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Duality of Mixed Hodge Structures of Algebraic Varieties[1)]

By

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Introduction

Let X be a compact complex manifold of pure dimension n and Y an analytic subset of X. Let $U = X - Y$. Then associated to the pair (X, Y) we have the following pair of exact sequences of rational cohomology groups

$$\begin{array}{ccccccccc} \rightarrow & H_c^i(U, \mathbf{Q}) & \rightarrow & H^i(X, \mathbf{Q}) & \rightarrow & H^i(Y, \mathbf{Q}) & \rightarrow & H_c^{i+1}(U, \mathbf{Q}) & \rightarrow \\ \leftarrow & H^{2n-i}(U, \mathbf{Q}) & \leftarrow & H^{2n-i}(X, \mathbf{Q}) & \leftarrow & H_Y^{2n-i}(X, \mathbf{Q}) & \leftarrow & H^{2n-i-1}(U, \mathbf{Q}) & \leftarrow \end{array}$$

which are dual to each other via Poincaré pairings (cf. (1.5)). On the other hand, when X is an algebraic variety (as we assume in the following), Deligne defined in [3] [4] the natural mixed ($\mathbf{Q}$-)Hodge structure on each term of the above sequences, in such a way that the morphisms are those of mixed Hodge structures. The purpose of this article is then to show that the duality mentioned above is also compatible with the mixed Hodge structures under a suitable definition. A result in a sense analogous to ours has been obtained by Herrera and Lieberman in [13] in which they showed that the above duality is compatible with 'infinitesimal Hodge filtrations' of X along Y. Duality of mixed Hodge structure itself was also mentioned in the introduction of [4] as according to N. Katz. However, since there seems no published articles on this subject, it would not be of little use to give a detailed exposition like the present one.

In Section 1 a precise statement of the theorem will be given and its proof is reduced to the case where we have to show that the pairing $\psi_Y\colon H^i(Y, \mathbf{Q}) \times H_Y^{2n-i}(X, \mathbf{Q}) \rightarrow \mathbf{Q}$ gives a duality of mixed Hodge structures under the assumption that Y is a divisor with only normal crossings in X. In this case we have

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the mixed Hodge structure on $H^i(Y, \boldsymbol{Q})$ (resp. $H_Y^{2n-i}(X, \boldsymbol{Q})$) as described in [10] or [20] (resp. coming directly from that on $H^{2n-i}(U, \boldsymbol{Q})$ as described in [3]). The problem is that apparently these two descriptions do not fit well in the framework of duality. Our proof then consists in constructing commutative diagrams (A) and (B) (cf. (1.9)) of certain complexes' sheaves which are 'dual' to each other, where simple and multiple residues of Herrera-Lieberman [12] and Herrera [11] respectively play an important role in defining morphisms. This will be carried out in Sections 2 and 3 together with the proof of the theorem. (See (1.9) for an outline.) In Section 4 we treat another problem on naturality of mixed Hodge structure, i.e., its compatibility with the spectral sequence of Fary associated to a descending sequence of analytic subsets of X (Proposition 4.6).

Note that the Hodge theory is applicable without further change to a wider class of complex spaces, i.e., those in the category $\mathscr{C}$ as defined in (1.2), so that our results are also valid for these spaces. In [7] we used the results of the present note in an application to fixed point sets of $\boldsymbol{C}^*$ actions on compact Kähler manifolds, which was the original motivation for this investigation.

Notations. Let $\mathscr{A}$ be an abelian category, $K^{\cdot}$ a complex in $\mathscr{A}$ and $P=\{P_n(K^{\cdot})\}$ (resp. $\{P^n(K^{\cdot})\}$) an increasing (resp. decreasing) filtration on $K^{\cdot}$. Then for any integer m, $K^{\cdot}[m]$ is the complex with $K^{\cdot}[m]^n=K^{m+n}$, and $P[m]$ is the filtration on $K^{\cdot}$ with $P[m]_n(K^{\cdot})=P_{n-m}(K^{\cdot})$ (resp. $P[m]^n(K^{\cdot})=P^{n+m}(K^{\cdot})$). For a topological space X we denote by $\mathscr{A}(X)$ the abelian category of sheaves of $\boldsymbol{C}$ vector spaces on X, and by $\mathscr{D}\mathscr{A}(X)$ its derived category.

§1. Mixed Hodge Structure and Duality

(1.1) Let $\mathscr{C}_0$ be the category in which objects are compact reduced complex spaces and arrows are morphisms of complex spaces. We define a subcategory $\mathscr{C}$ of $\mathscr{C}_0$ as follows; let $X\in \mathrm{Ob}\,\mathscr{C}_0$. Then X is in $\mathscr{C}$ if and only if there is a surjective morphism $f\colon Y\to X$ with Y a compact Kähler manifold. In [5, Lemma 4.6] and [6, Proposition 1.6] we have shown the following: Suppose that $X\in\mathscr{C}$. Then: 1) Every subspace of X is in $\mathscr{C}$. 2) Let $g\colon X\to Y$ be a surjective meromorphic map of compact complex spaces. Then $Y\in\mathscr{C}$. 3) Let $g\colon Y\to X$ be a projective morphism. Then $Y\in\mathscr{C}$. 4) Suppose that X is non-singular. Then the Hodge de Rham spectral sequence

(1) $$E_1^{p,q}=H^q(X, \Omega_X^p)\Rightarrow H^{p+q}(X, \boldsymbol{C})$$

degenerates at $E_1^{p,q}$, where Ω_X^p is the sheaf of germs of holomorphic p-forms on X. In particular we have the natural isomorphism

$$H^i = H^i(X, \boldsymbol{C}) \cong \bigoplus_{p+q=i} H^{p,q}$$

where $H^{p,q} = F^p H^i \cap \bar{F}^{q-1} H^i$, F (resp. $\bar{F}$) being the induced filtration from (1) on $H^i(X, \boldsymbol{C})$ (resp. complex conjugate of F).

(1.2) Let X be a complex space. A compactification X^* of X is a compact complex space containing X as a dense Zariski open subset. Two compactifications X_i^*, $i=1, 2$, of X are called equivalent if the identity, id: $X \to X$, extends to a bimeromorphic map id*: $X_1^* \to X_2^*$. We call a complex space with an equivalence class of compactifications a *meromorphic complex space*, or simply a *meromorphic space*. Let X (resp. Y) be a meromorphic space with an equivalence class $\mathfrak{C}_X$ (resp. $\mathfrak{C}_Y$) of compactifications. Then a morphism f: $X \to Y$ is called *meromorphic*, if f extends to a meromorphic map f^*: $X^* \to Y^*$ for any $X^* \in \mathfrak{C}_X$ and $Y^* \in \mathfrak{C}_Y$. Let $\mathscr{M}$ be the category of meromorphic spaces and meromorphic morphisms. We define the subcategory $\tilde{\mathscr{C}}$ of $\mathscr{M}$ as follows; a meromorphic space X with an equivalence class $\mathfrak{C}_X$ of compactifications is in $\tilde{\mathscr{C}}$ if and only if there is a compactification $X^* \in \mathfrak{C}_X$ with $X^* \in \mathscr{C}$.

(1.3) The concept of mixed Hodge structure was introduced by Deligne in [3].

(1.3.1) **Definition.** 1) Let n be an integer. Then a *$\boldsymbol{Q}$-Hodge structure of weight n* is a pair (H, F) consisting of a finite dimensional $\boldsymbol{Q}$-vector space H and a decreasing filtration $F = \{F^p H_{\boldsymbol{C}}\}$ of $H_{\boldsymbol{C}} = H \otimes_{\boldsymbol{Q}} \boldsymbol{C}$ such that $F^p H_{\boldsymbol{C}} \cap \bar{F}^{n-p+1} H_{\boldsymbol{C}} = \{0\}$ for all p, where $\bar{F}$ is the filtration conjugate to F. 2) A *mixed $\boldsymbol{Q}$-Hodge structure* is a triple (H, W, F) consisting of a $\boldsymbol{Q}$-vector space H as above, an increasing filtration $W = \{W_n H\}$ on H and a decreasing filtration F of $H_{\boldsymbol{C}}$ with the following property; for any $n \in \boldsymbol{Z}$ let $H^n = \mathrm{Gr}_W^n H = W_n H / W_{n-1} H$ and $F_{(n)}$ the filtration induced on $H_{\boldsymbol{C}}^n = H^n \otimes_{\boldsymbol{Q}} \boldsymbol{C}$ by F. Then the pair $(H^n, F_{(n)})$ is a $\boldsymbol{Q}$-Hodge structure of weight n in the sense of 1). In this case we also say that (H, W, F) is a mixed $\boldsymbol{Q}$-Hodge structure on H, or H has the mixed $\boldsymbol{Q}$-Hodge structure (H, W, F).

(1.3.2) **Example.** a) If (H, W, F) is a mixed $\boldsymbol{Q}$-Hodge structure, then for any integer r, $(H, W[-2r], F[r])$ is again a mixed $\boldsymbol{Q}$-Hodge structure which we shall denote simply by $H[r]$. b) Let $X \in \mathscr{C}$. Then by (1.1) 4) for every i the pair $(H^i(X, \boldsymbol{Q}), F)$ has the natural $\boldsymbol{Q}$-Hodge structure of weight i.

Let (H_i, W, F), $i=1, 2$, be mixed $\boldsymbol{Q}$-Hodge structures. Then a linear mapping $f: H_1 \to H_2$ is called a morphism of mixed $\boldsymbol{Q}$-Hodge structures if f (resp. $f_{\boldsymbol{C}}=f \otimes_{\boldsymbol{Q}} \boldsymbol{Q}: H_{1\boldsymbol{C}} \to H_{2\boldsymbol{C}}$) is compatible with the filtration W (resp. F). With morphisms thus defined mixed $\boldsymbol{Q}$-Hodge structures form an abelian category (MH) [3, 2.3.5]. In particular the kernel, image etc. of a morphism f in (MH) have the natural induced mixed $\boldsymbol{Q}$-Hodge structures. For a mixed $\boldsymbol{Q}$-Hodge structure (H, W, F) we call a subspace $E \subseteq H$ briefly a mixed $\boldsymbol{Q}$-Hodge substructure of H if $(E, W|_E, F|_E)$ is one.

(1.4) In [3] and [4] Deligne has defined for any algebraic variety X a natural mixed $\boldsymbol{Q}$-Hodge structure on its rational cohomology group $H^{\cdot}(X, \boldsymbol{Q})$, which is functorial in X. By the property of the category $\mathscr{C}$ listed in (1.1) together with [14] his construction extends without further change to the category $\tilde{\mathscr{C}}$ (cf. [4, 6.2]). Namely we have the following:

(1.4.1) **Proposition.** *For any meromorphic space $X \in \tilde{\mathscr{C}}$ there is a natural mixed $\boldsymbol{Q}$-Hodge structure on its rational cohomology group $H^{\cdot}(X, \boldsymbol{Q})$ which is functorial in X. Moreover if Z is a Zariski locally closed subset of X (i.e., its closure is analytic in X), then there is a natural mixed $\boldsymbol{Q}$-Hodge structure on the relative cohomology group $H^{\cdot}(X, Z, \boldsymbol{Q})$ which is functorial with respect to the pair (X, Z).*

For the latter statement see [4, 8.3.3], where it was also shown that the exact sequence of relative cohomology

$$\to H^i(X, Z, \boldsymbol{Q}) \to H^i(X, \boldsymbol{Q}) \to H^i(Z, \boldsymbol{Q}) \to$$

becomes one in (MH) if each term is given a mixed $\boldsymbol{Q}$-Hodge structure as in the above proposition ([4, 8.3.9]). Note that if Z is open, then $H^{\cdot}(X, Z, \boldsymbol{Q})$ is naturally isomorphic to the local cohomology group $H^{\cdot}_Y(X, \boldsymbol{Q})$, $Y=X-Z$, and the above sequence is isomorphic to the corresponding exact sequence of local cohomology. In particular this defines a natural mixed $\boldsymbol{Q}$-Hodge structure on the local cohomology group $H^{\cdot}_Y(X, \boldsymbol{Q})$. On the other hand, for any $U \in \tilde{\mathscr{C}}$ we may define the natural mixed $\boldsymbol{Q}$-Hodge structure on $H^{\cdot}_c(U, \boldsymbol{Q})$ (the cohomology with compact supports) in the following manner. Take any compactification $X \in \mathscr{C}$ of U and let $Y=X-U$. Then we have the natural isomorphism $H^{\cdot}_c(U, \boldsymbol{Q}) \cong H^{\cdot}(X, Y, \boldsymbol{Q})$. Then we define the structure to be that induced from $H^{\cdot}(X, Y, \boldsymbol{Q})$ by this isomorphism. By functoriality of the mixed Hodge structure this definition is independent of the choice of X (cf. the proof of [3, 3.2.11]).

(1.5) Let X be a compact complex manifold of pure dimension n and Y an analytic subset of X. Let $U=X-Y$. Then as in the introduction we have the following pair of exact sequences

$$(2)\qquad \begin{cases} \to H_c^i(U, \boldsymbol{Q}) \xrightarrow{\alpha_i} H^i(X, \boldsymbol{Q}) \xrightarrow{\beta_i} H^i(Y, \boldsymbol{Q}) \xrightarrow{\gamma_i} \\ \leftarrow H^{2n-i}(U, \boldsymbol{Q}) \xleftarrow{\alpha_i'} H^{2n-i}(X, \boldsymbol{Q}) \xleftarrow{\beta_i'} H_Y^{2n-i}(X, \boldsymbol{Q}) \xleftarrow{\gamma_i'} . \end{cases}$$

Suppose that $X \in \mathscr{C}$. Then by (1.4) each term of the above sequences has the natural mixed $\boldsymbol{Q}$-Hodge structure and the sequences are those in (MH). Indeed, we have the following precise information on the behavior of the filtration W under morphisms [4, 8.2.4];

$$(3)\qquad \begin{cases} W_i H_c^i(U, \boldsymbol{Q}) = H_c^i(U, \boldsymbol{Q}), & W_{i-1} H_c^i(U, \boldsymbol{Q}) = \operatorname{Im} \gamma_{i-1}, \\ W_{2n-i-1} H^{2n-i}(U, \boldsymbol{Q}) = \{0\}, & W_{2n-i} H^{2n-i}(U, \boldsymbol{Q}) = \operatorname{Im} \alpha_i', \end{cases}$$

where Im denotes the image. (From the proof below we infer readily that we need (3) only in the case where Y is a divisor with only normal crossings in X. Indeed, in this case (3) follows easily from the description in (3.10).)

On the other hand, we have the natural perfect bilinear pairings

$$\psi_U: H_c^i(U, \boldsymbol{Q}) \times H^{2n-i}(U, \boldsymbol{Q}) \to \boldsymbol{Q}$$

$$\psi_X: H^i(X, \boldsymbol{Q}) \times H^{2n-i}(X, \boldsymbol{Q}) \to \boldsymbol{Q}$$

$$\psi_Y: H^i(Y, \boldsymbol{Q}) \times H_Y^{2n-i}(X, \boldsymbol{Q}) \to \boldsymbol{Q}$$

which are compatible with the morphisms in the above sequences (cf. [12, 1.6, 1.7]). Here ψ_U (resp. ψ_X) is the usual Poincaré pairing which is defined as the composition of the cup product $H_c^{\cdot}(U, \boldsymbol{Q}) \times H^{2n-\cdot}(U, \boldsymbol{Q}) \to H_c^{2n}(U, \boldsymbol{Q})$ (resp. $H^{\cdot}(X, \boldsymbol{Q}) \times H^{2n-\cdot}(X, \boldsymbol{Q}) \to H^{2n}(X, \boldsymbol{Q})$) and the canonical linear map ν_c: $H_c^{2n}(U, \boldsymbol{Q}) \to \boldsymbol{Q}$ (resp. ν: $H^{2n}(X, \boldsymbol{Q}) \to \boldsymbol{Q}$) defined by the Poincaré duality. Similarly ψ_Y is the composition of $H^i(Y, \boldsymbol{Q}) \times H_Y^{2n-i}(X, \boldsymbol{Q}) \to H_Y^{2n}(X, \boldsymbol{Q}) \to H^{2n}(X, \boldsymbol{Q}) \xrightarrow{\nu} \boldsymbol{Q}$, where the first arrow is the modified cup product (cf. [12, 1.3]) and the second is the natural homomorphism. We denote by the same letters ψ_U, ψ_X, ψ_Y the pairings between the corresponding cohomology groups with coefficients in $\boldsymbol{C}$.

(1.6) Now we ask if the pairings in (1.5) are compatible with the mixed Hodge structures in some sense or other. For this purpose we give the following:

(1.6.1) **Definition.** 1) Let (H_i, F) be $\boldsymbol{Q}$-Hodge structures of weight n_i, $i=1, 2$, and ψ: $H_1 \times H_2 \to \boldsymbol{Q}$ a perfect $\boldsymbol{Q}$-bilinear pairing. Assume that $n_1+n_2=2n$ is even. Then ψ is said to be *strictly compatible* with the $\boldsymbol{Q}$-Hodge structures if

$\psi(F^pH_{1C}, F^qH_{2C})=0$ whenever $p+q>n$ and the induced pairings ψ^p: $\mathrm{Gr}_F^p H_{1C} \times \mathrm{Gr}_F^{n-p} H_{2C} \to C$ are perfect for all p. 2) Let (H_i, W, F), $i=1, 2$, be mixed Q-Hodge structures and ψ be as above. Let n be an integer. Then ψ is called *strictly compatible with level n* with the mixed Q-Hodge structures if $\psi(W_sH_1, W_tH_2)=0$ whenever $s+t<2n$ and if the induced pairings ψ^s: $\mathrm{Gr}_W^s H_1 \times \mathrm{Gr}_W^{2n-s} H_2 \to Q$ are perfect and strictly compatible in the sense of 1) with the Q-Hodge structures of weights s and $2n-s$ on respective spaces. In this case we also say that ψ gives a *duality of mixed Q-Hodge structures of level n*.

(1.6.2) *Remark*. Let (H_i, W, F) and ψ be as in 2) of the above definition. Then the following remarks follow easily from the above definition. *a*) For a mixed Q-Hodge structure $H=(H, W, F)$ we define its dual $H'=(H', W', F')$ $\in(MH)$ as follows: H' is the dual vector space of H, $W'_i=(W_{-1-i})^{\perp}$ and $F'^p=(F^{1-p})^{\perp}$, where $\perp$ denotes the orthogonal complement (cf. [3, 1.1.7]). Then ψ gives a duality of (H_i, W, F) of level n if and only if the natural isomorphism $H_1 \cong H'_2$ of vector spaces induced by ψ gives that of mixed Hodge structures $H_1[n] \cong H'_2$ in the notation of (1.3.1) a). *b*) Let E be any mixed Q-Hodge substructure of H_1 and E' the orthogonal complement of E in H_2 with respect to ψ. Then E' is a mixed Q-Hodge substructure of H_2 and the induced pairing ψ_E: $E \times H_2/E' \to Q$ gives a duality of mixed Q-Hodge structures of level n if ψ does, where H_2/E' has the induced mixed Q-Hodge structure.

