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Quasi-Invariance of Measures on an Infinite Dimensional Vector Space and the Continuity of the Characteristic Functions

By

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Introduction

Since the Bochner theorem was extended to the infinite dimensional case by Minlos [1] and Sazonov [2], the continuity of a characteristic function has been discussed mainly in connection with the carrier of the corresponding measure.

However, the study of the relation between the continuity of a characteristic function and the quasi-invariance of the corresponding measure has been rather neglected. In this paper we shall discuss this problem. Our main results are as follows.

Let E be a vector space, E' be its algebraical dual space, μ be a finite measure on E', and χ be the characteristic function of μ defined on E. Consider the weakest vector topology on E that makes χ continuous, and denote it with τ_μ.

Let T_μ be the set of all translations on E' under which μ is quasi-invariant. T_μ is regarded as a subset of E' by identifying any translation $x \to x+a$ on E' with a. Then we have the following

Theorem. (1) *We have $T_\mu \subset E^*_\mu$ $(\subset E')$, where E^*_μ is the topological dual space of E with respect to τ_μ.*

(2) *Let E^*_τ be the topological dual space of E with respect to a locally convex topology τ on E. Then, $E^*_\tau \subset T_\mu$ $(\subset E')$ implies that τ is weaker than τ_μ.*

Especially, if μ is E^*_τ-quasi-invariant and if χ is continuous in τ, then we have $T_\mu = E^*_\tau$ and $\tau = \tau_\mu$.

Using these results, we can estimate T_μ from the continuity of χ.

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Combining this with the relation between the continuity of χ and the carrier of μ, we can establish the relation between the carrier and quasi-invariance of μ. This fact implies various results. For instance, we can give another proof of the following results due to Xia Dao-Xing [3].

1. *Let X be a separable and metrizable locally convex vector space, and $Y(\subset X)$ be a complete metrizable topological vector space imbedded continuously in X. If some Borel measure on X is Y-quasi-invariant, then Y has a neighbourhood of* 0 *which is totally bounded in X.* (§5.)

2. *Let X be a Hilbert space, and $Y(\subset X)$ be another Hilbert space imbedded continuously in X. Then, there exists a Y-quasi-invariant measure on X if and only if the canonical imbedding $Y\rightarrow X$ is of the Hilbert-Schmidt type. Especially, Y must be separable even if X is not.* (§6.)

§1. Linearization of a Topology

In this section, as a preliminary discussion, we shall explain a method to linearize a topology on a vector space.

Proposition 1.1. *Let E be a vector space, and suppose that E becomes a topological additive group with respect to a topology τ on E. Furthermore, assume that τ satisfies the following condition;*

$$\forall x\in E,\quad \forall V\in\mathfrak{V},\quad \exists\alpha_0>0,\quad |\alpha|\leqq\alpha_0\Longrightarrow \alpha x\in V, \tag{1.1}$$

where $\mathfrak{V}$ is a fundamental system of neighbourhoods of 0 *with respect to τ.*

Then, putting

$$U_V=\bigcap_{|\alpha|\geqq 1}(\alpha V), \tag{1.2}$$

we obtain a topology τ' in which $\mathfrak{U}=\{U_V\}_{V\in\mathfrak{V}}$ is a fundamental system of neighbourhoods of 0*. The topology τ' is the weakest topology that is stronger than τ and compatible with the linear structure of E.*

Definition 1.1. The topology τ' given in Proposition 1.1 is called the *linearization* of τ.

Proof of Proposition 1.1. Suppose that a topology τ'' on E is stronger than τ and compatible with the linear structure of E. Then, for each $V\in\mathfrak{V}$, there exist $\alpha_0>0$ and a neighbourhood U of 0 in τ'' such that $|\alpha|\leqq\alpha_0$ implies $\alpha U\subset V$. Therefore we have $\alpha_0 U\subset U_V$. From the continuity of scalar multiplication,

$\alpha_0 U$ is a neighbourhood of 0 in τ'', so that τ'' is stronger than τ'.

The topology τ' is evidently stronger than τ. Next, we shall show that τ' is compatible with the linear structure of E.

Since the addition in E is continuous in τ, for each $V \in \mathfrak{B}$ there exists $V' \in \mathfrak{B}$ such that $V' + V' \subset V$. This implies $U_{V'} + U_{V'} \subset U_V$. Therefore, if we define a fundamental system of neighbourhoods of $x \in E$ by $\{x + U_V\}_{V \in \mathfrak{B}}$ (actually the topology τ' is defined in this way), the addition in E becomes continuous, so that E is a topological additive group with respect to τ'.

In order to prove the continuity of the scalar multiplication, it is sufficient to show the following three facts:

1) $\forall V \in \mathfrak{B}, \quad \exists \alpha_0 > 0, \quad \exists V_0 \in \mathfrak{B}, \quad |\alpha| \leqq \alpha_0 \Longrightarrow \alpha U_{V_0} \subset U_V,$

2) $\forall \alpha \neq 0, \quad \forall V \in \mathfrak{B}, \quad \exists V_0 \in \mathfrak{B}, \quad \alpha U_{V_0} \subset U_V,$

3) $\forall x \in E, \quad \forall V \in \mathfrak{B}, \quad \exists \alpha_0 > 0, \quad |\alpha| \leqq \alpha_0 \Longrightarrow \alpha x \in U_V.$

From (1.2), we have $\alpha U_V = \bigcap_{|\beta| \geqq 1} (\alpha \beta V) = \bigcap_{|\beta| \geqq |\alpha|} (\beta V)$, so that $|\alpha| \leqq 1$ implies $\alpha U_V \subset U_V$. Therefore the condition 1) is satisfied choosing $V_0 = V$ and $\alpha_0 = 1$.

