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S-Duality in τ-Cohomology Theories

By

Mitutaka MURAYAMA*

Introduction

The concept of S-duality was introduced in Spanier [12] and generalized to the equivariant case by Wirthmüller [16]. τ-cohomology theories [3] are G-cohomology theories for $G=\mathbf{Z}/2\mathbf{Z}$ with their own sign convention. In the present work we translate S-duality into a form suitable for τ-cohomology with respect to the sign convention, and discuss the duality between τ-cohomology and homology.

Notation and terminology in [3] are used freely.

Section 1 is a preparatory section. The sign convention is described there. In Section 2 we observe the existence of the duality isomorphisms and S-duals. The main results of this section are Theorems 2.2 and 2.7. In Section 3 we see mainly the relations of slant products (which induces duality) with suspensions $\sigma^{*,*}$, $\sigma_{*,*}$ and $\sigma(*, *)$. In Section 4, using the results in Section 3, we discuss some properties of S-duality of stable τ-maps.

In Section 5 we discuss the duality between τ-cohomology and homology, and the representation of τ-homology theories. The main results in this section are Theorems 5.2, 5.5 and Corollary 5.3. In Section 6 we see Atiyah-Poincaré type duality in τ-cohomology.

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§1. τ-Cohomology Theories

The main reference of this section is Araki-Murayama [3].

We work mainly on the category $\mathcal{T}op_0^\tau$ of τ-*spaces* (=spaces with invo-

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* Department of Mathematics, Osaka City University, Osaka 558, Japan

lutions) with *base points*. By *τ-spaces*, *τ-maps* and *τ-homotopies* we mean τ-spaces with base points, equivariant maps preserving base points and equivariant homotopies relative to the base points respectively, for simplicity. Involutions are denoted by τ in most place. By $[\ ,\ ]^\tau$ we denote the set of τ-homotopy classes.

Let ψ: $\mathcal{T}op_0^\tau \to \mathcal{T}op_0$ be the *forgetful functor* to forget involutions and ϕ: $\mathcal{T}op_0^\tau \to \mathcal{T}op_0$ be the *fixed-point functor* to restrict to fixed points. So ϕX is the set of fixed points of a τ-space X and $\phi f: \phi X \to \phi Y$ is the restriction of a τ-map $f: X \to Y$ to the fixed-point sets. The forgetful functor ψ induces the morphism $\psi_*: [X, Y]^\tau \to [X, Y]$ and the fixed-point functor ϕ induces the morphism $\phi_*: [X, Y]^\tau \to [\phi X, \phi Y]$.

Let $\boldsymbol{R}^{p,q}$ be the euclidian space with the involution such that $\tau(x_1,\dots, x_p, x_{p+1},\dots, x_{p+q}) = (-x_1,\dots, -x_p, x_{p+1},\dots, x_{p+q})$. Let $B^{p,q}$ and $S^{p,q}$ be the unit ball and unit sphere in $\boldsymbol{R}^{p,q}$. Let $\Sigma^{p,q} = B^{p,q}/S^{p,q}$. $\Sigma^{p,q}$ is identified with the one-point compactification of $\boldsymbol{R}^{p,q}$. We identify $\Sigma^{r,0} \wedge \Sigma^{p,q} = \Sigma^{r+p,q}$, $\Sigma^{p,q} \wedge \Sigma^{0,s} = \Sigma^{p,q+s}$ by the standard τ-homeomorphisms.

Let X and Y be τ-spaces. We endow the set $[\Sigma^{p,q}X, Y]^\tau$ with track addition along a fixed coordinate, where $\Sigma^{p,q}X = \Sigma^{p,q} \wedge X$ and the involution on $\Sigma^{p,q}X$ is induced by the diagonal action on $\Sigma^{p,q} \times X$. Then $[\Sigma^{p,q}X, Y]^\tau$ is a group for $q \geqq 1$ and abelian for $q \geqq 2$. Let J be the involution on $\Sigma^{1,0}$. After the identification $\Sigma^{1,0} \wedge \Sigma^{p-1,q} = \Sigma^{p,q}$ we have an involutive τ-map $J \wedge 1$: $\Sigma^{p,q} \to \Sigma^{p,q}$. Thus we have an induced involution

$$\rho = (J \wedge 1)^*: [\Sigma^{p,q}X, Y]^\tau \to [\Sigma^{p,q}X, Y]^\tau$$

for $p \geqq 1$. Clearly $\psi_*\rho = -1$ and $\phi_*\rho = 1$. Putting

$$\Lambda = \boldsymbol{Z}[\rho]/(1-\rho^2),$$

$[\Sigma^{p,q}X, Y]^\tau$ is a Λ-module for $p \geqq 1$ and $q \geqq 2$. Λ is identified with the Burnside ring $A(\boldsymbol{Z}/2\boldsymbol{Z})$ of $\boldsymbol{Z}/2\boldsymbol{Z}$ [3], Section 2, and $\Lambda = [\Sigma^{p,q}, \Sigma^{p,q}]^\tau$ for $p \geqq 1$ and $q \geqq 1$, [3], Theorem 12.5.

A *τ-complex* is a G-complex for $G = \boldsymbol{Z}/2\boldsymbol{Z}$, generated by τ, [3, 6]. Let $\mathcal{W}_0^\tau$ and $\mathcal{F}_0^\tau$ be the full subcategories of $\mathcal{T}op_0^\tau$ in which the objects are τ-spaces having τ-homotopy types of τ-complexes and finite τ-complexes, respectively. Let $\mathcal{C}\mathcal{W}_0^\tau$ and $\mathcal{C}\mathcal{F}_0^\tau$ be the full subcategories of $\mathcal{W}_0^\tau$ and $\mathcal{F}_0^\tau$ with τ-complexes and finite τ-complexes as objects, respectively. The base points of τ-complexes are vertices as usual.

A *reduced τ-cohomology theory* on the category $\mathscr{W}_o^\tau$ or on $\mathscr{F}_o^\tau$ is a system

$$\tilde{h}^{*,*}=\{\tilde{h}^{p,q}: (p, q)\in \boldsymbol{Z}\times\boldsymbol{Z}=RO(\boldsymbol{Z}/2\boldsymbol{Z})\}$$

of Λ-module-valued contravariant functors $\tilde{h}^{p,q}$ satisfying the following four axioms A1)–A4).

A1) Each $\tilde{h}^{p,q}$ is a τ-homotopy functor satisfying Wedge axiom and Mayer-Vietoris axiom on $\mathscr{C}\mathscr{W}_o^\tau$ or $\mathscr{C}\mathscr{F}_o^\tau$.

A2) Two kinds of suspension isomorphisms

$$\bar{\sigma}=\sigma^{1,0}: \tilde{h}^{p,q}(X)\cong\tilde{h}^{p+1,q}(\Sigma^{1,0}X)$$

and

$$\sigma=\sigma^{0,1}: \tilde{h}^{p,q}(X)\cong\tilde{h}^{p,q+1}(\Sigma^{0,1}X)$$

are defined as natural isomorphisms of Λ-module-valued functors.

A3) The following diagram

$$\begin{array}{ccccc} & \overset{\sigma}{\nearrow} & \tilde{h}^{p,q+1}(\Sigma^{0,1}X) & \xrightarrow{\bar{\sigma}} & \tilde{h}^{p+1,q+1}(\Sigma^{1,0}\Sigma^{0,1}X) \\ \tilde{h}^{p,q}(X) & & & & \downarrow{\scriptstyle \rho(T\wedge 1)^*} \\ & \underset{\bar{\sigma}}{\searrow} & \tilde{h}^{p+1,q}(\Sigma^{1,0}X) & \xrightarrow{\sigma} & \tilde{h}^{p+1,q+1}(\Sigma^{0,1}\Sigma^{1,0}X) \end{array}$$

is commutative for any X, where $T: \Sigma^{0,1}\Sigma^{1,0}\to\Sigma^{1,0}\Sigma^{0,1}$ is the τ-map switching factors.

A4) Let J be the involution of $\Sigma^{1,0}$, then

$$(J\wedge 1)^*=\rho \text{ times}: \tilde{h}^{p,q}(\Sigma^{1,0}X)\to\tilde{h}^{p,q}(\Sigma^{1,0}X).$$

Axioms A3) and A4) relate the ring Λ to sign conventions. Iterated suspension isomorphisms $\sigma^{s,t}: \tilde{h}^{p,q}(X)\cong\tilde{h}^{p+s,q+t}(\Sigma^{s,t}X)$ are defined as the composite $\sigma^{s,t}=\bar{\sigma}^s\circ\sigma^t$ after the canonical identification $\Sigma^{1,0}\wedge\cdots\wedge\Sigma^{1,0}\wedge\Sigma^{0,1}\wedge\cdots\wedge\Sigma^{0,1}=\Sigma^{s,t}$. We also use the notation $\sigma^{-s,-t}=(\sigma^{s,t})^{-1}$ for inverses of suspensions.

The associated unreduced τ-cohomology theory $h^{*,*}=\{h^{p,q}; (p, q)\in \boldsymbol{Z}\times\boldsymbol{Z}\}$ is defined as usual by $h^{p,q}(X, A)=\tilde{h}^{p,q}(X/A)$ and $h^{p,q}(X)=\tilde{h}^{p,q}(X_+)$, where $X_+=X\cup\{pt\}$.

Reduced τ-homology theories are defined in the obvious way and denoted by $\tilde{h}_{*,*}$. Suspensions in reduced τ-homology theories are denoted by $\sigma_{s,t}$.

Let $\boldsymbol{E}=\{E_n, \varepsilon_n: \Sigma^{1,1}E_n\to E_{n+1}\}$ be a *τ-spectrum* $(E_n\in\mathscr{W}_o^\tau)$, $(p, q)\in\boldsymbol{Z}\times\boldsymbol{Z}$, and $n>\max(-p, -q)$. For $X\in\mathscr{W}_o^\tau$ and $Y\in\mathscr{W}_o^\tau$ the Λ-homomorphism $\tilde{\varepsilon}_n$: $[\Sigma^{n+p,n+q}X, E_n\wedge Y]^\tau\to[\Sigma^{n+p+1,n+q+1}X, E_{n+1}\wedge Y]^\tau$ is defined as the composite

$$[\Sigma^{n+p,n+q}X, E_n \wedge Y]^\tau \xrightarrow{\Sigma^{1,1}_*} [\Sigma^{1,1}\Sigma^{n+p,n+q}X, \Sigma^{1,1}E_n \wedge Y]^\tau$$
$$\xrightarrow{(\varepsilon_n \wedge 1)^*} [\Sigma^{1,1}\Sigma^{n+p,n+q}X, E_{n+1} \wedge Y]^\tau$$
$$\xrightarrow{\rho^{n+p}(T\wedge 1)^*} [\Sigma^{n+p+1,n+q+1}X, E_{n+1} \wedge Y]^\tau$$

([3], (7.3)). Here, and henceforth, the τ-homeomorphisms $\Sigma^{p,q}\Sigma^{r,s}\,(=\Sigma^{p,0}\Sigma^{0,q}\cdot\Sigma^{s,0}\Sigma^{0,t}) \underset{\tau}{\approx} (\Sigma^{p,0}\Sigma^{s,0}\Sigma^{0,q}\Sigma^{0,t}=)\ \Sigma^{p+s,q+t}$ which are induced by the switching maps $\Sigma^{0,q}\Sigma^{r,0} \underset{\tau}{\approx} \Sigma^{r,0}\Sigma^{0,q}$ are generally denoted by T, for simplicity, [3], Section 7. Put

$$[X, \boldsymbol{E}\wedge Y]_{p,q}\,(=[X, \boldsymbol{E}\wedge Y]^{-p,-q}) = \varinjlim_n \{[\Sigma^{n+p,n+q}X, E_n \wedge Y]^\tau, \tilde{\varepsilon}_n\}$$

for each $(p, q)\in \boldsymbol{Z}\times\boldsymbol{Z}$. Then $\{[-, \boldsymbol{E}\wedge Y]^{p,q}; (p, q)\in \boldsymbol{Z}\times\boldsymbol{Z}\}$ is a reduced τ-cohomology theory on $\mathscr{F}_o^\tau$ for a fixed $Y\in\mathscr{W}_o^\tau$ together with the suspension isomorphisms $\sigma^{s,t}$, [3], (7.6) and Theorem 7.7, and $\{[X, \boldsymbol{E}\wedge -]_{p,q}; (p, q)\in \boldsymbol{Z}\times\boldsymbol{Z}\}$ is a reduced τ-homology theory on $\mathscr{W}_o^\tau$ for a fixed $X\in\mathscr{C}\mathscr{F}_o^\tau$ together with the suspension isomorphisms $\sigma_{r,s}$, [3], (13.3), Propositions 13.4 and 13.5. $[-, \boldsymbol{E}\wedge\Sigma^{0,0}]^{p,q}=[-, \boldsymbol{E}]^{p,q}$ is also denoted by $\tilde{\boldsymbol{E}}^{p,q}(-)$ or $\tilde{h}^{p,q}(-;\boldsymbol{E})$, and $[\Sigma^{0,0}, \boldsymbol{E}\wedge -]_{p,q}$ is denoted by $\tilde{\boldsymbol{E}}_{p,q}(-)$ or $\tilde{h}_{p,q}(-;\boldsymbol{E})$.

