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Some Calculations in the Unstable Adams-Novikov Spectral Sequence[1)]

By

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§1. Introduction

The unstable Adams-Novikov spectral sequence for a space X is a sequence of groups $\{E_r(X)\}$, $r=2, 3,\dots$, which converge to the homotopy groups of X, and whose E_2-term depends on the complex cobordism groups of X. We investigate this spectral sequence when X is the infinite special unitary group SU, or one of the finite groups $SU(n)$, or when X is an odd sphere S^{2n+1}.

The reader is referred to [2] for the construction and properties of the unstable Adams-Novikov spectral sequence. For some purposes, it is convenient to localize at a prime p, in which case the complex cobordism homology theory, based on the spectrum MU, is replaced by Brown-Peterson homology, based on the spectrum BP. We then have a useful spectral sequence with many of the properties of the stable Adams-Novikov spectral sequence. Namely, the filtrations are less than or equal to the filtrations in the unstable Adams spectral sequence based on mod-p homology. When X is a space for which $H_*(X; BP)$ is free over the coefficient ring $\pi_*(BP)$ and cofree as a coalgebra, then the E_2-term is isomorphic to an Ext group in an abelian category (see §2; also [2, §7]). Furthermore, this Ext group may be computed as the homology of an unstable cobar complex which we describe explicitly in Section 2. In particular, these considerations apply to the cases $X=SU$, $X=SU(n)$, or $X=S^{2n+1}$.

We first consider the situation where X is a p-local H-space with torsion-free homotopy and torsion-free homology. The results of Wilson [10] and the

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general properties of the spectral sequence imply that the spectral sequence collapses, with $E_2^{0,*}(X) \approx \pi_*(X)$. In this case, the homotopy groups of X (and even the homotopy type of X) are determined by the module of primitives $PH_*(X; BP)$ as an unstable Γ-comodule (see § 3). In particular, this is true for $X = SU$. We also give an explicit description of the generators of $E_2^{0,*}(SU)$.

In Section 4, we consider the case $X = SU(n)$. We compute $E_2^{1,*}(SU(n))$, which gives information about the pullback of the groups $\pi_{2k}(SU(k))$ to $\pi_{2k}(SU(n))$ for $n < k$. We also compute the kernel of the map $E_2^{2,*}(SU(n)) \to E_2^{2,*}(SU(n+1))$. In the case of an odd prime p, the non-zero elements in this kernel survive to $E_\infty^{2,*}(SU(n))$ to give elements in homotopy related to the unstable image of the J-homomorphism. There is a similar but more complicated result for $p=2$.

In Section 5, we give a vanishing line for $E_2^{s,t}(S^{2n+1})$, and also for $E_2^{s,t}(SU(n))$. We then show that in a range of dimensions, $\pi_*(S^{2n+1})$ and $\pi_*(SU(n))$ may be computed as Ext groups in the category of unstable Γ-comodules. These calculations agree with, and extend those of Zabrodsky [11], and Mimura, Nishida and Toda [7].

Throughout the paper, space means simply connected Hausdorff topological space with basepoint, and map means continuous function preserving basepoint. The homotopy relation for spaces and maps is $\simeq$. The smash product is denoted by $\wedge$. In an algebraic situation, homomorphism means that the algebraic structure is preserved, and $\approx$ means a homomorphism which is an isomorphism onto. The ring of integers is denoted by $\boldsymbol{Z}$, and the rational numbers by $\boldsymbol{Q}$. For a prime number p, the ring of integers localized at p is $\boldsymbol{Z}_{(p)}$. Except for 4.10 (3) all spaces are assumed to be p-localized. For an integer n, the ring of integers modulo n is $\boldsymbol{Z}_n$.

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§ 2. The Unstable Adams-Novikov Spectral Sequence

In this section, we summarize the main results of [2], which gives the construction and main properties of the unstable Adams-Novikov spectral sequence. When a prime p is fixed, BP refers to the Brown-Peterson spectrum associated with p. For a space X, the (reduced) homology groups of X with coefficients in BP are denoted by $H_*(X; BP)$. The coefficient ring $\pi_*(BP)$ is called A, and the ring of co-operations $\pi_*(BP \wedge BP)$ is called Γ (Γ is the BP-analogue of the dual of the Steenrod algebra).

The spectrum BP defines a functor $BP(\cdot)$ from spaces to spaces by

$$BP(X) = \lim_{n \to \infty} \Omega^n(BP_n \wedge X).$$

The unit in BP is a map $i: S \to BP$ of the sphere spectrum S to BP, which gives a map

$$\eta = \eta(X): X \to BP(X).$$

This gives rise to functors $D^s(\cdot)$ and a tower of fibrations

$$\cdots \to D^s(X) \xrightarrow{\delta^s} D^{s-1}(X) \to \cdots \to D^1(X) \xrightarrow{\delta^1} D^0(X) = X \tag{2.1}$$

as follows. Inductively on s,

$$\delta^s: D^s(X) \to D^{s-1}(X)$$

is the fibration over $D^{s-1}(X)$ induced from the path-space fibration over $D^{s-1}(BP(X))$ by the map $D^{s-1}(\eta)$. The homotopy exact couple of this tower is called the unstable Adams spectral sequence for X with respect to BP and its terms are denoted $\{E_r^{s,t}(X; BP)\}$. When the ring spectrum BP is assumed, we call this the spectral sequence for X, and write $\{E_r^{s,t}(X)\}$. From [2, § 7], we have the following.

Theorem 2.2. *For each simply-connected CW-space X, the spectral sequence $\{E_r^{s,t}(X; BP)\}$ converges to the homotopy groups of X localized at p. If $H_*(X; BP)$ is free as an A-module, and cofree as a coalgebra, then*

$$E_2^{s,t}(X; BP) \approx \mathrm{Ext}^s_{\mathscr{U}}(A[t], PH_*(X; BP)).$$

Here $\mathscr{U}$ is the category of unstable Γ-comodules (which will be described below). $A[t]$ stands for the free A-module on one generator of degree t; $PH_*(X; BP)$ stands for the submodule of primitives in $H_*(X; BP)$. Furthermore, these Ext *groups may be calculated as the homology of an unstable cobar complex $C^{*,*}(PH_*(X; BP))$, which will be described explicitly below.*

We next recall some facts about BP from [1] and [6]. First

$$H_*(BP; \mathbf{Z}_{(p)}) \approx \mathbf{Z}_{(p)}[m_1, m_2, \ldots]$$

$$H_*(BP \wedge BP; \boldsymbol{Z}_{(p)}) \approx H_*(BP)[t_1, t_2, \ldots]$$

where degree $(m_i) = 2(p^i - 1) =$ degree (t_i). The elements t_i are chosen so that

$$\eta_R(m_k) = \sum_{i=0}^{k} m_i t_{k-i}^{p^i} \tag{2.3}$$

where η_R is the right unit map. The elements m_0 and t_0 are to be interpreted as 1. Then the Hazewinkel elements v_i are defined recursively by the formula:

$$v_i = pm_i - \sum_{j=1}^{i-1} m_j v_{i-j}^{p^j}. \tag{2.4}$$

It is shown in [5] that the v_i are in the image of the Hurewicz homomorphism $\pi_*(BP) \to H_*(BP; \boldsymbol{Z}_{(p)})$, which is a monomorphism, so the v_i may be considered to be in $\pi_*(BP)$ also. Then

$$A = \pi_*(BP) \approx \boldsymbol{Z}_{(p)}[v_1, v_2, \ldots]$$
$$\Gamma = \pi_*(BP \wedge BP) \approx A[t_1, t_2, \ldots].$$

The structure maps for (A, Γ) consists of a product $\phi\colon \Gamma \otimes_A \Gamma \to \Gamma$, left and right unit maps $\eta_L, \eta_R\colon A \to \Gamma$, a counit map $\varepsilon\colon \Gamma \to A$, and a diagonal map $\psi\colon \Gamma \to \Gamma \otimes_A \Gamma$. The product ϕ and the left unit map η_L are built into the description of Γ as a polynomial algebra over A. The right unit map η_R is given above (2.3) for the m_i, and thereby, using (2.4), for the v_i also. The diagonal map ψ satisfies the formula

$$\sum_{i+j=n} m_i \psi(t_j)^{p^i} = \sum_{i+j+k=n} m_i t_j^{p^i} \otimes t_k^{p^{i+j}}. \tag{2.5}$$

The notation $M \otimes_A N$ requires that M be a right A-module, and that N be a left A-module. Γ is a right A-module by η_R, and a left A-module by η_L. The notation η_L is usually suppressed, and η_R is sometimes called η.