Next, suppose that a real number $\alpha \neq 0$ is given, and choose an integer n such that $|\alpha| < n$. Since the addition in E is continuous in τ, for each $V \in \mathfrak{B}$, there exists $V_n \in \mathfrak{B}$ such that

$$\underbrace{V_n + V_n + \cdots + V_n}_{\text{n terms}} \subset V.$$

Especially we have $n V_n \subset V$. Therefore we have

$$U_V = \bigcap_{|\beta| \geqq 1} (\beta V) \supset \bigcap_{|\beta| \geqq 1} (n \beta V_n) = \bigcap_{|\beta| \geqq n} (\beta V_n) \supset \bigcap_{|\beta| \geqq |\alpha|} (\beta V_n) = \alpha U_{V_n}.$$

This shows that the condition 2) is satisfied.

Lastly, the assumption (1.1) is equivalent with

$$\forall x \in E, \quad \forall V \in \mathfrak{B}, \quad \exists \alpha_0 > 0, \quad \alpha_0 x \in U_V.$$

Then, $|\alpha| \leqq \alpha_0$ implies $\alpha x = \alpha \alpha_0^{-1} (\alpha_0 x) \in \alpha \alpha_0^{-1} U_V \subset U_V$. Thus the fact 3) has been proved. Q. E. D.

§2. The Characteristic Topology

Let E be a vector space, E' be its algebraical dual space, and $\mathfrak{B}_E$ be the smallest σ-algebra of E' in which every element of E, regarded as a function on E', becomes measurable. For a probability measure μ on $(E', \mathfrak{B}_E)$, we shall define the characteristic topology on E as follows.

Let $\chi(\xi)$ be the characteristic function of μ.

Proposition 2.1. *There exists the weakest topology on E that makes $\chi(\xi)$ continuous and compatible with the linear structure of E.*

Proof. Since χ is a positive definite function, we have the inequality:

$$|\chi(\xi+\eta)-\chi(\xi)|\leqq\sqrt{2|1-\chi(\eta)|}\,. \tag{2.1}$$

Therefore, if we put

$$V_\varepsilon=\{\xi\in E;\ |1-\chi(\xi)|\leqq\varepsilon\}\,, \tag{2.2}$$

then $\xi\in V_\varepsilon$ and $\eta\in V_\varepsilon$ imply $\xi+\eta\in V_{\varepsilon+\sqrt{2\varepsilon}}$. In other words we have

$$V_\varepsilon+V_\varepsilon\subset V_{\varepsilon+\sqrt{2\varepsilon}}\,. \tag{2.3}$$

Thus, if we define a fundamental system of neighbourhoods of $\xi\in E$ by $\{\xi+V_\varepsilon\}_{\varepsilon>0}$, E becomes a topological group (not necessarily Hausdorff). Furthermore, on every finite dimensional subspace of E, χ is continuous with respect to the Euclid topology, so that we have

$$\forall\xi\in E,\quad \forall\varepsilon>0,\quad \exists\alpha_0>0,\quad |\alpha|\leqq\alpha_0 \Longrightarrow \alpha\xi\in V_\varepsilon\,. \tag{2.4}$$

Consequently, from Proposition 1.1, we can consider the linearization of this topology. The linearized topology is evidently the requested one in our Proposition. Q.E.D.

Definition 2.1. The topology mentioned in Proposition 2.1 is called the *characteristic topology* of μ and denoted with τ_μ.

In the topology τ_μ, a fundamental system of neighbourhoods of 0 is given by $\{U_\varepsilon\}_{\varepsilon>0}$, where

$$U_\varepsilon=\bigcap_{|\alpha|\geqq 1}(\alpha V_\varepsilon)=\{\xi\in E;\ \sup_{|\alpha|\leqq 1}|1-\chi(\alpha\xi)|\leqq\varepsilon\}\,. \tag{2.5}$$

The topology τ_μ is not necessarily Hausdorff. The condition $\xi\in\bigcap_{\varepsilon>0}U_\varepsilon$ is equivalent with $\chi(\alpha\xi)=1$ for $|\alpha|\leqq 1$. Therefore, from (2.1) we have $\chi(\alpha\xi)=1$ for every $\alpha\in\boldsymbol{R}^1$, so that we have $(\xi, x)=0$ for μ-almost all x. In fact, τ_μ is a Hausdorff topology on the factor space E/M, where

$$M=\{\xi\in E;\ (\xi, x)=0 \text{ for } \mu\text{-almost all } x\}\,. \tag{2.6}$$

Suppose that E becomes a topological vector space with respect to a topology τ. Then from the definition of τ_μ, we have

Proposition 2.2. *$\chi(\xi)$ is continuous in τ, if and only if τ is stronger than τ_μ.*

§3. The Topology of Measure Convergence

In this section, we shall prove that the characteristic topology is identical with the topology of measure convergence. First, we shall explain the latter.

Let μ be a probability measure on a measurable space $(X, \mathfrak{B})$, and $\mathfrak{F}$ be the set of all $\mathfrak{B}$-measurable real-valued functions on X. $\mathfrak{F}$ forms a vector space.