(1.7) Now our theorem is stated as follows.

(1.7.1) **Theorem.** *In the notation and assumption of* (1.5) *the pairings* ψ_U, ψ_X, ψ_Y *give dualities of mixed Q-Hodge structures of level n, defined naturally on each term of* (2) *by* (1.4).

Remark. Theorem is true even if X is a rational homology manifold, as one sees easily by using a resolution of X and reducing to the smooth case (cf. the proof of [4, 8.2.4 iv] and (1.8.3) below).

We shall first give an immediate corollary of the theorem. Let X (resp. Y) be a complex manifold of pure dimension m (resp. n) and f: $X \to Y$ a morphism. Let D_X^c: $H_c^{\cdot}(X, Q) \cong H^{2m-\cdot}(X, Q)'$ (resp. D_Y^c: $H_c^{\cdot}(Y, Q) \cong H^{2n-\cdot}(Y, Q)'$) be the Poincaré isomorphism, where $'$ denotes the dual space. Then we have the Gysin map with compact supports

$$f_*^c\colon H_c^{\cdot}(X, Q) \to H_c^{\cdot-2r}(Y, Q), \quad r=m-n$$

defined by $f_*^c = D_Y^c (f^*)' D_X^c$. If f is proper, we get also the usual Gysin map

$$f_*: H^{\cdot}(X, \boldsymbol{Q}) \to H^{\cdot-2r}(Y, \boldsymbol{Q})$$

defined by a similar formula. Then from Remark (1.6.2) a) and the above theorem we have the following:

(1.7.2) **Corollary.** *Suppose that $X, Y \in \tilde{\mathscr{C}}$ and $f: X \to Y$ is a morphism in $\tilde{\mathscr{C}}$. Then in the notation of Example (1.3.2) a) f_*^c induces an isomorphism of mixed $\boldsymbol{Q}$-Hodge structures*

$$H_c^{\cdot}(X, \boldsymbol{Q}) \cong H_c^{\cdot-2r}(X, \boldsymbol{Q})[-r].$$

If f is proper, then the same is true for f_*;

$$H^{\cdot}(X, \boldsymbol{Q}) \cong H^{\cdot-2r}(Y, \boldsymbol{Q})[-r].$$

(1.8) We make some preliminary reductions of the proof of the theorem.

(1.8.1) For ψ_X the result is well-known. For completeness we shall give a proof. As in Example (1.3.2) b) $H^k(X, \boldsymbol{Q})$ has a natural Hodge structure $(H^k(X, \boldsymbol{Q}), F)$ of weight k. Let $\mathscr{E}_X^{\cdot}$ (resp. $'\mathscr{D}_X^{\cdot}$) be the complex of sheaves of germs of complex valued C^∞-forms (resp. currents) on X. Let $\mathscr{E}_X^{p,q}$ (resp. $'\mathscr{D}_X^{p,q}$) be the sheaf of germs of C^∞-forms (resp. currents) of type (p, q) on X. Then we have the following commutative diagram

$$\begin{array}{ccccc} H^i(X, \boldsymbol{C}) & \times & H^{2n-i}(X, \boldsymbol{C}) & \xrightarrow{\psi_X} & \boldsymbol{C} \\ \downarrow e_i & & \downarrow e'_{2n-i} & & \| \\ H^i\Gamma(X, \mathscr{E}_X^{\cdot}) & \times & H^{2n-i}\Gamma(X, '\mathscr{D}_X^{\cdot}) & \xrightarrow{\psi'_X} & \boldsymbol{C} \end{array}$$

where the vertical arrows are de Rham isomorphisms (cf. [19] for e'_{2n-i}) and ψ'_X is induced by the natural pairing $\Gamma(X, \mathscr{E}_X^{\cdot}) \times \Gamma(X, '\mathscr{D}_X^{2n-\cdot}) \to \boldsymbol{C}$. Hence considering the filtration F_0 induced from F via e_i (resp. e'_{2n-i}) on $H^i\Gamma(X, \mathscr{E}_X^{\cdot})$ (resp. $H^{2n-i}\Gamma(X, '\mathscr{D}_X^{\cdot})$), it suffices to prove the corresponding assertion for ψ'_X. First we note that $F_0^p H^k\Gamma(X, \mathscr{K}_X^{\cdot}) = \operatorname{Im}(H^k\Gamma(X, \bigoplus_{s \geq p} \mathscr{K}_X^{s, \cdot-s}) \to H^k\Gamma(X, \mathscr{K}_X^{\cdot}))$, where $\mathscr{K}_X^{\cdot} = \mathscr{E}_X^{\cdot}$, or $'\mathscr{D}_X^{\cdot}$. Hence it is clear that $\psi'_X(F_0^p H^i\Gamma(X, \mathscr{E}_X^{\cdot}), F_0^q H^{2n-i} \cdot \Gamma(X, '\mathscr{D}_X^{\cdot})) = 0$ if $p+q > n$. Then we have to show that the induced pairing, $\psi_X'^p$: $\operatorname{Gr}_{F_0}^p H^i\Gamma(X, \mathscr{E}_X^{\cdot}) \times \operatorname{Gr}_{F_0}^{n-p} H^{2n-i}\Gamma(X, '\mathscr{D}_X^{\cdot}) \to \boldsymbol{C}$, is perfect. In fact, expressing (1) in terms of the complex $\Gamma(X, \mathscr{K}_X^{\cdot})$, $\mathscr{K}_X^{\cdot} = \mathscr{E}_X^{\cdot}$ or $'\mathscr{D}_X^{\cdot}$, we see that the degeneracy of (1) is equivalent to the first of the following isomorphisms

$$\operatorname{Gr}_{F_0}^s H^k\Gamma(X, \mathscr{K}_X^{\cdot}) \cong H^k \operatorname{Gr}_{F_0}^s \Gamma(X, \mathscr{K}_X^{\cdot}) \cong H^{k-s}(X, \Omega_X^s)$$

where the second is the standard Dolbeault isomorphism. Further by these isomorphisms $\psi_X'^p$ corresponds to the perfect pairing $H^{i-p}(X, \Omega_X^p) \times H^{n-i+p}(X, \Omega_X^{n-p}) \to \boldsymbol{C}$ giving the Serre duality and hence itself is perfect.

(1.8.2) We prove the theorem for ψ_U assuming that the theorem is true for ψ_Y. Consider the following pair of short exact sequences (cf. (2))

$$0 \to \operatorname{Im} \gamma_{i-1} \to H_c^i(U, \boldsymbol{Q}) \to \operatorname{Im} \alpha_i \to 0$$
$$0 \leftarrow \operatorname{Im} \gamma'_{i-1} \leftarrow H^{2n-i}(U, \boldsymbol{Q}) \leftarrow \operatorname{Im} \alpha'_i \leftarrow 0$$

where Im denotes the image. By the compatibility of the pairings with the sequences (2) it follows that ψ_U induces perfect pairings ψ'_U: $\operatorname{Im} \gamma_{i-1} \times \operatorname{Im} \gamma'_{i-1} \to \boldsymbol{Q}$ and ψ''_U: $\operatorname{Im} \alpha_i \times \operatorname{Im} \alpha'_i \to \boldsymbol{Q}$. Moreover (2) are exact sequences in (MH), and by (1.8.1) and the assumption ψ_X and ψ_Y give duality of mixed Hodge structures of level n. Hence from Remark (1.6.2), b) we deduce that ψ'_U and ψ''_U also give a duality of level n of the induced mixed Hodge structures on the corresponding terms. On the other hand, from (3) we get that $\operatorname{Im} \gamma_{i-1} = W_{i-1} H_c^i(U, \boldsymbol{Q})$ and $\operatorname{Im} \gamma'_{i-1} \cong H^{2n-i}(U, \boldsymbol{Q}) / W_{2n-i} H^{2n-i}(U, \boldsymbol{Q})$ (resp. $\operatorname{Im} \alpha_i \cong H_c^i(U, \boldsymbol{Q}) / W_{i-1} H_c^i(U, \boldsymbol{Q})$ and $\operatorname{Im} \alpha'_i = W_{2n-i} H^{2n-i}(U, \boldsymbol{Q})$). By the definition of duality in (1.6), from these we conclude that ψ_U itself gives a duality of mixed $\boldsymbol{Q}$-Hodge structures of level n.

(1.8.3) We reduce the proof for ψ_Y to the case where Y is a divisor with normal crossings in X. For this purpose take by Hironaka [14] a proper bimeromorphic morphism f: $\tilde{X} \to X$ such that $\tilde{X}$ is nonsingular, $\tilde{Y} = f^{-1}(Y)$ is a divisor with normal crossings in $\tilde{X}$ and that f gives an isomorphism of $\tilde{X} - \tilde{Y}$ and $X - Y$. Suppose that the pairing $\psi_{\tilde{Y}}$: $H^i(\tilde{Y}, \boldsymbol{Q}) \times H_{\tilde{Y}}^{2n-i}(\tilde{X}, \boldsymbol{Q}) \to \boldsymbol{Q}$ is a duality of mixed $\boldsymbol{Q}$-Hodge structures of level n. We consider the induced homomorphism

$$f^*: H^i(Y, \boldsymbol{Q}) \to H^i(\tilde{Y}, \boldsymbol{Q}) \quad (\text{resp. } H_Y^{2n-i}(X, \boldsymbol{Q}) \to H_{\tilde{Y}}^{2n-i}(\tilde{X}, \boldsymbol{Q})),$$

which turns out to be injective. Indeed the relation $\psi_{\tilde{Y}}(f^*a, f^*b) = \psi_Y(a, b)$, $a \in H^i(Y, \boldsymbol{Q})$, $b \in H_Y^{2n-i}(X, \boldsymbol{Q})$, which follows immediately from the definition, gives us the left inverse f_*: $H^i(\tilde{Y}, \boldsymbol{Q}) \to H^i(Y, \boldsymbol{Q})$ (resp. $H_{\tilde{Y}}^{2n-i}(\tilde{X}, \boldsymbol{Q}) \to H_Y^{2n-i}(X, \boldsymbol{Q})$) of f^* by a formula similar to f_*^c in (1.7). Now identify $H^i(Y, \boldsymbol{Q})$ as a subspace of $H^i(\tilde{Y}, \boldsymbol{Q})$ by means of f^*. Let H be the orthogonal complement of $H^i(Y, \boldsymbol{Q})$ in $H_{\tilde{Y}}^{2n-i}(\tilde{X}, \boldsymbol{Q})$ with respect to $\psi_{\tilde{Y}}$. Then by Remark (1.6.2), b) $\psi_{\tilde{Y}}$ induces a duality of mixed $\boldsymbol{Q}$-Hodge structures of level n of $H^i(Y, \boldsymbol{Q})$ and $H_{\tilde{Y}}^{2n-i}(\tilde{X}, \boldsymbol{Q})/H$. On the other hand, we have the natural isomorphism $H_Y^{2n-i}(X, \boldsymbol{Q}) \cong H_{\tilde{Y}}^{2n-i}(\tilde{X}, \boldsymbol{Q})/H$ in (MH) induced by f^* (cf. [3, 1.2.10 ii)]). It follows then that ψ_Y also induces a duality of mixed $\boldsymbol{Q}$-Hodge structures of level n. Note that in the above proof we may further assume that every irreducible component of $\tilde{Y}$ is nonsingular.

(1.9) By (1.8) what is left to show is that ψ_Y gives a duality of mixed $\boldsymbol{Q}$-Hodge structures of level n when Y is a divisor with normal crossings in X and with smooth irreducible components. This will be shown in the next two sections. Indeed, the purpose of these sections is to establish the following pair of commutative diagrams (A) and (B) of complexes in $\mathscr{A}(X)$, or in $\mathscr{D}\mathscr{A}(X)$ if one likes (which is in fact indispensable for $R\underline{\Gamma}_Y(\boldsymbol{C}_X)$ in (B)), with supports in Y

(A)
$$\begin{array}{ccccccc}
 & & \Omega^\cdot_{X|Y} & \xrightarrow{\quad r_\Omega \quad} & & \tilde{\Omega}^\cdot_Y & \\
 & \nearrow^{e_{X|Y}} & \downarrow{\scriptstyle j_{X|Y}} & \searrow^{\zeta_{X|Y}} & & \downarrow{\scriptstyle j_Y} & \searrow^{\zeta_Y} \\
\boldsymbol{C}_Y & & & & \mathscr{K}^\cdot_{X|Y} \xrightarrow{\quad r_{\mathscr{K}} \quad} & & \mathscr{K}^\cdot_Y \\
 & \searrow_{e'_{X|Y}} & & \nearrow_{\xi_{X|Y}} & & & \nearrow_{\xi_Y} \\
 & & \mathscr{E}^\cdot_{X|Y} & \xrightarrow{\quad r_{\mathscr{E}} \quad} & & \mathscr{E}^\cdot_Y &
\end{array}$$

(B)
$$\begin{array}{ccccccc}
 & & Q^\cdot_X(*Y)[-1] & \xleftarrow{\quad i_Q[-1] \quad} & & Q^\cdot_X\langle Y\rangle[-1] & \\
 & \swarrow^{\mu_Y} & \downarrow{\scriptstyle \mathrm{Res}} & \searrow^{H} & & \downarrow{\scriptstyle \mathrm{Res}_0} & \searrow^{H_0} \\
R\underline{\Gamma}_Y(\boldsymbol{C}_X) & & & & \mathscr{K}^\cdot_{Y^\infty} \xleftarrow{\quad i_{\mathscr{K}} \quad} & & \mathscr{K}^\cdot_X\langle Y\rangle \\
 & \nwarrow_{\rho_Y} & & \swarrow_{\eta_{Y^\infty}} & & & \swarrow_{\eta_X\langle Y\rangle} \\
 & & '\mathscr{D}^\cdot_{Y^\infty} & \xleftarrow{\quad i_{\mathscr{D}} \quad} & & '\mathscr{D}^\cdot_X\langle Y\rangle &
\end{array}$$

in which the morphisms are all quasi-isomorphic. (Definitions of each term and morphism will be given below.) Taking hypercohomology, these give rise to the following commutative diagrams ($\tilde{\mathrm{A}}$), ($\tilde{\mathrm{B}}$) of $\boldsymbol{C}$ vector spaces

($\tilde{\mathrm{A}}$)
$$\begin{array}{ccccccc}
 & & \boldsymbol{H}^i(Y, \Omega^\cdot_{X|Y}) & \longrightarrow & & \boldsymbol{H}^i(Y, \tilde{\Omega}^\cdot_Y) & \\
 & \nearrow & \downarrow & \searrow & & \downarrow & \searrow \\
H^i(Y, \boldsymbol{C}) & & & & H^i\Gamma(Y, \mathscr{K}^\cdot_{X|Y}) \longrightarrow & & H^i\Gamma(Y, \mathscr{K}^\cdot_Y) \\
 & \searrow & & \nearrow & & & \nearrow \\
 & & H^i\Gamma(Y, \mathscr{E}^\cdot_{X|Y}) & \longrightarrow & & H^i\Gamma(Y, \mathscr{E}^\cdot_Y) &
\end{array}$$

($\tilde{\mathrm{B}}$)
$$\begin{array}{ccccccc}
 & & \boldsymbol{H}^{2n-i}(Y, Q^\cdot_X(*Y)[-1]) & \longleftarrow & & \boldsymbol{H}^{2n-i}(Y, Q^\cdot_X\langle Y\rangle[-1]) & \\
 & \swarrow & \downarrow & \searrow & & \downarrow & \searrow^{H_0} \\
H^{2n-i}_Y(X, \boldsymbol{C}) & & & & H^{2n-i}\Gamma(Y, \mathscr{K}^\cdot_{Y^\infty}) \longleftarrow & & H^{2n-i}\Gamma(Y, \mathscr{K}^\cdot_X\langle Y\rangle) \\
 & \nwarrow & & \swarrow & & & \swarrow \\
 & & H^{2n-i}\Gamma(Y, '\mathscr{D}^\cdot_{Y^\infty}) & \longleftarrow & & H^{2n-i}\Gamma(Y, '\mathscr{D}^\cdot_X\langle Y\rangle) &
\end{array}$$

in which the morphisms are all isomorphic. Further we shall see that there are natural perfect $\boldsymbol{C}$-bilinear pairings between the corresponding terms of ($\tilde{\mathrm{A}}$) and ($\tilde{\mathrm{B}}$) that are compatible with the diagrams and coincide with ψ_Y on $H^i(Y, \boldsymbol{C})$

$\times H_Y^{2n-i}(X, \boldsymbol{C})$. On the other hand, Deligne's mixed Hodge structure on $H^i(Y, \boldsymbol{Q})$ (resp. $H_Y^{2n-i}(X, \boldsymbol{Q})$) comes from the natural bifiltered structure on $H^i\Gamma(Y, \mathscr{K}_Y^{\cdot})$ (resp. $\boldsymbol{H}^{2n-i}\Gamma(Y, Q_X^{\cdot}\langle Y\rangle[-1])$) by way of the above isomorphism. We show that there is a natural bifiltered structure on $H^{2n-i}\Gamma(Y, \mathscr{K}_X^{\cdot}\langle Y\rangle)$ such that $\tilde{H}_0$ is a bifiltered isomorphism and the perfect pairing between $H^i\Gamma(Y, \mathscr{K}_Y^{\cdot})$ and $H^{2n-i}\Gamma(Y, \mathscr{K}_X^{\cdot}\langle Y\rangle)$ mentioned above is strictly compatible with the bifiltered structures on both terms (cf. (3.9) and (3.11)). This would establish our assertion.

As is clear from the above explanation, for our purpose only parts of the above diagrams are actually necessary. We develop them here hoping that it helps to clarify the whole situation. In Section 2, mainly the left halves of the above diagrams will be constructed following Herrera-Lieberman [12] and Herrera [11], and then in Section 3 the right halves will be added and proof of the above assertions will be provided.