Each τ-cohomology theory $h^{*,*}$ is associated with two (non-equivariant) cohomology theories: the one is the *forgetful cohomology theory* ψh^* defined by $\psi h^n(-)=h^{0,n}(S^{1,0}\times -)\ (\cong h^{p,n-p}(S^{1,0}\times -))$, and the other is the *fixed-point cohomology theory* ϕh^* defined by $\phi h^n(-)=\varinjlim_p h^{p,n}(-)$. And the forgetful morphism ψ: $\{h^{p,q}\}\to\{\psi h^{p+q}\}$ and the fixed-point morphism ϕ: $\{h^{p,q}\}\to\{\phi h^q\}$ are defined. These are a kind of natural transformations of cohomology theories. (Cf., [3], §§ 2–3.)

Let $\boldsymbol{E}$ be a τ-spectrum. Applying the forgetful and fixed-point functors to each term and map of $\boldsymbol{E}$, we obtain spectra $\psi\boldsymbol{E}$ and $\phi\boldsymbol{E}$ called the *forgetful* and *fixed-point spectrum* respectively. The cohomology theories $h^*(\ ;\psi\boldsymbol{E})$ and $h^*(\ ;\phi\boldsymbol{E})$ represented by $\psi\boldsymbol{E}$ and $\phi\boldsymbol{E}$ coincide with the forgetful cohomology theory $\psi h^*(\ ;\boldsymbol{E})$ and the fixed-point cohomology theory $\phi h^*(\ ;\boldsymbol{E})$ of $h^{*,*}(\ ;\boldsymbol{E})$, respectively. The forgetful functor induces the homomorphisms ψ_*: $[\Sigma^{n-p,n-q}X, E_n]^\tau\to[\Sigma^{2n-p-q}X, (\psi\boldsymbol{E})_{2n}]$ which form the map of the direct systems. Taking the direct limits, we get a homomorphism

$$\psi_*: \tilde{\boldsymbol{E}}^{p,q}(X)\to\psi\tilde{\boldsymbol{E}}^{p+q}(X).$$

This homomorphism coincides with the forgetful morphism ψ for $\tilde{\boldsymbol{E}}^{*,*}$, [3], (7.10). Also the fixed-point functor induces the homomorphism ϕ_*: $\tilde{\boldsymbol{E}}^{p,q}(X)$

$\to \phi\tilde{E}^q(\phi X)$ which coincides with the fixed-point morphism ϕ for $\tilde{E}^{*,*}$, [3], (7.12).

An example of τ-spectrum is the τ-spectrum of stable τ-homotopy

$$\boldsymbol{SR}=\{\Sigma^{n,n},\ \varepsilon_n=T:\ \Sigma^{1,1}\Sigma^{n,n}\underset{\tau}{\approx}\Sigma^{n+1,n+1}\}.$$

In this case $\psi\boldsymbol{SR}$ and $\phi\boldsymbol{SR}$ are both the sphere spectra.

Proposition 1.1. *Let $X\in\mathscr{F}_o^\tau$ and $Y\in\mathscr{W}_o^\tau$. Then, for each $(p, q)\in\boldsymbol{Z}\times\boldsymbol{Z}$,*

$$[X, \boldsymbol{SR}\wedge Y]_{p,q}^\tau\cong[\Sigma^{n+p,n+q}X, \Sigma^{n,n}Y]^\tau$$

for large n.

This follows from [3], Proposition 13.12.

The cofibration sequence $S_+^{1,0}\to B_+^{1,0}\to\Sigma^{1,0}\to\Sigma^{0,1}S_+^{1,0}$ induces exact sequences

$$\begin{array}{ccccccc}
\cdots\to\tilde{h}^{p,q}(\Sigma^{1,0}X) & \to & \tilde{h}^{p,q}(B_+^{1,0}\wedge X) & \to & \tilde{h}^{p,q}(S_+^{1,0}\wedge X) & \to & \tilde{h}^{p,q+1}(\Sigma^{1,0}X)\to\cdots \\
\wr\|\sigma^{-1,0} & & \wr\| & & \wr\| & & \wr\|\sigma^{-1,0} \\
\cdots\to\tilde{h}^{p-1,q}(X) & \xrightarrow{\chi} & \tilde{h}^{p,q}(X) & \xrightarrow{\psi} & \psi\tilde{h}^{p+q}(X) & \xrightarrow{\delta} & \tilde{h}^{p-1,q+1}(X)\to\cdots
\end{array}$$

where the second row is called the *forgetful exact sequence* of $h^{*,*}$, [3], (5.1).

Proposition 1.2. *Let $\tilde{\Phi}$: $\tilde{h}^{*,*}\to\tilde{k}^{*+r,*+s}$ be a natural transformation of reduced τ-cohomology theories of degree (r, s). If*

$$\tilde{\Phi}:\ \tilde{h}^{p,q}(X)\to\tilde{k}^{p+r,q+s}(X)$$

is isomorphic for a fixed X and each $(p, q)\in\boldsymbol{Z}\times\boldsymbol{Z}$, then

$$\tilde{\Phi}:\ \tilde{h}^{p,q}(S_+^{1,0}\wedge X)\to\tilde{k}^{p+r,q+s}(S_+^{1,0}\wedge X)$$

and

$$\psi\tilde{\Phi}:\ \psi\tilde{h}^{p+q}(X)\to\psi\tilde{k}^{p+r+q+s}(X)$$

are isomorphic for any $(p, q)\in\boldsymbol{Z}\times\boldsymbol{Z}$.

Proof. Compare the forgetful exact sequences of $\tilde{h}^{*,*}(X)$ and $\tilde{k}^{*,*}(X)$. Then 5-lemma implies the result.

Similarly we obtain the following

Proposition 1.2′. *Let $\tilde{\Psi}$: $\tilde{h}_{*,*}\to\tilde{k}_{*+r,*+s}$ be a natural transformation of reduced τ-homology theories of degree (r, s). If*

$$\tilde{\Psi}:\ \tilde{h}_{p,q}(X)\to\tilde{k}_{p+r,q+s}(X)$$

is isomorphic for a fixed X and each $(p, q)\in\boldsymbol{Z}\times\boldsymbol{Z}$, then

$$\tilde{\Psi}: \tilde{h}_{p,q}(S_+^{1,0} \wedge X) \to \tilde{k}_{p+r,q+s}(S_+^{1,0} \wedge X)$$

is isomorphic for any $(p, q) \in \boldsymbol{Z} \times \boldsymbol{Z}$.

Propositions 1.2 and 1.2′ show that, if natural transformations $\tilde{\Phi}: \tilde{h}^{*,*} \to \tilde{k}^{*+r,*+s}$ and $\tilde{\Psi}: \tilde{h}_{*,*} \to \tilde{k}_{*+r,*+s}$ are isomorphic on $\Sigma^{0,0}$, then $\tilde{\Phi}: \tilde{h}^{*,*}(S_+^{1,0}) \cong \tilde{k}^{*+r,*+s}(S_+^{1,0})$ and $\tilde{\Psi}: \tilde{h}_{*,*}(S_+^{1,0}) \cong \tilde{k}_{*+r,*+s}(S_+^{1,0})$. Then comparison theorem for τ-(co-)homology theories has the following form. (Cf., [6], Chap. IV, 5 and [9], Comparison Theorem 2.14.)

Theorem 1.3. *Let* $\tilde{h}^{*,*}$ *and* $\tilde{k}^{*,*}$ *be reduced* τ*-cohomology theories on* $\mathscr{W}_o^\tau$ *or on* $\mathscr{F}_o^\tau$, *and* $\tilde{\Phi}: \tilde{h}^{*,*} \to \tilde{k}^{*+r,*+s}$ *be a natural transformation of reduced* τ*-cohomology theories of degree* (r, s). *If*

$$\tilde{\Phi}: \tilde{h}^{*,*}(\Sigma^{0,0}) \cong \tilde{k}^{*+r,*+s}(\Sigma^{0,0}),$$

then

$$\tilde{\Phi}: \tilde{h}^{*,*}(X) \cong \tilde{k}^{*+r,*+s}(X)$$

for any $X \in \mathscr{C}\mathscr{W}_o^\tau$ *or any* $X \in \mathscr{C}\mathscr{F}_o^\tau$.

Theorem 1.3′. *Let* $\tilde{h}_{*,*}$ *and* $\tilde{k}_{*,*}$ *be reduced* τ*-homology theories on* $\mathscr{W}_o^\tau$ *or on* $\mathscr{F}_o^\tau$, *and* $\tilde{\Psi}: \tilde{h}_{*,*} \to \tilde{k}_{*+r,*+s}$ *be a natural transformation of reduced* τ*-homology theories of degree* (r, s). *If*

$$\tilde{\Psi}: \tilde{h}_{*,*}(\Sigma^{0,0}) \cong \tilde{k}_{*+r,*+s}(\Sigma^{0,0}),$$

then

$$\tilde{\Psi}: \tilde{h}_{*,*}(X) \cong \tilde{k}_{*+r,*+s}(X)$$

for any $X \in \mathscr{C}\mathscr{W}_o^\tau$ *or any* $X \in \mathscr{C}\mathscr{F}_o^\tau$.

Next we state some isomorphisms of τ-homotopy groups.

Proposition 1.4. *Let* X *be a* τ*-space such that* i) X *is* m*-connected and* ii) ϕX *is* n*-connected. Let* (K, L) *be a pair of* τ*-complexes such that* $\dim(K-L) \leq m+1$ *and* $\dim(\phi K - \phi L) \leq n+1$. *Then any* τ*-map* $f: L \to X$ *can be extended equivariantly on* K.

The proof is similar to [3], Proposition 11.1.

Let $F(X, Y)$ be the base-point preserving function space from X to Y. Then $F(X, Y)$ is a τ-space with τ-action $(\tau f)(x) = \tau f(\tau x)$, $x \in X$.

Proposition 1.5. *Let* X *be a locally compact* τ*-complex and* Y *a* τ*-space*

Let r_{ϕ}: $\phi F(X, Y) \to F(\phi X, \phi Y)$ be the map obtained by restriction to ϕX. Assume that Y is m-connected. Then

$$r_{\phi *}: \pi_j(\phi F(X, Y)) \to \pi_j(F(\phi X, \phi Y))$$

is isomorphic if $j \leqq M$ and epimorphic if $j \leqq M+1$, where

$$M = \begin{cases} m - \dim(X - \phi X) & \text{if} \quad X \neq \phi X \\ \infty & \text{if} \quad X = \phi X . \end{cases}$$

Proof. As X is locally compact, we have

$$\pi_j(\phi F(X, Y)) \cong [\Sigma^{0,j} X, Y]^{\tau}$$

(cf., [6], Chap. III). Let i: $\phi X \to X$ be the inclusion. Consider

$$(1 \wedge i)^*: [\Sigma^{0,j} X, Y]^{\tau} \to [\Sigma^{0,j}(\phi X), Y]^{\tau} .$$

Then, applying Proposition 1.4 to the pair $(\Sigma^{0,j} X, \Sigma^{0,j}(\phi X))$ for surjectivity and to the pair $(\Sigma^{0,j} X \times I, \Sigma^{0,j} X \times \{0, 1\} \cup \Sigma^{0,j}(\phi X) \times I)$ for injectivity, we get the proof.