There is a formal group law associated with BP as follows. Let $\boldsymbol{CP}^\infty$ stand for infinite dimensional complex projective space, and let

$$\mu\colon \boldsymbol{CP}^\infty \times \boldsymbol{CP}^\infty \to \boldsymbol{CP}^\infty \tag{2.6}$$

be the map which classifies the tensor product of the canonical line bundles. Then (in unreduced homology),

$$H^*(\boldsymbol{CP}^\infty; BP) \approx A[\![X]\!]$$
$$H^*(\boldsymbol{CP}^\infty \times \boldsymbol{CP}^\infty; BP) \approx A[\![X_1, X_2]\!],$$

which are respectively the formal power series rings in one and two variables over A. The formal group law F is defined by the formal power series:

$$F(X_1, X_2)=\mu^*(X)=\sum a_{i,j} X_1^i X_2^j$$

where the coefficients $a_{i,j}$ belong to $A_{2i+2j-2}$. As in [1], let log be the formal power series (over $A\otimes\boldsymbol{Q}$):

$$\log(X)=\sum_{i\geqq 0} m_i X^{p^i}.$$

Let exp be the formal power series inverse to log, which satisfies $\log(\exp X)=X$. Then

$$F(X_1, X_2)=\exp(\log X_1+\log X_2).$$

The formal group law F is associative and commutative. Elements z_i from Γ may be substituted for the undeterminates, and we write $\sum^F z_i$ for $F(z_1, F(z_2, \ldots))$. Then formula (2.5) becomes

(2.7) $$\sum_j^F \psi(t_j)=\sum_{j,k}^F t_j\otimes t_k^{p^j}.$$

There is a canonical anti-automorphism $c\colon \Gamma\to\Gamma$, which satisfies $c\eta_L=\eta_R$ and $c\eta_R=\eta_L$. This gives a formal group law F^* conjugate to F, defined by the formula:

(2.8) $$\sum^{F^*} z_i=c(\sum^F c(z_i)).$$

Notation 2.9. Let $c(t_i)=h_i$.

The elements h_i satisfy the following formulas, which are obtained by applying c to (2.3), (2.5), and (2.12):

(2.10) $$m_n=\sum_{i=0}^n (h_{n-i})^{p^i}\eta(m_i)$$

(2.11) $$\sum_{i+j=n} \psi(h_j)^{p^i}\eta(m_i)=\sum_{i+j+k=n} h_k^{p^{i+j}}\otimes h_j^{p^i}\eta(m_i)$$

(2.12) $$\sum^{F^*} \psi(h_j)=\sum^{F^*} h_k^{p^j}\otimes h_j.$$

We also have $\Gamma\approx A[h_1, h_2,\ldots]$.

For each finite sequence of non-negative integers, $I=(i_1, i_2,\ldots, i_n)$, let

$$h^I=h_1^{i_1} h_2^{i_2}\cdots h_n^{i_n}.$$

The length of I is the integer $l(I)=i_1+\cdots+i_n$.

Definition 2.13. For each non-negatively graded, free left A-module M, let $U(M)$ be the sub-A-module of $\Gamma\otimes_A M$ spanned by all elements of the form $h^I\otimes_A m$ where $2l(I)<\text{degree}(m)$.

For an arbitrary non-negatively graded left A-module let

$$F_1 \xrightarrow{f} F_0 \to M \to 0$$

be exact with F_0 and F_1 free. Then define

$$U(M) = \operatorname{coker}(U(f)\colon U(F_1) \to U(F_0)).$$

It is easily verified that $U(M)$ is independent of F_1, F_0 and f.

Remark 2.14. In [2, (7.4)], the functor $U(\cdot)$ is defined in terms of another functor $G(\cdot)$ (specifically, $U(M)$ is the submodule of primitives in $G(M)$). The discussion of [2, (8.7)] shows that the two definitions of $U(M)$ agree.

There is a Γ-comodule structure on $\Gamma \otimes_A M$ by the map

$$\psi \otimes 1\colon \Gamma \otimes_A M \to \Gamma \otimes_A \Gamma \otimes_A M.$$

An easy induction using (2.12) shows that $\psi \otimes 1$ takes $U(M)$ to $U^2(M)$, and hence induces a map

$$\delta^U\colon U(M) \to U^2(M).$$

There is also a counit map $\varepsilon^U\colon U(M) \to M$ induced by the counit map in Γ. In the notation of [2, § 5], $(U, \delta^U, \varepsilon^U)$ is a cotriple on the category $\mathscr{A}$ of non-negatively graded left A-modules.

A module M in $\mathscr{A}$ with a U-structure will be called an unstable Γ-comodule (see [2, (7.4)]). This means that there is a map $\psi\colon M \to U(M)$ such that the following diagrams commute:

(2.15)
$$\begin{array}{ccc} M & \xrightarrow{\psi} & U(M) \\ {\scriptstyle\psi}\downarrow & & \downarrow{\scriptstyle\delta^U} \\ U(M) & \xrightarrow[U(\psi)]{} & U^2(M) \end{array} \qquad \begin{array}{ccc} M & \xrightarrow{\psi} & U(M) \\ & {\scriptstyle =}\searrow & \downarrow{\scriptstyle\varepsilon^U} \\ & & M \end{array}.$$

The category of unstable Γ-comodules will be called $\mathscr{U}$ ($\mathscr{U}$ is called $\mathscr{A}(U)$ in [2]). By construction $\mathscr{U}$ is an abelian category. To simplify notation, we shall write

$$\operatorname{Ext}_{\mathscr{U}}^{s,t}(M) \quad \text{for} \quad \operatorname{Ext}_{\mathscr{U}}^{s}(A[t], M)$$

for M in $\mathscr{U}$. These Ext groups may be calculated as the homology groups of an unstable cobar complex $C^{*,*}(M)$. Specifically, for each pair (s, t) of non-negative integers,

$$C^{s,t}(M) = U^s(M)_t.$$

As is customary, we write $[\gamma_1 | \cdots | \gamma_s]m$ for the element $\gamma_1 \otimes \cdots \otimes \gamma_s \otimes m$ in $U^s(M)$. The complementary degree t of such an element is the integer

$$t=\deg(\gamma_1)+\cdots+\deg(\gamma_s)+\deg(m).$$

The differential

$$d^s\colon C^{s,t}(M)\to C^{s+1,t}(M)$$

is given by

$$\begin{aligned} d^s[\gamma_1|\cdots|\gamma_s]m=&[1|\gamma_1|\cdots|\gamma_s]m\\ &+\sum(-1)^j[\gamma_1|\cdots|\gamma_j'|\gamma_j''|\cdots|\gamma_s]m\\ &+\sum(-1)^{s+1}[\gamma_1|\cdots|\gamma_s|\gamma']m''\end{aligned}$$

where $\psi(\gamma_j)=\sum\gamma_j'\otimes\gamma_j''$ and $\psi(m)=\sum\gamma'\otimes m''$. Then, from [2, (9.3)], we have

$$\mathrm{Ext}_{\mathscr{U}}^{s,t}(M)\approx H^{s,t}(C^{*,*}(M)).$$

Remark 2.16. As in the stable case, there is a smaller "normalized" cobar complex $\tilde{C}^{*,*}(M)$. This is obtained from the functor

$$\tilde{U}(M)=\ker\{\varepsilon^U\colon U(M)\to M\}$$

by

$$\tilde{C}^{s,*}(M)=\tilde{U}^s(M).$$

The inclusion $\tilde{C}(M)\to C(M)$ is a chain equivalence. This gives the following "easy" vanishing line.

Proposition 2.17. *If M is a k-connected unstable Γ-comodule, then* $\mathrm{Ext}_{\mathscr{U}}^{s,t}(M)=0$, *for* $t\leqq 2(p-1)s+k$.

The proof is immediate because $\tilde{C}^{s,t}(M)=0$ for $t\leqq 2(p-1)s+k$.

This vanishing line will be improved in Section 5.

§3. The Spectral Sequence for SU

When the space X is SU (=the infinite special unitary group), the unstable Adams-Novikov spectral sequence simplifies considerably, as shown below; compare with the unstable spectral sequence for SU based on mod p homology in [4, p. 191]. In fact, we have the following general result.

Theorem 3.1. *Let X be an H-space with torsion-free homology and torsion-free homotopy. Then the unstable Adams-Novikov spectral sequence for X collapses. That is,*

$$E_2^{s,t}(X) \approx \begin{cases} \pi_t(X), & for \quad s=0, \\ 0, & for \quad s>0. \end{cases}$$

Before giving the proof of Theorem 3.1, we recall the results of Wilson [10] which are the main ingredients of the proof.

Wilson's Spaces Y_k (3.2). There are indecomposable H-spaces $Y_k (k=1, 2, \ldots)$, which have the following properties.

(i) Y_k is $(k-1)$-connected, and $\pi_k(Y_k) = \boldsymbol{Z}_{(p)}$.

(ii) For $k \neq 2(p^n + p^{n-1} + \cdots + 1)$, $\Omega Y_{k+1} \simeq Y_k$. For $k = 2(p^n + p^{n-1} + \cdots + 1)$, $\Omega Y_{k+1} \simeq Y_k \times Y_{pk}$.

(iii) If $f\colon Y_k \to Y_k$ is a continuous map which induces an isomorphism $f_*\colon \pi_k(Y_k) \approx \pi_k(Y_k)$, then f is a homotopy equivalence.

