\operatorname{Cent}(K_0^0)$ is the centralizer of K_0^0 in G.*

Proof. Since $N(K_0)^0$ is a compact connected Lie group, there exist a torus group T and a simply connected semi-simple compact Lie group G' such that $\hat{N}_0=T\times G'$ is a finite covering group of $N(K_0)^0$ (c.f. L. Pontrjagin [8, §64]). Let $q_0\colon\hat{N}_0\to N(K_0)^0$ be the covering projection. Put $\hat{K}_0=q_0^{-1}(K_0^0)$. Since K_0^0 is a normal subgroup of $N(K_0)^0$, $\hat{K}_0$ is a normal subgroup of $\hat{N}_0$. Then $\hat{K}_0^0$ is also a normal subgroup of $\hat{N}_0$. Here we need the following

Assertion 6.3. *There exists a closed normal subgroup K_0' of $\hat{N}_0$ such that $\hat{N}_0$ is isomorphic to the product group $\hat{K}_0^0\times K_0'$ as a Lie group.*

Proof. There exist simply connected simple Lie groups G_i $(1\leq i\leq r)$ such that $G'=G_1\times\cdots\times G_r$. Since $\hat{K}_0^0$ is a compact connected Lie group, there exist simply connected simple Lie groups K_j $(1 \le j \le s)$ and a torus group T' such that $\tilde{K}_0 = T' \times K_1 \times \cdots \times K_s$ is a finite covering of $\hat{K}_0^0$. Let $p_0 \colon \tilde{K}_0 \to \hat{K}_0^0$ be the covering projection. Let $\rho_i \colon \hat{N}_0 = T \times G_1 \times \cdots \times G_r \to G_i$ be a projection on the direct factor G_i $(1 \le i \le r)$. Since $\hat{K}_0^0$ is a normal subgroup of $\hat{N}_0$, $\rho_i(\hat{K}_0^0)$ is a normal subgroup of G_i. Since G_i is a simple Lie group, $\rho_i(\hat{K}_0^0) = G_i$ or $\{1\}$. If $\rho_i(\hat{K}_0^0) = G_i$, $\rho_i(p_0(K_j))$ is a normal subgroup of G_i. Thus $\rho_i(p_0(K_j)) = G_i$ or $\{1\}$, for $1 \le i \le r$, $1 \le j \le s$.

Put $\rho'_i = \rho_i \circ p_0$. If $\rho'_i(K_{j_1}) = \rho'_i(K_{j_2})$ $(j_1 \ne j_2)$, then $\rho'_i(g_1) \cdot \rho'_i(g_2) = \rho'_i(g_1 \cdot g_2) = \rho'_i(g_2 \cdot g_1) = \rho'_i(g_2) \cdot \rho'_i(g_1)$ for $g_1 \in K_{j_1}$, $g_2 \in K_{j_2}$. Then $\rho'_i(K_{j_1})$ is a commutative normal subgroup of G_i, and $\rho'_i(K_{j_1}) = \{1\}$. If $\rho'_i(K_j) = G_i$, then $\rho'_i(T')$ is a normal subgroup of G_i, hence $\rho'_i(T') = \{1\}$. Therefore, if $\rho'_i(K_j) = G_i$, then $\rho'_i(T') = \{1\}$ and $\rho'_i(K_n) = \{1\}$ for $n \ne j$.

Assume $\rho'_{i_1}(K_j) = G_{i_1}$ and $\rho'_{i_2}(K_j) = G_{i_2}$ for $i_1 \ne i_2$. Let $\rho' \colon \tilde{K}_0 \to G_{i_1} \times G_{i_2}$ be a map defined by $\rho'(k) = (\rho'_{i_1}(k), \rho'_{i_2}(k))$. Since $\hat{K}_0^0$ is a normal subgroup of $\hat{N}_0$ and $\rho'(\tilde{K}_0) = \rho'(K_j)$, $\rho'(K_j)$ is a normal subgroup of $G_{i_1} \times G_{i_2}$. Then, for $x, y \in K_j$, there exists $k \in K_j$ such that $(\rho'_{i_1}(x), 1)\rho'(y)(\rho'_{i_1}(x)^{-1}, 1) = \rho'(k)$. Then $\rho'_{i_1}(xyx^{-1}) = \rho'_{i_1}(x)\rho'_{i_1}(y)\rho'_{i_1}(x)^{-1} = \rho'_{i_1}(k)$ and $\rho'_{i_2}(y) = \rho'_{i_2}(k)$. Since K_j, G_{i_n} $(n = 1, 2)$ are simply connected simple Lie groups, $\rho'_{i_n} \colon K_j \to G_{i_n}$ is an isomorphism between the Lie groups. Thus $xyx^{-1} = k = y$ for any $x, y \in K_j$, and K_j must be a commutative Lie group, which is a contradiction since K_j is a simple Lie group.