§2. S-Duality in the Stable τ-Homotopy Theory

The (p, q)-th *stable τ-homotopy group*, $(p, q) \in \boldsymbol{Z} \times \boldsymbol{Z}$, is denoted by

$$\{X, Y\}_{p,q} \; (= [X, \boldsymbol{SR} \wedge Y]_{p,q} = \varinjlim_n \, [\Sigma^{n+p,n+q} X, \Sigma^{n,n} Y]^{\tau}) .$$

$\{X, Y\}_{p,q}$ is also denoted by $\{X, Y\}^{-p,-q}$. By Proposition 1.1

$$\{X, Y\}_{p,q} \cong [\Sigma^{n+p,n+q} X, \Sigma^{n,n} Y]^{\tau}$$

for $X \in \mathscr{C}\mathscr{F}_o^{\tau}$, $Y \in \mathscr{W}_o^{\tau}$ and large n.

Let $\mathscr{C}\mathscr{F}_*^{\tau}$ be the full subcategory of $\mathscr{C}\mathscr{F}_o^{\tau}$ with objects X such that X and ϕX are path-connected. For $X, X' \in \mathscr{C}\mathscr{F}_*^{\tau}$ a τ-map

$$u: X \wedge X' \to \Sigma^{r_1,s_1} \cdots \Sigma^{r_k,s_k}, \quad r = r_1 + \cdots + r_k, \; s = s_1 + \cdots + s_k$$

is called a ((r, s)-)*duality τ-map* (or *R-duality τ-map*, $R = (r, s)$) if u: $X \wedge X' \to \Sigma^{r+s}$ and ϕu: $\phi X \wedge \phi X' \to \Sigma^s$ are duality maps in the sense of Spanier [12], page 360, and then X' is called an *(r, s)-dual* by means of u. For $X, X' \in \mathscr{C}\mathscr{F}_o^{\tau}$, X' is called an (*equivariant*) *S-dual* of X if some (iterated) suspension of X' is an (r, s)-dual of some (iterated) suspension of X for some (r, s). If u: $X \wedge X' \to \Sigma^{r,s}$ be an (r, s)-duality τ-map, then the τ-map $\bar{u}$: $X' \wedge X \to \Sigma^{r,s}$ defined by $\bar{u}(x', x) = u(x, x')$, $x \in X$, $x' \in X'$, is also an (r, s)-duality τ-map, [12], Lemma

(5.4). For a τ-map $u: X\wedge X'\to\Sigma^{r,s}$ and pairs $P=(p, q)$, $P'=(p', q')$ of non-negative integers we define

$$u_{P,P'}: \Sigma^{p,q}X\wedge\Sigma^{p',q'}X'\to\Sigma^{p,q}\Sigma^{p',q'}\Sigma^{r,s}$$

to be the composite

$$\Sigma^{p,q}X\wedge\Sigma^{p',q'}X' \underset{\tau}{\overset{1\wedge T'\wedge 1}{\approx}} \Sigma^{p,q}\Sigma^{p',q'}X\wedge X' \xrightarrow{1\wedge 1\wedge u} \Sigma^{p,q}\Sigma^{p',q'}\Sigma^{r,s}.$$

If u is an (r, s)-duality τ-map, then $u_{P,P'}$ is a $(P+P'+R)$-duality τ-map.

Here, and henceforth, T' denote switching maps in general.

For a τ-map $u: X\wedge X'\to\Sigma^{r,s}$ and $P=(p, q)\in \boldsymbol{Z}\times\boldsymbol{Z}$ we define

$$\Gamma_u^P: \{Y, X\}_{p,q}\to\{Y\wedge X', \Sigma^{r,s}\}_{p,q}$$

as follows: if $f: \Sigma^{n+p,n+q}Y\to\Sigma^{n,n}X$ represents an element $\{f\}\in\{Y, X\}_{p,q}$, then $\Gamma_u^P\{f\}$ is represented by the composite

$$\Sigma^{n+p,n+q}Y\wedge X' \xrightarrow{f\wedge 1} \Sigma^{n,n}X\wedge X' \xrightarrow{u_{N,O}} \Sigma^{n,n}\Sigma^{r,s},$$

where $N=(n, n)$ and $O=(0, 0)$. Then Γ_u^P is a well-defined Λ-homomorphism and coincides with the slant product $\{u\}/$, see Section 3. $\{\Gamma_u^P\}_{P\in\boldsymbol{Z}\times\boldsymbol{Z}}$ is a natural transformation of τ-cohomology theories with respect to Y.

Theorem 2.1. *Let $u: X\wedge X'\to\Sigma^{r,s}$ be an (r, s)-duality τ-map. Then*

$$\Gamma_u^P: \{Y, X\}_{p,q}\to\{Y\wedge X', \Sigma^{r,s}\}_{p,q}$$

is a Λ-isomorphism for any $Y\in\mathscr{C}\mathscr{F}_0^\tau$ and any $P=(p, q)\in\boldsymbol{Z}\times\boldsymbol{Z}$.

Proof. Consider the map $u_n: [\Sigma^{n+p,n+q}Y, \Sigma^{n,n}X]^\tau\to[\Sigma^{n+p,n+q}Y\wedge X', \Sigma^{n,n}\Sigma^{r,s}]^\tau$ in the definition of Γ_u^P, with $\varinjlim_n u_n=\Gamma_u^P$. Define $\lambda_n: \Sigma^{n,n}X\to F(X', \Sigma^{n,n}\Sigma^{r,s})$ by $\lambda_n(x)(x')=u_{N,O}(x, x')$, $x\in\Sigma^{n,n}X$, $x'\in X'$. Then the following diagram is commutative:

$$\begin{array}{ccc} [\Sigma^{n+p,n+q}Y, \Sigma^{n,n}X]^\tau & \xrightarrow{u_n} & [\Sigma^{n+p,n+q}Y\wedge X', \Sigma^{n,n}\Sigma^{r,s}]^\tau \\ & {\scriptstyle\lambda_{n*}}\searrow \quad \swarrow{\scriptstyle\mu_n} & \\ & [\Sigma^{n+p,n+q}Y, F(X', \Sigma^{n,n}\Sigma^{r,s})]^\tau & \end{array},$$

where μ_n is the isomorphism induced by a τ-homeomorphism

$$F(\Sigma^{n+p,n+q}Y\wedge X', \Sigma^{n,n}\Sigma^{r,s}) \underset{\tau}{\approx} F(\Sigma^{n+p,n+q}Y, F(X', \Sigma^{n,n}\Sigma^{r,s}))$$

taking f into $\bar{f}$ defined by $\bar{f}(y)(x')=f(y, x')$. Therefore u_n is isomorphic if and only if λ_{n*} is isomorphic.

We show that λ_{n*} is isomorphic for large n. Define $\nu_n: \Sigma^n(\phi X)\to F(\phi X',$

$\Sigma^{n+s})$ by $v_n(x)(x')=\phi u_{N,O}(x, x')$. Let r_ϕ: $\phi F(X', \Sigma^{n,n}\Sigma^{r,s})\to F(\phi X', \Sigma^{n+s})$ be the map obtained by restriction to $\phi X'$. Then the following diagram is commutative:

$$\begin{array}{ccc} \pi_j(\Sigma^n(\phi X)) & \xrightarrow{\phi\lambda_{n*}} & \pi_j(\phi F(X', \Sigma^{n,n}\Sigma^{r,s})) \\ & {\scriptstyle v_{n*}}\searrow \quad \swarrow{\scriptstyle r_{\phi*}} & \\ & \pi_j(F(\phi X', \Sigma^{n+s})) & \end{array}$$

By Proposition 1.5 $r_{\phi*}$ is isomorphic if $j\leqq(2n+r+s-1-\dim X')$. Recall that $\phi u_{N,O}$ is a duality map. By [12], (2.8) and the proof of Theorem (5.5), v_{n*} is isomorphic for $j<2(n+s-\dim\phi X')$ when n is large enough so that $\Sigma^n(\phi X)$ and $F(\phi X', \Sigma^{n+s})$ are 1-connected. Thus $\phi\lambda_{n*}$ is isomorphic for $j<2n-2\dim X'-1$. Recall that $\psi u_{N,O}$ is a duality map. Then $\psi\lambda_{n*}$: $\pi_j(\Sigma^{2n}X)\to\pi_j(F(X', \Sigma^{2n+r+s}))$ is isomorphic for $j<2(2n+r+s-\dim X')$ and large n. Then, by [3], Proposition 11.2 λ_{n*} is isomorphic for $n>2\cdot\dim X'+\dim Y+p+q+2$. Thus λ_{n*} is isomorphic for large n. q.e.d.

The duality isomorphism Γ_u^P: $\{Y, X\}_{p,q}\to\{Y\wedge X', \Sigma^{r,s}\}_{p,q}$, $P=(p, q)$, induces the homomorphisms

$$\psi(\Gamma_u^P)\colon \{\psi Y, \psi X\}_{p+q}\to\{\psi Y\wedge\psi X', \Sigma^{r+s}\}_{p+q}$$

and

$$\phi(\Gamma_u^P)\colon \{\phi Y, \phi X\}_{q}\to\{\phi Y\wedge\phi X', \Sigma^{s}\}_{q}$$

which correspond to $\Gamma_{\psi u}^{p+q}$ and $\Gamma_{\phi u}^{q}$ respectively, where $\{\ ,\ \}_n$ denotes the (non-equivariant) stable homotopy group. By the definition of a duality τ-map and [12], Lemma (5.8), we see that $\psi(\Gamma_u^P)=\Gamma_{\psi u}^{p+q}$ and $\phi(\Gamma_u^P)=\Gamma_{\phi u}^{q}$ are isomorphisms.

Adding the converse to the above results, we obtain the following

Theorem 2.2. *Let $X, X'\in\mathscr{CF}_*^\tau$ and $u\colon X\wedge X'\to\Sigma^{r,s}$ be a τ-map. Then the following are equivalent:*

(1) *u is a duality τ-map,*

(2) *Γ_u^P: $\{\Sigma^{0,0}, X\}_{p,q}\cong\{X', \Sigma^{r,s}\}_{p,q}$ for every $P=(p, q)\in\mathbf{Z}\times\mathbf{Z}$,*

(3) *Γ_u^P: $\{Y, X\}_{p,q}\cong\{Y\wedge X', \Sigma^{r,s}\}_{p,q}$ for any $Y\in\mathscr{CF}_o^\tau$ and $P=(p, q)\in\mathbf{Z}\times\mathbf{Z}$,*

(4) *$\Gamma_{\psi u}^n$: $\{\Sigma^0, \psi X\}_n\cong\{\psi X', \Sigma^{r+s}\}_n$ and*

$\Gamma_{\phi u}^n$: $\{\Sigma^0, \phi X\}_n\cong\{\phi X', \Sigma^{s}\}_n$ for every $n\in\mathbf{Z}$,

(5) *$\Gamma_{\psi u}^n$: $\{Y, \psi X\}_n\cong\{Y\wedge\psi X', \Sigma^{r+s}\}_n$ and*

$\Gamma_{\phi u}^n$: $\{Y, \phi X\}_n\cong\{Y\wedge\phi X', \Sigma^{s}\}_n$ for any $Y\in\mathscr{CF}_o$ and every $n\in\mathbf{Z}$,

where $\mathscr{C}\mathscr{F}_0$ denotes the category of finite pointed CW-complexes.

Proof. The implications (2)⇔(3) and (4)⇔(5) follow from comparison theorems (Theorem 1.3). The implications (1)⇒(2) and (1)⇒(3) are the result of Theorem 2.1.

Proof of (2)⇒(4). Since $\{\Gamma_u^p\}_{p\in \boldsymbol{Z}\times \boldsymbol{Z}}$ is a natural transformation of τ-cohomology theories, Proposition 1.2 implies that $\Gamma_{\psi u}^n$ is isomorphic for each $n\in \boldsymbol{Z}$. As the spectra $\psi \boldsymbol{SR}\wedge\psi X$ and $\psi \boldsymbol{SR}\wedge\Sigma^{r+s}$ are connective and X' is finite, we see that $\phi\colon \{\Sigma^{0,0}, X\}_{p,q}\cong\{\Sigma^0, \phi X\}_q$ and $\phi\colon \{X', \Sigma^{r,s}\}_{p,q}\cong\{\phi X', \Sigma^s\}_q$ for large p by [3], Proposition 5.4. Then (2) implies that $\Gamma_{\phi u}^q$ is isomorphic for each $q\in \boldsymbol{Z}$ because of $\phi(\Gamma_u^p)=\Gamma_{\phi u}^q$.

Proof of (4)⇒(1). By [12], Lemma (4.7) and Theorem (5.7) we see that ψu and ϕu are duality maps. This shows that u is a duality τ-map. q.e.d.