(iv) If X is a H-space with torsion-free homology and torsion-free homotopy, then X is a product of the Y_k's:

$$X \simeq \prod_\alpha Y_{k_\alpha}$$

(but not necessarily as H-spaces).

(v) Let BP_k be the k-th space in the Ω-spectrum for BP. Then there are maps i and j

$$Y_k \xrightarrow{i} BP_k \xrightarrow{j} Y_k$$

where $j \circ i$ is a homotopy equivalence. Moreover we may consider $BP_k = Y_k \times Z$ where Z is a space at least k-connected.

Proof of Theorem 3.1. For each of the spaces Y_k, consider the following homotopy-commutative diagram:

$$\begin{array}{ccccc} Y_k & \xrightarrow{\eta'} & BP(Y_k) & \xrightarrow{\mu'} & Y_k \\ {\scriptstyle i}\downarrow & & \downarrow{\scriptstyle BP(i)} & & \uparrow{\scriptstyle j} \\ BP_k & \xrightarrow{\eta} & BP(BP_k) & \xrightarrow{\mu} & BP_k \end{array}$$

where $\eta' = \eta(Y_k)$, $\eta = \eta(BP_k)$; the map μ is induced by the product in BP, and $\mu' = j \circ \mu \circ BP(i)$. The properties of BP as a ring spectrum imply that $\mu \circ \eta$ is a homotopy equivalence. Hence $\mu' \circ \eta'$ induces $\pi_k(Y_k) \approx \pi_k(Y_k)$, and so by (3.2, (iii)) must be a homotopy equivalence. By the construction (2.1) or [2, (2.4)] this implies that the spectral sequence for Y_k collapses. It follows that for any product of the Y_k's, the spectral sequence collapses. By (3.2, (iv)), X is such a space, so the spectral sequence collapses for X. Q. E. D.

Remark 3.3. Using the facts that SU is an H-space with torsion-free homology and torsion-free homotopy, we see that the spectral sequence collapses for SU. For an alternative approach, see Remark 3.8.

We proceed to analyze the image of the BP_*-Hurewicz homomorphism

$$\begin{aligned}\operatorname{im}\{\pi_*(SU)\to H_*(SU; BP)\} &\approx E_2^{0,*}(SU)\\ &\approx \operatorname{Ext}_{\mathscr{U}}^{0,*}(PH_*(SU; BP)).\end{aligned}$$

Here $PH_*(SU; BP)$ stands for the submodule of primitives in $H_*(SU; BP)$ considered as an unstable Γ-comodule. For later use, we give a description of $PH_*(SU; BP)$. Recall from [8] that there is a map

$$f\colon\ S^1 \wedge CP^\infty \to SU.$$

As in [1], let β_k, for $k=1, 2,\ldots$ be the natural generator of $H_{2k}(CP^\infty; BP)$. Let $f_*(\iota_1 \wedge \beta_k) = x_{2k+1}$, which is a primitive element in $H_{2k+1}(SU; BP)$. $H_*(SU; BP)$ is cofree as a coalgebra, and $PH_*(SU; BP)$ is the free A-module generated by $\{x_{2k+1}\}$, $k=1, 2,\ldots$.

To describe the unstable Γ-coaction on $PH_*(SU; BP)$ we proceed as follows. As in [1], by abuse of notation, let β_k also stand for the generator of $H_{2k}(CP^\infty; MU)$, and let b_k^{MU} be the generator of $H_{2k}(MU; MU)$. Then from [1, (11.4)] the formula for the coaction in $H_*(CP^\infty; MU)$ is

$$\psi(\beta_k) = \sum_j \big(\sum_s b_s^{MU}\big)_{k-j}^j \otimes \beta_j$$

where $(\sum_s b_s^{MU})_{k-j}^j$ stands for the terms of degree $2k-2j$ in $(\sum_s b_s^{MU})^j$. The Quillen idempotent induces a map from $H_*(MU; MU)$ to $H_*(BP; BP)$ which sends $\sum_s b_s^{MU}$ to $\sum_s^{F*} h_s$, as shown in [2, (8.3)]. Thus the formula for the coaction in $H_*(CP^\infty; BP)$ is

$$\psi(\beta_k) = \sum_j \big(\sum_s^{F*} h_s\big)_{k-j}^j \otimes \beta_j.$$

Then by naturality of f_*, the formula for the coaction in $H_*(SU; BP)$ is

(3.4) $$\psi(x_{2k+1}) = \sum_j \big(\sum_s^{F*} h_s\big)_{k-j}^j \otimes x_{2j+1}.$$

It would be awkward to calculate the groups $\operatorname{Ext}_{\mathscr{U}}^{0,*}(PH_*(SU; BP))$ directly. Instead, we do the following. Define an A-linear map

$$\phi\colon PH_*(SU; BP) \to PH_{*+2}(SU; BP)$$

by

$$\phi(x_{2i+1})=\sum_{k=1}^{i+1} ka_{i-k+1}x_{2k+1}$$

where $a_s=a_{s,1}$ (the $a_{i,j}$ are the BP-formal group law coefficients). The a_s may be computed recursively by $a_0=1$, and the formula (see [1, (10.1)]):

$$\sum_{n\geq 0} p^n m_n=(\sum_{s\geq 0} a_s)^{-1}.$$

Let ϕ^k stand for the k-th iterate of the map ϕ. The following is motivated by Toda's proof of the Bott Periodicity Theorem [8].

Proposition 3.5. *For each non-negative integer k, $\phi^k(x_3)$ generates $E_2^{0,2k+3}(SU)$.*

Proof. The Hopf construction applied to the map μ: $CP^\infty \times CP^\infty$ gives a map:

$$H(\mu)\colon S^1 \wedge CP^\infty \wedge CP^\infty \to S^1 \wedge CP^\infty.$$

The restriction of $H(\mu)$ to $S^1 \wedge S^2 \wedge CP^\infty$ will be called ζ. ζ induces a map in homology:

$$\zeta_*\colon H_*(CP^\infty; BP)\to H_{*+2}(CP^\infty; BP).$$

By dualizing the formal group law for BP, we see that

$$\zeta_*(\beta_i)=\sum_{k=1}^{i+1} ka_{i-k+1}\beta_i.$$

Consider the composite map (which will be called ζ_{k+1}):

$$S^{2k+3} \xrightarrow{\text{incl.}} S^{2k+1}\wedge CP^\infty \xrightarrow{S^{2k+1}(\zeta)} S^{2k-1}\wedge CP^\infty \xrightarrow{S^{2k-4}(\zeta)}$$
$$\cdots \xrightarrow{S^2(\zeta)} S^3\wedge CP^\infty \xrightarrow{\zeta} S^1\wedge CP^\infty \xrightarrow{f} SU.$$

The map ζ_{k+1} induces a map of the unstable Adams-Novikov spectral sequences

$$(\zeta_{k+1})_*\colon E_2^{0,2k+3}(S^{2k+3})\to E_2^{0,2k+3}(SU)$$

with $(\zeta_{k+1})_*(\iota_{2k+3})=\phi^k(x_3)$. This shows that $\phi^k(x_3)$ is a cycle. By examining the coefficient of x_{2m+1}, where m is the integer $k-1$ reduced modulo $p-1$, we see that $\phi^k(x_3)$ is not divisible by p. Therefore, $\phi^k(x_3)$ is a generator of $E_2^{0,2k+3}(SU)$. Q.E.D.

Remarks 3.6. (i) The above proof also shows that the composite map ζ_{k+1} is a generator of $\pi_{2k+1}(SU)$. This reproves a theorem of Toda [8].

(ii) Some examples of the generators of $E_2^{0,*}(SU)$ for the prime $p=2$ are the following

$$\phi(x_3)=2x_5-v_1x_3$$
$$\phi^2(x_3)=6x_7-6v_1x_5+3v_1^2x_3$$
$$\phi^3(x_3)=24x_9-36v_1x_7+30v_1^2x_5-(12v_2+9v_1^2)x_3\,.$$

Theorem 3.7. *Let Y be an H-space with torsion-free homology and torsion-free homotopy. Suppose that X is a space such that $H_*(X;BP)$ is cofree as a coalgebra over A, and that*

$$PH_*(X;BP)\approx PH_*(Y;BP)$$

as unstable Γ-comodules. Then $X\simeq Y$.

Proof. By (3.1), the unstable Adams-Novikov spectral sequence for Y collapses. Both X and Y satisfy the assumptions of (2.2), so $E_2^{*,*}(X)\approx E_2^{*,*}(Y)$. Thus the spectral sequence collapses for X too, and $\pi_*(X)$ is free over $\boldsymbol{Z}_{(p)}$. Also, $H_*(X;\boldsymbol{Z}_{(p)})$ is free over $\boldsymbol{Z}_{(p)}$ because of the isomorphisms:

$$\begin{aligned}H_*(BP;\boldsymbol{Z}_{(p)})\otimes_{\boldsymbol{Z}_{(p)}}H_*(X;\boldsymbol{Z}_{(p)})&\approx\pi_*(BP\wedge K(\boldsymbol{Z}_{(p)})\wedge X)\\&\approx H_*(BP;\boldsymbol{Z}_{(p)})\otimes_A H_*(X;BP)\,.\end{aligned}$$

Therefore, by [1, (5.23)], X is an H-space with torsion-free homotopy and torsion-free homology. By (3.2, (iv)), X and Y are each a product of the Y_k's. As $\pi_*(X)\approx\pi_*(Y)$, the factors which occur for X are the same as those which occur for Y. Q.E.D.