Index of notations: $\Omega_{X|Y}^{\cdot}$ $\mathscr{E}_{X|Y}^{\cdot}$ $e_{X|Y}$ $e'_{X|Y}$ $j_{X|Y}$ (2.1), $\mathscr{K}_{X|Y}^{\cdot}$ $\zeta_{X|Y}$ $\xi_{X|Y}$ (2.6), $\tilde{\Omega}_Y^{\cdot}$ $\mathscr{E}_Y^{\cdot}$ j_Y (3.2), $\mathscr{K}_Y^{\cdot}$ ζ_Y ξ_Y r_K $r_{\mathscr{E}}$ r_Ω (3.3), $Q_X^{\cdot}(*Y)$ Res (2.3), $'\mathscr{D}_{Y^\infty}^{\cdot}$ (2.0), $\mathscr{K}_{Y^\infty}^{\cdot}$ η_{Y^∞} (2.7), H (2.8), $Q_X^{\cdot}\langle Y\rangle$ i_Q (3.1), $'\mathscr{D}_X^{\cdot}\langle Y\rangle$ $\mathscr{K}_X^{\cdot}\langle Y\rangle$ $\eta_X\langle Y\rangle$ $i_{\mathscr{D}}$ $i_{\mathscr{K}}$ (3.5), H_0 Res_0 (3.6), μ_Y ρ_Y (2.4). Further for the pairings: ϕ_1 ϕ_2 (2.5), ϕ_3 (2.10), ϕ'_1 ϕ'_2 ϕ'_3 (3.8).

(1.10) We make the following remark for later reference. Let $\mathscr{A}$ be an abelian category. Let $K_i^{\cdot}$, $i=1, 2$, be finite complexes with finite filtrations $F_i=\{F_i^pK_i^{\cdot}\}$ in $\mathscr{A}$. Let $u: K_1^{\cdot}\to K_2^{\cdot}$ be a morphism compatible with F_i. Suppose that the associated graded morphism $\mathrm{Gr}_F^p u: \mathrm{Gr}_F^p K_1^{\cdot}\to \mathrm{Gr}_F^p K_2^{\cdot}$ is quasi-isomorphic, i.e., induces isomorphism in cohomology for every p. Then u itself is quasi-isomorphic. In particular if a morphism of double complexes $u: K_1^{\cdot\cdot}\to K_2^{\cdot\cdot}$ induces for each t a quasi-isomorphism $u_t: K_1^{\cdot t}\to K_2^{\cdot t}$, then u gives a quasi-isomorphism of the associated simple complexes $K_i^{\cdot}$. (Take $F_i^pK_i^{\cdot} = \bigoplus_{t\geq p} K_i^{\cdot-t,t}$.) As a special case if $u: K_1^{\cdot}\to K_2^{\cdot\cdot}$ is a morphism of a simple complex $K_1^{\cdot}$ into a double complex $K_2^{\cdot\cdot}$ with $u(K_1^{\cdot})\subseteq K_2^{\cdot 0}$ and which induces for each s a resolution $u_s: K_1^s\to K_2^{s,\cdot}$ of K_1^s, then u induces a quasi-isomorphism $K_1^{\cdot}\to K_2^{\cdot}$, where $K_2^{\cdot}$ is the simple complex associated with $K_2^{\cdot\cdot}$. Also, holds the assertion obtained by interchanging $K_1^{\cdot}$ and $K_2^{\cdot\cdot}$.

§2. Construction of the Diagrams

(2.0) a) Throughout Sections 2 and 3 we fix a compact complex manifold X

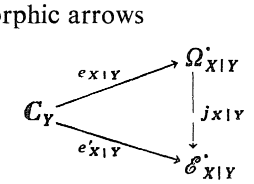
of pure dimension n, and a divisor Y on X with only normal crossings and whose irreducible components Y_i, $1 \leqq i \leqq r$, are nonsingular. Further we use the following notations: $U = X - Y$. $I = (i_1, \dots, i_q)$ an ordered q-tuple with $1 \leqq i_1 < \cdots < i_q \leqq r$. For any such I, $|I| = q$, $I_{\hat{j}} = (i_1, \dots, \hat{i}_j, \dots, i_q)$, $1 \leqq j \leqq q$, $\hat{i}_j$ implying the absense of i_j, $Y_I = Y_{i_1} \cap \cdots \cap Y_{i_q}$, $\delta_j^I \colon Y_I \to Y_{I_{\hat{j}}}$, $a_I \colon Y_I \to X$, $b_i \colon Y_i \to Y$, $a \colon Y \to X$ the natural inclusions. $Y_{(q)} = \coprod_{|I|=q} Y_I$, $\delta^{(q)} = \coprod_{|I|=q} \delta_j^I \colon Y_{(q)} \to Y_{(q-1)}$ and $a_{(q)} = \coprod a_I \colon Y_{(q)} \to X$.

b) As in Section 1 for a complex manifold Z we denote by $\Omega_Z^{\cdot}$ (resp. $\mathscr{E}_Z^{\cdot}$, ${}'\mathscr{D}_{Z,\cdot}$) the complex of sheaves of germs of holomorphic forms (resp. complex valued C^∞ forms, currents) on Z. $\mathcal{O}_Z$ is the structure sheaf of Z. If Z is of pure dimension m, we always put ${}'\mathscr{D}_Z^{\cdot} = {}'\mathscr{D}_{Z,2m-\cdot}$ to make the differential of degree $+1$. Let A be any closed subset of Z. Then $\Omega_{Z|A}^{\cdot}$ (resp. $\mathscr{E}_{Z|A}^{\cdot}$) will denote the sheaf-theoretic restriction of $\Omega_Z^{\cdot}$ (resp. $\mathscr{E}_Z^{\cdot}$) to A extended by zero to X, and ${}'\mathscr{D}_{A^\infty}^{\cdot}$ the subcomplex of ${}'\mathscr{D}_Z^{\cdot}$ of germs of currents with supports contained in A. Suppose that Z is compact. Then the natural pairing $\phi_Z \colon \Gamma(Z, \mathscr{E}_Z^{\cdot}) \times \Gamma(Z, {}'\mathscr{D}_Z^{2n-\cdot}) \to \boldsymbol{C}$ induces a natural $\boldsymbol{C}$-bilinear pairing $\phi_A \colon \Gamma(A, \mathscr{E}_{Z|A}^{\cdot}) \times \Gamma(Z, {}'\mathscr{D}_{A^\infty}^{2n-\cdot}) \to \boldsymbol{C}$. Since $\tilde{\phi}_A$ is compatible with the differentials of the complexes involved as well as ϕ_Z it induces a pairing

$$\phi_A \colon H^i\Gamma(A, \mathscr{E}_{Z|A}^{\cdot}) \times H^{2n-i}\Gamma(X, {}'\mathscr{D}_{Y^\infty}^{\cdot}) \to \boldsymbol{C}.$$

c) Let Z (resp. Z') be complex manifolds of pure dimension m (resp. n). Let $f \colon Z \to Z'$ be an embedding, $\bar{Z} = f(Z)$ and $q = n - m$. Then direct image of currents gives us the natural inclusion of complexes $f_*{}'\mathscr{D}_Z^{\cdot}[-2q] \to {}'\mathscr{D}_{\bar{Z}^\infty}^{\cdot}$ which we shall denote by $\hat{f}$, where $f_*{}'\mathscr{D}_Z^{\cdot}[-2q]$ is the (sheaf-theoretic) direct image of ${}'\mathscr{D}_Z^{\cdot}[-2q]$.

(2.1) We denote by $\boldsymbol{C}_X$ the constant sheaf on X with fiber the complex line $\boldsymbol{C}$. For any locally closed subset T of X, $\boldsymbol{C}_T$ denotes the constant sheaf on T with fiber $\boldsymbol{C}$ extended by zero to X. Let $e_X \colon \boldsymbol{C}_X \to \Omega_X^{\cdot}$ ($e'_X \colon \boldsymbol{C}_X \to \mathscr{E}_X^{\cdot}$) the natural augmentation and $j_X \colon \Omega_X^{\cdot} \to \mathscr{E}_X^{\cdot}$ the natural inclusion. We have $j_X e_X = e'_X$. Further by Poincaré lemma both e_X and e'_X give resolutions of $\boldsymbol{C}_X$. Hence, restricting to Y, e_X, e'_X and j_X induce the following commutative diagram of complexes with quasi-isomorphic arrows



where C_Y is considered as a complex with $C_Y^0=C_Y$ and $=0$ elsewhere. This gives rise to the commutative diagram (1) of (hyper) cohomology

$$(1) \qquad \begin{array}{ccc} & & H^i(Y, \Omega^{\cdot}_{X|Y}) \\ & \nearrow^{\tilde{e}_{X|Y}} & \downarrow j_{X|Y} \\ H^i(Y, C) & \searrow_{\tilde{e}'_{X|Y}} & H^i\Gamma(X, \mathscr{E}^{\cdot}_{X|Y}) \end{array}$$

in which the morphisms are all isomorphic.

(2.2) Let $''\mathscr{D}^{\cdot}_X$ be the complex of sheaves of germs of 'algebraic currents', i.e., $''\mathscr{D}^{\cdot}_X$ is defined by the presheaf of complex $V\to\mathrm{Hom}_C(\Gamma_c(V, \mathscr{E}^{2n-\cdot}_X), C)$ for $V \subseteq X$ open, where c denotes the compact support. Then $''\mathscr{D}^{\cdot}_X$ is a flabby resolution of C_X (cf. [16, 2.2]) and we have the natural inclusion $'\mathscr{D}^{\cdot}_X\to''\mathscr{D}^{\cdot}_X$. Let $\rho_Y\colon '\mathscr{D}^{\cdot}_{Y^\infty}\to''\mathscr{D}^{\cdot}_{Y^\infty}$ be the induced morphism of the subcomplexes of germs with supports in Y, or passing to the derived category $\mathscr{D}\mathscr{A}(X)$ of $\mathscr{A}(X)$ we may consider ρ_Y a map $'\mathscr{D}^{\cdot}_{Y^\infty}\to R\underline{\Gamma}_Y(C_X)$ in $\mathscr{D}\mathscr{A}(X)$ since $''\mathscr{D}^{\cdot}_X$ is flabby, where $R\underline{\Gamma}_Y$ is the derived functor of $\underline{\Gamma}_Y$ which takes the local sections with supports in Y. Now by a theorem of Poly [16], ρ_Y is isomorphic in $\mathscr{D}\mathscr{A}(X)$. Thus it gives a canonical isomorphism $\tilde{\rho}_Y\colon H^{2n-i}\Gamma(X, '\mathscr{D}^{\cdot}_{Y^\infty})\cong H^{2n-i}_Y(X, C)$, and from the definition of ρ_Y it follows readily that the pairing $\phi_Y\colon H^i\Gamma(Y, \mathscr{E}^{\cdot}_{X|Y})\times H^{2n-i}\Gamma(X, '\mathscr{D}^{\cdot}_{Y^\infty})\to C$ and ψ_Y (1.5) are compatible with $\tilde{e}_{X|Y}$ and $\tilde{\rho}_Y$.

(2.3) Let $\Omega^{\cdot}_X(*Y)$ be the complex of sheaves of germs of meromorphic forms on X whose polar loci are contained in Y. Then we put $Q^{\cdot}_X(*Y)=\Omega^{\cdot}_X(*Y)/\Omega^{\cdot}_X$. We have the following exact sequence of complexes

$$0\to\Omega^{\cdot}_X\to\Omega^{\cdot}_X(*Y)\to Q^{\cdot}_X(*Y)\to 0\,.$$

In [12] Herrera and Lieberman defined a natural complexes' homomorphism (called residue)

$$\mathrm{Res}\colon Q^{\cdot}_X(*Y)[-1]\to '\mathscr{D}^{\cdot}_{Y^\infty}$$

and for each i a homomorphism (called principal value)

$$\mathrm{PV}\colon \Omega^i_X(*Y)\to '\mathscr{D}^i_X.$$

(PV composed with the natural projection $'\mathscr{D}^{\cdot}_X\to'\mathscr{D}^{\cdot}_X/'\mathscr{D}^{\cdot}_{Y^\infty}$ is a complexes' homomorphism and this latter was actually called PV in [12].) These have the following local description. Let B be any polycylinder in X in which Y is defined by an equation $\varphi=0$ with $\varphi\in\Gamma(B, \mathcal{O}_B)$. First, to define PV let $\omega \in\Gamma(B, \Omega^{\cdot}_X(*Y))$. Then $\mathrm{PV}(\omega)\in\Gamma(B, '\mathscr{D}^{\cdot}_X)$ is given by

$$\mathrm{PV}(\omega)[\alpha]=\lim_{\delta\to 0}\int_{|\varphi|\geqq\delta}\omega\wedge\alpha, \qquad \alpha\in\Gamma_c(B, \mathscr{E}_B^{2n-\cdot})$$

(the point being that the right hand side exists) where the integration is taken over the semianalytic set $|\varphi|\geqq\delta$ in B with the natural orientation coming from the complex structure. Next, for any $\bar{\omega}\in\Gamma(B, Q_X^{\cdot}(*Y))$ take a representative $\omega\in\Gamma(B, \Omega_X^{\cdot}(*Y))$. Then $\mathrm{Res}(\bar{\omega})\in\Gamma(B, {}'\mathscr{D}_{Y^\infty}^{\cdot+1})$ is given by

$$\mathrm{Res}(\bar{\omega})[\beta]=\lim_{\delta\to 0}\int_{|\varphi|=\delta}\omega\wedge\beta, \qquad \beta\in\Gamma_c(B, \mathscr{E}_X^{2n-1-\cdot})$$

with the integration taken over the semianalytic set $|\varphi|=\delta$ with the opposite orientation to the one induced by the domain $|\varphi|\geqq\delta$. Also we denote by V the natural inclution $\Omega_X^{\cdot}\to{}'\mathscr{D}_X^{\cdot}$.

(2.4) In [12] it was further shown that the maps induced by Res on hypercohomology groups fit into an interesting commutative triangle which we shall now recall. (For the more details see [12].) Let $\mathscr{S}_X^{\cdot}$ be the complex of sheaves of semianalytic cochains on X. Let $I\colon \Omega_X^{\cdot}\to\mathscr{S}_X^{\cdot}$ be the homomorphism defined by 'integration'. Since $\mathscr{S}_X^{\cdot}$ is a flabby resolution of $\mathbf{C}_X$, I induces a morphism in $\mathscr{D}\mathscr{A}(X)$, $R\underline{\Gamma}_Y(I)\colon R\underline{\Gamma}_Y(\Omega_X^{\cdot})\to R\underline{\Gamma}_Y(\mathbf{C}_X)$, or since Y is of codimension 1 and hence $R\underline{\Gamma}_Y(\Omega_X^{\cdot})\cong Q_X^{\cdot}(*Y)[-1]$ by a theorem of Grothendieck, a morphism $\mu_Y\colon Q_X^{\cdot}(*Y)[-1]\to R\underline{\Gamma}_Y(\mathbf{C}_X)$. Then we have the following diagram in $\mathscr{D}\mathscr{A}(X)$

(2)
$$\begin{array}{ccc} & \overset{\mu_Y}{\longleftarrow} & Q_X^{\cdot}(*Y)[-1] \\ R\underline{\Gamma}_Y(\mathbf{C}_X) & & \big\downarrow \mathrm{Res} \\ & \underset{\rho_Y}{\longleftarrow} & {}'\mathscr{D}_{Y^\infty}^{\cdot} \end{array}$$

which is in fact commutative by [12, Th. 5.1] (noting that $\mathscr{H}^k Q_X^{\cdot}(*Y)_x\cong\mathbf{H}^k(x, Q_X^{\cdot}(*Y))\cong\varinjlim_{x\in V}\mathbf{H}^k(V, Q_X^{\cdot}(*Y))$ so that (2) follows by passing to the limit). Since μ_Y is isomorphic by a theorem of Grothendieck (cf. [12, Corollary 2.4]) and ρ_Y is isomorphic by Poly (2.2), Res is (quasi)-isomorphic. Hence passing to the hypercohomology we get the following commutative diagram

(3)
$$\begin{array}{ccc} & \overset{\bar{\mu}_Y}{\longleftarrow} & \mathbf{H}^{2n-i}(X, Q_X^{\cdot}(*Y)[-1]) \\ H_Y^{2n-i}(X, \mathbf{C}) & & \big\downarrow \mathrm{Res} \\ & \underset{\bar{\rho}_Y}{\longleftarrow} & H^{2n-i}\Gamma(X, {}'\mathscr{D}_{Y^\infty}^{\cdot}) \end{array}$$

in which the morphisms are all isomorphic.

(2.5) We define a pairing between the triangles (1) and (3). First, we define

$$\phi_1 : \boldsymbol{H}^i(X, \Omega^{\cdot}_{X|Y}) \times \boldsymbol{H}^{2n-i}(X, Q^{\cdot}_X(*Y)[-1]) \to \boldsymbol{C}$$

as follows. Let u: $\Omega^{\cdot}_{X|Y} \otimes_{\mathcal{O}_X} Q^{2n-\cdot}_X(*Y)[-1] \to Q^{2n}_X(*Y)[-1]$ be the $\mathcal{O}_X$-linear mapping induced by the exterior product. This gives rise to a natural bilinear pairing ϕ_1': $\boldsymbol{H}^i(X, \Omega^{\cdot}_{X|Y}) \times \boldsymbol{H}^{2n-i}(X, Q^{\cdot}_X(*Y)[-1]) \to \boldsymbol{H}^{2n}(X, Q^{\cdot}_X(*Y)[-1])$ (cf. [12, 1.1–1.4]). Define a $\boldsymbol{C}$-linear mapping T: $\boldsymbol{H}^{2n}(X, Q^{\cdot}_X(*Y)[-1]) \to \boldsymbol{C}$ by $T = \nu_0 \tilde{\mu}_Y$, where $\tilde{\mu}_Y$: $\boldsymbol{H}^{2n}(X, Q^{\cdot}_X(*Y)[-1]) \to H^{2n}_Y(X, \boldsymbol{C})$ is as in (2.4) and ν: $H^{2n}_Y(X, \boldsymbol{C}) \to \boldsymbol{C}$ is given by the Poincaré duality. Then define $\phi_1 = T\phi_1'$. Next let ϕ_2: $H^i\Gamma(X, \mathscr{E}^{\cdot}_{X|Y}) \times H^{2n-i}\Gamma(X, '\mathscr{D}^{\cdot}_{Y^\infty}) \to \boldsymbol{C}$ be the pairing ϕ_Y defined in (2.0) b). Then the compatibility of ϕ_1 and ϕ_2 with $\tilde{j}_{X|Y}$ and $\tilde{\mathrm{R}}\mathrm{es}$ was shown in the proof of [12, 5.7(c)]. Since ψ_Y and ϕ_2 are compatible with $\tilde{\rho}_Y$ and $\tilde{e}_{X|Y}$ by (2.2), by the commutativity we have obtained the following: There is a natural perfect $\boldsymbol{C}$-bilinear pairing between the triangles (1) and (3).