Remark 2.3. The above theorems show that Wirthmüller's definition of a duality [16] is equivalent to our definition under Propositions 1.1 and 1.2, i.e., let $\{u\}\in\{X\wedge X', \Sigma^{0,0}\}^{r,s}$ be an (r, s)-duality in the sense of [16] for X, $X'\in\mathscr{C}\mathscr{F}_0^\tau$, then there exists a duality τ-map u showing that X and X' are S-duals of each other, and the converse follows from Theorem 2.1.

Next we show the existence of equivariant S-duality.

Proposition 2.4. *Let X and X' be τ-subcomplexes of $\Sigma^{r,s+1}$ such that X, $X'\in\mathscr{C}\mathscr{F}_*^\tau$, $X\cap X'=\emptyset$ and the inclusion $X'\to\Sigma^{r,s+1}-X$ (which may not preserve the base points) is a τ-homotopy equivalence. Then there exists an (r, s)-duality τ-map*

$$u\colon X\wedge X'\to\Sigma^{r,s}$$

showing that X' is an (r, s)-dual of X.

Proof. The proof is almost the same as [11], p. 180. As X and X' are closed in $\Sigma^{r,s+1}$ and $X\cap X'=\emptyset$, $\Sigma^{s+1}-\phi X-\phi X'\neq\emptyset$. We choose a point $a\in\Sigma^{s+1}-\phi X-\phi X'$. Then $\Sigma^{r,s+1}-\{a\}\underset{\tau}{\approx}\boldsymbol{R}^{r,s+1}$ and we have an embedding of X and X' as disjoint τ-subsets of $\boldsymbol{R}^{r,s+1}$. We define a τ-map (which doesn't preserve the base points)

$$v\colon X\times X'\to\Sigma^{r,s}$$

by $v(x, x')=(x-x')/\|x-x'\|$, where $\|\ \|$ denotes the standard τ-invariant norm in $\boldsymbol{R}^{r,s+1}$. Suppose we obtained a pointed τ-map $u\colon X\wedge X'\to\Sigma^{r,s}$ such that v is τ-homotopic to the composite $X\times X'^p\to X\wedge X'\xrightarrow{u}\Sigma^{r,s}$, then by [11], page

181, ψu and ϕu are duality map, and u is a duality τ-map. To show the existence of the τ-map u we use the following

Lemma 2.5. *Let X be a τ-complex and Y a pointed τ-space. If any map of X to Y and any map of ϕX to ϕY are null-homotopic, then any τ-map is τ-homotopic to the constant map.*

The proof is the same as [3], Proposition 11.1.

By [11], page 181, any map of $X \vee X'$ to Σ^{r+s} and any map of $\phi X \vee \phi X'$ to Σ^s are null-homotopic. Then any τ-map of $X \vee X'$ to $\Sigma^{r,s}$ is τ-homotopic to the constant map. In particular, by τ-homotopy extension property of τ-complex pair $(X \times X', X \vee X')$ v is τ-homotopic to a τ-map which sends $X \vee X'$ to the base point of $\Sigma^{r,s}$. Thus we obtained the required (r, s)-duality τ-map u: $X \wedge X' \to \Sigma^{r,s}$. q.e.d.

Proposition 2.6. *Let $X \in \mathscr{C}\mathscr{F}^{\tau}_{*}$. Then there exists an (r, s)-dual $X' \in \mathscr{C}\mathscr{F}^{\tau}_{*}$ of X for some (r, s).*

Proof. For a finite τ-complex X there is a finite simplicial τ-complex K having the τ-homotopy type of X [2], Section 3. K can be embedded equivariantly to a simplicial τ-complex $\Sigma^{r,s+1}$ for some (r, s). Take the τ-subcomplex $X' \subset \Sigma^{r,s+1}$ complementary to K as an (r, s)-dual of X in a similar way to [10]. (We can assume $X' \in \mathscr{C}\mathscr{F}^{\tau}_{*}$ by replacing $\Sigma^{r,s+1}$ with $\Sigma^{r,s+2}$, if necessary. Then X' is replaced by $\Sigma X'$.) Then, by Proposition 2.4 there is a duality τ-map $K \wedge X' \to \Sigma^{r,s}$, and replacing K to X by the τ-homotopy equivalence we complete the proof. q.e.d.

Theorem 2.7. *For any finite pointed τ-complex X there exists an equivariant S-dual of X.*

Proof. For any $X \in \mathscr{C}\mathscr{F}^{\tau}_{0}$, ΣX belongs to $\mathscr{C}\mathscr{F}^{\tau}_{*}$. Then the theorem follows the above proposition. q.e.d.

The following theorem is an equivariant version of Atiyah [4], Proposition (3.2), and the proof is the same as [4].

Theorem 2.8. *Let M be a compact smooth τ-manifold, and i: $(M, \partial M) \to (B^{r,s}, S^{r,s})$ an embedding such that $i(M)$ is transversal to $S^{r,s}$ and $B^s - \phi(i(M)) \neq \emptyset$. Let ν be the normal bundle of i. Then the Thom complex $T(\nu)$ of ν is an equivariant S-dual of $M/\partial M$. (If $\partial M = \emptyset$, $M/\emptyset$ denotes $M \cup \{pt\}$ as usual.)*

Remark 2.9. Any compact smooth τ-manifold can be embedded equivariantly to $B^{r,s}$ transversal to $S^{r,s}$ for some (r, s), cf., [7], (10.3) and [14], Corollary 1.10.

Remark 2.10. If $M/\partial M \in \mathscr{C}\mathscr{F}^{\tau}_{*}$, then there is an (r, s)-*duality* τ-map $M/\partial M \wedge T(\nu) \to \Sigma^{r,s}$ by Proposition 2.4.

Proposition 2.11. *$S^{1,0}_{+}$ is an equivariant S-dual of itself.*

Proof. $S^{1,0}$ is a 0-dim compact smooth τ-manifold. Then the above theorem implies the result.

§ 3. Suspensions and Duality

In this section we discuss relations among duality and suspensions $\sigma^{*,*}$, $\sigma_{*,*}$ and $\sigma(*, *)$ (defined below).

First we describe *slant products* / in τ-cohomology, [3], (13.15).

From now on we often use the notation Σ^R, $R=(r, s)$, to denote $\Sigma^{r,s}$, for simplicity.

Let $\boldsymbol{E}=\{E_p, \varepsilon^E_p\}$, $\boldsymbol{F}=\{F_q, \varepsilon^F_q\}$ and $\boldsymbol{G}=\{G_r, \varepsilon^G_r\}$ be τ-spectra, and $\boldsymbol{\mu}=\{\mu_{p,q}: E_p \wedge F_q \to G_{p+q}\}$ be a τ-pairing, [3], Section 8. A slant product

$$/ : [X \wedge X', \boldsymbol{E} \wedge Z]^{r,s} \otimes [Y, \boldsymbol{F} \wedge X]_{p,q} \to [Y \wedge X', \boldsymbol{G} \wedge Z]_{p-r,q-s}$$

(instead of $\tilde{\boldsymbol{E}}^{r,s}(X \wedge X') \otimes \tilde{\boldsymbol{F}}_{p,q}(X) \to \tilde{\boldsymbol{G}}^{r-p,s-q}(X'))$ is defined as follows: Let $u: \Sigma^{m-r,m-s} X \wedge X' \to E_m \wedge Z$ and $f: \Sigma^{n+p,n+q} Y \to F_n \wedge X$ represent elements $\{u\} \in [X \wedge X', \boldsymbol{E} \wedge Z]^{r,s}$ and $\{f\} \in [Y, \boldsymbol{F} \wedge X]_{p,q}$, respectively. Define $\mu'_{m,n}(u, f)$: $\Sigma^{m-r+n+p,m-s+n+q} Y \wedge X' \to G_{m+n} \wedge Z$ to be the composite $\Sigma^{M-R+N+P} Y \wedge X' \xrightarrow{\rho(m-s)(n+p)T \wedge 1} \Sigma^{M-R} \Sigma^{N+P} Y \wedge X' \xrightarrow{1 \wedge f \wedge 1} \Sigma^{M-R} F_n \wedge X \wedge X' \xrightarrow{1 \wedge T'} \Sigma^{M-R} X \wedge X' \wedge F_n \xrightarrow{u \wedge 1} E_m \wedge Z \wedge F_n \xrightarrow{1 \wedge T''} E_m \wedge F_n \wedge Z \xrightarrow{\mu_{m,n} \wedge 1} G_{m+n} \wedge Z$, $M-R=(m-r, m-s)$, $N+P=(n+p, n+q)$. Put $\mu_{m,n}(u, f)=(-\rho)^{sn} \mu'_{m,n}(u, f)$. Then $\{u\}/\{f\} \in [Y \wedge X', \boldsymbol{G} \wedge Z]_{p-r,q-s}$ is defined by $\{\mu_{m,n}(u, f)\}$. Thus Γ^P_u coincides with the slant product $\{u\} / : \{Y, X\}_{p,q} \to \{Y \wedge X', \Sigma^R\}_{p,q}$, $u: X \wedge X' \to \Sigma^R$, $P=(p, q)$.

Slant products satisfy the compatibility with suspensions: For $x \in [X \wedge X', \boldsymbol{E} \wedge Z]^{r,s}$ and $y \in [Y, \boldsymbol{F} \wedge X]_{p,q}$

(3.1) $$(\sigma^{a,b} x)/(\sigma_{a,b} y) = x/y \in [Y \wedge X', \boldsymbol{G} \wedge Z]_{p-r,q-s} ,$$

(3.2) $$(\bar{\sigma}^{a,b} x)/y = \alpha(A, P) \cdot \bar{\sigma}^{a,b}(x/y) \in [Y \wedge \Sigma^{a,b} X', \boldsymbol{G} \wedge Z]_{p-r-a,q-s-b} ,$$

$$\alpha(A, P) = (-\rho)^{bq} \rho^{(a+b)(p+q)} , \quad A=(a, b) , \quad P=(p, q) ,$$

(3.3) $$(\sigma_{a,b}x)/y=\sigma_{a,b}(x/y)\in[Y\wedge X',\ \mathbf{G}\wedge\Sigma^{a,b}Z]_{p-r+a,q-s+b},$$

(3.4) $$x/(\sigma^{a,b}y)=\sigma^{a,b}(x/y)\in[\Sigma^{a,b}Y\wedge X',\ \mathbf{G}\wedge Z]_{p-r-a,q-s-b},$$

where $\bar{\sigma}^{a,b}$: $[X\wedge X',\ \mathbf{E}\wedge Z]_{p,q}\to[X\wedge\Sigma^{a,b}X',\ \mathbf{E}\wedge Z]_{p-a,q-b}$ is defined to be the composite: $[X\wedge X',\mathbf{E}\wedge Z]_{p,q}\xrightarrow{\sigma^{a,b}}[\Sigma^{a,b}X\wedge X',\mathbf{E}\wedge Z]_{p-a,q-s}\xrightarrow{(T'\wedge 1)^*}[X\wedge\Sigma^{a,b}X',\ \mathbf{E}\wedge Z]_{p-a,q-b}$. (These can be shown similarly to [3], § 8.)

We use the notation $\Gamma^P(x)$ to denote the slant product

$$x/ : [Y,\ \mathbf{F}\wedge X]_{p,q}\to[Y\wedge X',\ \mathbf{G}\wedge Z]_{p-r,q-s}, \quad x\in[X\wedge X',\ \mathbf{E}\wedge Z]^{r,s}.$$

Then, for a τ-map u: $X\wedge X'\to\Sigma^R$, $\Gamma^P_u=\Gamma^P(\{u\})$ ($\mathbf{E}=\mathbf{F}=\mathbf{G}=\mathbf{SR}$).

Let $A=(a,\ b)$ be a pair of non-negative integers. The suspension isomorphisms

$$\sigma(A)\colon \{X,\ Y\}_{p,q}\cong\{\Sigma^A X,\ \Sigma^A Y\}_{p,q}$$

of stable τ-homotopy groups are defined as follows: If f: $\Sigma^{N+P}X\to\Sigma^N Y$ represents an element $\{f\}\in\{X,\ Y\}_{p,q}$, $N=(n,\ n)$, $P=(p,\ q)$, then $\sigma(A)\{f\}$ is represented by $\Sigma^A f$ defined as the composite

$$\Sigma^{N+P}\Sigma^A X\underset{\tau}{\overset{T'\wedge 1}{\approx}}\Sigma^A\Sigma^{N+P}X\xrightarrow{1\wedge f}\Sigma^A\Sigma^N Y\underset{\tau}{\overset{T''\wedge 1}{\approx}}\Sigma^N\Sigma^A Y.$$

From now on we use the notations σ^A and σ_A to denote $\sigma^{a,b}$ and $\sigma_{a,b}$ respectively, for simplicity.