Thus we may assume that $\rho'_j(K_j) = G_j$ and $\rho'_i(K_j) = \{1\}$ $(i \ne j)$ for $1 \le j \le s$, $1 \le i \le r$. For $i > s$, $\rho_i(\hat{K}_0^0) = \rho'_i(\tilde{K}_0) = \rho'_i(T')$ which is a commutative normal subgroup of G_i, hence $\rho'_i(T') = \{1\}$. Then $p_0(T')$ is a subgroup of T, and there exists a torus subgroup S of T such that $T = p_0(T') \times S$. Put $K' = S \times G_{s+1} \times \cdots \times G_r$. Then $\hat{N}_0 = \hat{K}_0^0 \times K'_0$, and Assertion 6.3 follows.

Proof of Lemma 6.2 *continued.* By Assertion 6.3, there exists a closed normal subgroup K'_0 of $\hat{N}_0$ such that $\hat{N}_0 = \hat{K}_0^0 \times K'_0$. Since K_0^0 is a connected group, $q_0(\hat{K}_0^0) = K_0^0$. Then $N(K_0)^0 = q_0(\hat{N}_0) = q_0(\hat{K}_0^0) \cdot q_0(K'_0) = K_0^0 \cdot q_0(K'_0)$. Note that $q_0(K'_0)$ is contained in $\mathrm{Cent}(K_0^0)$. Thus, for $a \in N(K_0)^0 \cap N(H)$, there exists $a' \in K_0^0$ and $n \in \mathrm{Cent}(K_0^0)$ such that $a = a' \cdot n$. Since $N(H) \subset N(H^0)$ and $H^0 \subset K_0^0$, $H^0 = aH^0a^{-1} = a'nH^0n^{-1}a'^{-1} = a'H^0a'^{-1}$. Thus $a' \in N(H^0)$ and Lemma 6.2 follows.

For $a \in Q_1$, we define a map $h \colon M \to M$ as follows:

$$h(\alpha^{-1}(gH, r)) = \alpha^{-1}((ga(r)^{-1}, r)) \qquad \text{for} \quad (gH, r) \in G/H \times (0, 1),$$

$h([g, 0])=[ga(i)^{-1}, 0]$ for $[g, 0]\in\pi^{-1}(i)$ $(i=0, 1)$.

Lemma 6.4. *h is a smooth map.*

Proof. Choose $a_0\in(N(H)\cap N(K_0))^0\subset N(H)^0\cap N(K_0)^0$ such that $a(0)^{-1}=a_0H$. There exists a neighborhood W_0 of $1K_0$ in G/K_0 such that $\pi_0^{-1}(W_0)\cdot a_0$ is contained in $a_0\cdot\pi_0^{-1}(U_0)$. Since $a(r)=a(0)$ for $0\leq r\leq 1/4$, $h(p_{0,1/2}^{-1}(gW_0))$ is contained in $p_{0,1/2}^{-1}(ga_0U_0)$. Let $\tilde{h}_1$: $W_0\times D_{1/2}(V_0)\to U_0$ be a map given by the composition $\rho_1\circ\phi_{0,ga_0}\circ h\circ\phi_{0,g}^{-1}$, and let $\tilde{h}_2$: $W_0\times D_{1/2}(V_0)\to D_{1/2}(V_0)$ be a map given by the composition $\rho_2\circ\phi_{0,ga_0}\circ h\circ\phi_{0,g}^{-1}$. Note that