Remark 3.8. It is possible by a lengthy calculation to determine the coaction in the space $Y=Y_3\times Y_5\times\cdots\times Y_{2p-1}$, and then to show that

$$PH_*(SU;BP)\approx PH_*(Y;BP)$$

as unstable Γ-comodules. Thus (3.7) implies that

$$SU\simeq Y_3\times Y_5\times\cdots\times Y_{2p-1}\,.$$

Making use of (3.2, (iv)), we have that $\Omega^2Y_3\simeq S^1\times Y_{2p-1}$, and that for $2\leqq j\leqq p-1$, $\Omega^2Y_{2j+1}\simeq Y_{2j-1}$. Thus

$$\begin{aligned}\Omega^2(SU)&\simeq S^1\times Y_{2p-1}\times Y_3\times\cdots\times Y_{2p-3}\\&\simeq S^1\times SU\end{aligned}$$

which is the (complex) Bott periodicity theorem.

§4. Calculations of $E_2^{1,*}(SU(n))$

Let $SU(n)$ be the spectral unitary group in n variables. $H_*(SU(n);BP)$ is free as an A-module, and cofree as a coalgebra. The submodule of primitives $PH_*(SU(n); BP)$ will be called $M(n)$. By (2.2), we have

$$E_2^{s,t}(SU(n)) \simeq \operatorname{Ext}_{\mathcal{U}}^{s,t}(M(n)).$$

As an A-module, $M(n)$ is freely generated by the elements $x_3, x_5, \dots, x_{2n-1}$ defined in Section 3. The unstable Γ-comodule structure $\psi: M(n) \to UM(n)$ is given by formula (3.4). The groups $\operatorname{Ext}_{\mathcal{U}}^{s,t}(M(n))$ are the homology of the unstable cobar complex. By sparseness, we have

$$E_2^{s,t}(SU(n)) = 0, \qquad \text{for } t \text{ even}.$$

Consider the fibration

$$SU(n) \to SU(n+1) \to S^{2n+1}.$$

Passing to BP_*-homology, and then taking primitives, we obtain a short exact sequence of unstable Γ-comodules:

$$0 \to M(n) \to M(n+1) \to A[2n+1] \to 0.$$

This induces a long exact sequence of Ext groups, which, after identifying them as E_2-terms, becomes

$$\cdots \to E_2^{s,t}(SU(n)) \to E_2^{s,t}(SU(n+1)) \to E_2^{s,t}(S^{2n+1}) \xrightarrow{\delta} \cdots \tag{4.1}$$

where δ has bidegree (1, 0). The indexing (s, t) is such that t–s is the homotopy dimension, whereas in [2], as in the stable case (e.g. [6]), t–s is stem dimension.

Proposition 4.2.

(i) $E_2^{s,t}(SU(n)) \approx E_2^{s,t}(SU(n+1))$, *for* $t < 2n+2s$.

(ii) $E_2^{0,t}(SU(n)) \approx 0$, *for* $t \geqq 2n$,
$E_2^{0,2i+1}(SU(n)) \approx \boldsymbol{Z}_{(p)}$, *for* $1 \leqq i < n$.

(iii) $E_2^{1,2n+1}(SU(n)) \approx \boldsymbol{Z}_{(p)}/(n!)\boldsymbol{Z}_{(p)}$.

(iv) *For $n < k$, the inclusion $SU(n) \to SU(k)$ induces a monomorphism*

$$E_2^{1,2k+1}(SU(n)) \to E_2^{1,2k+1}(SU(k)).$$

Proof. Statements (i) and (ii) follow immediately from the long exact sequences (4.1), the easy vanishing line (2.17) for $E_2^{s,t}(S^{2n+1})$, and the calculation of $E_2^{0,*}(SU)$. Part (iii) follows from (3.5) and (4.1). Part (iv) follows from (4.1) and the fact that $E_2^{0,t}(S^{2n+1}) = 0$, for $t \neq 2n+1$. Q.E.D.

The calculation of the rest of the 1-line for $SU(n)$ is the calculation of which elements of $E_2^{1,2k+1}(SU(k))$ pull back to $E_2^{1,2k+1}(SU(n))$, for $n < k$. The difficulty in working directly with the cobar complex $C^{*,*}(M(n))$ is the formal sum which occurs in the expression for the differential. That is,

$$\begin{aligned} d(x_{2k+1}) &= 1\otimes x_{2k+1} - \psi(x_{2k+1}) \\ &= -\sum_{j=1}^{k-1} (\textstyle\sum_s^{F^*} h_s)_{k-j}^j \otimes x_{2j+1}\,. \end{aligned}$$

To overcome this difficulty, we introduce a map $\bar{e}\colon \Gamma\to\boldsymbol{Q}$ as follows. From Section 2, we see that $\Gamma\otimes_{\boldsymbol{Z}}\boldsymbol{Q}$ is a polynomial algebra over $\boldsymbol{Q}$ generated by the m_k and the $\eta_R(m_k)$, $k=1, 2,\dots$.

Definition 4.3. Let $\bar{e}\colon \Gamma\otimes_{\boldsymbol{Z}}\boldsymbol{Q}\to\boldsymbol{Q}$ be the homomorphism defined on the generators by

$$\bar{e}(m_k)=1/p^k$$
$$\bar{e}(\eta_R(m_k))=0$$

for $k=1, 2,\dots$. Also, let $\bar{e}\colon \Gamma\to\boldsymbol{Q}$ denote the restriction of this map to Γ.

Lemma 4.4. *For the Hazewinkel elements v_i in Γ, we have $\bar{e}(v_1)=1$, $\bar{e}(v_i)=0$ for $i>1$.*

Proof. $v_1=pm_1$, so $\bar{e}(v_1)=1$. For $i>1$, the v_i are defined by formula (2.4), and it follows by induction on i that $\bar{e}(v_i)=0$ for $i>1$. Q.E.D.

In the unstable cobar complex $C^{*,*}(S^{2n+1})$, we have $C^{0,*}(S^{2n+1})\approx A$. For a in A, $d(a)=\eta_R(a)-a$, so

$$\bar{e}(d(a))=-\bar{e}(a)\,.$$

Therefore, $\bar{e}(d(A))\subset\boldsymbol{Z}_{(p)}$. Passing to the homology of the unstable cobar complex, we obtain a well-defined map

$$e\colon E_2^{1,*}(S^{2n+1})\to\boldsymbol{Q}/\boldsymbol{Z}_{(p)}$$

which is (up to sign) an unstable analogue of the complex Adams e-invariant localized at p. The description of $E_2^{1,*}(S^{2n+1})$ in [2, §9] shows that e is a monomorphism. Thus the order of any element α in $E_2^{1,*}(S^{2n+1})$ is the same as the order of $e(\alpha)$ in $\boldsymbol{Q}/\boldsymbol{Z}_{(p)}$.

Definition 4.5. For each pair (k, j) of positive integers, we define a rational number $b_{k,j}$ as follows. For each $j\geqq 1$, let

$$\left(Y+\frac{Y^p}{p}+\frac{Y^{p^2}}{p^2}+\cdots\right)^j=\sum_{k\geqq j} b_{k,j}Y^k$$

in the formal power series ring $\boldsymbol{Q}[\![Y]\!]$.

The properties of $\bar{e}$ which will be used in the proof of (4.7) are the following. Recall from (2.9) that $h_k=c(t_k)$, where c is the canonical anti-automorphism.

Lemma 4.6. (i) $\bar{e}(h_k)=1/p^k$,

(ii) $\bar{e}(\sum^{F*} h_s)^j_{k-j}=b_{k,j}$.

Proof. Apply the canonical anti-automorphism c to (2.3) to obtain

$$m_k=\eta_R(m_k)+h_1^{p^{k-1}}\eta_R(m_{k-1})+\cdots+h_k.$$

As $\bar{e}$ is 0 on the image of η_R, part (i) follows. For part (ii), we have

$$\begin{aligned}\bar{e}(\textstyle\sum^{F*} h_s)^j_{k-j}&=\bar{e}(\textstyle\sum h_s)^j_{k-j}=\left(1+\dfrac{Y^{p-1}}{p}+\dfrac{Y^{p^2-1}}{p^2}+\cdots\right)^j_{k-j}\\&=\left(Y+\dfrac{Y^p}{p}+\dfrac{Y^{p^2}}{p^2}+\cdots\right)^j_k\\&=b_{k,j}.\end{aligned}$$

Q.E.D.