We compare $\sigma(A)$: $\{X,\ Y\}_{p,q}\to\{\Sigma^A X,\ \Sigma^A Y\}_{p,q}$ with $\sigma^A\circ\sigma_A$.

As to the definitions of $\sigma^{*,*}$ and $\sigma_{*,*}$ in τ-(co-)homology represented by τ-spectra we refer to [3], (7.5), (7.6) and (13.3).

Proposition 3.1. *Let $A=(a,\ b)$ and $C=(c,\ d)$ be pairs of non-negative integers and $x\in\{X,\ Y\}_{p,q}$. Then*

$$\sigma(A)x=\alpha(A,\ P)\cdot\sigma^A\circ\sigma_A x,$$

$$\sigma^C\circ\sigma(A)=T'^*\circ\sigma(A)\circ\sigma^C \quad \textit{and} \quad T'_*\circ\sigma_C\circ\sigma(A)=\sigma(A)\circ\sigma_C$$

where $P=(p,\ q)$, $\alpha(A,\ P)=(-\rho)^{bq}\rho^{(a+b)(p+q)}$ and T': $\Sigma^C\Sigma^A\underset{\tau}{\approx}\Sigma^A\Sigma^C$.

Proof. Let f: $\Sigma^{N+P}X\to\Sigma^N Y$ represent x. Compare $\Sigma^A f$ with $\sigma^A_n\circ\sigma^n_A f$ ($=n$-stage of $\sigma^A\circ\sigma_A$). Computing permutations of suspension parameters and difference of conventional signs, we see that $[\Sigma^A f]=\alpha(A,\ P)\cdot[\sigma^A_n\circ\sigma^n_A f]\in[\Sigma^{N+P}X,\ \Sigma^N Y]^\tau$, where "$[f]$" denotes the τ-homotopy class of f. Then we obtain $\sigma(A)x=\alpha(A,\ P)\cdot\sigma^A\circ\sigma_A x$. Similarly we obtain the other results. q.e.d.

Remark 3.2. *Let* A, C, T' and $\alpha(\ ,\ )$ be as above. Then

$$\sigma^A\circ\sigma_C=\sigma_C\circ\sigma^A,$$

$$T'_*\circ\sigma_C\circ\sigma_A=\alpha(A, C)\cdot\sigma_A\circ\sigma_C \quad\text{and}\quad T'^*\circ\sigma^A\circ\sigma^C=\alpha(A, C)\cdot\sigma^C\circ\sigma^A.$$

(Recall the sign conventions A3) and A4).)

Henceforth $\bar{\sigma}(A)$ denotes the composite

$$\{X\wedge Y, Z\}_{p,q}\xrightarrow{\sigma(A)}\{\Sigma^A X\wedge Y, \Sigma^A Z\}_{p,q}\xrightarrow{(T'\wedge 1)^*}\{X\wedge\Sigma^A Y, \Sigma^A Z\}_{p,q},$$

$T'\colon X\wedge\Sigma^A\underset{\tau}{\approx}\Sigma^A X$. Then $\bar{\sigma}(A)=\alpha(A, P)\cdot\bar{\sigma}^A\circ\sigma_A$, and $\{u_{A,O}\}=\sigma(A)\{u\}$, $\{u_{O,A}\}=\bar{\sigma}(A)\{u\}$ for $u\colon X\wedge X'\to\Sigma^R$.

Proposition 3.3. *Let* $x\in\{X\wedge X', Z\}^{r,s}$ *and* $A=(a, b)$ *be a pair of non-negative integers. Then commutativity holds in each diagram*

$$\begin{array}{ccc}
\{Y, X\}_{p,q} & \xrightarrow{\Gamma^P(x)} & \{Y\wedge X', Z\}_{p-r,q-s} \\
\downarrow{\scriptstyle\sigma(A)} & & \downarrow{\scriptstyle\sigma(A)} \\
\{\Sigma^A Y, \Sigma^A X\}_{p,q} & \xrightarrow{\Gamma^P(\sigma(A)x)} & \{\Sigma^A Y\wedge X', \Sigma^A Z\}_{p-r,q-s},
\end{array}$$

$$\begin{array}{ccc}
\{Y, X\}_{p,q} & \xrightarrow{\Gamma^P(x)} & \{Y\wedge X', Z\}_{p-r,q-s} \\
 & {\scriptstyle\Gamma^P(\bar{\sigma}(A)x)}\searrow & \downarrow{\scriptstyle\bar{\sigma}(A)} \\
 & & \{Y\wedge\Sigma^A X', \Sigma^A Z\}_{p-r,q-s},
\end{array}$$

where $P=(p, q)$.

Proof. By Proposition 3.1, (3.1)$\sim$(3.4) and Remark 3.2, we see that $\Gamma^P(\sigma(A)x)\circ\sigma(A)=\alpha(A, R)\cdot\alpha(A, P)\cdot(\sigma^A\circ\sigma_A x/\sigma_A\circ\sigma^A\ )=\alpha(A, P-R)\cdot(\sigma_A x/\sigma^A\ )=\alpha(A, P-R)\sigma^A\circ\sigma_A(x/\ )=\sigma(A)\circ\Gamma^P(x)$, and $\bar{\sigma}(A)\circ\Gamma^P(x)=\alpha(A, P-R)\cdot\bar{\sigma}^A\circ\sigma_A(x/\ )=\alpha(A, P-R)\cdot\bar{\sigma}^A\circ(\sigma_A x/\ )=\alpha(A, P-R)\cdot\alpha(A, P)\cdot(\bar{\sigma}^A\circ\sigma_A x/\ )=\alpha(A, R)(\bar{\sigma}^A\circ\sigma_A x/\ )=\bar{\sigma}(A)x/\ =\Gamma^P(\bar{\sigma}(A)x)$, $R=(r, s)$. (Remark that $\alpha(A, P-R)=\alpha(A, P)\cdot\alpha(A, R)$.)

q. e. d.

Next we discuss relations among iterated suspensions and slant products. As is easily seen, for $A=(a, b)$, $A'=(a', b')$, $C=(c, d)$ and $C'=(c', d')$ commutativity holds in each diagram

$$\begin{array}{ccc}
\{X, Y\}_{p,q} & \xrightarrow{\sigma(C+A)} & \{\Sigma^{C+A}X, \Sigma^{C+A}Y\}_{p,q} \\
 & {\scriptstyle\sigma(C)\circ\sigma(A)}\searrow & \Vert\wr\ {\scriptstyle(T\wedge 1)^*\circ(T\wedge 1)_*} \\
 & & \{\Sigma^C\Sigma^A X, \Sigma^C\Sigma^A Y\}_{p,q},
\end{array}\tag{3.5}$$

$$
(3.6)\quad
\begin{array}{ccc}
 & \overset{\sigma(C+A)\circ\bar{\sigma}(C'+A')}{\nearrow} & \{\Sigma^{C+A}X\wedge\Sigma^{C'+A'}X',\ \Sigma^{C+A}\Sigma^{C'+A'}Z\}_{p,q} \\
 & & \Vert\wr\ (T\wedge 1\wedge T\wedge 1)^{*}\circ(T\wedge T\wedge 1)_{*} \\
\{X\wedge X',\ Z\}_{p,q} & \xrightarrow{\sigma(C)\circ\sigma(A)\circ\bar{\sigma}(C')\circ\bar{\sigma}(A')} & \{\Sigma^{C}\Sigma^{A}X\wedge\Sigma^{C'}\Sigma^{A'}X',\ \Sigma^{C}\Sigma^{A}\Sigma^{C'}\Sigma^{A'}Z\}_{p,q} \\
 & & \Vert\wr\ \alpha(A,C')\cdot(1\wedge T'\wedge 1)_{*} \\
 & \underset{\sigma(C)\circ\bar{\sigma}(C')\circ\sigma(A)\circ\bar{\sigma}(A')}{\searrow} & \{\Sigma^{C}\Sigma^{A}X\wedge\Sigma^{C'}\Sigma^{A'}X',\ \Sigma^{C}\Sigma^{C'}\Sigma^{A}\Sigma^{A'}Z\}_{p,q},
\end{array}
$$

$T'\colon \Sigma^{A}\Sigma^{C'}\underset{\tau}{\approx}\Sigma^{C'}\Sigma^{A}$. Then, by naturality of Γ we obtain

Proposition 3.4. *Let $A=(a, b)$, $A'=(a', b')$, $C=(c, d)$ and $C'=(c', d')$ be pairs of non-negative integers, $x\in\{X\wedge X', Z\}^{r,s}$ and $P=(p, q)\in \boldsymbol{Z}\times\boldsymbol{Z}$. Then commutativity holds in the diagram*

$$
\begin{array}{ccc}
\{Y,\ \Sigma^{C+A}X\}_{p,q} & \xrightarrow{\Gamma^{P}(\sigma(C+A)\circ\bar{\sigma}(C'+A')x)} & \{Y\wedge\Sigma^{C'+A'}X',\ \Sigma^{C+A}\Sigma^{C'+A'}Z\}_{p-r,q-s} \\
\Vert\wr\ (T\wedge 1)_{*} & & \Vert\wr\ (T\wedge 1)^{*}\circ(T\wedge T\wedge 1)_{*} \\
\{Y,\ \Sigma^{C}\Sigma^{A}X\}_{p,q} & \xrightarrow{\Gamma^{P}(\sigma(C)\circ\sigma(A)\circ\bar{\sigma}(C')\circ\bar{\sigma}(A')x)} & \{Y\wedge\Sigma^{C'}\Sigma^{A'}X',\ \Sigma^{C}\Sigma^{A}\Sigma^{C'}\Sigma^{A'}Z\}_{p-r,q-s} \\
 & \underset{\Gamma^{P}(\sigma(C)\circ\bar{\sigma}(C')\circ\sigma(A)\circ\bar{\sigma}(A')x)}{\searrow} & \Vert\wr\ \alpha(A,C')\circ(1\wedge T'\wedge 1)_{*} \\
 & & \{Y\wedge\Sigma^{C'}\Sigma^{A'}X',\ \Sigma^{C}\Sigma^{C'}\Sigma^{A}\Sigma^{A'}Z\}_{p-r,q-s}.
\end{array}
$$

Remark 3.5. Let $\boldsymbol{E}$ be a τ-spectrum. Define $\sigma(A)\colon [X, \boldsymbol{E}\wedge Y]_{p,q}\to [\Sigma^{A}X, \boldsymbol{E}\wedge\Sigma^{A}Y]_{p,q}$, $A=(a, b)$, $a\geqq 0$, $b\geqq 0$, as follows: If $f\colon \Sigma^{M+P}X\to E_{m}\wedge Y$ represents $\{f\}\in[X, \boldsymbol{E}\wedge Y]_{p,q}$, $M=(m, m)$, $P=(p, q)$, then $\sigma(A)\{f\}$ is represented by the composite: $\Sigma^{M+P}\Sigma^{A}X\underset{\tau}{\approx}\Sigma^{A}\Sigma^{M+P}X\xrightarrow{1\wedge f}\Sigma^{A}E_{m}\wedge Y\underset{\tau}{\approx}E_{m}\wedge\Sigma^{A}Y$. This is a generalization of the suspension isomorphism $\sigma(A)$ of stable τ-homotopy groups, and Propositions 3.1, 3.3, and 3.4 hold when we replace $\{\ ,\ \}_{*,*}=[\ ,\boldsymbol{SR}\wedge\ ]_{*,*}$ by $[\ ,\boldsymbol{E}\wedge\ ]_{*,*}$.