Definition 3.1. A sequence $\{f_n\} \subset \mathfrak{F}$ is said to *converge to* 0 *in measure*, if

(3.1) $$\forall \alpha > 0, \quad \mu(\{x \in X; |f_n(x)| \geqq \alpha\}) \to 0 \qquad (n \to \infty).$$

Definition 3.2. Consider a topology on $\mathfrak{F}$ whose fundamental system of neighbourhoods of 0 is given by $\{U_{\alpha,\varepsilon}\}_{\alpha>0,\varepsilon>0}$, where

(3.2) $$U_{\alpha,\varepsilon} = \{f \in \mathfrak{F}; \mu(\{x \in X; |f(x)| \geqq \alpha\}) < \varepsilon\}.$$

This topology is called the *topology of measure convergence.*

A sequence $\{f_n\} \subset \mathfrak{F}$ converges to 0 in measure, if and only if it converges to 0 in the topology of measure convergence.

$U_{\alpha,\varepsilon}$ is increasing with respect to both α and ε, so that $\{U_{1/n,1/n}\}_{n=1,2,\dots}$ becomes a fundamental system of neighbourhoods of 0.

$\mathfrak{F}$ is a topological vector space with respect to the topology of measure convergence. First, $|f(x)+g(x)| \geqq \alpha+\beta$ implies $|f(x)| \geqq \alpha$ or $|g(x)| \geqq \beta$, so that we have $U_{\alpha,\varepsilon} + U_{\beta,\delta} \subset U_{\alpha+\beta,\varepsilon+\delta}$, hence the addition in $\mathfrak{F}$ is continuous. Next, $|f(x)| \geqq \alpha$ is equivalent with $|cf(x)| \geqq |c|\alpha$, so that we have $cU_{\alpha,\varepsilon} = U_{|c|\alpha.\varepsilon}$. Combining this with the fact:

$$\forall f \in \mathfrak{F}, \quad \forall \varepsilon > 0, \quad \exists \alpha > 0, \quad f \in U_{\alpha,\varepsilon},$$

we see that the scalar multiplication in $\mathfrak{F}$ is also continuous.

The topology of measure convergence is not necessarily Hausdorff. The condition $f \in \bigcap_{\alpha>0,\varepsilon>0} U_{\alpha,\varepsilon}$ is equivalent with $f(x)=0$ for μ-almost all x. This means that the topology of measure convergence is Hausdorff on the factor space $\mathfrak{F}/\mathfrak{M}$, where

(3.3) $$\mathfrak{M} = \{f \in \mathfrak{F}; f(x)=0 \text{ for } \mu\text{-almost all } x\}.$$

Here, we shall give another representation of the topology of measure convergence.

Let $F(t)$ be a bounded continuous function on $[0, \infty)$. For $f \in \mathfrak{F}$, we shall put

$$I_F(f) = \int_X F(|f(x)|) d\mu(x). \tag{3.4}$$

Lemma. 1) *$I_F(f)$ is continuous at $f=0$ in the topology of measure convergence.*

2) *If we assume that*

$$\forall \alpha > 0, \quad \inf_{t \geqq \alpha} F(t) > F(0), \tag{3.5}$$

then $I_F(f_n) \to I_F(0)$ implies that $\{f_n\}$ converges to 0 in measure.

Proof. First, we shall prove 1). Since we have

$$|I_F(f) - I_F(0)| \leqq \int_X |F(|f(x)|) - F(0)| d\mu(x), \tag{3.6}$$

we can estimate as follows:

$$\begin{aligned} &|I_F(f) - I_F(0)| \\ &\quad \leqq \sup_{t \leqq \delta} |F(t) - F(0)| + \sup_{t \geqq \delta} |F(t) - F(0)| \cdot \mu(\{x \in X;\ |f(x)| \geqq \delta\}). \end{aligned} \tag{3.7}$$

Since $F(t)$ is continuous, for any given $\varepsilon > 0$, there exists $\delta > 0$ such that the first term in (3.7) becomes smaller than ε. Putting $\sup_{t \geqq 0} |F(t)| = M$, we see that for every $f \in U_{\delta, \varepsilon/2M}$, the second term in (3.7) becomes smaller than ε. Thus, $f \in U_{\delta, \varepsilon/2M}$ implies $|I_F(f) - I_F(0)| < 2\varepsilon$, so that 1) has been proved.

Next, we shall prove 2). From the assumption (3.5), we have $F(t) \geqq F(0)$, so that we get

$$\begin{aligned} &I_F(f) - I_F(0) \\ &\quad \geqq \int_{|f(x)| \geqq \alpha} (F(|f(x)|) - F(0)) d\mu(x) \\ &\quad \geqq (\inf_{t \geqq \alpha} F(t) - F(0)) \cdot \mu(\{x \in X;\ |f(x)| \geqq \alpha\}). \end{aligned} \tag{3.8}$$

Therefore, for any given $\varepsilon > 0$ and $\alpha > 0$, $f \in U_{\alpha, \varepsilon}$ can be derived from

$$I_F(f) - I_F(0) < \varepsilon(\inf_{t \geqq \alpha} F(t) - F(0)),$$

so that 2) has been proved. Q.E.D.