(2.6) Define a double complex $\mathscr{K}^{\cdot\cdot}_{X|Y}$ in $\mathscr{A}(X)$ as follows:

$$\begin{aligned} \mathscr{K}^{s,t}_{X|Y} &= \bigoplus_{|I|=t+1} \mathscr{E}^s_{X|Y_I}, \qquad & s, t \geqq 0 \\ &= \ 0, & \text{otherwise} \end{aligned}$$

where the differential d': $\mathscr{K}^{\cdot,\cdot}_{X|Y} \to \mathscr{K}^{\cdot+1,\cdot}_{X|Y}$ is induced from the complexes $\mathscr{E}^{\cdot}_{X|Y_I}$ and d'': $\mathscr{K}^{\cdot,t}_{X|Y} \to \mathscr{K}^{\cdot,t+1}_{X|Y}$ is defined by

$$d'' = \sum_{j=1}^{t+1} (-1)^{t+j} \bigoplus_I \delta^{I*}_j$$

δ^{I*}_j: $\mathscr{E}^{\cdot}_{X|Y_I} \to \mathscr{E}^{\cdot}_{X|Y_{\hat{I}_j}}$ being the restriction mappings induced by δ^I_j. Let $\mathscr{K}^{\cdot}_{X|Y}$ be the associated simple complex. Then define $\xi_{X|Y}$: $\mathscr{E}^{\cdot}_{X|Y} \to \mathscr{K}^{\cdot}_{X|Y}$ by the composition of the restriction $\mathscr{E}^{\cdot}_{X|Y} \to \bigoplus_{i=1}^{r} \mathscr{E}^{\cdot}_{X|Y_i} = \mathscr{K}^{\cdot,0}_{X|Y}$ and the natural inclusion $\mathscr{K}^{\cdot,0}_{X|Y} \to \mathscr{K}^{\cdot}_{X|Y}$. Then define $\zeta_{X|Y}$: $\Omega^{\cdot}_{X|Y} \to \mathscr{K}^{\cdot}_{X|Y}$ by $\zeta_{X|Y} = \xi_{X|Y} j_{X|Y}$. Thus we obtain the following commutative triangle of complexes

$$\begin{array}{ccc} \Omega^{\cdot}_{X|Y} & \xrightarrow{\ \zeta_{X|Y}\ } & \\ {\scriptstyle j_{X|Y}}\downarrow & & \boldsymbol{K}^{\cdot}_{X|Y} \\ \mathscr{E}^{\cdot}_{X|Y} & \xrightarrow{\ \xi_{X|Y}\ } & \end{array}$$

On the other hand, by a Mayer-Vietoris argument one gets readily that the sequence

$$0 \longrightarrow \mathscr{E}^s_{X|Y} \xrightarrow{\xi^s_{X|Y}} \mathscr{K}^{s,0}_{X|Y} \xrightarrow{d''} \cdots \xrightarrow{d''} \mathscr{K}^{s,r}_{X|Y} \longrightarrow 0$$

is exact for every $s \geqq 0$, which in turn implies that $\xi_{X|Y}$ is a quasi-isomorphism (cf. (1.10)). Since $j_{X|Y}$ is also quasi-isomorphic (2.1), so is $\zeta_{X|Y}$, and passing to hypercohomology we have the following commutative diagram

(4)
$$\begin{array}{ccc} \boldsymbol{H}^i(Y, \Omega^{\cdot}_{X|Y}) & \xrightarrow{\zeta_{X|Y}} & \\ \downarrow \tilde{j}_{X|Y} & & H^i\Gamma(Y, \mathscr{K}^{\cdot}_{X|Y}) \\ H^i\Gamma(Y, \mathscr{E}^{\cdot}_{X|Y}) & \xrightarrow{\xi_{X|Y}} & \end{array}$$

in which the morphisms are all isomorphic.

(2.7) Define the double complex $\mathscr{K}^{\cdot\cdot}_{Y^\infty}$ in $\mathscr{A}(X)$ as follows.

$$\mathscr{K}^{s,t}_{Y^\infty} = \bigoplus_{|I|=-t+1} {}'\mathscr{D}^s_{Y_I^\infty}, \qquad s \geqq 0, t \leqq 0$$
$$= 0, \qquad \text{otherwise}$$

where the differential $d'\colon \mathscr{K}^{\cdot,\cdot}_{Y^\infty} \to \mathscr{K}^{\cdot+1,\cdot}_{Y^\infty}$ is induced from the complex ${}'\mathscr{D}^{\cdot}_{Y_I^\infty}$, and $d''\colon \mathscr{K}^{\cdot,t}_{Y^\infty} \to \mathscr{K}^{\cdot,t+1}_{Y^\infty}$ is defined by

$$d'' = \sum_{j=0}^{-t+1} (-1)^{t+j} \bigoplus_I \delta^I_{j*}$$

$\delta^I_{j*}\colon {}'\mathscr{D}^{\cdot}_{Y_I^\infty} \to {}'\mathscr{D}^{\cdot}_{Y_{I_j}^\infty}$ being the natural inclusion induced by δ^I_j. Let $\mathscr{K}^{\cdot}_{Y^\infty}$ be the associated simple complex. Let $u\colon \mathscr{K}^{\cdot,0}_{Y^\infty} = \bigoplus_{i=1}^r {}'\mathscr{D}^{\cdot}_{Y_i^\infty} \to {}'\mathscr{D}^{\cdot}_{Y^\infty}$ be defined by $u(\alpha_1,\dots,\alpha_r) = \sum_{i=1}^r \alpha_i$. Composed with the natural projection $\mathscr{K}^{\cdot}_{Y^\infty} = \bigoplus \mathscr{K}^{\cdot\cdot}_{Y^\infty} \to \bigoplus \mathscr{K}^{\cdot,0}_{Y^\infty}$ this gives a morphism of complexes

$$\eta_{Y^\infty}\colon \mathscr{K}^{\cdot}_{Y^\infty} \to {}'D^{\cdot}_{Y^\infty}.$$

This is quasi-isomorphic since the following sequence

$$0 \longrightarrow \mathscr{K}^{s,-r+1}_{Y^\infty} \xrightarrow{d''} \cdots \xrightarrow{d''} \mathscr{K}^{s,0}_{Y^\infty} \xrightarrow{\eta_{Y^\infty}} {}'\mathscr{D}^s_{Y^\infty} \longrightarrow 0$$

i.e., the sequence

$$0 \longrightarrow {}'\mathscr{D}^s_{Y^\infty_{(1,\dots,r)}} \longrightarrow \cdots \longrightarrow \bigoplus_{i=1}^r {}'\mathscr{D}^s_{Y_i^\infty} \xrightarrow{\eta_{Y^\infty}} {}'\mathscr{D}^s_{Y^\infty} \longrightarrow 0$$

is exact for every s (cf. (1.10) and [16, 3.2] or the proof of Lemma (3.7.1) below).

(2.8) The construction of PV and Res in (2.3) has been generalized by Herrera [11] (cf. also [17, § 5]) to higher codimension. In our case it gives for every I the complexes' homomorphism

$$h_I\colon Q^{\cdot}_X(*Y)[-q] \longrightarrow {}'D^{\cdot}_{Y_I^\infty}.$$

This is defined by the following local formula; let B be any polydisc in X such that each Y_i is defined on B by an equation $\varphi_i=0$ with $\varphi_i \in \Gamma(B, \mathcal{O}_X)$. Let $J=(j_1,\dots, j_{r-q})$ be the $(r-q)$-tuple such that $\{i_1,\dots, i_q, j_1,\dots, j_{r-q}\}=\{1,\dots, r\}$ and put $\varphi_J=\varphi_{j_1}\cdots\varphi_{j_{r-q}}$. Then for any $\bar{\omega} \in \Gamma(B, Q_X^{\cdot -q}(*Y))$

$$h_I(\bar{\omega})=\mathrm{PV}_{\varphi_J}\,\mathrm{Res}_{\varphi_{i_1},\dots,\varphi_{i_q}}(\omega) \in \Gamma(B, {}'\mathscr{D}^{\cdot}_{Y_I^\infty})$$

i.e., for any $\beta \in \Gamma_c(B, \mathscr{E}_X^{2n-\cdot})$

$$h_I(\bar{\omega})[\beta]=\lim_{\delta\to 0}\int_{B(\delta)} \omega\wedge\beta\,, \quad B(\delta)=\{|\varphi_J|>\delta,\ |\varphi_{i_\mu}|=\delta,\ 1\leqq \mu\leqq q\}$$

where $\omega \in \Gamma(B, \Omega_X^{\cdot -q}(*Y))$ is any representative of $\bar{\omega}$, and the integration is taken over the semianalytic set $B(\delta)$ with a suitable orientation.

Using h_I we define a complexes' homomorphism

$$H\colon Q_X^{\cdot}(*Y)[-1] \to \mathscr{K}^{\cdot}_{Y^\infty}$$

as follows; $H=\bigoplus_t h_t$, $h_t\colon Q_X^{\cdot}(*Y)[-1]\to \mathscr{K}_{Y^\infty}^{\cdot -t,t}$, $h_t=\bigoplus_{|I|=-t+1} h_I[t]$, $h_I[t]$: $Q_X^{\cdot}(*Y)[-1]\to {}'\mathscr{D}^{\cdot}_{Y_I^\infty}[-t]$, where $h_I[t]$ is the translation to the left by t of h_I defined above.

(2.9) **Lemma.** *The following diagram* (5) *is commutative*

$$\text{(5)} \qquad \begin{array}{ccc} Q_X^{\cdot}(*Y)[-1] & \xrightarrow{\ H\ } & \mathscr{K}^{\cdot}_{Y^\infty} \\ \Big\downarrow{\scriptstyle \mathrm{Res}} & \swarrow{\scriptstyle \eta_{Y^\infty}} & \\ {}'\mathscr{D}^{\cdot}_{Y^\infty} & & \end{array}$$

Moreover the morphisms are all quasi-isomorphic in this triangle.

Proof. Let x be any point of X. Take a polydisc B in X which contains x and in which every irreducible component Y_i of Y is defined by a single equation $\varphi_i=0$. Let $\varphi=\varphi_1\cdots\varphi_r$. Let $\omega \in \Gamma(B, \Omega_X^p(*Y))$ and $\alpha \in \Gamma_c(B, \mathscr{E}_X^{2n-p-1})$ be any elements. Then by the above definitions what we have to show amounts to the following equality

$$\text{(6)} \qquad \lim_{\delta\to 0}\int_{|\varphi|=\delta} \omega\wedge\alpha=\sum_{i=1}^{r}\lim_{\delta\to 0}\int_{|\varphi_i|=\delta}\omega\wedge\alpha\,.$$

We shall show (6). Put $\theta=\omega\wedge\alpha$. By our assumption, taking B small enough we can take coordinates $(z_1,\dots, z_n)$ of B in such a way that $z_i=\varphi_i$ for $1\leqq i\leqq s$ for some $s\leqq r$ and $\varphi_i=1$ for $s+1\leqq i\leqq r$. For any subset J of $\mathfrak{S}=\{1,\dots, s\}$ let $\varphi_J=\prod_{i\in J}\varphi_i$ and $I_J(\theta)=\lim_{\delta\to 0}\int_{|\varphi_J|=\delta}\theta$. First we write $\theta=\theta'+\theta''$ with θ' (resp. θ'') of bidegree $(n, n-1)$ (resp. $(n-1, n)$). Then the proof of [12, Proposition

6.5 (9)] shows that $I_J(\theta'')=0$ for any J. Thus to prove (6) we may assume that $\theta=\theta'$. Then write

$$\theta=\sum_{i=1}^{n}\theta_i,\ \theta_i=k_i/z_i^{\alpha_i}dz_i\wedge dz(i)\wedge d\bar{z}(i)\,, \tag{7}$$

where $\alpha_i\geqq 0$, $dz(i)\wedge d\bar{z}(i)=\prod_{j\neq i}dz_j\wedge d\bar{z}_j$, and k_i are C^∞ semi-meromorphic forms on B whose polar loci are contained in $\bigcup_{j\neq i}Y_j$. Then as in the proof of [12, Proposition 6.5 (8), (10)] we get that for any J

$$I_J(\theta)=\sum_{i\in J}\lim_{\delta\to 0}2\pi\sqrt{-1}/(\alpha_i-1)!\int_{B\cap Y_i\cap\{|\varphi_i|\geqq\delta\}}a_i^*(D_i^{(\alpha_i-1)}k_i)dz(i)d\bar{z}(i)\,, \tag{8}$$

where $D_i^\alpha=\partial^\alpha/\partial z_i^\alpha$ and the integrals in the sum are actually finite by [12]. From this, taking $J=\mathfrak{S}$, $\{1\},\dots,\{s\}$ (6) follows immediately. Finally the last assertion follows from the commutativity, (2.4) and (2.7).

Remark. In the above proof, if $\alpha_1=\cdots=\alpha_s=1$ in (7), (8) gives for each $i\in\mathfrak{S}$ the following:

$$I_i(\theta)=2\pi\sqrt{-1}\lim_{\delta\to 0}\int_{B\cap Y_i\cap\{|\varphi_i|\geqq\delta\}}a_i^*k_idz(i)\wedge d\bar{z}(i)\,. \tag{9}$$

As a corollary we get the following commutative diagram of hypercohomology groups with isomorphic arrows

$$\begin{array}{ccc} \boldsymbol{H}^{\cdot}(X, Q_X^{\cdot}(*Y)[-1]) & \xrightarrow{\tilde{H}} & \\ \downarrow{\scriptstyle \widetilde{\mathrm{Res}}} & & H^{\cdot}\Gamma(X, \mathscr{K}_{Y^\infty}^{\cdot}) \\ H^{\cdot}\Gamma(X, {}'\mathscr{D}_{Y^\infty}^{\cdot}) & \xleftarrow{\tilde{\eta}_{Y^\infty}} & \end{array} \tag{10}$$

(2.10) We define a $\boldsymbol{C}$-bilinear pairing

$$\phi_3\colon H^i\Gamma(Y, \mathscr{K}_{X|Y}^{\cdot})\times H^{2n-i}\Gamma(Y, \mathscr{K}_{Y^\infty}^{\cdot})\to\boldsymbol{C}$$

as follows; let $K_{X|Y}^{s,t}=\Gamma(Y, \mathscr{K}_{X|Y}^{s,t})=\bigoplus_{|I|=t+1}\Gamma(Y_I, \mathscr{E}_{X|Y_I}^s)$ and $K_{Y^\infty}^{2n-s,-t}=\Gamma(Y, \mathscr{K}_{Y^\infty}^{2n-s,-t})=\bigoplus_{I=-t+1}\Gamma(Y_1, {}'\mathscr{D}_{Y_I^\infty}^s)$. Then by (2.0) b) there is a natural $\boldsymbol{C}$-bilinear pairing $\tilde{\phi}_3\colon K_{X|Y}^{s,t}\times K_{Y^\infty}^{2n-s,-t}\to\boldsymbol{C}$. Following the definition one checks immediately that $\tilde{\phi}_3(d'\alpha, \beta)=\tilde{\phi}_3(\alpha, d'\beta)$, $\alpha\in K_{X|Y}^{s-1,t}$, $\beta\in K_{Y^\infty}^{2n-s,-t}$, $\tilde{\phi}_3(d''\alpha, \beta)=\tilde{\phi}_3(\alpha, d''\beta)$, $\alpha\in K_{X|Y}^{s,t-1}$, $\beta\in K_{Y^\infty}^{2n-s,-t}$ so $\tilde{\phi}_3$ induces a natural bilinear pairing between the cohomology groups of the associated simple complexes $K_{X|Y}^{\cdot}=\Gamma(Y, \mathscr{K}_{X|Y}^{\cdot})$ and $K_{Y^\infty}^{\cdot}=\Gamma(Y, \mathscr{K}_{Y^\infty}^{\cdot})$, which is by definition ϕ_3. Moreover from the definitions of $\xi_{X|Y}$, η_{Y^∞} and the direct image of currents it follows that

$$\phi_2(\alpha, \eta_{Y^\infty}(\beta))=\phi_3(\xi_{X|Y}(\alpha), \beta),\quad \alpha\in H^i\Gamma(Y, \mathscr{E}_{X|Y}^{\cdot}),\ \beta\in H^{2n-i}\Gamma(Y, \mathscr{K}_{Y^\infty}^{\cdot})\,.$$

Combining this with (2.5) we have proved the following: There is a natural perfect pairings between the triangles (4) and (10).

§3. Construction of the Diagrams (continued) and Proof of Theorem

(3.0) We denote by $(z_1,\dots, z_n)_s$, $0\leqq s\leqq r$, local coordinates $z_1,\dots, z_n$ of X with domain V such that $V\cap Y=\{z_1\cdots z_s=0\}$. We call such coordinates briefly *normal s-coordinates* (around x if x is the center of these coordinates).

(3.1) The logarithmic de Rham complex $\Omega_X^{\cdot}\langle Y\rangle$ of X along Y is a subcomplex of $\Omega_X^{\cdot}(*Y)$, defined locally as follows [3, 3.1.2]; let $x\in X$ be any point and $(z_1,\dots, z_n)_s$ be normal s-coordinates around x. Then

(1) $$\Omega_X^{\cdot}\langle Y\rangle_x=\{\sum_{1\leqq i_1<\cdots<i_k\leqq s} dz_{i_1}/z_{i_1}\wedge\cdots\wedge dz_{i_k}/z_{i_k}\wedge\alpha_{i_1\cdots i_k},\quad \alpha_{i_1\cdots i_k}\in\Omega_{X,x}^{\cdot-k},\ k\leqq s\}.$$

Clearly $\Omega_X^{\cdot}\subseteqq\Omega_X^{\cdot}\langle Y\rangle$ and put $Q_X^{\cdot}\langle Y\rangle=\Omega_X^{\cdot}\langle Y\rangle/\Omega_X^{\cdot}$. Let $i_\Omega\colon\Omega_X^{\cdot}\langle Y\rangle\to\Omega_X^{\cdot}(*Y)$ be the natural inclusion. Then we have the induced inclusion $i_Q\colon Q_X^{\cdot}\langle Y\rangle\to Q_X^{\cdot}(*Y)$ and the following commutative diagram of complexes

(2) $$\begin{array}{ccccccccc} 0 & \longrightarrow & \Omega_X^{\cdot} & \longrightarrow & \Omega_X^{\cdot}\langle Y\rangle & \longrightarrow & Q_X^{\cdot}\langle Y\rangle & \longrightarrow & 0 \\ & & \| & & \downarrow{\scriptstyle i_\Omega} & & \downarrow{\scriptstyle i_Q} & & \\ 0 & \longrightarrow & \Omega_X^{\cdot} & \longrightarrow & \Omega_X^{\cdot}(*Y) & \longrightarrow & Q_X^{\cdot}(*Y) & \longrightarrow & 0\,. \end{array}$$

Since i_Ω is quasi-isomorphic ([3, 3.1.11]), so is i_Q.