Next we discuss composition of stable τ-maps. Let $x\in\{X, Y\}_{p,q}$, $y\in\{Y, Z\}_{r,s}$. We define $y\circ x\in\{X, Z\}_{p+r,q+s}$ as follows: Let $f\colon \Sigma^{M+P}X\to\Sigma^{M}Y$, $g\colon \Sigma^{N+R}Y\to\Sigma^{N}Z$ represent x, y respectively, $M=(m, m)$, $P=(p, q)$, $N=(n, n)$, $R=(r, s)$. Define $\xi(f, g)$ by the composite: $\Sigma^{M+P+N+R}X\xrightarrow{T^{1}}\Sigma^{N+R}\Sigma^{M+P}X\xrightarrow{1\wedge f}\Sigma^{N+R}\Sigma^{M}Y\xrightarrow{T^{2}}\Sigma^{M+N+R}Y\xrightarrow{T^{3}}\Sigma^{M}\Sigma^{N+R}Y\xrightarrow{1\wedge g}\Sigma^{M}\Sigma^{N}Z\xrightarrow{\rho^{k}\tilde{T}}\Sigma^{1,1}\cdots\Sigma^{1,1}\Sigma^{N}Z\xrightarrow{\bar{\varepsilon}}\Sigma^{M+N}Z$, where $T^{1}=\rho^{(m+p)(n+s)}T\wedge 1$, $T^{2}=\rho^{m(n+s)}T\wedge 1$, $T^{3}=\rho^{(n+r)m}T\wedge 1$, $\tilde{T}=(T\wedge 1)\cdots(T\wedge 1)$, $k=k(m)=m(m-1)/2$, $\bar{\varepsilon}=\varepsilon_{m+n-1}\cdots\varepsilon_{n}$. Then $y\circ x$ is represented by $\rho^{k}\xi(f, g)$. Note that the diagram

$$
\begin{array}{ccc}
\Sigma^{N+R}\Sigma^{M}Y & \xrightarrow{T^{2}} & \Sigma^{M+N+R}Y \\
 & \underset{\alpha(N+R,M)T'\wedge 1}{\searrow} & \downarrow T^{3} \\
 & & \Sigma^{M}\Sigma^{N+R}Y
\end{array}
$$

is τ-homotopy commutative for large m, n, we see that $y \circ x = y/x$. By (3.1)–(3.4) and Proposition 3.3, we also see the following compatibility with suspensions: For $x \in \{X, Y\}_{p,q}$, $y \in \{Y, Z\}_{r,s}$ and $A=(a, b)$, $a \geqq 0$, $b \geqq 0$,

$$(\sigma_A y) \circ x = \sigma_A(y \circ x), \quad (\sigma^A y) \circ (\sigma_A x) = y \circ x,$$
$$y \circ (\sigma^A x) = \sigma^A(y \circ x), \quad (\sigma(A)y) \circ (\sigma(A)x) = \sigma(A)(y \circ x). \tag{3.7}$$

For a τ-spectrum $\boldsymbol{E}=\{E_n\}$ we define τ-pairings $\boldsymbol{\nu}=\{\nu_{m,n}\}: \boldsymbol{SR} \wedge \boldsymbol{E} \to \boldsymbol{E}$ and $\boldsymbol{\nu}'=\{\nu'_{m,n}\}: \boldsymbol{E} \wedge \boldsymbol{SR} \to \boldsymbol{E}$ by $\nu_{m,n}=\bar{\varepsilon} \circ \tilde{T}: \Sigma^{m,m}E_n \to \Sigma^{1,1} \cdots \Sigma^{1,1}E_n \to E_{m+n}$ and $\nu'_{m,n}=(-\rho)^{mn}\nu_{n,m} \circ T': E_m \wedge \Sigma^{n,n} \xrightarrow{T'} \Sigma^{n,n}E_m \to E_{m+n}$. Let $x \in \{X, Z\}_{r,s}$. Then, using $\boldsymbol{\nu}$, we have a map $x_*: [Y, \boldsymbol{E} \wedge X]_{p,q} \to [Y, \boldsymbol{E} \wedge Z]_{p+r,q+s}$ defined by $x_*(y) = x/y$ for $y \in [Y, \boldsymbol{E} \wedge X]_{p,q}$. Let $f: \Sigma^{M+R}X \to \Sigma^M Z$ represent x, $M=(m, m)$, $R=(r, s)$, then $\rho^k \cdot \sigma_{-M} \circ f_* \circ \sigma_{M+R}: [Y, \boldsymbol{E} \wedge X]_{p,q} \to [Y, \boldsymbol{E} \wedge Z]_{p+r,q+s}$, $k=m(m-1)/2$, coincides with x_*, i.e., for any representative f of x

$$x_* = \rho^k \sigma_{-M} \circ f_* \circ \dot{\sigma}_{M+R}, \quad k=k(m)=m(m-1)/2. \tag{3.8}$$

We use 1_X also to denote the identity map of X. Note that $1_X \in \{X, X\}_{0,0}$ is represented by $\rho^k 1_{\Sigma^M X}$, see [3], Section 8, Example 1. Similarly let $y \in \{Y, X\}_{p,q}$, then, using $\boldsymbol{\nu}'$, we have a map $y^*: [X, \boldsymbol{E} \wedge Z]_{r,s} \to [Y, \boldsymbol{E} \wedge Z]_{p+r,q+s}$ defined by $y^*(x)=x/y$ for $x \in [X, \boldsymbol{E} \wedge Z]_{r,s}$, and $y^* = \rho^k \sigma^{-M-P} g^* \sigma^M$ for $g: \Sigma^{M+P}Y \to \Sigma^M X$, $\{g\}=y$.

Let $x \in \{X, Y\}_{p,q}$, $z \in \{Z, W\}_{r,s}$. Then $x \wedge 1_Z \in \{X \wedge Z, Y \wedge Z\}_{p,q}$, $1_Y \wedge z \in \{Y \wedge Z, Y \wedge W\}_{r,s}$, $x \wedge 1_W \in \{X \wedge W, Y \wedge W\}_{p,q}$ and $1_X \wedge z \in \{X \wedge Z, X \wedge W\}_{r,s}$, where $x \wedge 1_Z$ is represented by $f \wedge 1_Z$ for a representative f of x, and $1_Y \wedge z$ is represented by the composite: $\Sigma^{N+R}Y \wedge Z \underset{\tau}{\approx} \Sigma^{N+R}Z \wedge Y \xrightarrow{g \wedge 1} \Sigma^N Z \wedge Y \underset{\tau}{\approx} \Sigma^N Y \wedge Z$ for a representative $g: \Sigma^{N+R}Z \to \Sigma^N Z$ of z. Then, by definition, we see easily that

$$(1_Y \wedge z) \circ (x \wedge 1_Z) = \alpha(P, R) \cdot (x \wedge 1_W) \circ (1_X \wedge z) \in \{X \wedge Z, Y \wedge W\}_{p+r,q+s}. \tag{3.9}$$

We also see the following

Proposition 3.6. *Let* $x \in \{X \wedge X', \Sigma^{0,0}\}^{r,s}$, $y \in \{Y, X\}_{p,q}$ *and* $z \in \{Z, Y\}_{p',q'}$. *Then* $z \wedge 1_{X'} \in \{Z \wedge X', Y \wedge X'\}_{p',q'}$ *and*

$$\Gamma(x)(y \circ z) = x/(y \circ z) = (x/y) \circ (z \wedge 1_{X'}) = (\Gamma(x)y) \circ (z \wedge 1_{X'}).$$

In particular, $y \wedge 1_{X'} \in \{Y \wedge X', X \wedge X'\}_{p,q}$ *and*

$$\Gamma(x)y = x/y = (x/1_{X \wedge X'}) \circ (y \wedge 1_{X'}) = x \circ (y \wedge 1_X).$$

§4. S-Duals of Stable τ-Maps

Let $X, X' \in \mathscr{CF}_0^\tau$ and $R=(r, s) \in \boldsymbol{Z} \times \boldsymbol{Z}$. An element $x \in \{X \wedge X', \Sigma^{0,0}\}^{r,s}$ is called to be an *R-duality* when the maps

$$\Gamma^{*,*}(x)\colon \{\Sigma^{0,0}, X\}_{*,*} \to \{X', \Sigma^{0,0}\}_{*-r,*-s},$$
$$\Gamma^{*,*}(\bar{x})\colon \{\Sigma^{0,0}, X'\}_{*,*} \to \{X, \Sigma^{0,0}\}_{*-r,*-s}$$

are isomorphisms, Wirthmüller [16] (cf., Remark 2.3), where $\bar{x}$ denotes $T'^*x \in \{X' \wedge X, \Sigma^{0,0}\}^{r,s}$, $T'\colon X' \wedge X \underset{\tau}{\approx} X \wedge X'$. We also call an element $x \in \{X \wedge X', \Sigma^{a,b}\}^{r,s}$ to be a duality if $\Gamma^{*,*}(x)$ and $\Gamma^{*,*}(\bar{x})$ are isomorphisms. Then $\Gamma^{*,*}(x)\colon \{Y, X\}_{*,*} \to \{Y \wedge X', \Sigma^{a,b}\}_{*-r,*-s}$ and $\Gamma^{*,*}(\bar{x})\colon \{Y, X'\}_{*,*} \to \{Y \wedge X, \Sigma^{a,b}\}_{*-r,*-s}$ are isomorphisms for any $Y \in \mathscr{CF}_0^\tau$ (Theorem 1.3), and an R-duality τ-map gives an R-duality $\{u\} \in \{X \wedge X', \Sigma^{r,s}\}_{0,0}$ (Theorem 2.1).

Let $x \in \{X \wedge X', \Sigma^{a,b}\}^{r,s}$ and $y \in \{Y \wedge Y', \Sigma^{a,b}\}^{r',s'}$ be dualities. The duality isomorphism

$$D(x, y)\colon \{X, Y\}_{p,q} \to \{Y', X'\}_{p+r-r',q+s-s'}$$

is defined to be the composite

$$\{X, Y\}_{p,q} \xrightarrow{\Gamma^{P}(y)} \{X \wedge Y', \Sigma^{a,b}\}_{p-r',q-s'} \overset{T'^*}{\cong} \{Y' \wedge X, \Sigma^{a,b}\}_{p-r',q-s'}$$
$$\xrightarrow{(\Gamma^{P+R-R'}(\bar{x}))^{-1}} \{Y', X'\}_{p+r-r',q+s-s'}, \quad T'\colon Y' \wedge X \underset{\tau}{\approx} X \wedge Y'.$$

Clearly $D(x, y)^{-1} = D(\bar{y}, \bar{x})$, and $D(\sigma_{-a,-b}x, \sigma_{-a,-b}y) = D(x, y)$ by (3.3).

Using the results in Section 3, we discuss compatibility of duality isomorphisms D with suspensions.

Proposition 4.1. *Let $x \in \{X \wedge X', \Sigma^{0,0}\}^{r,s}$ and $y \in \{Y \wedge Y', \Sigma^{0,0}\}^{r',s'}$ be dualities and $A=(a, b)$ a pair of non-negative integers. Then commutativity holds in each diagram*

$$\begin{array}{ccc}
\{X, Y\}_{p,q} & \xrightarrow{D(x,y)} & \{Y', X'\}_{p+r-r',q+s-s'} \\
\sigma(A)\downarrow & \nearrow D(\sigma(A)x,\sigma(A)y) & \\
\{\Sigma^A X, \Sigma^A Y\}_{p,q} & &
\end{array}$$

$$\begin{array}{ccc}
\{X, Y\}_{p,q} & \xrightarrow{D(x,y)} & \{Y', X'\}_{p+r-r',q+s-s'} \\
& \searrow D(\bar{\sigma}(A)x,\bar{\sigma}(A)y) & \downarrow\sigma(A) \\
& & \{\Sigma^A Y', \Sigma^A X'\}_{p+r-r',q+s-s'}
\end{array}$$

.

Proof. Clearly $\overline{\sigma(A)x}=\bar{\sigma}(A)\bar{x}$. Let $T'': Y'\wedge\Sigma^A X\underset{\tau}{\approx}\Sigma^A X\wedge Y'$, $T': Y'\wedge X\underset{\tau}{\approx}X\wedge Y'$. Then $T''^*\circ\sigma(A)=\bar{\sigma}(A)\circ T'^*$. By Proposition 3.3 $D(\sigma(A)x, \sigma(A)y)\circ\sigma(A)=\Gamma(\bar{\sigma}(A)\bar{x})^{-1}\circ T''^*\circ\Gamma(\sigma(A)y)\circ\sigma(A)=\Gamma(\bar{\sigma}(A)\bar{x})^{-1}\circ T''^*\circ\sigma(A)\circ\Gamma(y)=\Gamma(\bar{\sigma}(A)x)^{-1}\circ\bar{\sigma}(A)\circ T'^*\circ\Gamma(y)=\Gamma(\bar{x})^{-1}\circ T'^*\circ\Gamma(y)=D(x, y)$. Similarly we obtain the other.

q. e. d.