$$\begin{aligned}(h\circ\phi_{0,g}^{-1})((x, rke_0))&=h([g\sigma_0(x)k, re_0])\\&=h(\alpha^{-1}((g\sigma_0(x)kH, r^2)))\\&=\alpha^{-1}((g\sigma_0(x)ka_0H, r^2))\\&=[g\sigma_0(x)ka_0, re_0]\end{aligned}$$

for $x\in W_0$, $k\in K_0$, $0<r\leq 1/2$. Since $a_0\in N(K_0)$, $ka_0K_0=a_0K_0$. Then

$$\tilde{h}_1(x, v)=a_0^{-1}\sigma_0(x)a_0K_0 \quad \text{for} \quad (x, v)\in W_0\times D_{1/2}(V_0), \quad \text{and}$$
$$\tilde{h}_2(x, rke_0)=\sigma_{0,ga_0}(g\sigma_0(x)a_0K_0)^{-1}g\sigma_0(x)ka_0\cdot re_0$$

for $x\in W_0$, $k\in K_0$, $0\leq r\leq 1/2$. Thus $\tilde{h}_1$ is a smooth map and $\tilde{h}_2$ is smooth on $W_0\times(D_{1/2}(V_0)-0)$. We shall prove that $\tilde{h}_2$ is smooth on $W_0\times 0$, hence h is smooth on $\pi^{-1}(0)$. This is trivial in the case $\dim S(V_0)=0$.

Let $\xi_{a_0,g}$: $W_0\to G$ be a map given by $\xi_{a_0,g}(x)=\sigma_{0,ga_0}(g\sigma_0(x)a_0K_0)^{-1}g\sigma_0(x)$. Then $\xi_{a_0,g}$ is a smooth map. By Lemma 6.2, there exist $a_0'\in N(H^0)\cap K_0^0$ and $n\in\mathrm{Cent}(K_0^0)$ such that $a_0=na_0'$. Then $\tilde{h}_2(x, rke_0)=\xi_{a_0,g}(x)kna_0'\cdot rke_0$ $=\xi_{a_0,g}(x)nka_0'\cdot re_0$ for $x\in W_0$, $k\in K_0^0$ and $0\leq r\leq 1/2$. Note that $N(H^0)\cap K_0^0=N(H^0, K_0^0)$.

Assertion 6.5. *For $a\in N(H^0, K_0^0)$, let ψ_a: $D(V_0)\to D(V_0)$ be a map defined by $\psi_a(rke_0)=rkae_0$ for $0\leq r\leq 1$, $k\in K$. Then ψ_a is a diffeomorphism. Moreover, let ψ: $N(H^0, K_0^0)\to\mathrm{Diff}^\infty(D(V_0))$ be a map given by $\psi(a)=\psi_a$, then ψ is continuous.*

Proof. If $\dim S(V_0)=0$, then $K_0^0\subset H$ and $\psi_a=1_{D(V_0)}$. In this case, the proof is trivial. We assume $\dim S(V_0)>0$. Since $S(V_0)=K_0/H$ is connected, K_0^0 acts transitively on $S(V_0)$. Let L be the ineffective kernel of the action $K_0^0\times S(V_0)\to S(V_0)$. Put $\bar{K}=K_0^0/L$ and $\bar{H}=(H\cap K_0^0)/L$. Then $\bar{K}$ acts transitively and effectively on $S(V_0)$ and $\bar{H}$ is an isotropy subgroup of this action. By Lemma 4.3, $\bar{K}$, $\bar{H}$ and $N(\bar{H}, \bar{K})/\bar{H}$ are G, H and $N(H)/H$ in Lemma 4.3, respectively. Hence $\overline{H}$ is connected. Since the identity component of $H \cap K_0^0$ is H^0, $\overline{H}=H^0 \cdot L/L$. For $a \in N(H^0, K_0^0)$, the left coset aL is an element of $N(\overline{H}, \overline{K})$. Then a defines an element $\tilde{a} \in N(\overline{H}, \overline{K})/\overline{H}$. Note that $\psi_a(rke_0)=rkae_0=rk\tilde{a}e_0$ for $0 \leq r \leq 1$, $k \in K_0^0$.