Thus, if a bounded continuous function $F(t)$ on $[0, \infty)$ satisfies (3.5), $\{f_n\}$

converges to 0 in measure if and only if $I_F(f_n)$ converges to $I_F(0)$. In other words, a fundamental system of neighbourhoods of 0 in the topology of measure convergence is given by

$$W_\varepsilon = \{f\,;\, I_F(f) < I_F(0) + \varepsilon\}\,. \tag{3.9}$$

Note that whenever $F(t)$ is monotonically increasing, $F(t)$ satisfies (3.5). Furthermore, if we assume that $F(t)$ is a convex function with $F(0)=0$, then $d(f, g) = I_F(f-g)$ becomes a metric on $\mathfrak{F}$, and the topology of measure convergence can be defined by the metric d. (Accurately, d is a pseudo-metric on $\mathfrak{F}$, and becomes a metric on $\mathfrak{F}/\mathfrak{M}$, where $\mathfrak{M}$ is given by (3.3).)

Especially, choosing $F(t) = t/(1+t)$, we obtain the metric d given as follows:

$$d(f, g) = \int_X \frac{|f(x)-g(x)|}{1+|f(x)-g(x)|}\, d\mu(x)\,. \tag{3.10}$$

Proposition 3.1. *Let μ and ν be probability measures on $(X, \mathfrak{B})$. If $\nu \lesssim \mu$ (ν is absolutely continuous with respect to μ), then the topology of measure convergence corresponding to μ is stronger than that corresponding to ν.*

Corollary. *If $\mu \sim \nu$, then two topologies of measure convergence are identical with each other.*

Proof. Since a neighbourhood of 0 in the topology of measure convergence is given by (3.2), it is sufficient to show that

$$\forall \varepsilon > 0\,, \quad \exists \delta > 0\,, \quad \mu(B) < \delta \Longrightarrow \nu(B) < \varepsilon\,. \tag{3.11}$$

We shall prove the contraposition. Assume that

$$\exists \varepsilon > 0\,, \quad \forall n\,, \quad \exists B_n \in \mathfrak{B}\,, \quad \mu(B_n) < \frac{1}{2^n} \quad \text{and} \quad \nu(B_n) \geqq \varepsilon\,.$$

Then, putting $B = \overline{\lim}\, B_n \; (= \bigcap_{k=1}^{\infty} \bigcup_{n=k}^{\infty} B_n)$, we get $\mu(B) = 0$ and $\nu(B) \geqq \varepsilon$. This contradicts with $\nu \lesssim \mu$. Q. E. D.

Now, we shall return to the case of $(E', \mathfrak{B}_E)$. Every element of E, regarded as a function on E', is $\mathfrak{B}_E$-measurable, so that we can imbed E into $\mathfrak{F}$, where $\mathfrak{F}$ is the set of all $\mathfrak{B}_E$-measurable real-valued functions on E'.

Proposition 3.2. *For a probability measure on $(E', \mathfrak{B}_E)$, the characteristic topology on E is identical with the restriction of the topology of measure convergence on E.*

Proof. Since we have

$$|1-\chi(\xi)| \leqq \int_{E'} |1-\exp(i(\xi, x))| d\mu(x)$$
$$\leqq \int_{E'} |1-\cos(\xi, x)| d\mu(x)$$
$$+ \int_{E'} |\sin(\xi, x)| d\mu(x),$$

putting $F_1(t) = 1-\cos t$ and $F_2(t) = |\sin t|$, we have

$$|1-\chi(\xi)| \leqq I_{F_1}(\xi) + I_{F_2}(\xi).$$

Therefore from Lemma 1), $\chi(\xi)$ is continuous with respect to the topology of measure convergence. In other words, the topology of measure convergence is stronger than the characteristic topology on E.

Conversely, if ξ belongs to U_ε in (2.5), integrating $1-\chi(\alpha\xi)$ with respect to α, we get

$$\varepsilon > \left| \frac{1}{2} \int_{-1}^{1} (1-\chi(\alpha\xi))\, d\alpha \right| \tag{3.12}$$
$$= \int_{E'} \left(1 - \frac{\sin(\xi, x)}{(\xi, x)}\right) d\mu(x).$$

However, $F(t) = 1-\sin t/t$ is bounded and continuous on $[0, \infty)$ and satisfies the assumption (3.5), so that from Lemma 2), we see that the characteristic topology is stronger than the topology of measure convergence. Q.E.D.

Corollary. *Let μ and ν be probability measures on $(E', \mathfrak{B}_E)$. If $\nu \lesssim \mu$, then the characteristic topology of μ is stronger than that of ν. Especially if $\nu \sim \mu$, then two characteristic topologies are identical with each other.*

§4. Proof of our Main Theorems

For a probability measure μ on $(E', \mathfrak{B}_E)$, let E_μ^* be the topological dual space of E with respect to the characteristic topology τ_μ. (Though τ_μ is not necessarily Hausdorff nor locally convex, we can define E_μ^* as the set of all continuous linear functions in the topology τ_μ.)

For an element x of E', the translated measure μ_x is defined by

$$\mu_x(B) = \mu(B-x), \quad \forall B \in \mathfrak{B}_E. \tag{4.1}$$

Let T_μ be the set of all translations on E' under which μ is quasi-invariant, namely put

(4.2) $$T_\mu=\{x\in E';\ \mu_x\sim\mu\}.$$

T_μ forms an additive group, but it is not a vector space in general.