(3.2) Define the complex $\Sigma_{Y/X}^{\cdot}$ by $\Sigma_{Y/X}^{\cdot}=\mathcal{H}om_{\mathcal{O}_X}(\Omega_X^{n-\cdot}\langle Y\rangle, \Omega_X^n)$. Regarding the natural injection $\mathcal{H}om_{\mathcal{O}_X}(\Omega_X^{n-\cdot}\langle Y\rangle, \Omega_X^n)\to\mathcal{H}om(\Omega_X^{n-\cdot}, \Omega_X^n)\cong\Omega_X^{\cdot}$ as an inclusion, we consider $\Sigma_{Y/X}^{\cdot}$ a subcomplex of $\Omega_X^{\cdot}$. Various characterizations of $\Sigma_{Y/X}^{\cdot}$ are given in the following:

(3.2.1) **Lemma.** *Let V be any open subset of X. Then for $\omega\in\Gamma(V, \Omega_X^{\cdot})$ the following conditions are equivalent.*

0) $\omega\in\Gamma(V, \Sigma_{Y/X}^{\cdot})$.

1) $a_i^*\omega=0$ *for every* $1\leqq i\leqq r$.

2) *Let $x\in V$ be any point and $(z_1,\dots, z_n)_s$ normal s-coordinates around x. Then we may write*

$$\omega_x=\sum_{1\leqq k\leqq s}\sum_{\substack{i_1<\cdots<i_k\\ i_{k+1}<\cdots<i_s}} z_{i_1}\cdots z_{i_k}dz_{i_{k+1}}\wedge\cdots\wedge dz_{i_s}\wedge\alpha_{i_1\dots i_k;i_{k+1}\dots i_s},$$
$$\{i_1,\dots, i_s\}=\{1,\dots, s\},\quad \alpha_{i_1\dots i_k;i_{k+1}\dots i_s}\in\Omega_{X,x}^{p-s+k}.$$

3) *In the notation of* 2), $\omega_x \in z_1 \cdots z_s \Omega_X^{\cdot}\langle Y \rangle$.

In fact, implications 1)←2)↔3) are clear and that 1)→2) is easily seen by induction on r. Finally the equivalence of 0) and 2) follows from (1) by elementary calculations which we leave to the reader. In view of this lemma we define a subcomplex $\mathscr{E}\Sigma_{Y/X}^{\cdot}$ of $\mathscr{E}_X^{\cdot}$ by the following; for any open $V \subseteq X$ as above $\omega \in \Gamma(V, \mathscr{E}_X^{\cdot})$ belongs to $\Gamma(V, \mathscr{E}\Sigma_{Y/X}^{\cdot})$ if and only if $a_i^* \omega = 0$ for all i. Then as for $\Sigma_{Y/X}^{\cdot}$ we have that for any point $x \in X$ and any normal s-coordinates $(z_1, \dots, z_n)_s$ around x, $\omega \in \mathscr{E}\Sigma_{Y/X,x}^{\cdot}$ if and only if ω_x is written in the form

$$\omega_x = \sum_{1 \leqq k \leqq l \leqq m \leqq s} \sum_{\substack{i_1 < \cdots < i_k, i_{l+1} < \cdots < i_m \\ i_{k+1} < \cdots < i_l, i_{m+1} < \cdots < i_s}} z_{i_1} \cdots z_{i_k} \bar{z}_{i_{k+1}} \cdots \bar{z}_{i_l} dz_{i_{l+1}}$$

$$\wedge \cdots \wedge dz_{i_m} \wedge d\bar{z}_{i_{m+1}} \wedge \cdots \wedge d\bar{z}_{i_s} \wedge \beta_{i_1 \dots i_k ; i_{k+1} \dots i_l ; i_{l+1} \dots i_m ; i_{m+1} \cdots i_s},$$

for some $\beta_{i_1 \dots i_k ; i_{k+1} \dots i_l ; i_{l+1} \dots i_m ; i_{m+1} \dots i_s} \in \mathscr{E}_{X,x}^{p-s+k+l}$.

(3.2.2) **Lemma.** *$\Sigma_{Y/X}^{\cdot}$ and $\mathscr{E}\Sigma_{Y/X}^{\cdot}$ are resolutions of $\mathbf{C}_U$ with respect to the natural augmentations e_U: $\mathbf{C}_U \to \Sigma_{Y/X}^{\cdot}$ (resp. e'_U: $\mathbf{C}_U \to \mathscr{E}\Sigma_{Y/X}^{\cdot}$).*

Proof. Let $x \in X$ be any point and $(z_1, \dots, z_n)_s$ be normal s-coordinates around x. Let $\alpha \in \mathscr{E}_{X,x}^p$, $p \geqq 1$, be closed. We may assume that α is defined on the unit polydisc $B = \{|z_i| < 1\}$. Notations: $I = (i_1, \dots, i_a)$, $1 \leqq i_1 < \cdots < i_a \leqq n$, $J = (j_1, \dots, j_b)$, $1 \leqq j_1 < \cdots < j_b \leqq n$, $dz_I = dz_{i_1} \wedge \cdots \wedge dz_{i_a}$, $d\bar{z}_J = d\bar{z}_{j_1} \wedge \cdots \wedge d\bar{z}_{j_b}$. Then write $\alpha = \sum_{I,J} a_{IJ} dz_1 \wedge d\bar{z}_J$, $a + b = p$. Let $\tilde{a}_{IJ} = \int_0^1 r^{p-1} a_{IJ}(rz, r\bar{z}) dr$ and

$$(3) \qquad \beta = \sum_{\substack{I,J \\ a+b=p}} \Big(\sum_{k=1}^{a} (-1)^{k-1} \tilde{a}_{IJ} z_{i_k} dz_{i_1} \wedge \cdots \wedge d\hat{z}_{i_k} \wedge \cdots \wedge dz_{i_s} \wedge d\bar{z}_J$$

$$+ \sum_{l=1}^{b} (-1)^{l-1} \tilde{a}_{IJ} \bar{z}_{j_l} dz_I \wedge d\bar{z}_{j_1} \wedge \cdots \wedge d\hat{\bar{z}}_{j_l} \wedge \cdots \wedge d\bar{z}_{j_t} \Big).$$

Then the Poincaré lemma says that $\alpha = d\beta$ (cf. [9, A6]). Furthermore by (3) and the characterization of $\Sigma_{Y/X}^{\cdot}$ (resp. $\mathscr{E}\Sigma_{Y/X}^{\cdot}$) in 2) of Lemma (3.2.1) (resp. just before the lemma) one gets immediately that if $\alpha \in \Sigma_{Y/X,x}^{\cdot}$ (resp. $\mathscr{E}\Sigma_{Y/X,x}^{\cdot}$), then $\beta \in \Sigma_{Y/X,x}^{\cdot}$ (resp. $\mathscr{E}\Sigma_{Y/X,x}^{\cdot}$) too. Finally if $\alpha \in \Sigma_{Y|X,x}^0$ (resp. $\mathscr{E}\Sigma_{Y/X,x}^0$) and $d\alpha = 0$, then α is a constant and $= 0$ when $x \in Y$. Q.E.D.

Now we put $\tilde{\Omega}_Y^{\cdot} = \Omega_X^{\cdot}/\Sigma_{Y/X}^{\cdot}$, $\mathscr{E}_Y^{\cdot} = \mathscr{E}_X^{\cdot}/\mathscr{E}\Sigma_{Y/X}^{\cdot}$. ($\mathscr{E}_Y^{\cdot}$ coincides with the complex of germs of C^∞ forms on Y in the sense of Bloom-Herrera [1] as follows easily from the definition.) Let e_Y: $\mathbf{C}_Y \to \tilde{\Omega}_Y^{\cdot}$ be the natural augmentation. Let $j_{Y/X}$: $\Sigma_{Y/X}^{\cdot} \to \mathscr{E}\Sigma_{Y/X}^{\cdot}$ and j_X: $\Omega_X^{\cdot} \to \mathscr{E}_X^{\cdot}$ be the natural, and j_Y: $\tilde{\Omega}_Y^{\cdot} \to \mathscr{E}_Y^{\cdot}$ be the induced inclusions (cf. Lemma 3.2.1). Then we have the following commutative diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \boldsymbol{C}_U & \longrightarrow & \boldsymbol{C}_X & \longrightarrow & \boldsymbol{C}_Y & \longrightarrow & 0 \\
 & & \big\downarrow{\scriptstyle e_U} & & \big\downarrow{\scriptstyle e_X} & & \big\downarrow{\scriptstyle e_Y} & & \\
0 & \longrightarrow & \Sigma^{\cdot}_{Y/X} & \longrightarrow & \Omega^{\cdot}_X & \longrightarrow & \tilde{\Omega}^{\cdot}_Y & \longrightarrow & 0 \\
 & & \big\downarrow{\scriptstyle j_{Y/X}} & & \big\downarrow{\scriptstyle j_X} & & \big\downarrow{\scriptstyle j_Y} & & \\
0 & \longrightarrow & \mathscr{E}\Sigma^{\cdot}_{Y/X} & \longrightarrow & \mathscr{E}^{\cdot}_X & \longrightarrow & \mathscr{E}^{\cdot}_Y & \longrightarrow & 0\,.
\end{array}$$

By the above lemma all the morphisms in this diagram are quasi-isomorphic.

(3.3) We define a double complex $\mathscr{K}_Y^{\cdot\cdot}$ on X as follows;

$$\mathscr{K}_Y^{s,t} = \bigoplus_{|I|=t+1} a_{I*}\mathscr{E}^s_{Y_I} = a_{(t+1)*}\mathscr{E}^s_{Y^{(t+1)}}, \quad s,\ t \geqq 0,$$

$$=0, \quad \text{otherwise}$$

where the differential d': $\mathscr{K}_Y^{\cdot,\cdot} \to \mathscr{K}_Y^{\cdot+1,\cdot}$ is induced from the complexes $\mathscr{E}^{\cdot}_{Y_I}$ and d'': $\mathscr{K}_Y^{\cdot,t} \to \mathscr{K}_Y^{\cdot,t+1}$ is given by the formula

$$d'' = \sum_{j=1}^{t+1} (-1)^{t+j} \bigoplus_I a_{I*}\delta_j^{I*}\,.$$

Let $\mathscr{K}^{\cdot}_Y$ be the associated simple complex. Then the restriction maps $\mathscr{E}^{\cdot}_{X|Y_I} \to \mathscr{E}^{\cdot}_{Y_I}$ induces the morphism of double complexes $\mathscr{K}^{\cdot\cdot}_{X|Y} \to \mathscr{K}^{\cdot\cdot}_Y$ and hence that of the associated simple complexes $r_{\mathscr{K}}$: $\mathscr{K}^{\cdot}_{X|Y} \to \mathscr{K}^{\cdot}_Y$. Define ξ_Y: $\mathscr{E}^{\cdot}_Y \to \mathscr{K}^{\cdot}_Y$ as the composition of the restriction map $\mathscr{E}^{\cdot}_Y \to \mathscr{K}^{\cdot}_Y{}^0$ and the natural inclusion $\mathscr{K}^{\cdot}_Y{}^0 \to \mathscr{K}^{\cdot}_Y$, and then put $\zeta_Y = \xi_Y j_Y$. Let r_Ω: $\Omega^{\cdot}_{X|Y} \to \tilde{\Omega}^{\cdot}_Y$ and $r_{\mathscr{E}}$: $\mathscr{E}^{\cdot}_{X|Y} \to \mathscr{E}^{\cdot}_Y$ be the natural restriction maps. Then we get easily the commutativity: $r_{\mathscr{E}} j_{X|Y} = j_Y r_\Omega$, and $r_{\mathscr{K}} \xi_{X|Y} = \xi_Y r_{\mathscr{E}}$ etc.

(3.4) From (2.1) (2.6) and (3.3) we obtain the commutative diagram (A). We shall see that the morphisms are all quasi-isomorphic in (A). First note that this is already true for the left half of the diagram ((2.1) and (2.6)). So by the commutativity it is enough to show that r_Ω and $r_{\mathscr{K}}$ are quasi-isomorphic since so is j_Y as has been shown in (3.2). For r_Ω this follows from the following commutative diagram

$$\begin{array}{ccc}
 & \overset{e_{X|Y}}{\nearrow} & \Omega^{\cdot}_{X|Y} \\
\boldsymbol{C}_Y & & \big\downarrow{\scriptstyle r_\Omega} \\
 & \underset{e_Y}{\searrow} & \tilde{\Omega}^{\cdot}_Y
\end{array}$$

where $e_{X|Y}$ and e_Y give resolutions of $\boldsymbol{C}_Y$ (cf. (2.1) and (3.2)). On the other hand, for $r_{\mathscr{K}}$ we have for each t a similar commutative diagram

$$\begin{array}{ccc} & \overset{e_t}{\nearrow} & \bigoplus_{|I|=t+1} \mathscr{E}^{\cdot}_{X|Y_I} = \mathscr{K}^{\cdot t}_{X|Y} \\ \boldsymbol{C}_{Y_{(t+1)}} & & \big\downarrow r_{\mathscr{K}} \\ & \underset{\bar{e}_t}{\searrow} & a_{(t+1)*}\mathscr{E}^{\cdot}_{Y_{(t+1)}} = \mathscr{K}^{\cdot t}_{Y} \end{array}$$

with e_t (resp. $\bar{e}_t$) giving a resolution of $\boldsymbol{C}_{Y_{(t+1)}}$. Hence $r_{\mathscr{K}}$ is quasi-isomorphic (cf. (1.10)). (Remark: It can also be directly verified that ξ_Y is quasi-isomorphic. Cf. [10, §4].) Finally passing to the hypercohomology we obtain the diagram $(\tilde{A})$.

(3.5) Let $'\bar{\mathscr{D}}^{\cdot}_Y$ be the subcomplex of $'\mathscr{D}^{\cdot}_X\,[2]$ which annihilates $\mathscr{E}\Sigma^{2n-2-\cdot}_{Y/X}$ and $'\mathscr{D}^{\cdot}_Y$ the sheaf-theoretic restriction of $'\bar{\mathscr{D}}^{\cdot}_Y$ to Y. Then $'\mathscr{D}^{\cdot}_Y$ is nothing but the complex of sheaves of germs of currents on Y in the sense of Bloom-Herrera [1]. As in (2.0) c) a (resp. b_i) induces the natural injection $\hat{a}\colon a_*{}'\mathscr{D}^{\cdot}_Y[-2]\to {}'\mathscr{D}^{\cdot}_{Y^\infty}$ (resp. $\hat{b}_i\colon b_{i*}{}'\mathscr{D}^{\cdot}_{Y_i}\to {}'\mathscr{D}^{\cdot}_Y$). Now define a complex $'\mathscr{D}^{\cdot}_X\langle Y\rangle$ (resp. a double complex $\mathscr{K}^{\cdot\cdot}_X\langle Y\rangle$) in $\mathscr{A}(X)$ as follows;

$$'\mathscr{D}^{\cdot}_X\langle Y\rangle = {}'\mathscr{D}^{\cdot}_Y[-2]$$

(resp.

$$(4)\qquad \mathscr{K}^{s,t}_X\langle Y\rangle = \bigoplus_{|I|=-t+1} a_{I*}{}'\mathscr{D}^{s+2t-2}_{Y_I} = a_{(-t+1)*}{}'\mathscr{D}^{s+2t-2}_{Y_{(-t+1)}},\quad s\geqq 0,\ t\leqq 0,$$
$$=0,\quad \text{otherwise}$$

where the differential $d'\colon \mathscr{K}^{\cdot\cdot}_X\langle Y\rangle\to\mathscr{K}^{\cdot+1,\cdot}_X\langle Y\rangle$ is induced from that of $'\mathscr{D}^{\cdot}_{Y_I}$ and $d''\colon \mathscr{K}^{\cdot t}_X\langle Y\rangle\to\mathscr{K}^{\cdot,t+1}_X\langle Y\rangle$ is defined by

$$d''=\sum_{j=1}^{-t+1}(-1)^{t+j}\bigoplus_I a_{I*}(\hat{\delta}^I_j)$$

$\hat{\delta}^I_j\colon {}'\mathscr{D}^{\cdot}_{Y_I}\to {}'\mathscr{D}^{\cdot+2}_{Y_{I_j^\vee}}$ being the injection induced by δ^I_j (cf. (2.1))).

Define $\eta_X\langle Y\rangle\colon \mathscr{K}^{\cdot}_X\langle Y\rangle\to {}'\mathscr{D}^{\cdot}_X\langle Y\rangle$ as the composition of the natural projection $\mathscr{K}^{\cdot}_X\langle Y\rangle\to\mathscr{K}^{\cdot 0}_X\langle Y\rangle$ and the map $\sum_i\hat{b}_i\colon \mathscr{K}^{\cdot 0}_X\langle Y\rangle\to {}'\mathscr{D}^{\cdot}_X\langle Y\rangle$, where $(\sum_{i=1}^r\hat{b}_i)(\omega)=\sum_{i=1}^r(\hat{b}_i\omega)$. Let $i_{\mathscr{D}}\colon {}'\mathscr{D}^{\cdot}_X\langle Y\rangle\to {}'\mathscr{D}^{\cdot}_{Y^\infty}$ be the inclusion given by $\hat{a}$ and define $i_{\mathscr{K}}\colon \mathscr{K}^{\cdot}_X\langle Y\rangle\to\mathscr{K}^{\cdot}_{Y^\infty}$ by $i_{\mathscr{K}}=\bigoplus_{|I|=-t+1}\hat{a}_I\colon \bigoplus_I a_{I*}{}'\mathscr{D}^{\cdot}_{Y_I}[2t-2]\to\bigoplus_I {}'\mathscr{D}^{\cdot}_{Y_I^\infty}$. From these definitions we have easily that $\eta_{Y^\infty}i_{\mathscr{K}}=i_{\mathscr{D}}\eta_X\langle Y\rangle$.