Let x, y and $A=(a, b)$ be as above, and $A'=(a', b')$, $C=(c, d)$, $C'=(c', d')$ be pairs of non-negative integers. By Proposition 4.1 we have

$$D(\sigma(A)\bar{\sigma}(A')x, \sigma(A)\bar{\sigma}(A')y)\circ\sigma(A)=\sigma(A')\circ D(x, y). \tag{4.1}$$

Similarly to Proposition 4.1, by (3.5) and Proposition 3.4 we see that

$$\begin{aligned}D(\sigma(C, A)x, \sigma(C, A)y)\circ\sigma(C, A)&=D(\sigma(C+A)x, \sigma(C+A)y)\circ\sigma(C+A)\\&=D(x, y),\end{aligned}\tag{4.2}$$

$$\begin{aligned}&D(\sigma(\bar{C}', \bar{A}')x, \sigma(\bar{C}', \bar{A}')y)\\&=(T\wedge 1)^*\circ(T\wedge 1)_*\circ D(\bar{\sigma}(C'+A')x, \bar{\sigma}(C'+A')y)\\&=(T\wedge 1)^*\circ(T\wedge 1)_*\circ\sigma(C'+A')\circ D(x, y)\\&=\sigma(C', A')\circ D(x, y),\end{aligned}\tag{4.3}$$

$$\begin{aligned}&D(\sigma(C, \bar{C}', A, \bar{A}')x, \sigma(C, \bar{C}', A, \bar{A}')y)\circ\sigma(C)\\&=\sigma(C')\circ D(\sigma(A, \bar{A}')x, \sigma(A, \bar{A}')y)\end{aligned}\tag{4.4}$$

and

$$\begin{aligned}&D(\sigma(C, \bar{C}', A, \bar{A}')x, \sigma(C, \bar{C}', A, \bar{A}')y)\\&=D(\sigma(C, A, \bar{C}', \bar{A}')x, \sigma(C, A, \bar{C}', \bar{A}')y),\end{aligned}\tag{4.5}$$

where $\sigma(C, A)$, $\sigma(\bar{C}', \bar{A}')$, $\sigma(A, \bar{A}')$ and $\sigma(C, \bar{C}', A, \bar{A}')$ denote $\sigma(C)\circ\sigma(A)$, $\bar{\sigma}(C')\circ\bar{\sigma}(A')$, $\sigma(A)\circ\bar{\sigma}(A')$ and $\sigma(C)\circ\bar{\sigma}(C')\circ\sigma(A)\circ\bar{\sigma}(A')$ respectively. Thus we obtain

Proposition 4.2. *Let* $x\in\{X\wedge X', \Sigma^{0,0}\}^{r,s}$ *and* $y\in\{Y\wedge Y', \Sigma^{0,0}\}^{r',s'}$ *be dualities, and* $A=(a, b)$, $A'=(a', b')$, $C=(c, d)$ *and* $C'=(c', a')$ *be pairs of non-negative integers. Then the following diagram is commutative:*

$$\begin{array}{ccccccc}
 & \{X, Y\} & & \xrightarrow{D(x,y)} & & \{Y', X'\} & \\
\swarrow^{\sigma(A)} & \downarrow{\scriptstyle\sigma(C+A)} & & & & \downarrow{\scriptstyle\sigma(C'+A')} & \searrow^{\sigma(A')} \\
\{\Sigma^A X, \Sigma^A Y\} & & & \xrightarrow{D(\sigma(A,\bar{A}'))} & & & \{\Sigma^{A'}Y', \Sigma^{A'}X'\} \\
\downarrow{\scriptstyle\sigma(C)} & \{\Sigma^{C+A}X, \Sigma^{C+A}Y\} & & \xrightarrow{D(\sigma(C+A,\overline{C'+A'}))} & & \{\Sigma^{C'+A'}Y', \Sigma^{C'+A'}X'\} & \downarrow{\scriptstyle\sigma(C')} \\
 & \wr\!\Vert\,\tilde{T} & & & & \tilde{T}\,\wr\!\Vert & \\
\{\Sigma^C\Sigma^A X, \Sigma^C\Sigma^A Y\} & & & \xrightarrow[D(\sigma(C,\bar{C}',A,\bar{A}'))]{} & & & \{\Sigma^{C'}\Sigma^{A'}Y', \Sigma^{C'}\Sigma^{A'}X'\},
\end{array}$$

where $\tilde{T}=(T\wedge 1)^*\circ(T\wedge 1)_*$, $\{\ ,\ \}_{p,q}$ *etc. are denoted by* $\{\ ,\ \}$, *and* $D(\sigma(\ ))$ *denotes* $D(\sigma(\ )x, \sigma(\ )y)$.

From (3.1)–(3.4) and the definition of $D(x, y)$ we obtain

Proposition 4.3. *Let* $x\in\{X\wedge X', \Sigma^{0,0}\}^{r,s}$ *and* $y\in\{Y\wedge Y', \Sigma^{0,0}\}^{r',s'}$ *be dualities,* $z\in\{X, Y\}_{p,q}$ *and* $A=(a, b)$ *be a pair of non-negative integers. Then* $D(x, y)$ *satisfies the compatibility with suspensions* σ^A *and* σ_A:

$$D(\sigma^A x, y)\circ\sigma^A z=\alpha(A, P+R-R')\cdot D(x, y)z\in\{Y', X'\}_{p+r-r',q+s-s'},$$
$$D(x, \sigma^A y)\circ\sigma_A z=D(x, y)\in\{Y', X'\}_{p+r-r',q+s-s'},$$
$$D(\bar{\sigma}^A x, y)z=\sigma_A\circ D(x, y)z\in\{Y', \Sigma^A X'\}_{p+r-r'+a,q+s-s'+b},$$
$$D(x, \bar{\sigma}^A y)z=\alpha(A, P)\cdot\sigma^A\circ D(x, y)z\in\{\Sigma^A Y', X'\}_{p+r-r'-a,q+s-s'-b},$$

where $P=(p, q)$, $R=(r, s)$, $R'=(r', s')$ *and* $\alpha(A, C)=(-\rho)^{bd}\rho^{(a+c)(b+d)}$ *for* $C=(c, d)$.

Next we see the relation of compositions of stable τ-maps and their duals.

Proposition 4.4. *Let* $x\in\{X\wedge X', \Sigma^{0,0}\}^{r,s}$, $y\in\{Y\wedge Y', \Sigma^{0,0}\}^{r',s'}$ *and* $z\in\{Z\wedge Z', \Sigma^{0,0}\}^{r'',s''}$ *be dualities, and* $u\in\{X, Y\}_{p,q}$, $v\in\{Y, Z\}_{p',q'}$. *Then* $v\circ u\in\{X, Z\}_{p+p',q+q'}$ and

$$D(x, z)(v\circ u)=\alpha(P, P'+R'-R'')\cdot(D(x, y)u)\circ(D(y, z)v)$$
$$\in\{Z', X'\}_{p+p'+r-r'',q+q'+s-s''},$$

where $P=(p, q)$, $P'=(p', q')$, $R'=(r', s')$, $R''=(r'', s'')$ *and* $\alpha(P, C)=(-\rho)^{qd}\rho^{(p+q)(c+d)}$ *for* $C=(c, d)$.

Proof. Let $T_1\colon X\wedge Y'\underset{\tau}{\approx}Y'\wedge X$, $T_2\colon Y\wedge Z'\underset{\tau}{\approx}Z'\wedge Y$, $T_3\colon X\wedge Z'\underset{\tau}{\approx}Z'\wedge X$, $T_4\colon Y\wedge Y'\underset{\tau}{\approx}Y'\wedge Y$, and $u'=D(x, y)u$, $v'=D(y, z)v$. Then, by definition, $y/u=T_1^*(\bar{x}/u')$ and $z/v=T_2^*(\bar{y}/v')$. By Proposition 3.6, we see that $\Gamma(z)(v\circ u)=z/(v\circ u)=(z/v)\circ(u\wedge 1_{Z'})=(T_2^*(\bar{y}/v'))\circ(u\wedge 1_{Z'})=(T_4^*\bar{y})\circ(1_Y\wedge v')\circ(u\wedge 1_{Z'})=y\circ(1_Y\wedge v')\circ(u\wedge 1_{Z'})$, and $T_3^*\circ\Gamma(\bar{x})(u'\circ v')=T_3^*(\bar{x}/(u'\circ v'))=(T_1^*(\bar{x}/u'))\circ(1_X\wedge v')=(y/u)\circ(1_X\wedge v')=y\circ(u\wedge 1_{Y'})\circ(1_X\wedge v')$. Then, by (3.9), we complete the proof.

We reduce duality τ-maps to stable τ-maps in $\{X\wedge X', \Sigma^{0,0}\}^{*,*}$. For $X\in\mathscr{C}\mathscr{F}_0^\tau$ there exists an S-dual X' by Theorem 2.7. Then we can choose the duality τ-map having the form $u\colon\Sigma^{0,1}X\wedge X'\to\Sigma^{0,1}\Sigma^{r,s}$. In fact, let $u'\colon\Sigma^{0,1}X\wedge X'\to\Sigma^{r,s+1}$ be a duality τ-map obtained by Propositions 2.4 and 2.6, then $s\geqq 0$ from the construction. We define u by $\rho^r T\circ u'\colon\Sigma^{0,1}X\wedge X'\to\Sigma^{0,1}\Sigma^{r,s}$. This is the required one. For this u we define an (r, s)-duality $\langle u\rangle\in\{X\wedge X', \Sigma^{0,0}\}^{r,s}$ by

(4.6) $$\langle u\rangle=\sigma_{-r,-s}\circ\sigma(0, 1)^{-1}\{u\}.$$

For a τ-map $u: X\wedge X'\to\Sigma^R$, $R=(r, s)$, and $u_{P,P'}: \Sigma^P X\wedge\Sigma^{P'}X'\to\Sigma^P\Sigma^{P'}\Sigma^R$, we see easily that $\{u_{P,P'}\}=\sigma(P)\circ\bar{\sigma}(P')\{u\}$. Thus, for a duality τ-map $u_{P,P'}: \Sigma^P X\wedge\Sigma^{P'}X'\to\Sigma^P\Sigma^{P'}\Sigma^R$, we observe that $\langle u_{P,P'}\rangle=\sigma_{-R}\circ\bar{\sigma}(P')^{-1}\circ\sigma(P)^{-1}\{u_{P,P'}\}=\langle u\rangle$ is an R-duality.

Theorem 4.5. *Let X' and X'' be S-duals of $X\in\mathscr{C}\mathscr{F}_o^\tau$ so that $x\in\{X\wedge X', \Sigma^{0,0}\}^{r,s}$ and $x'\in\{X\wedge X'', \Sigma^{0,0}\}^{r',s'}$ are dualities. Then a stable τ-homotopy equivalence $\{f\}\in\{X', X''\}_{r'-r,s'-s}$, $f: \Sigma^{n+r'-r,n+s'-s}X'\to\Sigma^{n,n}X''$, is canonically determined by $\{f\}=D(x', x)1_X$ for large n.*

Proof. Put $v=D(x, x')1_X\in\{X'', X'\}_{r-r',s-s'}$. Then, by Proposition 4.4 we see that v is the inverse of $\{f\}$. q.e.d.

§5. Duality between τ-Cohomology and Homology

First we see the following τ-cohomology version of [16], Proposition 1.2, and the proof is the same as [16]. (Use Comparison Theorem 1.3′.)

Proposition 5.1. *Let $x\in\{X\wedge X', \Sigma^{0,0}\}^{r,s}$ be an (r, s)-duality, and $Y\in\mathscr{C}\mathscr{F}_\iota^\tau$, $Z\in\mathscr{C}\mathscr{W}_o^\tau$. Then*

$$\Gamma^P(x) : \{Y, Z\wedge X\}_{p,q}\to\{Y\wedge X', Z\}_{p-r,q-s}$$

is a Λ-isomorphism for each $P=(p, q)\in\boldsymbol{Z}\times\boldsymbol{Z}$.