The matrix $B=[b_{k,j}]$ is lower triangular, with diagonal entries $b_{k,k}=1$. Hence there is a well-defined inverse matrix $C=[c_{k,j}]$ which is also lower triangular. For each $2\leqq n\leqq k$, let $\omega_k(n)$ be the integer defined by

$$\omega_k(n)=\underset{n\leqq j\leqq k}{\text{maximum}}\,\{\text{order } c_{k,j} \text{ in } \boldsymbol{Q}/\boldsymbol{Z}_{(p)}\}.$$

Theorem 4.7. *For each* $2\leqq n\leqq k$,

$$E_2^{1,2k+1}(SU(n))\approx\omega_k(n)\boldsymbol{Z}_{(p)}/k!\boldsymbol{Z}_{(p)}.$$

Proof. From Section 2 (see also [2]), we know that $E_2^{*,*}(SU(n))$ may be calculated as the homology of the unstable cobar complex $C^{*,*}(M(n))$. Let g_k be the element $-d(x_{2k+1})$ in $C^{1,2k+1}(M(k+1))$; that is

$$\begin{aligned}g_k&=\psi(x_{2k+1})-1\otimes x_{2k+1}\\&=\sum_{j=1}^{k-1}(\textstyle\sum^{F*} h_s)^j_{k-j}\otimes x_{2j+1}\\&=\sum_{j=1}^{k-1}\gamma_{k,j}\otimes x_{2j+1}\end{aligned}$$

where $\gamma_{k,j}=(\sum^{F*} h_s)^j_{k-j}$. The long exact sequence (4.1) shows that the homology class of g_k generates $E_2^{1,2k+1}(SU(k))$.

Next, let integers $\tau_k(n)$ for $1\leqq n<k$ and rational numbers $a_{k,j}(n)$ for $1\leqq n\leqq k$, all $1\leqq j$, be defined as follows. First, let $a_{k,j}(k)=\bar{e}(\gamma_{k,j})$. Then recursively for $1\leqq n<k$, let

$$\tau_k(n)=\text{order of } a_{k,n}(n+1) \quad\text{in}\quad \boldsymbol{Q}/\boldsymbol{Z}_{(p)}$$

$$a_{k,j}(n)=\tau_k(n)(a_{k,j}(n+1)-a_{k,n}(n+1)a_{n,j}(n)).$$

We shall show by downward induction on n that for $1\leqq n\leqq k$, there are elements $g_k(n)$ in $UM(n)$ of the form

$$g_k(n)=\sum_{j=1}^{n-1}\gamma_{k,j}(n)\otimes x_{2j+1}$$

where the $\gamma_{k,j}(n)$ are in Γ with $\bar{e}(\gamma_{k,j}(n))=a_{k,j}(n)$, and such that the homology class of $g_k(n)$ in $E_2^{1,2k+1}(SU(n))$ represents the generator. This is true for $n=k$, because $g_k(k)=g_k$. Then assume inductively for $n<k$ that there is an element $g_k(n+1)$ as asserted. Consider the map

$$E_2^{1,2k+1}(SU(n+1))\xrightarrow{\rho_*}E_2^{1,2k+1}(S^{2n+1}).$$

Then $\rho_*(g_k(n+1))=\gamma\otimes x_{2n+1}$, where γ is in Γ, with $\bar{e}(\gamma)=a_{k,n}(n+1)$. Then $\tau_k(n)=$order of $\gamma\otimes x_{2n+1}$ in $E_2^{1,2k+1}(S^{2n+1})$, so there is an element $a_k(n+1)$ in A, with

$$d(a_k(n+1))=\tau_k(n)\cdot\gamma\otimes x_{2n+1}.$$

Then the element

$$\begin{aligned}g_k(n)&=\tau_k(n)g_k(n+1)-d(a_k(n+1)x_{2n+1})\\&=\sum_{j=1}^{n-1}\gamma_{k,j}(n)\otimes x_{2j+1}\end{aligned}$$

generates $E_2^{1,2k+1}(SU(n))$, and satisfies

$$\bar{e}(\gamma_{k,j}(n))=a_{k,j}(n).$$

Next we define rational numbers $b_{k,j}(n)$ for all pairs of integers (k, j) with $k\geqq n$ as follows. First, $b_{k,j}(k)=b_{k,j}$ as defined in (4.5). Then recursively for $n<k$, let

$$b_{k,j}(n)=b_{k,j}(n+1)-b_{k,n}(n+1)b_{n,j}.$$

Observe that the result of row-reducing the k-th row of the matrix $B=[b_{k,j}]$ by using rows $k-1$, $k-2,\ldots, n+1$ takes the k-th row

$$\langle b_{k,1}, b_{k,2},\ldots, b_{k,k-1}, 1, 0,\ldots\rangle$$

to the row

$$\langle b_{k,1}(n+1),\ldots, b_{k,n}(n+1), 0,\ldots, 0, 1, 0,\ldots\rangle.$$

Hence, in the matrix C (=the inverse of B), we have $c_{k,n}=-b_{k,n}(n+1)$. In particular, these two rational numbers have the same order in $\boldsymbol{Q}/\boldsymbol{Z}_{(p)}$. We assert that

$$a_{k,j}(n)=\omega_k(n)\cdot b_{k,j}(n)$$

which we shall show by downward induction on n. The statement is true for $n=k$, as $a_{k,j}(k)=b_{k,j}(k)$, and $\omega_k(k)=1$. Assume inductively that

$$a_{k,j}(n+1)=\omega_k(n+1)b_{k,j}(n+1).$$

Then

$$\begin{aligned}a_{k,j}(n)&=\tau_k(n)\,(a_{k,j}(n+1)-a_{k,n}(n+1)a_{n,j})\\&=\tau_k(n)\omega_k(n+1)\,(b_{k,j}(n+1)-b_{k,n}(n+1)b_{n,j})\\&=\tau_k(n)\omega_k(n+1)b_{k,j}(n)\\&=\operatorname{order}(a_{k,n}(n+1))\cdot\omega_k(n+1)\cdot b_{k,j}(n)\\&=\operatorname{order}(\omega_k(n+1)b_{k,n}(n+1))\cdot\omega_k(n+1)\cdot b_{k,j}(n)\\&=\max\{\omega_k(n+1),\ \operatorname{order}(b_{k,n}(n+1)\}b_{k,j}(n)\\&=\omega_k(n)b_{k,j}(n).\end{aligned}$$

Finally, we consider the map

$$(i_{n,k})_*\colon E_2^{1,2k+1}(SU(n))\to E_2^{1,2k+1}(SU(k))$$

induced by the inclusion $i_{n,k}\colon SU(n)\to SU(k)$. By (4.2, (iv)), $(i_{n,k})_*$ is a monomorphism, and by the above,

$$(i_{n,k})_*g_k(n)=\omega_k(n)\cdot g_k.$$

Thus $E_2^{1,2k+1}(SU(n))\approx\omega_k(n)\boldsymbol{Z}_{(p)}/k!\boldsymbol{Z}_{(p)}$ as asserted. Q.E.D.

From these calculations, we can determine the kernel of the homomorphism

$$E_2^{2,*}(SU(n))\to E_2^{2,*}(SU(n+1)).$$

Recall from [2], that $E_2^{1,2k+1}(S^{2n+1})$ is shown to be a cyclic group of order $\sigma_k(n)$ where $\sigma_k(n)$ is as follows. If $k-n\not\equiv 0 \bmod 2p-2$, $\sigma_k(n)=1$. Otherwise, write $k-n=(2p-2)\cdot p^m\cdot q$, where q is prime to p. Then for an odd prime p,

$$\sigma_k(n)=\min\{p^n,\ p^{m+1}\}$$

and for $p=2$,

$$\sigma_k(n)=\begin{cases}2, & \text{if } k-n \text{ is odd},\\ 2, & \text{if } k=3,\ n=1,\\ 4, & \text{if } k-n=2,\ n\geqq 2,\\ \min\{2^n,\ 2^{m+2}\}, & \text{otherwise}.\end{cases}$$

Corollary 4.8. *For $k<n$, the kernel of*

$$E_2^{2,2k+1}(SU(n)) \to E_2^{2,2k+1}(SU(n+1))$$

is a cyclic group of order $\sigma_k(n)/\tau_k(n)$, where $\sigma_k(n)$ is as above, and $\tau_k(n)$ is the integer defined in the course of the proof of Theorem 4.7.

Proof. From the long exact sequence (4.1), we must determine the cokernel of

$$E_2^{1,2k+1}(SU(n+1)) \to E_2^{1,2k+1}(S^{2n+1}).$$

We have calculated that for $k<n$, the order of the image is $\tau_k(n)$. Therefore the cokernel has order $\sigma_k(n)/\tau_k(n)$ as asserted. Q.E.D.