Theorem 4.1. *We have $T_\mu\subset E_\mu^*$.*

Proof. Suppose that $x\in T_\mu$. Since $\mu_x\sim\mu$, the characteristic topology of μ_x is identical with that of μ. If we denote the characteristic function of μ with $\chi(\xi)$, then the characteristic function of μ_x is given by $\exp(i(\xi, x))\chi(\xi)$. Therefore, in the characteristic topology τ_μ of μ, the function $\exp(i(\xi, x))\chi(\xi)$ must be continuous. Since $\chi(\xi)$ is continuous in τ_μ, $\exp(i(\xi, x))$ must be continuous in τ_μ at every ξ such that $\chi(\xi)\neq 0$, especially at $\xi=0$. Namely, for some neighbourhood U_ε in (2.5), we have

(4.3) $$\xi\in U_\varepsilon \Longrightarrow |1-\exp(i(\xi, x))|<1.$$

However, $\xi\in U_\varepsilon$ implies $\alpha\xi\in U_\varepsilon$ for $|\alpha|\leqq 1$, so we have $|1-\exp(i\alpha(\xi, x))|<1$ for $|\alpha|\leqq 1$, hence we have $|(\xi, x)|<\pi/3$.

Thus, $|(\xi, x)|$ is bounded on U_ε. This means that x is continuous in τ_μ, namely $x\in E_\mu^*$.

This holds for every $x\in T_\mu$, so we have $T_\mu\subset E_\mu^*$. Q.E.D.

Corollary. *If E is a topological vector space in some topology τ, and if $\chi(\xi)$ is continuous in τ, then we have $T_\mu\subset E_\tau^*$, where E_τ^* is the topological dual space of E with respect to τ.*

Proof. Since χ is continuous in τ, τ_μ is weaker than τ, so that we have $E_\mu^*\subset E_\tau^*$. Therefore, we get the corollary from Theorem 4.1. Q.E.D.

This corollary enables us to tell about the quasi-invariance of a measure in terms of the continuity of the characteristic function. For instance, for a measure on $\boldsymbol{R}^\infty$, if the characteristic function is continuous in the (l^2)-norm on $\boldsymbol{R}_0^\infty$, then we must have $T_\mu\subset(l^2)$.

For a subset A of E, we shall put

(4.4) $$\|x\|_A=\sup_{\xi\in A}|(\xi, x)|,\quad \forall x\in E'.$$

Similarly for a subset A' of E', we shall put

(4.5) $$\|\xi\|_{A'}=\sup_{x\in A'}|(\xi, x)|,\quad \forall\xi\in E.$$

These values may be ∞. The set of all x such that $\|x\|_A<\infty$ forms a linear subspace of E', and $\|\cdot\|_A$ becomes a semi-norm on this space. Similar arguments hold for $\|\cdot\|_{A'}$.

Theorem 4.2. *Let Y be a linear subspace of E'. Suppose that Y is a complete metrizable topological vector space with respect to a topology stronger than the weak topology on E'. If $Y \subset T_\mu$, then for some neighbourhood V of 0 in Y, $\|\xi\|_V$ is finite for all $\xi \in E$, and the semi-normed topology defined by $\|\cdot\|_V$ is weaker than the characteristic topology τ_μ.*

Proof. For a neighbourhood U_ε of 0 in the characteristic topology, let A_ε be its polar set, namely put

$$A_\varepsilon = \{x \in E';\ |(\xi, x)| \leqq 1 \quad \text{for} \quad \xi \in U_\varepsilon\}. \tag{4.6}$$

Then, A_ε is evidently a weakly closed subset of E', so that $A_\varepsilon \cap Y$ is a closed subset of Y.

Since $\{U_{1/n}\}_{n=1,2,\dots}$ is a fundamental system of neighbourhoods of 0 in τ_μ, $x \in E_\mu^*$ is equivalent with $x \in A_{1/n}$ for some n, namely we have

$$\bigcup_{n=1}^{\infty} A_{1/n} = E_\mu^* \supset T_\mu \supset Y.$$

Thus, we get $\bigcup_{n=1}^{\infty} (A_{1/n} \cap Y) = Y$. Since Y is a complete metric space, from Baire's category theorem, some $A_{1/n} \cap Y$ has an inner point in Y. Namely, we have

$$\begin{aligned} &\exists y_0 \in Y, \quad \exists V \text{ (neighbourhood of 0 in } Y), \\ &y_0 + V \subset A_{1/n} \cap Y. \end{aligned} \tag{4.7}$$

We shall show that this V is a requested one in our theorem. If $y \in V$ and $\xi \in U_{1/n}$, then we have

$$|(\xi, y)| \leqq |(\xi, y + y_0)| + |(\xi, y_0)| \leqq 2.$$

Therefore $\xi \in U_{1/n}$ implies $\|\xi\|_V \leqq 2$. Hence, $\xi \in \alpha U_{1/n}$ implies $\|\xi\|_V \leqq 2\alpha$. This shows that $\|\xi\|_V$ is finite for all $\xi \in E$ and that the topology defined by $\|\cdot\|_V$ is weaker than τ_μ. Q.E.D.