(3.6) We show that the maps Res: $Q^{\cdot}_X(*Y)[-1]\to {}'\mathscr{D}^{\cdot}_{Y^\infty}$ and $H\colon Q^{\cdot}_X(*Y)[-1]\to\mathscr{K}^{\cdot}_{Y^\infty}$, when restricted to $Q^{\cdot}_X\langle Y\rangle[-1]$, factors through $'\mathscr{D}^{\cdot}_X\langle Y\rangle$ and $\mathscr{K}^{\cdot}_X\langle Y\rangle$ respectively. First, we take any $I=(i_1,\dots,i_q)$, $1\leqq q\leqq r$. Let $D_I=\bigcup_{j\notin I}(Y_I\cap Y_j)$. This is a divisor with normal crossings in Y_I. We then define a complexes' homomorphism

$$\mathrm{res}_I\colon Q^{\cdot}_X\langle Y\rangle \to a_{I*}\Omega^{\cdot}_{Y_I}\langle D_I\rangle[-q]$$

as follows; let $x\in X$ be any point and $(z_1,\ldots, z_n)_s$ be normal s-coordinates around x. For any $\bar{\omega}\in Q^{\cdot}_X\langle Y\rangle_x$ take a representative $\omega\in \Omega^{\cdot}_X\langle Y\rangle_x$ and write

$$\omega = dz_{i_1}/z_{i_1}\wedge\cdots\wedge dz_{i_q}/z_{i_q}\wedge\omega_I+\sum_{i=1}^{q}\omega_i,$$
$$\omega_I\in\Omega^{\cdot-q}_X\langle\bigcup_{j\notin I}Y_j\rangle_x,\quad \omega_i\in\Omega^{\cdot}_X\langle\bigcup_{j\neq i}Y_j\rangle_x.$$

Then by definition

$$\mathrm{res}_I(\bar{\omega})=a_I^*\omega_I\in\Omega^{\cdot-q}_{Y_I}\langle D_I\rangle_x. \tag{5}$$

In fact, as in the arguments used in the classical definition of residue by Leray [15] one checks easily that $\mathrm{res}_I(\bar{\omega})$ is independent of the various choices made above, depending only on $\bar{\omega}$, and that res_I is a complexes' homomorphism. When $q=1$, from (5), Section 2 (6), Section 2 (7) and Section 2 (9) we obtain the following;

$$(1/2\pi\sqrt{-1})\,\mathrm{Res}=\sum_{i=1}^{r}(\hat{a}_i\cdot a_{i*}(\mathrm{PV}_i)\cdot\mathrm{res}_i),$$

where $\hat{a}_i$ is as in (2.0) c), $\mathrm{res}_i=\mathrm{res}_{\{i\}}$ and $\mathrm{PV}_i\colon \Omega^{\cdot}_{Y_i}\langle D_i\rangle\to{}'\mathscr{D}^{\cdot}_{D_i^\infty}\subseteq{}'\mathscr{D}^{\cdot}_{Y_i}$ is a principal value on Y_i. Since $a_i=ab_i$, this is equivalent to saying that the following diagram is commutative

$$Q^{\cdot}_X\langle Y\rangle[-1]\xrightarrow{\bigoplus_{i=1}^r\mathrm{res}_i}\bigoplus_{i=1}^{r}a_{i*}\Omega^{\cdot}_{Y_i}\langle D_i\rangle[-2]\xrightarrow{a_*(\sum_{i=1}^r b_i\cdot\mathrm{PV}_i)}a_*{}'\mathscr{D}^{\cdot}_Y[-2]\equiv{}'\mathscr{D}^{\cdot}_X\langle Y\rangle\xrightarrow{\hat{a}}{}'\mathscr{D}^{\cdot}_{Y^\infty},$$
$$Q^{\cdot}_X\langle Y\rangle[-1]\xrightarrow{(1/2\pi\sqrt{-1})\mathrm{Res}}{}'\mathscr{D}^{\cdot}_{Y^\infty}.$$

On the other hand, iterating the arguments used to deduce Section 2, (9) we obtain a similar formula for h_I

$$(1/2\pi\sqrt{-1})^q h_I=\hat{a}_I\,\mathrm{PV}_I\,\mathrm{res}_I,$$

i.e., the following diagram is commutative

$$Q^{\cdot}_X\langle Y\rangle[-q]\xrightarrow{\mathrm{res}_I}a_{I*}\Omega^{\cdot}_{Y_I}\langle D_I\rangle[-2q]\xrightarrow{a_{I*}(\mathrm{PV}_I)}a_{I*}{}'\mathscr{D}^{\cdot}_{Y_I}[-2q]\xrightarrow{\hat{a}_I}{}'\mathscr{D}^{\cdot}_{Y_I^\infty}$$
$$Q^{\cdot}_X\langle Y\rangle[-q]\xrightarrow{1/(2\pi\sqrt{-1})^q h_I}{}'\mathscr{D}^{\cdot}_{Y_I^\infty}$$

where PV_I is the principal value on Y_I. Now we define $\mathrm{Res}_0\colon Q^{\cdot}_X\langle Y\rangle[-1]$

$\to {}'\mathscr{D}^{\cdot}_X\langle Y\rangle$ and $H_0\colon Q^{\cdot}_X\langle Y\rangle[-1]\to\mathscr{K}^{\cdot}_X\langle Y\rangle$ as follows;

$$\mathrm{Res}_0=2\pi\sqrt{-1}a_*(\sum_{i=1}^{r}\hat{b}_i\,\mathrm{PV}_i)\cdot\mathrm{res}_i$$

and

(6) $$H_0=\bigoplus_t h_0^t,\quad h_0^t=\bigoplus_{|I|=-t+1} h_I^0\colon Q^{\cdot}_X\langle Y\rangle[-1]\to\mathscr{K}_X^{\cdot -t,t}\langle Y\rangle$$

with $h_I^0\colon Q^{\cdot}_X\langle Y\rangle[-1]\to a_{I*}{}'\mathscr{D}^{\cdot}_{Y_I}[t-2]$ defined by $h_I^0=(2\pi\sqrt{-1})^{-t}a_{I*}(\mathrm{PV}_I)\cdot\mathrm{res}_I$. Then we have the commutativity $\mathrm{Res}\cdot i_Q=i_{\mathscr{D}}\cdot\mathrm{Res}_0$, and $i_{\mathscr{K}}\cdot H_0=H\cdot i_Q$. This proves our assertion.

(3.7) Combining (2.4), (2.9), (3.5) and (3.6) we obtain the commutative diagram (B). Further we have to show the following:

(3.7.1) **Lemma.** *The morphisms in* (B) *are all quasi-isomorphic.*

Proof. The assertion is already true for the left half of the diagram by (2.4) and (2.9), and $i_Q[-1]$ is quasi-isomorphic by (3.1). Thus, by the commutativity it is enough to show that $i_{\mathscr{K}}$, $\eta_X\langle Y\rangle$ and H_0 are quasi-isomorphic. For $i_{\mathscr{K}}$ it suffices to show that the complexes $\mathscr{K}_X^{\cdot,t}\langle Y\rangle$ and $\mathscr{K}_{Y^\infty}^{\cdot,t}$ are quasi-isomorphic by $i_{\mathscr{K}}$ for every t (cf. (1.10)). Let $q=-t+1$. Then since for each I the complex ${}'\mathscr{D}^{\cdot}_{Y_I}$ is a resolution of $\boldsymbol{C}_{Y_I}$ [18, § 19], the cohomology group $\mathscr{H}^{\cdot}(\mathscr{K}_X^{\cdot,t}\langle Y\rangle)$ concentrates in degree $2q$, where it is isomorphic to $\bigoplus_{|I|=q}\boldsymbol{C}_{Y_I}$. On the other hand, by (2.2) $\mathscr{H}^{\cdot}({}'\mathscr{D}^{\cdot}_{Y_I^\infty})$ is isomorphic to the sheaves $\mathscr{H}^{\cdot}_{Y_I^\infty}(\boldsymbol{C}_X)$, and hence $\mathscr{H}^{\cdot}(\mathscr{K}_{Y_I^\infty}^{\cdot,t})$ also concentrates in degree $2q$ where it is isomorphic to $\bigoplus_{|I|=q}\mathscr{H}^{2q}_{Y_I}(\boldsymbol{C}_X)\cong\bigoplus_{|I|=q}\boldsymbol{C}_{Y_I}$. Thus it suffices to show that for each I, $\hat{a}_I$: $a_{I*}{}'\mathscr{D}_Y^{\cdot -2q}\to{}'\mathscr{D}^{\cdot}_{Y^\infty}$ induces an injection $\mathscr{H}^{2q}(\hat{a}_I)$ on cohomology in degree $2q$. In fact one sees readily that $\mathscr{H}^{2q}(\hat{a}_I)(1)\neq 0$, $\hat{a}_I(1)$ being the current defined by the analytic variety Y_I. We shall give a proof for H_0 in (3.9). So it remains to show that $\eta_X\langle Y\rangle$ is quasi-isomorphic.

We have to show that the following sequence is exact for every s (cf. (1.10) and (4))

$$0\to a_{(r)*}{}'\mathscr{D}^s_{Y(r)}[-2r]\xrightarrow{d_r}\cdots\xrightarrow{d_1}a_{(1)*}{}'\mathscr{D}^s_{Y(1)}[-2]\xrightarrow{\eta_X\langle Y\rangle}a_*{}'\mathscr{D}^s_Y[-2]\to 0,$$

where d_i are induced by $\delta^{(i)}$. Since the sheaves involved are fine and the morphisms are linear over C^∞ functions, it is enough to show that for any open subset $V\subseteq X$ the sequence

(7) $$0\to\Gamma_c(V,a_{(r)*}{}'\mathscr{D}^s_{Y(r)}[-2r])\to\cdots\to\Gamma_c(V,a_{(1)*}{}'\mathscr{D}^s_{Y(1)}[-2])$$

$$\to \Gamma_c(V,\, a_*{}'\mathscr{D}_Y^s[-2]) \to 0$$

is exact. This sequence is, up to signs of differential, the topological transpose of the Frechet complex

$$0 \leftarrow \Gamma(V,\, a_{(r)*}\mathscr{E}_{Y_{(r)}}^{2n-s}) \leftarrow \cdots \leftarrow \Gamma(V,\, a_{(1)*}\mathscr{E}_{Y_{(1)}}^{2n-s}) \leftarrow \Gamma(V,\, a_*\mathscr{E}_Y^{2n-s}) \leftarrow 0$$

which is exact by (3.4) (following from the quasi-isomorphy of ξ_Y). Thus (7) itself is exact (cf. [19, Lemma 1]). Q.E.D.

Hence passing to hypercohomology we obtain ($\tilde{\mathrm{B}}$) with all the arrows isomorphic.

(3.8) We define $\boldsymbol{C}$-bilinear pairings

$$\phi_1'\colon H^i(Y,\, \tilde{\Omega}_Y^{\cdot}) \times H^{2n-i}(Y,\, Q_X^{\cdot}\langle Y\rangle[-1]) \to \boldsymbol{C}$$
$$\phi_2'\colon H^i\Gamma(Y,\, \mathscr{E}_Y^{\cdot}) \times H^{2n-i}\Gamma(Y,\, {}'\mathscr{D}_X^{\cdot}\langle Y\rangle) \to \boldsymbol{C}$$
$$\phi_3'\colon H^i\Gamma(Y,\, \mathscr{K}_Y^{\cdot}) \times H^{2n-i}\Gamma(Y,\, \mathscr{K}_X^{\cdot}\langle Y\rangle) \to \boldsymbol{C}$$

by formulae similar to ϕ_1, ϕ_2 (2.5) and ϕ_3 (2.10) respectively: ϕ_1' is the composite $\boldsymbol{H}^i(Y,\, \tilde{\Omega}_Y^{\cdot}) \times \boldsymbol{H}^{2n-i}(Y,\, Q_X^{\cdot}\langle Y\rangle[-1]) \to \boldsymbol{H}^{2n-1}(Y,\, Q_X^{\cdot}\langle Y\rangle) \to \boldsymbol{H}^{2n}(X,\, \Omega_X^{\cdot}) \to \boldsymbol{C}$ and ϕ_2' (resp. ϕ_3') is induced by the natural pairing $\phi_2'\colon \Gamma(Y,\, \mathscr{E}_Y^{\cdot}) \times \Gamma(Y,\, {}'\mathscr{D}_X^{2n-\cdot}\langle Y\rangle)\,(=\Gamma(Y,\, {}'\mathscr{D}_Y^{2n-\cdot-2})) \to \boldsymbol{C}$ (resp. $\tilde{\phi}_3'\colon K_Y^{s,t} \times K_X^{2n-s,-t}\langle Y\rangle \to \boldsymbol{C}$, where $K_Y^{s,t} = \Gamma(Y,\, \mathscr{K}_Y^{s,t}) = \Gamma(Y_{(t+1)},\, \mathscr{E}_{Y_{(t+1)}}^s)$ and $K_X^{2n-s,t}\langle Y\rangle = \Gamma(Y,\, \mathscr{K}_X^{2n-s,-t}\langle Y\rangle) = \Gamma(Y_{(t+1)},\, {}'\mathscr{D}_{Y_{(t+1)}}^{2n-s-2t-2})$). The detail is left to the reader. Furthermore from the definitions of $r_{\mathscr{E}}$, $i_{\mathscr{D}}$ (resp. $r_{\mathscr{K}}$, $i_{\mathscr{K}}$) and the definition of direct image of currents it follows immediately that $\phi_2(\alpha,\, i_{\mathscr{D}}\beta) = \phi_2'(r_{\mathscr{E}}\alpha,\, \beta)$ (resp. $\phi_3(\alpha,\, i_{\mathscr{K}}\beta) = \phi_3'(r_{\mathscr{K}}\alpha,\, \beta)$), $\alpha \in H^i\Gamma(Y,\, \mathscr{E}_{X|Y}^{\cdot})$ (resp. $H^i\Gamma(Y,\, \mathscr{K}_{X|Y}^{\cdot})$), $\beta \in H^{2n-i}\Gamma(Y,\, {}'\mathscr{D}_X^{\cdot}\langle Y\rangle)$ (resp. $H^{2n-i}\Gamma(Y,\, \mathscr{K}_X^{\cdot}\langle Y\rangle)$). Also, comparing the maps of which ϕ_1 and ϕ_1' are composites we get that $\phi_1(\alpha,\, i_Q\beta) = \phi_1'(r_\Omega\alpha,\, \beta)$, $\alpha \in \boldsymbol{H}^i(Y,\, \Omega_{X|Y}^{\cdot})$, $\beta \in \boldsymbol{H}^{2n-i}(Y,\, Q_X^{\cdot}\langle Y\rangle[-1])$. Summarizing (2.5), (2.10) and the above we have the following:

Proposition. *There are natural perfect pairings between the corresponding terms of* ($\tilde{\mathrm{A}}$) *and* ($\tilde{\mathrm{B}}$), *compatible with the diagrams in an obvious sense.*

(3.9) We define an increasing (resp. decreasing) filtration W (resp. F) on the complexes $\Omega_X^{\cdot}\langle Y\rangle$, $Q_X^{\cdot}\langle Y\rangle[-1]$ and $\mathscr{K}_X^{\cdot}\langle Y\rangle$ as follows; let $x \in X$ be any point and $(z_1,\dots,z_n)_s$ be normal s-coordinates around x. Then

$$W_k\Omega_X^{\cdot}\langle Y\rangle_x = \{\textstyle\sum dz_{i_1}/z_{i_1} \wedge \cdots \wedge dz_{i_t}/z_{i_t} \wedge \alpha_{i_1\dots i_t},$$
$$t \leqq k,\ 1 \leqq i_1 < \cdots < i_t \leqq s,\ \alpha_{i_1\dots i_t} \in \Omega_{X,x}^{\cdot-t}\}$$

$$F^p\Omega_X^{\cdot}\langle Y\rangle = \Omega_X^{\cdot}\langle Y\rangle \quad \text{if} \quad \cdot \geqq p \qquad \text{and} \qquad = 0 \quad \text{if} \quad \cdot < p.$$

Let W' (resp. F') be the filtration on $Q^{\cdot}_X\langle Y\rangle$ induced from W (resp. F) on $\Omega^{\cdot}_X\langle Y\rangle$ by the natural quotient map $\Omega^{\cdot}_X\langle Y\rangle\to Q^{\cdot}_X\langle Y\rangle$. Define W' (resp. F') on $Q^{\cdot}_X\langle Y\rangle[-1]$ by $W'_k(Q^{\cdot}_X\langle Y\rangle[-1])=W'_k(Q^{\cdot-1}_X\langle Y\rangle)$ (resp. $F'^p(Q^{\cdot}_X\langle Y\rangle[-1])=F'^p(Q^{\cdot-1}_X\langle Y\rangle)$. Then we define W and F on $Q^{\cdot}_X\langle Y\rangle[-1]$ by

$$(8)\qquad \begin{cases} W=W'[-1] \\ F=F'. \end{cases}$$

Since $W'[-1]_k=W'_{k+1}$ by definition, $W_kQ^{\cdot}_X\langle Y\rangle[-1]$ consists of elements which are linear combination of forms of type

$$dz_{i_1}/z_{i_1}\wedge\cdots\wedge dz_{i_t}/z_{i_t}\wedge\bar{\alpha}_{i_1\dots i_t},\quad t\leqq k+1,\ 1\leqq i_1<\cdots<i_t\leqq s,$$
$$\bar{\alpha}_{i_1\dots i_t}\in Q^{\cdot-1-t}_X\langle Y\rangle.$$

For any $m>0$ let ${}'\mathscr{D}^{p,q}_{Y(m)}$ be the sheaf of germs of currents of type (p, q) on $Y_{(m)}$. Then define

$$(9)\qquad \begin{cases} W_k\mathscr{K}^{\cdot}_X\langle Y\rangle=\bigoplus_{t\geqq -k}\mathscr{K}^{\cdot-t,t}\langle Y\rangle \\ F^p\mathscr{K}^{\cdot}_X\langle Y\rangle=\bigoplus_{s+t=\cdot}F^p\mathscr{K}^{\cdot}_X\langle Y\rangle\cap\mathscr{K}^{s,t}_X\langle Y\rangle \\ \qquad\qquad =\bigoplus_{s+t=\cdot}\ \bigoplus_{m\geqq p+t-1}a_{(-t+1)*}{}'\mathscr{D}^{m,\cdot-m+2t-2}_{Y(-t+1)}. \end{cases}$$

(3.9.1) **Lemma.** *H_0: $Q^{\cdot}_X\langle Y\rangle[-1]\to\mathscr{K}^{\cdot}_X\langle Y\rangle$ induces a bifiltered quasi-isomorphism H_{0*}: $(Q^{\cdot}_X\langle Y\rangle[-1], W, F)\to(\mathscr{K}^{\cdot}_X\langle Y\rangle, W, F)$.*

Proof. First, from (6), (5), (8) and (9) it follows easily that H_0 is compatible with the filtrations W and F. Then we have to show that

$$\mathrm{Gr}^p_F\,\mathrm{Gr}^k_W(H_0)\colon \mathrm{Gr}^p_F\,\mathrm{Gr}^k_W(Q^{\cdot}_X\langle Y\rangle[-1])\to\mathrm{Gr}^p_F\,\mathrm{Gr}^k_W(\mathscr{K}^{\cdot}_X\langle Y\rangle)$$

is quasi-isomorphic. First, from (5), (8) and (9) we get that res_I, $|I|=k+1$, induce an isomorphism

$$\mathrm{Gr}^p_F\,\mathrm{Gr}^k_W(\bigoplus_I\mathrm{res}_I)\colon \mathrm{Gr}^p_F\,\mathrm{Gr}^k_W(Q^{\cdot}_X\langle Y\rangle[-1])\cong a_{(k+1)*}\Omega^{p-1-k}_{Y(k+1)}.$$

Here the right hand side should be considered as a complex concentrated in degree $p+1$. On the other hand, from (4) and (9) we derive

$$\mathrm{Gr}^p_F\,\mathrm{Gr}^k_W\,\mathscr{K}^{\cdot}_X\langle Y\rangle\cong\mathrm{Gr}^p_F\,\mathscr{K}^{\cdot+k,-k}_X\langle Y\rangle\cong a_{(k+1)*}{}'\mathscr{D}^{p-k-1,\cdot}_{Y(k+1)}[-p-1].$$

Then from (5) and the definition (6) of H_0, we get that with respect to the above isomorphisms $\mathrm{Gr}^p_F\,\mathrm{Gr}^k_W(H_0)$ corresponds to the natural augmentations $V_{(k+1)}$: $a^{p-k-1}_{Y(k+1)*}\Omega^{p-k-1}_{Y(k+1)}\to a_{(k+1)*}{}'\mathscr{D}^{p-k-1,\cdot}_{Y(k+1)}[-p-1]$, which is quasi-isomorphic by the Dolbeault-Grothendieck lemma. Q.E.D.