Remark 4.9. The results of Bousfield [3] concerning products in the unstable homotopy spectral sequence with coefficients in a ring may be generalized to a ring spectrum (in place of the ring). The statements of [3, (8.2)] apply to our situation. In particular, the differentials are seen to act as derivations with respect to the action of $E_r^{*,*}(S^{2n+1})$ on $E_r^{*,*}(SU(n))$. The coboundary map

$$\delta\colon E_2^{s,t}(S^{2n+1}) \to E_2^{s+1,t}(SU(n))$$

has the form $\delta(\alpha) = \pm\alpha\otimes d'(x_{2n+1})$, for α in $E_2^{s,t}(S^{2n+1})$. Thus δ is map of spectral sequences. For an odd prime p, the elements of $E_2^{1,2k+1}(S^{2n+1})$ are all permanent cycles [2], and their coboundaries are permanent cycles in $E_2^{2,2k+1}(SU(n))$. When non-zero, these represent non-zero elements of $\pi_*(SU(n))$. A similar but more complicated statement holds for $p=2$, taking into account the differential d^3 on $E_2^{1,*}(S^{2n+1})$.

Examples 4.10. Some examples (of the upper left-hand corners) of the matrices of (4.5) are the following:

(1) For $p=2$,

$$B=\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ \frac{1}{2} & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 \\ \frac{1}{4} & \frac{1}{4} & \frac{3}{2} & 1 & 0 & 0 \\ 0 & \frac{1}{2} & \frac{3}{4} & 2 & 1 & 0 \\ 0 & \frac{1}{4} & \frac{7}{8} & 3 & \frac{5}{2} & 1 \end{pmatrix}$$

$$C=\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ -\frac{1}{2} & 1 & 0 & 0 & 0 & 0 \\ \frac{1}{2} & -1 & 1 & 0 & 0 & 0 \\ -\frac{7}{8} & \frac{5}{4} & -\frac{3}{2} & 1 & 0 & 0 \\ \frac{13}{8} & -\frac{9}{4} & -\frac{9}{4} & -2 & 1 & 0 \\ -\frac{15}{16} & \frac{5}{2} & \frac{1}{4} & \frac{5}{2} & -\frac{5}{2} & 1 \end{pmatrix}.$$

(2) For $p=3$,

$$B=\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ \frac{1}{3} & 0 & 1 & 0 & 0 & 0 \\ 0 & \frac{2}{3} & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & \frac{1}{9} & 0 & \frac{4}{3} & 0 & 1 \end{pmatrix}$$

$$C=\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ -\frac{1}{3} & 0 & 1 & 0 & 0 & 0 \\ 0 & -\frac{2}{3} & 0 & 1 & 0 & 0 \\ \frac{1}{3} & 0 & -1 & 0 & 1 & 0 \\ 0 & \frac{7}{9} & 0 & -\frac{4}{3} & 0 & 1 \end{pmatrix}.$$

(3) The methods of this section also apply to the unstable Adams-Novikov spectral sequence based on MU (instead of BP). In this case, the definition of $\bar{e}$ becomes

$$\bar{e}(m_k)=1/k+1$$
$$\bar{e}(\eta(m_k))=0$$

where the $m_k=[CP^k]/k+1$ in $\pi_{2k}(MU)$. The matrix B of (4.5) is defined by

$$\left(Y+\frac{Y^2}{2}+\frac{Y^3}{3}+\cdots\right)^j=\sum_{k\geqq j} b_{k,j}Y^k.$$

The matrix C is the inverse of B, and then $\omega_n(k)$ is the least common multiple of the orders of $c_{k,j}$ in $\boldsymbol{Q}/\boldsymbol{Z}$ as $n\leqq j\leqq k$. The analogue to Theorem 4.7 becomes:

$$E_2^{1,2k+1}(SU(n))\approx\omega_k(n)\boldsymbol{Z}/k!\boldsymbol{Z}$$

For this, the integral case,

$$B=\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0\\ \frac{1}{2} & 1 & 0 & 0 & 0 & 0\\ \frac{1}{3} & 1 & 1 & 0 & 0 & 0\\ \frac{1}{4} & \frac{11}{12} & \frac{3}{2} & 1 & 0 & 0\\ \frac{1}{5} & \frac{5}{6} & \frac{7}{4} & 2 & 1 & 0\\ \frac{1}{6} & \frac{137}{180} & \frac{15}{8} & \frac{17}{6} & \frac{5}{2} & 1 \end{pmatrix}$$

$$C=\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0\\ -\frac{1}{2} & 1 & 0 & 0 & 0 & 0\\ \frac{1}{6} & -1 & 1 & 0 & 0 & 0\\ -\frac{1}{24} & \frac{7}{12} & -\frac{3}{2} & 1 & 0 & 0\\ \frac{1}{120} & -\frac{1}{4} & \frac{5}{4} & -2 & 1 & 0\\ -\frac{1}{720} & \frac{31}{360} & -\frac{3}{4} & \frac{13}{6} & -\frac{5}{2} & 1 \end{pmatrix}.$$

§5. Some Calculations in $E_2^{*,*}(S^{2n+1})$

In this section, we describe a method of making resolutions of unstable Γ-comodules which are convenient for calculations. We use these resolutions to establish vanishing lines for unstable Ext groups. In particular, we give a vanishing line for $E_2^{*,*}(S^{2n+1})$. We also make some calculations of $E_2^{2,*}(S^{2n+1})$ in low stems, which together with the calculations of $E_2^{1,*}(S^{2n+1})$ of [2], give some of the unstable groups $\pi_*(S^{2n+1})$.

Throughout this section, a prime p is fixed, BP is the Brown-Peterson spectrum associated with p, $\pi_*(BP)=A$, and $\pi_*(BP\wedge BP)=\Gamma$. The category of connected A-modules is called $\mathscr{A}$. The category $\mathscr{U}$ of unstable Γ-comodules is defined in Section 2 by the cotriple $(U, \varepsilon^U, \delta^U)$ on $\mathscr{A}$. Let $J: \mathscr{U}\to\mathscr{A}$ be the forgetful functor. Then for M in $\mathscr{A}$ and N in $\mathscr{U}$, there are natural isomorphisms α and β:

$$\text{(5.1)} \qquad \mathrm{Hom}_{\mathscr{U}}(N, U(M)) \underset{\beta}{\overset{\alpha}{\rightleftarrows}} \mathrm{Hom}_{\mathscr{A}}(J(N), M).$$

If $f: N\to U(M)$ is a map in $\mathscr{U}$, then $\alpha(f)=\varepsilon^U\circ f: J(N)\to M$ is a map in $\mathscr{A}$. If $g: J(N)\to M$ is a map in $\mathscr{A}$, then $\beta(g)=U(g)\circ\psi: N\to U(M)$ is a map in $\mathscr{U}$. Specifically, if x is in N, with $\psi(x)=\sum_i \gamma_i\otimes x_i$, then

$$\text{(5.2)} \qquad \beta(g)(x)=\sum_i \gamma_i\otimes g(x_i).$$

In particular, a map $f: N\to U(M)$ in $\mathscr{U}$ is determined by the map $g=\varepsilon^U\circ f$ by formula (5.2).

Recall that $A[t]$ is the free A-module on one generator ι_t of degree t, with trivial Γ-coaction. Then for any A-module M,

$$\mathrm{Hom}_{\mathscr{U}}(A[t])\approx \mathrm{Hom}_{\mathscr{A}}(J(A[t]), M)\approx M_t.$$

Acyclic Resolutions (5.3). Suppose that M is in $\mathscr{U}$. Then an acyclic resolution of M by models is a sequence

$$0\to M \xrightarrow{\partial_{-1}} U(M^0) \xrightarrow{\partial_0} U(M^1) \xrightarrow{\partial_1} U(M^2) \to\cdots$$

which is acyclic, and the maps $\partial_{-1}, \partial_0, \partial_1,\dots$ are in $\mathscr{U}$. From this, we obtain a complex

$$M^0 \xrightarrow{d^0} M^1 \xrightarrow{d^1} M^2 \to \cdots$$

where, for each $i\geqq 0$, d is the map

$$d^i(x)=\varepsilon^U\circ\partial_i(1\otimes x).$$

Then, by standard homological algebra in the abelian category $\mathscr{U}$, we have:

$$\text{(5.4)} \qquad \mathrm{Ext}_{\mathscr{U}}^{s,t}(M)\approx(\ker d^s/\mathrm{im}\, d^{s-1})_t.$$

Our next step is to find convenient resolutions for certain kinds of modules.

Definition 5.5. An A-module M is called quasi-free if $M\approx F/R$ where F is a free A-module with homogeneous basis $\{x_\alpha\}$, and R is a sub-A-module of F which is generated by homogeneous elements $\{r\}$, each of which is of the form

$$r=\sum_{\alpha} c_\alpha x_\alpha$$

where the c_α are in $\boldsymbol{Z}_{(p)}$.

The usefulness of quasi-free modules is the following.

Proposition 5.6. *Let M be a quasi-free A-module. Let n be a fixed positive integer, and let N be the sub-A-module of M spanned by the elements of M_m for $m<n$. Then*

$$M\approx N\oplus M/N$$

as A-modules.