Consider the case $\overline{K}=SU(n)$ $(n \geq 2)$, $\overline{H}=SU(n-1)$ and $N(\overline{H}, \overline{K})/\overline{H}=U(1)$. In this case, $V_0=C^n$ and $U(1)$ acts on V_0 as a scalar multiplication. Thus $\psi_a(rke_0)=\tilde{a} \cdot rke_0$ for $rke_0 \in D(V_0)$. Hence ψ_a is a diffeomorphism. It is easy to see that ψ is continuous.

Next consider the case $\overline{K}=Sp(n)$ $(n \geq 1)$, $\overline{H}=Sp(n-1)$ and $N(\overline{H}, \overline{K})/\overline{H}=Sp(1)$. In this case, $V_0=H^n$ and $Sp(1)$ acts on V_0 as a scalar multiplication on the right. Then $\psi_a(v)=v \cdot \tilde{a}$ for $v \in D(V_0)$, hence ψ_a is a diffeomorphism and ψ is continuous. Similarly we can see that ψ_a is a diffeomorphism and ψ is continuous in the other cases, and Assertion 6.5 follows.

Proof of Lemma 6.4 *continued.* Since $\tilde{h}_2(x, v)=\xi_{a_0,g}(x)n \cdot \psi_{a'_0}(v)$, by Assertion 6.5, $\tilde{h}_2$ is a smooth map. Thus $\tilde{h}_1$ and $\tilde{h}_2$ are smooth maps, hence h is smooth on $\pi^{-1}(0)$. Similarly we can see that h is smooth on $\pi^{-1}(1)$. By the definition, h is smooth on $\pi^{-1}((0, 1))$, and this completes the proof of Lemma 6.4.

Let $\hat{L}(a)$ be a smooth map $h: M \to M$ in Lemma 6.4, for $a \in Q_1$. Since $\hat{L}(a^{-1})=\hat{L}(a)^{-1}$, h is a diffeomorphism of M. By the definition, h is an equivariant map. Thus we have a map $\hat{L}: Q_1 \to \mathrm{Diff}_G^\infty(M)$. Note that $\hat{L}$ is an abstract group homomorphism.

Lemma 6.6. $\hat{L}: Q_1 \to \mathrm{Diff}_G^\infty(M)$ *is continuous.*

Proof. Let W_i be a neighborhood of $1K_i$ in G/K_i such that $\overline{W}_i \subset U_i$ $(i=0, 1)$, and let W_2 be a neighborhood of $1H$ in G/H such that $\overline{W}_2 \subset U_2$. Put $A_i=\{n \in N(K_i)^0;\ n^{-1}\overline{W}_i n \subset U_i\}$. Then A_i is an open neighborhood of the identity in $N(K_i)^0$. Let $q_i: \hat{N}_i \to N(K_i)^0$ be a finite covering such that $\hat{N}_i$ is a direct product $T_i \times G'_i$. Here T_i is a torus group and G'_i is a simply connected semi-simple compact Lie group. Put $\hat{K}_i=q_i^{-1}(K_i^0)$. By Assertion 6.3, there exists a closed normal subgroup K'_i of $\hat{N}_i$ such that $\hat{N}_i=\hat{K}_i^0 \times K'_i$. Let s_i be a smooth local cross section of q_i defined on an open neighborhood B_i of the identity in $N(K_i)^0$. Since $\pi_2: (N(H) \cap N(K_i))^0 \to ((N(H) \cap N(K_i))/H)^0$ is a fibration, there exists a smooth local cross section t_i of π_2 defined on an open neighborhood E_i of $1H$ such that $t_i(E_i) \subset A_i \cap B_i$.

Put $O=\{a \in Q_1;\ a(i)^{-1} \in E_i\ (i=0,\ 1)\}$. Then O is an open neighborhood of the identity. Since $\hat{L}$ is a group homomorphism, it is enough to show that $\hat{L}$ is continuous on O. Let C denote one of the sets $\{p_{i,1/2}^{-1}(g\overline{W}_i)\ (i=0,\ 1,\ g \in G),\ \alpha^{-1}(g\overline{W}_2 \times [1/5,\ 4/5])\ (g \in G)\}$. By Lemma 2.3, if $\hat{L}_C\colon O \xrightarrow{L} \mathrm{Diff}_G^\infty(M)^0 \xrightarrow{j_C^*} C^\infty(C,\ M)$ is continuous for any C, then $\hat{L}$ is continuous, where $j_C\colon C \hookrightarrow M$ is an inclusion map.