Let $\boldsymbol{E}=\{E_n; n\in Z\}$ be a τ-spectrum. A decomposition

$$[X, \boldsymbol{E}]_{p,q}=\varinjlim_n [\Sigma^{N+P}X, E_n]^\tau=\varinjlim_m \varinjlim_n [\Sigma^{M+N+P}X, \Sigma^M E_n]^\tau,$$

$P=(p, q)$, $M=(m, m)$, $N=(n, n)$, implies

(5.1) $$[X, \boldsymbol{E}]_{p,q}=\varinjlim_n \{\Sigma^{N+P}X, E_n\}_{0,0}$$

(cf., [3], the proof of Proposition 13.5). Then we obtain the following

Theorem 5.2. *Let $x\in\{X\wedge X', \Sigma^{0,0}\}^{r,s}$ be an (r, s)-duality, $\boldsymbol{E}=\{E_n; n\in\boldsymbol{Z}\}$ a τ-spectrum, $Y\in\mathscr{C}\mathscr{F}_o^\tau$ and $P=(p, q)\in\boldsymbol{Z}\times\boldsymbol{Z}$. Then there exists a duality isomorphism*

$$\Gamma^P(x, \boldsymbol{E}): [Y, \boldsymbol{E}\wedge X]_{p,q}\cong[Y\wedge X', \boldsymbol{E}]_{p-r,q-s}.$$

Proof. By Proposition 5.1 we obtain the isomorphism

$$\sigma_{r,s}\circ\Gamma^P(x): \{\Sigma^{N+P}Y, E_n\}_{0,0}\cong\{\Sigma^{N+P}X, E_n\wedge\Sigma^R\}_{0,0},$$

$N=(n, n)$, $R=(r, s)$. Then, taking the direct limit and by (5.1) we obtain the isomorphism to be the composite

$$[Y, \boldsymbol{E}\wedge X]_{p,q}\cong[Y\wedge X', \boldsymbol{E}\wedge\Sigma^{R}]_{p,q}\overset{\sigma^{-r,-s}}{\cong}[Y\wedge X', \boldsymbol{E}]_{p-r,q-s}. \qquad \text{q.e.d.}$$

As $S_{+}^{1,0}$ is an S-dual of itself and there is a (0, 0)-duality, Proposition 2.11, putting $Y=\Sigma^{0,0}$, we obtain

Corollary 5.3. *Let* $x\in\{X\wedge X', \Sigma^{0,0}\}^{r,s}$ *be an* (r, s)*-duality and* $\boldsymbol{E}$ *a* τ*-spectrum. Then*

$$\Gamma^{P}(x, \boldsymbol{E})\colon \tilde{h}_{p,q}(X; \boldsymbol{E})\cong\tilde{h}^{r-p,s-q}(X'; \boldsymbol{E}).$$

In particular

$$\Gamma^{P}(x, \boldsymbol{E})\colon h_{p,q}(S^{1,0}; \boldsymbol{E})\cong h^{-p,-q}(S^{1,0}; \boldsymbol{E}).$$

Let $\tilde{h}_{*,*}=\{\tilde{h}_{p,q}, (p, q)\in\boldsymbol{Z}\times\boldsymbol{Z}\}$ be a reduced τ-homology theory on $\mathscr{W}_{o}^{\tau}$. For each $X\in\mathscr{CF}_{o}^{\tau}$, there is an S-dual X' of X and a duality $x\in\{X\wedge X', \Sigma^{0,0}\}^{r,s}$ by Theorem 2.7 and (4.6). Put

$$\tilde{h}^{p,q}(X)=\tilde{h}_{r-p,s-q}(X') \tag{5.2}$$

for each $(p, q)\in\boldsymbol{Z}\times\boldsymbol{Z}$. By Theorem 4.5, $\tilde{h}^{p,q}(X)$ is uniquely determined up to canonical isomorphisms. Let $f\colon X\to Y$ be a τ-map in $\mathscr{CF}_{o}^{\tau}$. Choose an S-dual Y' and a duality $y\in\{Y\wedge Y', \Sigma^{0,0}\}^{r',s'}$. Then we see that $D(x, y)\colon\{X, Y\}_{p,q}\cong\{Y', X'\}_{p+r-r',q+s-s'}$. Put $u=D(x, y)\{f\}$. Define

$$f^{*}\colon \tilde{h}^{p,q}(Y)\to\tilde{h}^{p,q}(X) \tag{5.3}$$

by the following: Let $f'\colon\Sigma^{M+P+R-R'}Y'\to\Sigma^{M}X'$ represent u, $M=(m, m)$, $P=(p, q)$, $R=(r, s)$, $R'=(r', s')$. Then f^{*} is defined by $\rho^{k}\sigma_{-M}\circ f'_{*}\circ\sigma_{M+P+R-R'}$, $k=k(m)$, cf. (3.8). This definition is independent of the choice of representatives of u. Suspensions $\sigma^{a,b}$ are defined by $\sigma(a, b)x$ and (5.2).

In order to show that $\tilde{h}^{p,q}(Z)\xrightarrow{g^{*}}\tilde{h}^{p,q}(Y)\xrightarrow{f^{*}}\tilde{h}^{p,q}(X)$ is exact for a τ-cofibration sequence $X\xrightarrow{f}Y\xrightarrow{g}Z$, we use the following τ-cohomology version of [16], Proposition 4.1, and [12], Theorem (6.10).

Proposition 5.4. *Let* $x\in\{X\wedge X', \Sigma^{0,0}\}^{r,s}$ *and* $y\in\{Y\wedge Y', \Sigma^{0,0}\}^{r,s}$ *be* (r, s)*-dualities. Let* $f\colon X\to Y$ *and* $f'\colon Y'\to X'$ *be* τ*-maps such that* $\{f'\}=D(x, y)\{f\}$. *Then there exists an* $(r, s+1)$*-duality* $w\in\{C_{f}\wedge C_{f'}, \Sigma^{0,1}\}^{r,s}$ *compatible with the* τ*-cofibration sequence of* f *and* f', *i.e.,*

$$\begin{array}{ccccccccc}
\widetilde{SR}_{*,*}(X) & \xrightarrow{f_*} & \widetilde{SR}_{*,*}(Y) & \xrightarrow{i_*} & \widetilde{SR}_{*,*}(C_f) & \xrightarrow{p_*} & \widetilde{SR}_{*,*}(\Sigma X) & \xrightarrow{\Sigma f_*} & \widetilde{SR}_{*,*}(\Sigma Y) \\
\downarrow{\scriptstyle \bar{\sigma}(U)x/} & & \downarrow{\scriptstyle \bar{\sigma}(U)y/} & & \downarrow{\scriptstyle w/} & & \downarrow{\scriptstyle \sigma(U)x/} & & \downarrow{\scriptstyle \sigma(U)y/} \\
\widetilde{SR}^{*,*}(\Sigma X') & \xrightarrow{\Sigma f'^*} & \widetilde{SR}^{*,*}(\Sigma Y') & \xrightarrow{p'^*} & \widetilde{SR}^{*,*}(C_{f'}) & \xrightarrow{i'^*} & \widetilde{SR}^{*,*}(X') & \xrightarrow{f'^*} & \widetilde{SR}^{*,*}(Y')
\end{array}$$

and the dual diagram for $\bar{x}$, $\bar{y}$, $\bar{w}$ *commute, where* $U=(0, 1)$.

Let $x \in \{X \wedge X', \Sigma^{0,0}\}^{r,s}$ and $y \in \{Y \wedge Y', \Sigma^{0,0}\}^{r',s'}$ be given dualities. Let $m=\max(r, r', s, s')$, $M=(m, m)$. Then $\bar{\sigma}^{M-R}x$ and $\bar{\sigma}^{M-R'}y$, $R=(r, s)$, $R'=(r', s')$, are both (m, m)-dualities. And choose a representative f' of $D(\bar{\sigma}^{M-R}x, \bar{\sigma}^{M-R'}y)\{f\}$. Then we can apply the above Proposition to a given $f: X \to Y$, and by (5.2), (5.3) we see that $\tilde{h}^{p,q}(Z) \xrightarrow{g^*} \tilde{h}^{p,q}(Y) \xrightarrow{f^*} \tilde{h}^{p,q}(X)$ is exact for the cofibration sequence $X \xrightarrow{f} Y \xrightarrow{g} Z$. Thus we obtain

Proposition 5.5. *By the above Definitions* (5.2) *and* (5.3) $\tilde{h}^{*,*}=\{\tilde{h}^{p,q}, (p, q) \in \boldsymbol{Z} \times \boldsymbol{Z}\}$ *is a reduced* τ*-cohomology theory on* $\mathscr{C}\mathscr{F}_0^\tau$.

By [2], Theorem 3.4 for $G=\boldsymbol{Z}/2\boldsymbol{Z}$ any reduced τ-cohomology theory on $\mathscr{C}\mathscr{F}_0^\tau$ is represented by an Ω-τ-spectrum $\boldsymbol{E}$, i.e.,

$$\tilde{h}^{p,q}(X) \cong \tilde{h}^{p,q}(X; \boldsymbol{E}), \quad X \in \mathscr{C}\mathscr{F}_0^\tau, \qquad (p, q) \in \boldsymbol{Z} \times \boldsymbol{Z}.$$

Then, by (5.2) we obtain a sequence of natural isomorphisms:

$$\tilde{h}_{p,q}(X) \cong \tilde{h}^{r-p,s-q}(X') \cong \tilde{h}^{r-p,s-q}(X'; \boldsymbol{E}) \cong \tilde{h}_{p,q}(X; \boldsymbol{E}).$$

Thus we obtain an equivariant version of G. W. Whitehead [15] as follows.

Theorem 5.6. *A reduced* τ*-homology theory on* $\mathscr{C}\mathscr{F}_0^\tau$ *is represented by a suitable* Ω-τ*-spectrum.*

Corollary 5.7. *A reduced* τ*-homology theory on* $\mathscr{C}\mathscr{W}_0^\tau$ *is represented by a suitable* Ω-τ*-spectrum.*

Proof. If an Ω-τ-spectrum $\boldsymbol{E}$ represents $\tilde{h}_{*,*}|_{\mathscr{C}\mathscr{F}_0^\tau}$, then $\boldsymbol{E}$ represents $\tilde{h}_{*,*}$ by Theorem 1.3′. Thus the corollary follows from the above theorem.

q.e.d.

§6. Atiyah-Poincaré Duality in τ-(Co-) Homology

In this section we discuss Atiyah-Poincaré-type duality for real-complex orientable τ-cohomology theories [1].

A compact smooth τ-manifold is called a *weakly real-complex manifold* if the normal bundle ν of an equivariant embedding $(M, \partial M) \to (B^{a,b}, S^{a,b})$ transversal to $S^{a,b}$ is a real-complex vector bundle (=Real vector bundle in the sense of Atiyah [5]) for some (a, b). Let $h^{*,*}$ be a real-complex orientable τ-cohomology theory and M be a weakly real-complex manifold with r-dimensional normal real-complex vector bundle ν of an embedding $(M, \partial M)\to(B^{a,b}, S^{a,b})$, $r\geqq 1$. Then there is the Thom isomorphism

$$\Phi: h^{*,*}(M)\cong \tilde{h}^{*+r,*+r}(T(\nu)).$$

On the other hand, by Theorem 2.8 and Corollary 5.3 we obtain the duality isomorphism

$$D: \tilde{h}_{a-r-p,b-r-q}(M/\partial M)\cong \tilde{h}^{p+r,q+r}(T(\nu)),$$

where $\tilde{h}^{*,*}$ is represented by a τ-spectrum. If $\dim M=m+n$ and $\dim \phi M=n$, then $a-r=m$ and $b-r=n$. Combining these isomorphisms we obtain the following

Theorem 6.1. *Let $h^{*,*}$ be a real-complex orientable τ-cohomology theory and M a weakly real-complex manifold such that $\dim M=m+n$ and $\dim \phi M=n$. Then there exists a duality isomorphism*

$$D_M=D^{-1}\circ\Phi: h^{p,q}(M)\cong h_{m-p,n-q}(M, \partial M)$$

for every $(p, q)\in \boldsymbol{Z}\times\boldsymbol{Z}$.

Typical example of real-complex orientable τ-cohomology theory is $\boldsymbol{MR}^{*,*}$, [8]. Thus we obtain

Corollary 6.2. *Let M be a weakly real-complex manifold such that $\dim M=m+n$ and $\dim M=n$. Then there exists a duality isomorphism*

$$D_M: \boldsymbol{MR}^{p,q}(M)\cong \boldsymbol{MR}_{m-p,n-q}(M, \partial M)$$

for every $(p, q)\in\boldsymbol{Z}\times\boldsymbol{Z}$.

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