(3.10) Suppose now that $X \in \mathscr{C}$ (cf. (1.1)). We shall give a description of the mixed $\boldsymbol{Q}$-Hodge structure on the spaces $H^{\cdot}(Y, \boldsymbol{Q})$ and $H^{\cdot}_Y(X, \boldsymbol{Q})$.

(3.10.1) *Mixed Hodge structure on* $H^i(Y, \boldsymbol{Q})$ (cf. [10, §4] and [20, 3.5]). Let $\mathscr{E}^{p,q}_{Y_I}$ be the sheaves of germs of C^∞ forms of type (p, q) on Y_I. Then define filtrations W and F on $\mathscr{K}^{\cdot}_Y$ as follows;

$$W_k \mathscr{K}^{\cdot}_Y = \bigoplus_{t \geqq -k} \mathscr{K}^{\cdot-t,t}_Y$$

$$F^p \mathscr{K}^{\cdot}_Y = \bigoplus_{s,t} F^p \mathscr{K}^{\cdot}_Y \cap \mathscr{K}^{s,t}_Y = \bigoplus_{s,t} \bigoplus_{\cdot \geqq p} a_{(t+1)*} \mathscr{E}^{\cdot, s-\cdot}_{Y_{(t+1)}}.$$

We denote by the same letters W and F the filtrations induced on $K^{\cdot}_Y = \Gamma(Y, \mathscr{K}^{\cdot}_Y)$. Now by ($\tilde{\text{A}}$) we have the natural isomorphism $H^{\cdot}(Y, \boldsymbol{C}) \cong H^{\cdot}\Gamma(Y, \mathscr{K}^{\cdot}_Y)$. Let W and F still be the filtrations induced on $H^{\cdot}(Y, \boldsymbol{C})$ by this isomorphism. Then W comes from a filtration on $H^{\cdot}(Y, \boldsymbol{Q})$ (still denoted by W), and the triple $(H^i(Y, \boldsymbol{Q}), W[i], F)$ is the desired mixed $\boldsymbol{Q}$-Hodges structure on $H^i(Y, \boldsymbol{Q})$.

(3.10.2) *Mixed Hodge structure on* $H^{2n-i}_Y(X, \boldsymbol{Q})$. Denote by the same letters W and F the filtrations on $H^{\cdot}(Y, Q^{\cdot}_X\langle Y\rangle[-1])$ induced from the corresponding filtrations on $Q^{\cdot}_X\langle Y\rangle[-1]$ defined in (3.9). On the other hand, from ($\tilde{\text{B}}$) we get the natural isomorphism $H^{\cdot}_Y(X, \boldsymbol{C}) \cong \boldsymbol{H}^{\cdot}(Y, Q^{\cdot}_X\langle Y\rangle[-1])$. Shift W and F to $H^{\cdot}_Y(X, \boldsymbol{C})$ by this isomorphism. Then we have the following; W comes from a filtration on $H^{\cdot}_Y(X, \boldsymbol{Q})$ (still denoted by W) and the triple $(H^{2n-i}_Y(X, \boldsymbol{Q}), W[2n-i], F)$ is the desired mixed $\boldsymbol{Q}$-Hodge structure on $H^{2n-i}_Y(X, \boldsymbol{Q})$.

Proof is analogous to that of [3, 3.2.5], so we shall be brief. The above isomorphism $H^{\cdot}_Y(X, \boldsymbol{C}) \cong \boldsymbol{H}^{\cdot}(Y, Q^{\cdot}_X\langle Y\rangle[-1])$ comes from the isomorphism $Q^{\cdot}_X\langle Y\rangle[-1] \cong R\underline{\Gamma}_Y(\boldsymbol{C}_X)$ in the derived category $\mathscr{D}\mathscr{A}(X)$. Recall first that the canonical filtration τ of a complex $(K^{\cdot}, d)$ is defined as follows [3, 1.4.6]; $\tau_k(K^{\cdot}) = 0$ for $\cdot > k$, $= \operatorname{Ker} d$ for $\cdot = k$ and $= K^{\cdot}$ for $\cdot < k$. Then the spectral sequence (10) associated to $(R\underline{\Gamma}_Y(\boldsymbol{C}_X), \tau[-2])$ and $\Gamma(X, \quad)$,

$$E_1^{p,q} = H^{2p+q}(X, \mathscr{H}^{-p+2}_Y(\boldsymbol{C}_X)) \Rightarrow H^{p+q}_Y(X, \boldsymbol{C}) \tag{10}$$

is, up to the renumbering $E_1^{p,q} \Rightarrow E_{r+1}^{2p+q,-p+2}$, nothing but the local-global spectral sequence of local cohomology $E_2^{p,q} = H^p(X, \mathscr{H}^q_Y(\boldsymbol{C}_X)) \Rightarrow H^{p+q}_Y(X, \boldsymbol{C})$ (cf. [3, 1.4.8]). Under our assumption of normal crossings we have $\mathscr{H}^0_Y(\boldsymbol{C}_X) = \mathscr{H}^1_Y(\boldsymbol{C}_X) = 0$, and $\mathscr{H}^i_Y(\boldsymbol{C}_X) \cong R^{i-1} j_* \boldsymbol{C}_U \cong a_{(i-1)*} \boldsymbol{C}_{Y_{(i-1)}}$ for $i \geqq 2$, where $j: U \to X$ is the inclusion (cf. [3, 1.8.2]). Hence we get that in (10) $E_1^{p,q} = 0$ for $p \geqq 1$ and $\leqq -r$, and $= H^{2p+q}(Y_{(-p+1)}, \boldsymbol{C})$ for $-r < p < 1$. On the other hand, taking τ for the complex $K^{\cdot}=Q_X^{\cdot}\langle Y\rangle[-1]$ there is a natural filtered quasi-isomorphism $(Q_X^{\cdot}\langle Y\rangle[-1], \tau[-2])\cong(Q_X^{\cdot}\langle Y\rangle[-1], W)$, $\tau[-2]_k=\tau_{k+2}$, induced by the identity of $Q_X^{\cdot}\langle Y\rangle$ (cf. (8) and [3, 3.1.8]). Thus we have

$$(11)\qquad (Q_X^{\cdot}\langle Y\rangle[-1], W)\cong(R\underline{\Gamma}_Y(\boldsymbol{C}_X), \tau[-2]),$$

where $\cong$ denotes a filtered quasi-isomorphism. Hence (10) is also associated to the left hand side of (11). This proves the first assertion since the filtration on the abutment of (10) clearly comes from that on $H_Y^{p+q}(X, \boldsymbol{Q})$. Further by (11) F on $Q_X^{\cdot}\langle Y\rangle[-1]$ induces a filtration on $E_1^{p,q}$ of (10), and from (8) it follows that this defines on it a (pure) Hodge structure of weight $2p+q$. Thus $(R\underline{\Gamma}_Y(\boldsymbol{Q}_X), (R\underline{\Gamma}_Y(\boldsymbol{Q}_X), \tau[-2]), (Q_X^{\cdot}\langle Y\rangle[-1], W, F))$ is a cohomological mixed $\boldsymbol{Q}$-Hodge complex ([4, 8.1.6]), so by [4, 8.1.9] $(H^{2n-i}(X, \boldsymbol{Q}), W[2n-i], F)$ is a mixed $\boldsymbol{Q}$-Hodge structure on $H^{2n-i}(X, \boldsymbol{Q})$.

Finally it remains to check that the above definition coincides with that of Deligne in [4]. First of all as in the proof of [3, 3.2.5] (or as above) the bi-filtered complex $(Q_X^{\cdot}\langle Y\rangle, W', F')$ in (3.9) defines a mixed $\boldsymbol{Q}$-Hodge structure on $H_Y^i(X, \boldsymbol{Q}[1])$ and this fits into the long exact sequence of mixed Hodge structures $(\otimes\,\boldsymbol{C})$

$$\to \boldsymbol{H}^i(X, \Omega_X^{\cdot}\langle Y\rangle)\to \boldsymbol{H}^i(X, Q_X^{\cdot}\langle Y\rangle)\to \boldsymbol{H}^{i+1}(X, \Omega_X^{\cdot})\to$$

which is isomorphic to the bottom line of Section 1, (2)$\otimes\boldsymbol{C}$. On the other hand, from (8) if we denote by the same letters W' and F' (resp. W and F) the filtration on $H^{i+1}(X, \boldsymbol{C})\cong H^i(X, \boldsymbol{C}[1])$ induced from $(Q_X^{\cdot}\langle Y\rangle, W', F')$ (resp. $(Q_X^{\cdot}\langle Y\rangle[-1], W, F)$), then we have

$$(H^i(X, \boldsymbol{C}), W'[-2], F'[1])=(H^i(X, \boldsymbol{C}), W, F)$$

(cf. (1.3.1 a)). Hence our mixed Hodge structure fits into the exact sequence Section 1, (2) in (MH), as well as the one defined in (1.4). It follows that both structure is identical.

(3.11) *Proof of Theorem.* We put $K_Y^{\cdot}=\Gamma(Y, \mathscr{K}_Y^{\cdot})$ and $K_X^{\cdot}\langle Y\rangle=\Gamma(Y, \mathscr{K}_X^{\cdot}\langle Y\rangle)$. By Lemma (3.9.1) we get the bifiltered isomorphism

$$(H^{\cdot}(X, Q_X^{\cdot}\langle Y\rangle[-1]), W, F)\cong(H^{\cdot}(K_X^{\cdot}\langle Y\rangle), W, F).$$

Hence we may consider the mixed Hodge structure on $H_Y^{\cdot}(X, \boldsymbol{Q})$ coming from the right hand side by virtue of ($\tilde{\text{B}}$) (cf. (3.10.2)). From (3.8) we have the following commutative diagram of perfect pairings

$$
\begin{array}{ccc}
H^i(K_Y^{\cdot}) \times H^{2n-i}(K_X^{\cdot}\langle Y\rangle) & \xrightarrow{\phi_3'} & \\
\wr\| \qquad\qquad \wr\| & & \boldsymbol{C} \\
H^i(Y, \boldsymbol{C}) \times H_Y^{2n-i}(X, \boldsymbol{C}) & \xrightarrow{\psi_Y} &
\end{array}
$$

Thus in (1.9) it is enough to show the corresponding assertion for ϕ_3', $(H^i(K_Y^{\cdot}), W[i], F)$ and $(H^{2n-i}(K_X^{\cdot}\langle Y\rangle), W[2n-i], F)$. From the definition of ϕ_3' it follows immediately that

$$\phi_3'(W_k H^i(K_Y^{\cdot}), W_l H^{2n-i}(K_X^{\cdot}\langle Y\rangle))=0 \quad \text{if} \quad k+l<0, \quad \text{and}$$

$$\phi_3'(F^p W_k H^i(K_Y^{\cdot}), F^q W_{-k} H^{2n-i}(K_X^{\cdot}\langle Y\rangle))=0 \quad \text{if} \quad p+q>n.$$

In particular it induces a bilinear pairing

$$\phi_3'(p, k)\colon \mathrm{Gr}_F^p \mathrm{Gr}_W^k H^i(K_Y^{\cdot}) \times \mathrm{Gr}_F^{n-p} \mathrm{Gr}_W^{-k} H^{2n-i}(K_X^{\cdot}\langle Y\rangle) \to \boldsymbol{C}.$$

Since $W_m = W[i]_{m+i}$ and $W_{-m} = W[2n-i]_{-m+2n-i}$, we have only to show that this is perfect (cf. Def. (1.6.1)). Now by Deligne [4, 7.2.8] each of these spaces is naturally isomorphic to the E_∞ of the spectral sequence

$$E_1^{a,b} = H^{a+b}\,\mathrm{Gr}_W^a \mathrm{Gr}_F^p K^{\cdot} \Rightarrow H^{a+b}\,\mathrm{Gr}_F^p K^{\cdot} \tag{12}$$

associated with the filtered complex $(\mathrm{Gr}_F^t K^{\cdot}, W)$, with $t=p, n-p$ and $K^{\cdot} = K_Y^{\cdot}, K_X^{\cdot}\langle Y\rangle$, i.e.

$$E_\infty^{k,i-k} \cong \mathrm{Gr}_F^p \mathrm{Gr}_W^k H^i(K_Y^{\cdot}), \qquad \text{for} \quad (\mathrm{Gr}_F^p K_Y^{\cdot}, W),$$

$$E_\infty^{-k,2n-i+k} \cong \mathrm{Gr}_F^{n-p} \mathrm{Gr}_W^{-k} H^{2n-i}(K_X^{\cdot}\langle Y\rangle), \qquad \text{for} \quad (\mathrm{Gr}_F^{n-p} K_X^{\cdot}\langle Y\rangle, W).$$

Thus, by the biregularity of (12) it suffices to show that the natural pairing

$$\phi_3'\langle p, k\rangle\colon E_1^{k,i-k} \times E_1^{-k,2n-i+k} \to \boldsymbol{C}$$

inducing $\phi_3'(p, k)$ at E_∞ is perfect. Indeed, we have the natural isomorphisms $E_1^{k,i-k}(\mathrm{Gr}_F^p K_Y^{\cdot}) \cong H^i(\mathrm{Gr}_W^k \mathrm{Gr}_F^p K_Y^{\cdot}) \cong H^i\Gamma(Y_{(k+1)}, \mathscr{E}_{Y_{(k+1)}}^{p,\cdot}) = H^q(Y_{(k+1)}, \Omega_{Y_{(k+1)}}^p)$, $q = i-p$, and $E_1^{-k,2n-i+k}(\mathrm{Gr}_F^{n-p} K_X^{\cdot}\langle Y\rangle) \cong H^{2n-i}(\mathrm{Gr}_W^{-k} \mathrm{Gr}_F^{n-p} K_X^{\cdot}\langle Y\rangle) \cong H^{2n-i}\Gamma(Y_{(k+1)}, {}'\mathscr{D}_{Y_{(k+1)}}^{n-p-k-1,\cdot-n+p-k-1}) \cong H^{n-k-1-c}(Y_{(k+1)} \Omega_{Y_{(k+1)}}^{n-k-1-p})$ and $\phi_3'\langle p, k\rangle$ corresponds by these isomorphisms to the natural perfect pairing giving the Serre duality on $Y_{(k+1)}$. Q.E.D.

§4. Fary Spectral Sequence and Mixed Hodge Structure

(4.1) We start in an abstract setting. Let $\mathscr{A}$ and $\mathscr{A}'$ be abelian categories and $T\colon \mathscr{A} \to \mathscr{A}'$ be a covariant left exact functor. Let K be an object of $\mathscr{A}$ and F a finite decreasing filtration on K with $F^0(K)=K$ and $F^{m+1}(K)=0$ for some $m\geqq 0$. Then one has the usual spectral sequence of the hypercohomology of T applied to the filtered object (K, F):

$$(1)\qquad E_1^{p,q}=R^{p+q}T(\mathrm{Gr}_F^p K)\Rightarrow R^{p+q}T(K),$$

where Gr denotes the associated graded object. This is calculated as follows. Take any T-acyclic filtered resolution $(K^{\cdot}, F)$ of (K, F) in the sense that there is an exact sequence $0\to K\to K^0\to K^1\to$ such that it induces for every p an exact sequence $0\to\mathrm{Gr}_F^p K\to\mathrm{Gr}_F^p K^0\to\mathrm{Gr}_F^p K^1\to$ and that $\mathrm{Gr}_F^p K^n$ are T-acyclic for all p and n. Then by definition, (1) is up to isomorphisms the spectral sequence of the filtered complex $(M^{\cdot}, F)=(TK^{\cdot}, TF)$, where $TF^p(TK^{\cdot})=T(F^p(K^{\cdot}))$ (cf. [3, 1.4]).