Theorem 4.3. *Suppose that E is a locally convex vector space with respect to a topology τ. If $E_\tau^* \subset T_\mu$, then τ is weaker than τ_μ.*

Proof. Let U be a convex symmetric neighbourhood of 0 in τ. Defining $\|\cdot\|_U$ on E' by (4.4), we shall put

$$E_U^* = \{x \in E';\ \|x\|_U < \infty\}. \tag{4.8}$$

Evidently we have $E_U^* \subset E_\tau^*$, so that we have $E_U^* \subset T_\mu$. On the other hand,

E_U^* is a Banach space with respect to $\|\cdot\|_U$, and its topology is stronger than the weak topology on E'. Therefore, there exists a neighbourhood V of 0 mentioned in Theorem 4.2. Since E_U^* is a normed space, we can suppose that V is the unit ball in E_U^*. Then, from Theorem 4.2, the topology defined by $\|\cdot\|_V$ is weaker than τ_μ, namely the polar set

$$B=\{\xi\in E;\ \|\xi\|_V\leqq 1\}$$

contains a neighbourhood of 0 in τ_μ. However, since V is the polar set of U, B is the bipolar set of U. Therefore, remembering that U is a convex symmetric set, we have $B=\overline{U}$ (=the closure of U). Thus, $\overline{U}$ contains a neighbourhood of 0 in τ_μ.

This fact holds for every convex symmetric neighbourhood U in τ. Since the topology of a topological group is regular, this means that τ is weaker than τ_μ. Q. E. D.

Corollary. *Suppose that E is a locally convex topological vector space with respect to a topology τ. For a probability measure μ on $(E', \mathfrak{B}_E)$, if μ is E_τ^*-quasi-invariant and if the characteristic function χ is continuous in τ, then we have $T_\mu=E_\tau^*$ and $\tau=\tau_\mu$.*

Proof. Since μ is E_τ^*-quasi-invariant, we have $T_\mu\supset E_\tau^*$ and τ is weaker than τ_μ.

Since χ is continuous in τ, we have $T_\mu\subset E_\tau^*$ and τ_μ is weaker than τ.

Q. E. D.

For instance, for a Borel measure μ on $\boldsymbol{R}^\infty$, if μ is (l^2)-quasi-invariant and if the characteristic function is continuous in the (l^2)-norm, then we have $T_\mu=(l^2)$ and the characteristic topology τ_μ is identical with the topology of (l^2).

Assume that $T_\mu=E_\mu^*$, then μ is E_μ^*-quasi-invariant and χ is continuous in τ_μ. The corollary insists that such a situation (that μ is E_τ^*-quasi-invariant and χ is continuous in τ) occurs only for the characteristic topology τ_μ. Furthermore, such a situation occurs for τ_μ if and only if $T_\mu=E_\mu^*$.

§5. Totally Boundedness

As applications of the theorems in Section 4, in this and the following sections, we shall prove the results 1 and 2 mentioned in Introduction.

First we shall prove:

Theorem 5.1. *Let X be a separable and metrizable locally convex vector space, and $Y(\subset X)$ be a complete metrizable topological vector space imbedded continuously in X. If some Borel measure on X is Y-quasi-invariant, then Y has a neighbourhood of* 0 *which is totally bounded in X.*

Proof. Let X^* be the topological dual space of X. Then, X can be imbedded into $(X^*)'$, the algebraical dual space of X^*. We can define the measurable space $((X^*)', \mathfrak{B}_{X^*})$ as in Section 2, replacing E with X^*. Denoting the Borel algebra of X with $\mathfrak{B}$, we shall first prove:

$$\mathfrak{B} = \mathfrak{B}_{X^*} \cap X. \tag{5.1}$$

For every $\xi \in X^*$, since (ξ, x) is continuous on X, it is $\mathfrak{B}$-measurable, so that we have $\mathfrak{B} \supset \mathfrak{B}_{X^*} \cap X$.

Conversely, we shall show that every open set of X belongs to $\mathfrak{B}_{X^*} \cap X$. Since X is separable and metrizable, every open set O of X satisfies Lindelöf's property (that is, an open covering of O contains necessarily a countable covering of O). Therefore, there exist a sequence of points $\{x_n\}$ and a sequence of convex symmetric neighbourhoods $\{U_n\}$ of 0 such that

$$O = \bigcup_{n=1}^{\infty} (x_n + U_n). \tag{5.2}$$

We shall show $U_n \in \mathfrak{B}_{X^*} \cap X$. We can suppose that U_n is a closed neighbourhood. Then, U_n is the bipolar set of itself, namely we have

$$U_n = \{x \in X;\ |(\xi, x)| \leqq 1 \quad \text{for every} \quad \xi \in A_n\}, \tag{5.3}$$

where $$A_n = \{\xi \in X^*;\ |(\xi, x)| \leqq 1 \quad \text{for every} \quad x \in U_n\}.$$

Thus, U_n is a weakly closed set. However, a weakly open set of X belongs to $\mathfrak{B}_{X^*} \cap X$, because from Lindelöf's property it can be written as a countable union of weak neighbourhoods, which belong evidently to $\mathfrak{B}_{X^*} \cap X$.

Hence U_n, so the given O also, belongs to $\mathfrak{B}_{X^*} \cap X$. This completes the proof of (5.1).

From (5.1), a Borel measure on X can be identified with a measure on $((X^*)', \mathfrak{B}_{X^*})$. Since Y is continuously imbedded in X, it is also imbedded in $(X^*)'$, and the topology of Y is stronger than the weak topology of $(X^*)'$. Therefore, if the measure μ is Y-quasi-invariant, then from Theorem 4.2, there exists a neighbourhood V of 0 in Y such that the semi-normed topology defined by $\|\cdot\|_V$ is weaker than the characteristic topology τ_μ. Then, the unit ball in $\|\cdot\|_V$ contains a neighbourhood of 0 in τ_μ.