First consider the case $C=p_{0,1/2}^{-1}(g\overline{W}_i)$. Let $\beta_1\colon \hat{N}_0=\hat{K}_0^0 \times K_0' \to \hat{K}_0^0$ and $\beta_2\colon \hat{N}_0 \to K_0'$ be the projection on the first factor and the second factor respectively. Let L_1 be the composition

$$O \xrightarrow{r} E_0 \xrightarrow{t_0} A_0 \cap B_0 \xrightarrow{(\xi_g,\, q_0 \circ \beta_2 \circ s_0)} C^\infty(\overline{W}_0,\ G) \times \mathrm{Cent}(K_0^0) \xrightarrow{m} C^\infty(\overline{W}_0,\ G).$$

Here r, ξ and m are given by $r(a)=a(0)^{-1}$, $\xi_g(a_0)(x)=\xi_{a_0,g}(x)$ and $m(f,\ n)(x)=f(x)\cdot n$, respectively. Put $a_0=(t_0 \circ r)(a)$ for $a \in O$. Then $\pi_0(\xi_{g,a_0}(x))=\pi_0(a_0^{-1})$ for $x \in \overline{W}_0$ and $\pi_0((q_0 \circ \beta_2 \circ s_0)(a_0))=\pi_0(a_0)$. Therefore $L_1(a) \in K_0$ for any $a \in O$, and $L_1(O) \subset C^\infty(\overline{W}_0,\ K_0)$. Let L_2 be the composition

$$O \xrightarrow{r} E_0 \xrightarrow{t_0} A_0 \cap B_0 \xrightarrow{q_0 \circ \beta_1 \circ s_0} N(H^0,\ K_0^0) \xrightarrow{\psi} \mathrm{Diff}^\infty(D_{1/2}(V_0)).$$

By Assertion 6.5, L_2 is continuous. Let L_3 be the composition

$$\begin{aligned} O &\xrightarrow{(L_1, L_2)} C^\infty(\overline{W}_0,\ K_0) \times \mathrm{Diff}^\infty(D_{1/2}(V_0)) \\ &\xrightarrow{(\rho_1^*, \rho_2^*)} C^\infty(\overline{W}_0 \times D_{1/2}(V_0), K_0) \times C^\infty(\overline{W}_0 \times D_{1/2}(V_0),\ D_{1/2}(V_0)) \\ &\xrightarrow{\kappa} C^\infty(\overline{W}_0 \times D_{1/2}(V_0),\ K_0 \times D_{1/2}(V_0)) \\ &\xrightarrow{\mu_*} C^\infty(\overline{W}_0 \times D_{1/2}(V_0),\ D_{1/2}(V_0)), \end{aligned}$$

where μ is given by $\mu(k,\ v)=k \cdot v$, and κ is the map in Lemma 2.2. Then L_3 is continuous, and $L_3(a)=\tilde{h}_2$. Let $\gamma\colon A_0 \to C^\infty(\overline{W}_0,\ U_0)$ be a map defined by $\gamma(a_0)(x)=a_0^{-1}\sigma_0(x)a_0K_0$. γ is a restriction map to A_0 of a map $\bar{\gamma}\colon N(K_0) \to C^\infty(G/K_0,\ G/K_0)$ given by $\bar{\gamma}(n)(gK_0)=n^{-1}gnK_0$. Since $\bar{\gamma}$ is a continuous map, γ is continuous. Let L_4 be the composition

$$O \xrightarrow{r} E_0 \xrightarrow{t_0} A_0 \xrightarrow{\gamma} C^\infty(\overline{W}_0,\ U_0) \xrightarrow{\rho_1^*} C^\infty(\overline{W}_0 \times D_{1/2},\ U_0).$$

Then L_4 is continuous and $L_4(h)=\tilde{h}_1$. L_C is the composition

$$\begin{aligned} O &\xrightarrow{(L_4, L_3)} C^\infty(\overline{W}_0 \times D_{1/2}(V_0),\ U_0) \times C^\infty(\overline{W}_0 \times D_{1/2}(V_0),\ D_{1/2}(V_0)) \\ &\xrightarrow{\kappa} C^\infty(\overline{W}_0 \times D_{1/2}(V_0),\ U_0 \times D_{1/2}(V_0)) \\ &\xrightarrow{(\phi_{0,g})^*(\phi_{0,g})_*} C^\infty(C,\ p_{0,1/2}^{-1}(gU_0)) \hookrightarrow C^\infty(C,\ M). \end{aligned}$$

Thus L_C is continuous.