(4.2) As in [3, 1.3] we define $Z_r^{p,q}$ and $B_r^{p,q}$, $1\leqq r\leqq\infty$, for the above complex $(M^{\cdot}, F)$ by the following:

$$Z_r^{p,q}=\mathrm{Ker}\,(d\colon F^p(M^{p+q})\to M^{p+q+1}/F^{p+r}(M^{p+q+1}))$$
$$M^{p+q}/B_r^{p,q}=\mathrm{Coker}\,(d\colon F^{p-r+1}(M^{p+q-1})\to M^{p+q}/F^{p+1}(M^{p+q})).$$

Let $\bar{Z}_r^{p,q}$ (resp. $\bar{B}_r^{p,q}$) be the natural image of $Z_r^{p,q}$ (resp. $B_r^{p,q}$) in $M^{p+q}/B_1^{p,q}$, which are contained in $E_1^{p,q}\cong\bar{Z}_1^{p,q}$. Then we have the sequence of inclusions

$$\bar{Z}_1^{p,q}\supseteqq\cdots\supseteqq\bar{Z}_\infty^{p,q}\supseteqq\bar{B}_\infty^{p,q}\supseteqq\cdots\supseteqq\bar{B}_1^{p,q}=\{0\}$$

and the natural isomorphisms

$$(2)\qquad E_r^{p,q}\underset{\mathrm{def}}{=}Z_r^{p,q}/Z_r^{p,q}\cap B_r^{p,q}\cong\bar{Z}_r^{p,q}/\bar{B}_r^{p,q},\quad \infty\geqq r\geqq 1\,.$$

We see readily that $\bar{Z}_r^{p,q}$ and $\bar{B}_r^{p,q}$ can be described as follows (cf. [8, I.4.7] for $r=2$). For any triple (s, t, u) with $0\leqq s<t<u\leqq m+1$ consider the long exact sequence

$$(3)_{stu}\qquad \xrightarrow{\alpha_{i;stu}} R^iT(F^s(K)/F^u(K))\xrightarrow{\beta_{i;stu}} R^iT(F^s(K)/F^t(K))$$
$$\xrightarrow{\delta_{i;stu}} R^{i+1}T(F^t(K)/F^u(K))\longrightarrow$$

coming from the short exact sequence

$$0\to F^t(K)/F^u(K)\to F^s(K)/F^u(K)\to F^s(K)/F^t(K)\to 0\,.$$

Then in view of the isomorphisms $R^iT(F^t(K)/F^u(K))\cong H^i(\mathrm{Gr}_F^{t,u}(M^{\cdot}))$ with $\mathrm{Gr}_F^{t,u}(M^{\cdot})\cong F^tM^{\cdot}/F^uM^{\cdot}$ etc., we get that

$$(4)\qquad \bar{Z}_r^{p,q}=\mathrm{Ker}\,\delta_{p+q;p,p+1,p+r}\quad\text{and}\quad \bar{B}_r^{p,q}=\mathrm{Im}\,\delta_{p+q-1;p-r+1,p,p+1}$$

where we put $F^p(K)=K$ for $-\infty\leqq p<0$ and $=0$ for $m+1<p\leqq\infty$. With the last convention we consider $(3)_{stu}$ also for general values of s, t, u with $s<t<u$.

(4.3) Put $\delta_r^{p,q}=\delta_{p+q;p,p+1,p+r}$ and $\delta'^{p,q}_r=\delta_{p+q;p-r+1,p,p+1}$ so that we have $\delta_r^{p,q}\cdot\delta'^{p,q}_r=0$ and $E_r^{p,q}\cong\operatorname{Ker}\delta_r^{p,q}/\operatorname{Im}\delta'^{p,q}_r$. We then consider the following commutative diagram of exact sequences

$$\begin{array}{ccccc}
 & & \downarrow & & \\
 & & R^{p+q-1}T(K/F^p(K)) & & \\
 & & \downarrow{\scriptstyle\delta} & \searrow{\scriptstyle\delta'^{p,q}_\infty} & \\
\to R^{p+q}T(F^{p+1}(K)) & \xrightarrow{\alpha} & R^{p+q}T(F^p(K)) & \xrightarrow{\beta} & R^{p+q}T(\mathrm{Gr}_F^p(K)) \\
 & \searrow{\scriptstyle\alpha_{p+1}} & \downarrow{\scriptstyle\alpha_p} & & \xrightarrow{\delta_\infty^{p,q}} R^{p+q+1}T(F^{p+1}(K))\longrightarrow \\
 & & R^{p+q}T(K) & & \\
 & & \downarrow & &
\end{array}$$

where the horizontal and vertical lines are $(3)_{p,p+1,\infty}$ and $(3)_{0p\infty}$ respectively, and $\alpha_{p+1}=\alpha_{p+q;0,p+1,\infty}$. From this together with (4) and (2) it follows that we have the natural isomorphisms

$$\begin{aligned}
(5)\qquad \mathrm{Gr}_F^p\,R^{p+q}T(K)&=\operatorname{Im}\alpha_p/\operatorname{Im}\alpha_{p+1}\cong R^{p+q}T(F^p(K))/(\operatorname{Im}\alpha+\operatorname{Im}\delta)\\
&\cong\operatorname{Im}\beta/\operatorname{Im}\delta'^{p,q}_\infty\cong\operatorname{Ker}\delta_\infty^{p,q}/\operatorname{Im}\delta'^{p,q}_\infty=\bar Z_\infty^{p,q}/\bar B_\infty^{p,q}\cong E_\infty^{p,q},
\end{aligned}$$

where we denote by the same letter F the filtration induced on the abutment $R^{p+q}T(K)$. Similarly consider the following commutative diagram for $2\leqq r<\infty$

$$\begin{array}{ccccc}
\to R^{p+q-1}T(F^{p-r+2}/F^p) & \xrightarrow{\alpha} & R^{p+q-1}T(F^{p-r+1}/F^p) & \xrightarrow{\beta} & R^{p+q-1}T(F^{p-r+1}/F^{p-r+2}) \\
 & \searrow{\scriptstyle\delta'^{p,q}_{r-1}} & \downarrow{\scriptstyle\delta'^{p,q}_r} & & \xrightarrow{\delta} R^{p+q}T(F^{p-r+2}/F^p)\to \\
 & & R^{p+q}T(F^p/F^{p+1}) & &
\end{array}$$

where $F^p=F^p(K)$ and the top line is $(3)_{p-r+1,p-r+2,p}$ so that $\delta=\delta_{r-1}^{p,q-1}$. From this it follows that $\bar Z_{r-1}^{p-r+1,q-r+2}=\operatorname{Ker}\delta=\operatorname{Im}\beta\cong R^{p+q-1}T(F^{p-r+1}/F^p)/\operatorname{Im}\alpha$. Hence $\delta'^{p,q}_r$ induces a map $\bar\delta'^{p,q}_r\colon Z_{r-1}^{p-r+1,q-r+2}\to\operatorname{Ker}\delta_{r-1}^{p,q}/\operatorname{Im}\delta'^{p,q}_{r-1}\cong E_{r-1}^{p,q}$ such that $\operatorname{Im}\bar\delta'^{p,q}_r\cong\operatorname{Im}\delta'^{p,q}_r/\operatorname{Im}\delta'^{p,q}_{r-1}$. Thus we have the isomorphisms

$$\begin{aligned}
(6)\qquad E_r^{p,q}&\cong\operatorname{Ker}\delta_r^{p,q}/\operatorname{Im}\delta'^{p,q}_r\\
&\cong\{\operatorname{Ker}(\delta_r^{p,q}|_{\operatorname{Ker}\delta_{r-1}^{p,q}})/\operatorname{Im}\delta'^{p,q}_{r-1}\}/(\operatorname{Im}\delta'^{p,q}_r/\operatorname{Im}\delta'^{p,q}_{r-1})\\
&\cong\operatorname{Ker}d_{r-1}^{p,q}/\operatorname{Im}d_{r-1}^{p-r,q-r+1}\underset{\text{def}}{=}H(E_{r-1}^{p,q}),
\end{aligned}$$

where in general $d_r^{p,q}\colon E_r^{p,q}\to E_{r+1}^{p+r,q-r+1}$ is the differential of the spectral sequence (1).

(4.4) Now assume that $\mathscr{A}'$ is the abelian category of $\boldsymbol{Q}$-vector spaces and linear mappings. Then we call (1) a spectral sequence in (MH) if the following conditions are satisfied; 1) $E_r^{p,q}$, $1 \leqq r \leqq \infty$, and $R^iT(K)$ are all finite dimensional and have natural mixed $\boldsymbol{Q}$-Hodge structures, 2) the differentials $d_r^{p,q}$ of (1) are compatible with these mixed $\boldsymbol{Q}$-Hodge structures, 3) $F^pR^iT(K)$ are mixed Hodge substructures of $R^iT(K)$ for all p and i, and finally 4) the natural isomorphisms $H(E_r^{p,q}) \cong E_{r+1}^{p,q}$ and $E_\infty^{p,q} \cong \mathrm{Gr}^p R^{p+q}T(K)$ are those in (MH), where the right hand side carries a natural mixed Hodge structure induced from $F^pR^{p+q}T(K)$. (Caution: Here and in the following F has nothing to do with the Hodge filtration of a mixed Hodge structure.)

Now from (5) and (6) we derive easily the following:

(4.4.1) **Lemma.** *Suppose that each $R^iT(F^s(K)/F^t(K))$ is finite dimensional and carry a mixed $\boldsymbol{Q}$-Hodge structure such that $(3)_{stu}$ are exact sequences in (MH) for all s, t, u. Then we can define natural mixed $\boldsymbol{Q}$-Hodge structures on $E_r^{p,q}$, $1 \leqq r \leqq \infty$, and $R^iT(K)$ in such a way that (1) is a spectral sequence in (MH) in the sense defined above.*

Proof. First we define mixed $\boldsymbol{Q}$-Hodge structures on $E_r^{p,q}$ by means of the isomorphisms $E_r^{p,q} \cong \mathrm{Ker}\,\delta_r^{p,q}/\mathrm{Im}\,\delta'^{p,q}_r$, where $\mathrm{Ker}\,\delta_r^{p,q}$ and $\mathrm{Im}\,\delta'^{p,q}_r$ have natural mixed $\boldsymbol{Q}$-Hodge structures since $\delta_r^{p,q}$ and $\delta'^{p,q}_r$ are in (MH) by our assumption. Next again by our assumption $R^iT(K) = R^iT(F^0K/F^{m+1}K)$ is given a mixed $\boldsymbol{Q}$-Hodge structure. Further since $F^pR^iT(K) = \mathrm{Im}\,\alpha_{i;0,p,\infty}$, it is a mixed Hodge substructure of $R^iT(K)$. Finally from (5) and (6) the conditions 2) and 4) follow easily. Q.E.D.

Note that the mixed Hodge structure on $E_1^{p,q}$ is defined via the natural isomorphism $E_1^{p,q} \cong R^{p+q}T(\mathrm{Gr}^p K)$, as follows from the above proof.

(4.5) Let X be a compact complex space with $X \in \mathscr{C}$ (1.1). Let $A_0 = \emptyset \subseteqq A_1 \subseteqq \cdots \subseteqq A_{m+1} = X$ be an increasing sequence of analytic subspaces of X. Let $U_{s,t} = A_t - A_s$, $s < t$, and $U_s = U_{s,m+1} = X - A_s$. Then we get that $U_{s,t} = U_s - U_t$. Let L be any sheaf of abelian groups on X. Then for any locally closed subset U of X we denote by L_U the sheaf which is zero outside U and coincides with $L|_U$ on U (cf. [8, 2.9.1]). With this notation let $\boldsymbol{Q}_{s,t} = \boldsymbol{Q}_{U_{s,t}}$ and $\boldsymbol{Q}_p = \boldsymbol{Q}_{U_p}$, $\boldsymbol{Q}$ being a constant sheaf $\boldsymbol{Q}_X$ on X. Then we have the decreasing filtration

$$F\colon \boldsymbol{Q}_0 = \boldsymbol{Q}_X \supseteqq \boldsymbol{Q}_1 \supseteqq \cdots \supseteqq \boldsymbol{Q}_m \supseteqq \boldsymbol{Q}_{m+1} = 0$$

of $\boldsymbol{Q}_X$ by the subsheaves $\boldsymbol{Q}_p$. Thus in (4.1) if we let $\mathscr{A} = \mathscr{A}(X)$, $T = \Gamma(X, \ )$ and $(K, F) = (\boldsymbol{Q}_X, F)$, then the spectral sequence (1) becomes the following

(7) $$E_1^{p,q}:=H_c^{p+q}(U_{p,p+1},\boldsymbol{Q})\Rightarrow H^{p+q}(X,\boldsymbol{Q}).$$

In fact, since $U_{s,t}$ is closed in U_s, we have in general the isomorphisms $F^s(\boldsymbol{Q})/F^t(\boldsymbol{Q})\cong\boldsymbol{Q}_{s,t}$ (cf. [8, 2.9.3]) so that $R^{p+q}T(F^s(\boldsymbol{Q})/F^t(\boldsymbol{Q}))\cong H_c^{p+q}(U_{s,t},\boldsymbol{Q})$. Up to the renumbering $E_r^{p,q}\to E_{r+1}^{q,p}$, (7) is nothing but the Fary spectral sequence associated with the descending sequence $\{A_t\}$ [2, IV.12].

From (7) and (1.4) it follows that $E_1^{p,q}$ and the abutment $H^{p+q}(X,\boldsymbol{Q})$ of this sequence have the natural mixed $\boldsymbol{Q}$-Hodge structures. Then using Lemma (4.4.1) we shall show the following:

(4.6) **Proposition.** *The spectral sequence* (7) *is one in* (*MH*) *such that on* $E_1^{p,q}$ *and* $H^{p+q}(X,\boldsymbol{Q})$ *the mixed* $\boldsymbol{Q}$*-Hodge structures coincide with those defined above by* (1.4). *In particular if* (7) *degenerates, then we have the isomorphisms in* (*MH*)

$$H_c^{p+q}(U_{p,p+1},\boldsymbol{Q})\cong\mathrm{Gr}_F^p H^{p+q}(X,\boldsymbol{Q}).$$

Proof. Since $R^iT(F^s(\boldsymbol{Q})/F^t(\boldsymbol{Q}))\cong H_c^i(U_{s,t},\boldsymbol{Q})$ as above, we can define the natural mixed $\boldsymbol{Q}$-Hodge structure on $R^iT(F^s(\boldsymbol{Q})/F^t(\boldsymbol{Q}))$ by this isomorphism in view of (1.4). Then it follows immediately that on $E_1^{p,q}$ and $H^i(X,\boldsymbol{Q})$ the mixed $\boldsymbol{Q}$-Hodge structures coincide with those given just before the lemma. Hence by (4.4.1) it is enough to show that the exact sequence $(3)_{stu}$ are compatible with the given mixed $\boldsymbol{Q}$-Hodge structure. Since $U_{s,t}=A_t-A_s$, by the above isomorphism $(3)_{stu}$ corresponds to the exact sequence of relative cohomology associated to the triple (A_s, A_t, A_u)

$$\xrightarrow{\alpha_i} H^i(A_u,A_s,\boldsymbol{Q})\xrightarrow{\beta_i} H^i(A_t,A_s,\boldsymbol{Q})\xrightarrow{\delta_i} H^{i+1}(A_u,A_t,\boldsymbol{Q})\xrightarrow{\alpha_{i+1}}$$

where $\alpha_i=\alpha_{i;stu}$, $\beta_i=\beta_{i;stu}$ etc. Firstly by the functoriality of the mixed Hodge structures (1.4), α_i and β_i are morphisms in (*MH*). Next, we decompose δ_i into $\delta_i=\bar{\delta}_i h_i$, where h_i: $H^i(A_t,A_s,\boldsymbol{Q})\to H^i(A_t,\boldsymbol{Q})$ is the restriction map and $\bar{\delta}_i$: $H^i(A_t,\boldsymbol{Q})\to H^i(A_u,A_t,\boldsymbol{Q})$ is the connection homomorphism in the exact sequence of relative cohomology associated with the pair (A_u, A_t). Since h_i and $\bar{\delta}_i$ are morphisms in (*MH*) by (1.4), δ_i also is. Hence the proposition is proved.

Remark. Presumably Lemma (4.4.1) could also be applicable to the spectral sequence

$$E_1^{p,q}\colon H_{U_{p,p+1}}^{p+q}(X,\boldsymbol{Q})\Rightarrow H^{p+q}(X,\boldsymbol{Q}),$$

which is the 'Poincaré dual' of (7).

References

[1] Bloom, T. and Herrera, M., De Rham cohomology of an analytic space, *Invent. Math.*, **7** (1969), 275–296.

[2] Bredon, G., *Sheaf theory,* McGraw-Hill, New York, 1967.

[3] Deligne, P., Théorie de Hodge II, *Publ. Math. IHES,* **40** (1972), 5–57.

[4] ———, Théorie de Hodge III, *Publ. Math. IHES,* **44** (1975), 5–77.

[5] Fujiki, A., Closedness of the Douady spaces of compact Kähler spaces, *Publ. RIMS, Kyoto Univ.,* **14** (1978), 1–52.

[6] ———, On automorphism groups of compact Kähler manifolds, *Invent. Math.,* **44** (1978), 225–258.

[7] ———, Fixed points of the actions on compact Kähler manifolds, *Publ. RIMS, Kyoto Univ.,* **15** (1979), 797–826.

[8] Godement, R., *Topologie algébrique et théorie des faisceaux.* Hermann, Paris, 1958.

[9] Goldberg, S. I., *Curvature and Homology,* Academic Press, New York, 1962.

[10] Griffiths, Ph. and Schmid, W., Recent developments in Hodge theory: a discussion of techniques and results, in discrete subgroups of Lie groups and applications to moduli, *Bombay Colloquium,* 1973, Oxford Univ. Press, (1975), 31–128.

[11] Herrera, M., Les courants residue multiple, *Bull. Soc. Math. France, Memoire,* **38** (1974), 27–30.

[12] Herrera, M. and Lieberman, D., Residues and principal values on complex spaces, *Math. Ann.,* **194** (1971), 259–294.

[13] ———, Duality and the de Rham cohomology of infinitesimal neighborhoods, *Invent. Math.,* **13** (1971), 97–124.

[14] Hironaka, H., *Bimeromorphic smoothing of a complex analytic spaces,* Math. Inst. Warwick Univ., England, 1971.

[15] Leray, J., Probléme de Cauchy III, *Bull. Soc. Math. France,* **87** (1959), 81–180.

[16] Poly, J.-P., Sur l'homologie des courants a support dans un ensemble semi-analytique, *Bull. Soc. Math. France, Memoire,* **38** (1974), 35–43.

[17] Ramis, J.-P. et Ruget, G., Résidus et dualité, *Invent. Math.,* **26** (1974), 89–131.

[18] de Rham, G., *Variétés differentiables,* 2e. ed., Hermann, Paris, 1960.

[19] Serre, J.-P., Un théorème de dualité, *Comment. Math. Helv.,* **29** (1955), 9-26.

[20] Steenbrink, J., Limits of Hodge structures, *Invent. Math.,* **31** (1976), 229–257.