Proof. Write $M\approx F/R$ as given in the definition (5.5). Then let

$$\begin{aligned} F_0 &= \text{span}\,\{x_\alpha \mid \text{degree}\,(x_\alpha)<n\} \\ F_1 &= \text{span}\,\{x_\alpha \mid \text{degree}\,(x_\alpha)\geqq n\} \\ R_0 &= \text{span}\,\{r \mid \text{degree}\,(r)<n\} \\ R_1 &= \text{span}\,\{r \mid \text{degree}\,(r)\geqq n\}\,. \end{aligned}$$

Then it is immediate that $F=F_0\oplus F_1$, $R=R_0\oplus R_1$, and $N\approx F_0/R_0$, $M/N\approx F_1/R_1$. Thus

$$\begin{aligned} M &\approx F_0/R_0\oplus F_1/R_1 \\ &\approx N\oplus M/N\,. \end{aligned}$$

Q.E.D.

Proposition 5.7. *Let M be an unstable Γ-comodule, which is quasi-free as an A-module, and suppose that* connectivity $(M)=k-1$, *with* $k\geqq 2p-1$. *Then*

$$\begin{aligned} &\text{Ext}_{\mathscr{U}}^{2s,t}(M)=0\,, && \text{for}\quad t<2(p-1)ps+k\,, \\ &\text{Ext}_{\mathscr{U}}^{2s+1,t}(M)=0\,, && \text{for}\quad t<2(p-1)(ps+1)+k\,. \end{aligned}$$

Proof. By downward induction on the connectivity of M. If M is highly connected, (5.7) holds by the Miller-Zahler vanishing line for stable Γ-comodules ([6], [12]). Assume inductively that (5.7) holds for connectivity $\geqq k$ and let M be an unstable Γ-comodule, quasi-free as an A-module, with connectivity $(M)=k-1$, and $k\geqq 2p-1$. We construct an acyclic resolution of M by models as follows.

Let $S^{2p-2}M$ be the $(S^{2p-2}M)_{2p-2+n}=M_n$. For x in M, let $S^{2p-2}x$ denote the corresponding element in $S^{2p-2}M$. The unstable Γ-comodule structure on M is a map $\psi\colon M\to U(M)$. Recall from (2.13) that $U(M)$ is spanned by the elements $h^I \otimes x$ where x is in M, and $2l(I) < \text{degree}(x)$. Thus coker ψ is spanned by the same $h^I \otimes x$ but excluding the $1 \otimes x$. We define an A-linear map

$$f\colon \operatorname{coker} \psi \to S^{2p-2}M$$

by

$$f(h_1 \otimes x) = S^{2p-2}x$$
$$f(h^I \otimes x) = 0, \quad \text{for} \quad h^I \neq h_1 .$$

It is easy to see that f is well defined.

Let N be the sub-A-module of coker ψ spanned by the elements of degree strictly less than $2(p-1)p+k$. Then by (5.6),

$$\operatorname{coker} \psi \approx N \oplus (\operatorname{coker} \psi / N) .$$

Let $M^1 = S^{2p-2}M \oplus (\operatorname{coker} \psi / N)$, and let

$$(f \oplus \lambda)\colon \operatorname{coker} \psi \to M^1$$

be the map where f is as above and λ is the natural projection. Then $g = \beta(f \oplus \lambda)$ is a map

$$g\colon \operatorname{coker} \psi \to U(M^1) .$$

We claim that (i): g is one-one; and (ii): g is subjective in degrees $< 2(p-1)p + k$. Note that M quasi-free (as an A-module) implies that $U(M)$ and coker ψ are quasi-free also. Consider

$$\begin{array}{ccccc} \operatorname{coker} \psi & \xrightarrow{g} & U(M^1) & \xrightarrow{\varepsilon} & M^1 \\ \approx\downarrow & & & & \downarrow\approx \\ N \oplus (\operatorname{coker} \psi / N) & & & & S^{2p-2}M \oplus (\operatorname{coker} \psi / N) . \end{array}$$

Suppose that x_1 is in N and x_2 is in $(\operatorname{coker} \psi / N)$ and that $g(x_1 \oplus x_2) = 0$. Then the projection of $\varepsilon g(x_1 \oplus x_2)$ on coker ψ/N is x_2, so $x_2 = 0$. N is spanned by elements of the form $h^I \otimes x$ which have degree $< 2(p-1)p+k$. Thus N is spanned by elements $h_1^n \otimes x$, where $0 < n < p$. Then

$$g(h_1^n \otimes x) = (n h_1^{n-1} \otimes S^{2p-2}x) + h_1^n \otimes \varepsilon g(1 \otimes x) .$$

Statements (i) and (ii) now follow readily by induction on n.

We next construct an acyclic resolution of M by models

$$0 \to M \xrightarrow{\partial_{-1}} U(M) \xrightarrow{\partial_0} U(M^1) \xrightarrow{\partial_1} U(M^2) \longrightarrow \cdots$$

as follows. The map $\partial_{-1}=\psi$, the coaction map for M. The map ∂_0 is the composite

$$U(M) \xrightarrow{\lambda} \operatorname{coker}\psi \xrightarrow{g} U(M^1)$$

where λ is the natural quotient map and g is defined above. Let $M^2=\operatorname{coker}\partial_0$, and for $i>2$, let $M^i=U^{i-2}(M^2)$. For $i\geqq 1$, the ∂_i are the maps in the cobar resolution for M^2. Then by (5.4),

$$\operatorname{Ext}_{\mathscr{U}}^{s,t}(M)\approx(\ker d^s/\operatorname{im} d^{s-1})_t$$

of the sequence

$$M \xrightarrow{d^0} M^1 \xrightarrow{d^1} M^2 \xrightarrow{d^2} \cdots .$$

By construction, $M_t^1=0$, for $t<2(p-1)+k$ and $M_t^2=0$, for $t<2(p-1)+k$. Therefore

$$\operatorname{Ext}_{\mathscr{U}}^{1,t}(M)=0, \quad \text{for} \quad t<2(p-1)+k,$$
$$\operatorname{Ext}_{\mathscr{U}}^{2,t}(M)=0, \quad \text{for} \quad t<2(p-1)p+k.$$

For $s\geqq 3$, we have isomorphisms

$$\operatorname{Ext}_{\mathscr{U}}^{s,t}(M)\approx\operatorname{Ext}_{\mathscr{U}}^{s-2,t}(M^2).$$

The statement of (5.7) follows by induction. Q.E.D.

Corollary 5.8. *For* $n\geqq p-1$,

$$E_2^{2s,t}(S^{2n+1})=0, \quad \textit{for} \quad t<2(p-1)ps+2n+1,$$
$$E_2^{2s+1,t}(S^{2n+1})=0, \quad \textit{for} \quad t<2(p-1)(ps+1)+2n+1.$$

Definition 5.9. For each pair of positive integers (m, k) with $m<p$, let $M_m(k)$ be the free A-module with generators

$$\{x_{2m+1+2(p-1)i}\}, \quad 0\leqq i<k.$$

The Γ-coaction on $M_m(k)$ is given by formula (3.4). (The coaction is defined because Γ is sparse, i.e. $\Gamma_t=0$ if $t\not\equiv 0 \bmod (2p-2)$.)

Corollary 5.10.

$$\operatorname{Ext}_{\mathscr{U}}^{2s+1,t}(M_m(k))=0, \quad \textit{for} \quad t<(2p-2)(ps+k)+2m+1,$$
$$\operatorname{Ext}_{\mathscr{U}}^{2s+2,t}(M_m(k))=0, \quad \textit{for} \quad t<2(p-1)(ps+k+1)+2m+1.$$

Proof. The short exact sequence of unstable Γ-comodules

$$0\to M_m(k)\to M_m(k+1)\to A[2m+1+2(p-1)k]\to 0$$

induces a long exact sequence of Ext groups. Then (5.10) follows by induction, using (3.3) and (5.8). Q.E.D.

Corollary 5.11. *Let $n \leqq p-1$. Then for each $s \geqq 0$,*

$$E_2^{2s+1,t}(S^{2n+1})=0, \quad \text{for} \quad t<2(p-1)(ps+1)+2n+1,$$
$$E_2^{2s+2,t}(S^{2n+1})=0, \quad \text{for} \quad t<2(p-1)(ps+2)+2n+1.$$

Proof. This is the statement of (5.10) with $k=1$. Q.E.D.

Remarks 5.12. (i) In [7], spaces $B_m(k)$ are constructed, with

$$PH_*(B_m(k); BP) \approx M_m(k).$$

(In [7], $B_m(k)$ is denoted $B_m^k(p)$.) As a consequence of [7, (3.5)], the unstable Adams-Novikov spectral sequence for $SU(n)$ breaks up into a direct sum of $(p-1)$ spectral sequences, with

$$E_2(SU(n)) \approx \bigoplus_{1 \leqq m < p} \operatorname{Ext}_{\mathscr{U}}(M_m(k(n, m)))$$

where $k(n, m)=\left[\dfrac{n-m-1}{p}\right]+1$.