Now consider the case $C=\alpha^{-1}(g\overline{W}_2 \times [1/5,\ 4/5])$. Let $m\colon g\overline{W}_2 \times N(H)/H$

$\to G/H$ be a map defined by $m(gH, nH)=ghH$, and let ρ: $G/H\times[1/5, 4/5]\to [1/5, 4/5]$ be the projection on the second factor. Then $\hat{L}_C$ is given by the composition

$$\begin{aligned} O &\xrightarrow{i^*\circ\delta_*} C^\infty([1/5, 4/5], N(H)/H) \\ &\xrightarrow{(1_{g\overline{W}_2})_!} C^\infty(g\overline{W}_2\times[1/5, 4/5], g\overline{W}_2\times N(H)/H) \\ &\xrightarrow{m_*} C^\infty(g\overline{W}_2\times[1/5, 4/5], G/H) \\ &\xrightarrow{\rho^\sharp} C^\infty(g\overline{W}_2\times[1/5, 4/5], G/H\times[1/5, 4/5]) \\ &\xrightarrow{\alpha^*\circ(\alpha^{-1})_*} C^\infty(C, \alpha^{-1}(G/H\times[1/5, 4/5]))\hookrightarrow C^\infty(C, M), \end{aligned}$$

where i: $[1/5, 4/5]\hookrightarrow[0, 1]$ is the inclusion map and δ: $N(H)/H\to N(H)/H$ is a map given by $\delta(a)=a^{-1}$. By Lemma 2.2, $\hat{L}_C$ is continuous.

We can see that L_C is continuous in the case $C=p_{1,1/2}^{-1}(g\overline{W}_1)$ similarly as in the case $C=p_{0,1/2}^{-1}(g\overline{W}_0)$, and this completes the proof of Lemma 6.6.

Proof of Proposition 6.1. From Lemma 6.6, $\hat{L}(Q_1)$ is contained in $\mathrm{Diff}_G^\infty(M)_0$. Then, by the definition, $\hat{L}(Q_1)$ is contained in S, and $\hat{L}=L^{-1}$. Combining Lemma 4.5, Proposition 4.6 and Lemma 6.6, $\hat{L}$: $S\to Q_1$ is an isomorphism between topological groups, and this completes the proof of Proposition 6.1.

Proof of Theorem. By Corollary 3.6, $\mathrm{Diff}_G^\infty(M)_0$ has the same homotopy type as $\mathrm{Ker}\,P$. Combining Lemma 5.1, Lemma 5.2 and Proposition 6.1, $\mathrm{Ker}\,P$ has the same homotopy type as Q_0. Note that Q_0 has the same homotopy type as the path space $\Omega(N(H)/H; (N(H)\cap N(K_0))/H, (N(H)\cap N(K_1))/H)_0$. This completes the proof of our Theorem.

§7. Concluding Remarks

From our Theorem, we have the following

Corollary 7.1. (1) *If* $K_0=K_1=G$, *then* $\mathrm{Diff}_G^\infty(M)_0$ *has the same homotopy type as* $(N(H)/H)^0$.

(2) *If* $N(H)/H$ *is a finite group, then* $\mathrm{Diff}_G^\infty(M)_0$ *is contractible.*

Remark 7.2. In K. Abe and K. Fukui [1], we have proved that $\mathrm{Diff}_G^\infty(M)_0$ is perfect if M is a G-manifold with one orbit type and $\dim M/G\geq 1$. But, by using Proposition 3.1, we can see that $\mathrm{Diff}_G^\infty(M)_0$ is not perfect in the case $M/G=[0, 1]$.

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