On the other hand, since the measure μ lies on X, for any given $\varepsilon>0$ there exists a totally bounded set B such that $\mu(B)>1-\varepsilon$. (The Borel measure μ on X can be identified with a Borel measure on $\bar{X}$, the completion of X. Since $\bar{X}$ is complete, metrizable and separable, every Borel measure on $\bar{X}$ lies on a countable union of compact sets. Cf. Parthasarathy [4].)

Considering the semi-norm $\|\cdot\|_B$ on X^*, we have

$$\begin{aligned}|1-\chi(\xi)| &\leqq \int_X |1-\exp(i(\xi, x))|d\mu(x)\\ &\leqq \int_B |(\xi, x)|d\mu(x)+2\mu(B^{\mathrm{C}})\\ &\leqq \|\xi\|_B+2\varepsilon,\end{aligned}$$

so that we have

$$\|\xi\|_B<\varepsilon \Longrightarrow |1-\chi(\xi)|<3\varepsilon. \tag{5.4}$$

Since $\|\xi\|_B<\varepsilon$ is equivalent with $\|\xi\|_{B/\varepsilon}<1$, (5.4) implies that each neighbourhood of 0 in τ_μ contains the unit ball in $\|\cdot\|_A$ where A is some totally bounded set of X.

Thus, the unit ball in $\|\cdot\|_V$ contains the unit ball in $\|\cdot\|_A$. Therefore we have

$$V\subset C(A), \tag{5.5}$$

where $C(A)$ is the convex closed hull of A. Since A is totally bounded, $C(A)$, hence V also is totally bounded in X. Q.E.D.

Corollary. *Let X be a complete, metrizable and separable locally convex vector space. (For instance, suppose that X is a separable Banach space or a separable Fréchet space.) Let Y be a closed subspace of X. If some Borel measure on X is Y-quasi-invariant, then Y must be finite dimensional. In other words, T_μ never contains an infinite dimensional closed subspace.*

Proof. Since Y is closed, Y becomes a complete metrizable topological vector space with respect to the induced topology from X. Therefore from Theorem 5.1, some neighbourhood of 0 in Y must be totally bounded in X, hence in Y. This means that Y is locally totally bounded. This is impossible unless Y is finite dimensional. Q.E.D.

Example. Let (l^p) be the set of all p-th power summable sequences:

$$(l^p)=\{x=(x_n);\ \sum_{n=1}^{\infty}|x_n|^p<\infty\}. \tag{5.6}$$

(l^p) is monotonically increasing, namely $p'<p$ implies $(l^{p'})\subset(l^p)$. However, if $p<\infty$, no Borel measure on (l^p) is $(l^{p'})$-quasi-invariant, because the unit ball in $(l^{p'})$ is not totally bounded in (l^p). (Putting $e_k=(\delta_{k,n})_{n=1,2,\dots}\in(l^{p'})$, we have $\|e_k\|_{p'}=1$ and $\|e_k-e_j\|_p\geqq 1$ for $k\neq j$.)

For $p=\infty$, though (l^∞) is a Banach space, it is not separable, so Theorem 5.1 can not be applied. In fact, some Borel measure on $\boldsymbol{R}^\infty$ is (l^1)-quasi-invariant and lies on (l^∞) (cf. [5]). However the space (l_0^∞), the set of all sequences such that $\lim_{n\to\infty} x_n=0$, is separable in $\|\cdot\|_\infty$, so that Theorem 5.1 can be applied, thus no Borel measure on (l_0^∞) is (l^1)-quasi-invariant.

§6. Hilbert-Schmidt Imbedding

In this section, we shall prove the result 2 in Introduction. Namely we shall prove:

Theorem 6.1. *Let X be a Hilbert space, and $Y(\subset X)$ be another Hilbert space imbedded continuously in X. There exists a Y-quasi-invariant measure on $(X, \mathfrak{B}_{X^*}\cap X)$ if and only if the canonical imbedding $Y\to X$ is of the Hilbert-Schmidt type. Especially, Y must be separable even if X is not.*

Proof. Suppose that the topology of X is defined by the inner product $(\ ,\ )_X$, and the topology of Y is defined by $(\ ,\ )_Y$. Consider the gaussian measure g corresponding to $(\ ,\ )_Y$, then g is Y-quasi-invariant and lies on X, if $(\ ,\ )_X$ is of the Hilbert-Schmidt type with respect to $(\ ,\ )_Y$ on Y (cf. [6]). Thus, the sufficiency part of Theorem 6.1 has been proved.

We shall prove the necessity part of Theorem 6.1. Let $(\ ,\ )'_X$ and $(\ ,\ )'_Y$ be the dual inner products of $(\ ,\ )_X$ and $(\ ,\ )_Y$ respectively defined on X^*, the topological dual space of X. Imbedding X into $(X^*)'$, we can suppose that a measure μ on $((X^*)', \mathfrak{B}_{X^*})$ is Y-quasi-invariant and lies on X. Since the topology of Y is stronger than the weak topology of $(X^*)'$, from Theorem 4.2 we see that the topology defined by $(\ ,\ )'_Y$ is weaker than the characteristic topology τ_μ.