(ii) The vanishing line (5.11) cannot be improved on S^3 as $\beta_1^r \alpha_1^2 \neq 0$ in $E_2(S^3)$ for all r. (β_1 and α_1 denote the classes in E_2 which represent the corresponding homotopy elements.) This is not the best possible for higher spheres; see Theorem 5.16, below.

Corollary 5.13. *For $t-s<2(p-1)(p(p-1)+k)+2m-1$,*

$$\operatorname{Ext}_{\mathscr{U}}^{s,t}(M_m(k))=E_\infty^{s,t}(B_m(k))$$

and there are no extensions if $t-s<2(p-1)(p(p-1)+k-1)+2m$.

Proof. Sparseness. Q.E.D.

Example 5.14. If $n<p-1$, and $q<2p(p-1)+2n+1$,

$$\pi_q(SU(n)) \approx E_2(SU(n))$$

and all non-zero elements occur in filtrations 0, 1 and 2. Compare with [12; (5.7.4)].

Remarks 5.15. The above applies to any space X such that $H_*(X; BP)$ is free as an A-module and cofree as a coalgebra, and such that $PH_*(X; BP)$ as an unstable Γ-comodule is isomorphic to a direct sum of $M_m(k)$'s. In particular, for p an odd prime, by [7; (4.1)], these methods apply to $Spin(n)$ and to $Sp(n)$.

We leave the details to the reader.

To illustrate the method (5.7) of resolving unstable Γ-comodules, we show the following (see [9; (7.1)]).

Theorem 5.16. *Let a prime p be fixed. For q in the range $2n+1<q<2p^2+2n-4$, the p-localized group $\pi_q(S^{2n+1})$ is zero except in the following cases:*

(i) *For $0<k<p$,*

$$\pi_{(2p-2)k+2n}(S^{2n+1})\approx \boldsymbol{Z}_p .$$

Also (the case when $k=p$),

$$\begin{aligned}\pi_{(2p-2)p+2n}(S^{2n+1})&\approx \boldsymbol{Z}_p, && \text{for} \quad n=1,\\ &\approx \boldsymbol{Z}_{p^2}, && \text{for} \quad n>1.\end{aligned}$$

(ii) *For $n<k<p$*

$$\pi_{(2p-2)k+2n-1}(S^{2n+1})\approx \boldsymbol{Z}_p .$$

Also (the case when $k=p$),

$$\begin{aligned}\pi_{(2p-2)p+2n-1}(S^{2n+1})&\approx \boldsymbol{Z}_p, && \text{for} \quad n=1, \text{ or } n\geqq p,\\ &\approx \boldsymbol{Z}_{p^2}, && \text{for} \quad 1<n<p.\end{aligned}$$

Proof. We prove (5.16) for $n<p$. The proof for $n\geqq p$ is similar. For the range $2n+1<t-s<2p^2+2n-4$, by (5.11), we have $E_2^{s,t}(S^{2n+1})=0$ for $s>2$. Hence in this range,

$$\begin{aligned}\pi_q(S^{2n+1})&\approx E_2^{1,q+1}(S^{2n+1}), && \text{for} \quad q \text{ even},\\ &\approx E_2^{2,q+2}(S^{2n+1}), && \text{for} \quad q \text{ odd}.\end{aligned}$$

We shall calculate $E_2^{*,*}(S^{2n+1})$ in this range by an acyclic resolution of $A[2n+1]$ as follows. Let M^0 be the free A-module with one generator x of degree $2n+1$. Let M^1 be the free A-module with one generator y of degree $(2p-2)+2n+1$. Let M^2 be the A-module generated by elements $z_1, z_2,\dots, z_{p-n}$, where degree$(z_i)=(2p-2)(n+i)+2n+1$, modulo the ideal of relations:

$$pz_1$$

$$(n+k)v_1z_k-p(n+k+1)z_{k+1}.$$

We note that these relations imply that in M^2,

$$p^iz_i=0, \qquad \text{for} \quad 1\leqq i<p-n,$$

$$p^{p-n+1}z_{p-n}=0.$$

Let M^3 be the sub-A-module of $\Gamma\otimes_A M^2$ generated by the elements $\{h_1^k\otimes z_i\}$, where $1\leqq i\leqq p-n$ and $k>0$, and with relations induced from those in M^2. Let

$$\partial_{-1}\colon A[2n+1]\to M^0$$

be the map with $\partial_{-1}(\iota_{2n+1})=1\otimes x$. By [2, § 8], in the range of dimensions under consideration, $U(M^0)$ has an A-basis consisting of the elements:

$$\begin{aligned} &h_1^k\otimes x, && \text{for } 0\leqq k\leqq n, \\ &h_1^k v_1\otimes x, && \text{for } n\leqq k<p-1. \end{aligned}$$

Let $\partial_0\colon U(M^0)\to U(M^1)$ be the A-linear map defined by

$$\begin{aligned} &\partial_0(h_1^k\otimes x)=kh_1^{k-1}\otimes y, && \text{for } 0\leqq k\leqq n, \\ &\partial_0(h_1^k v_1\otimes x)=kv_1h_1^{k-1}\otimes y-p(k+1)h_1^k\otimes y, && \text{for } n\leqq k<p-1 \end{aligned}$$

Let $\partial_1\colon U(M^1)\to U(M^2)$ be the A-linear map defined by

$$\begin{aligned} &\partial_1(h_1^k\otimes y)=0, && \text{for } 0\leqq k<n, \\ &\partial_1(h_1^k\otimes y)=\sum_{i=0}^{k-n}\binom{k}{i}h_1^i\otimes z_{k-n-i+1}, && \text{for } n\leqq k<p-1. \end{aligned}$$

Finally, let $\partial_2\colon U(M^2)\to U(M^3)$ be the composite

$$U(M^2)\xrightarrow{\lambda}U(M^2)/\operatorname{Im}\partial_1\approx M^3\xrightarrow{\psi}U(M^3)$$

where λ is the natural quotient map and ψ is the coaction map for M^3.

It is readily verified that

$$0\to A[2n+1]\xrightarrow{\partial_{-1}}U(M^0)\xrightarrow{\partial_0}U(M^1)\xrightarrow{\partial_1}U(M^2)\xrightarrow{\partial_2}U(M^3)$$

is an acyclic resolution of $A[2n+1]$ in the range of dimensions under consideration. Thus, by (5.4), in the range $t-s<2p^2+2n-4$, $E_2^{s,t}(S^{2n+1})$ may be calculated as the homology of the complex

$$M^0\xrightarrow{d^0}M^1\xrightarrow{d^1}M^2\xrightarrow{d^2}M^3.$$

The maps d^i are given in (5.3), which in this case become:

$$d^0(v_1^kx)=pkv_1^{k-1}x$$

$$d^1(v_1^ky)=\begin{cases} 0, & \text{if } n>1, \\ 0, & \text{if } n=1,\ k<p-1, \\ p^{p-1}z_{p-1}, & \text{if } n=1,\ k=p-1. \end{cases}$$

Furthermore, the kernel of d^2 is generated by the elements $\{v_1^m p^{k-1} z_k\}$ for $m \geqq 0$, $k \geqq 1$. The relations in M^3 imply that

$$v_1^k z_1 = p^k z_{k+1}, \qquad \text{for} \quad k < p-n-1,$$
$$v_1^{p-n-1} z_1 = p^{p-n} z_{p-n}$$

(to within a factor of a unit in $\boldsymbol{Z}_{(p)}$). Thus in the range $t < 2p^2 + 2n - 3$, $E_2^{1,t}(S^{2n+1})$ is generated by the classes $\{v_1^k y\}$ (except when $n=1$, and $k=p-1$, in which case $pv_1^{p-1} y$ is the generator). These classes have the orders asserted in (5.14, (i)). Also, in the range $t < 2p^2 + 2n - 2$, $E_2^{2,t}(S^{2n+1})$ is generated by the classes $\{v_1^k z_1\}$ for $0 \leqq k < p-n-1$, and by $v_1^{p-n-1} z_1/p$. These classes have the orders asserted in (5.16), (ii). Q. E. D.

Remarks 5.17. In the range $q < 2p^2 - 5$, the double suspension homomorphism

$$\pi_{2n+1+q}(S^{2n+1}) \rightarrow \pi_{2n+3+q}(S^{2n+3})$$

is the multiplication by p except in the following cases, when the double suspension sends a generator to a generator.

(1) $q = (2p-2)k - 1$ and $0 < k < p$, all n,
$q = (2p-2)p - 1$ and $n > 1$.

(2) $q = (2p-2)k - 2$, and $n \geqq p-1$.

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Note added in proof: In order to compute the matrix C in Section 4 it is convenient to use the following observation: For a polynomial $A(Y)=Y+a_2Y^2+\cdots$ let $[A(Y)]$ be the matrix with the entries in the j-th column given by the coefficients of $(A(Y))^j$. Then $[B(Y)]$ $[A(Y)]=[AB(Y)]$.