On the other hand, since μ lies on X, the characteristic function is continuous in Sazonov topology of $(\ ,\ )'_X$, hence Sazonov topology is stronger than τ_μ. (Cf. [2], Sazonov topology is the topology defined by the family of all inner products which are of the Hilbert-Schmidt type with respect to $(\ ,\ )'_X$.) Thus, the topology defined by $(\ ,\ )'_Y$ is weaker than Sazonov topology, namely $(\ ,\ )'_Y$

is continuous in Sazonov topology. This means that $(\ , \ )'_Y$ is Hilbert-Schmidt with respect to $(\ , \ )'_X$. Therefore, on the space Y, the dual inner product $(\ , \ )_X$ must be Hilbert-Schmidt with respect to $(\ , \ )_Y$. This implies that the canonical imbedding $Y \to X$ is of the Hilbert-Schmidt type, so that the proof has been completed. Q. E. D.

This theorem can be generalized to the case that Y is a complete metrizable topological vector space.

Proposition 6.1. *Let X be a Hilbert space, $Y (\subset X)$ be a complete metrizable topological vector space imbedded continuously in X. Then, there exists a Y-quasi-invariant measure on $(X, \mathfrak{B}_{X^*} \cap X)$ if and only if $(\ , \ )_X$ is of the Hilbert-Schmidt type with respect to some continuous inner product $(\ , \ )_H$ on Y.*

Proof. The sufficiency part can be proved by considering the gaussian measure corresponding to $(\ , \ )_H$.

For the proof of the necessity part, similarly with the proof of Theorem 6.1, we see that there exists a neighbourhood V of 0 in Y such that $\| \cdot \|_V$ is continuous in τ_μ. Since τ_μ is weaker than Sazonov topology of $(\ , \ )'_X$, this implies that $\| \cdot \|_V$ is continuous in Sazonov topology, therefore we have $\|\xi\|_V \leqq \|\xi\|'_H$ for some inner product $(\ , \ )'_H$ which is of the Hilbert-Schmidt type with respect to $(\ , \ )'_X$.

On the space Y, considering the dual inner product $(\ , \ )_H$ of $(\ , \ )'_H$, we see that $(\ , \ )_X$ is of the Hilbert-Schmidt type with respect to $(\ , \ )_H$ and that $(\ , \ )_H$ is continuous in Y. Q. E. D.

The reason why we discuss a measure on a Hilbert space is that we know exactly the continuity of the characteristic function by Sazonov's theorem.

Even if X is not a Hilbert space, we can estimate T_μ by similar discussions, if we know some results on the continuity of the characteristic function. For instance, we can prove:

Proposition 6.2. *For a sequence $a = (a_n)$ of positive numbers, we shall define the space $(l^p)_a$ as follows:*

$$(l^p)_a = \{x = (x_n);\ \sum_{n=1}^{\infty} a_n |x_n|^p < \infty\} . \tag{6.1}$$

(If $a_n \equiv 1$, then we get the usual (l^p).)

For $1 \leqq p < \infty$, if an $(l^{p'})$-quasi-invariant Borel measure exists on $(l^p)_a$, then we must have $\sum_{n=1}^{\infty} a_n < \infty$.

Remark. Considering the case $a_n \equiv 1$, we obtain again the result of the example in Section 5.

Proof. Suppose that a Borel measure μ on $(l^p)_a$ is $(l^{p'})$-quasi-invariant. From Theorem 4.2, if we denote the unit ball of $(l^{p'})$ with V, the semi-norm $\|\cdot\|_V$ is continuous in the characteristic topology τ_μ on $\boldsymbol{R}_0^\infty$.

Denoting the unit ball of $(l^p)_a$ with B, we have

$$\lim_{n\to\infty} \mu(nB) = 1. \tag{6.2}$$

On the other hand, we have

$$\begin{aligned} |1-\chi(\xi)| &\leqq \int_{\boldsymbol{R}^\infty} |1-\exp(i(\xi, x))| d\mu(x) \\ &\leqq \|\xi\|_n + 2\mu(nB^{\mathbf{C}}), \end{aligned}$$

where

$$\|\xi\|_n = \int_{nB} |(\xi, x)| d\mu(x). \tag{6.3}$$

Therefore, the topology defined by the family of semi-norms $\|\cdot\|_n$ is stronger than τ_μ. Thus, the semi-norm $\|\cdot\|_V$ is continuous in the topology defined by $\{\|\cdot\|_n\}$, so that we have

$$\exists C>0, \quad \exists n, \quad \|\xi\|_V \leqq C\|\xi\|_n. \tag{6.4}$$

Putting $e_k = (\delta_{kn})_{n=1,2,\ldots} \in \boldsymbol{R}_0^\infty$, we have

$$|(e_k, x)| = |x_k|,$$

so that $x \in V$ implies $|(e_k, x)| \leqq 1$, thus we have $\|e_k\|_V = 1$. Substituting this into (6.4), we get $1 \leqq C\|e_k\|_n$, namely we get

$$1 \leqq C \int_{nB} |x_k| d\mu(x). \tag{6.5}$$

Taking the p-th power of the both hand sides, and applying Hölder's inequality, we get

$$1 \leqq C^p \int_{nB} |x_k|^p d\mu(x). \tag{6.6}$$

Multiplying a_k and summing up the both hand sides with k, we have

$$\sum_{k=1}^{\infty} a_k \leqq C^p \int_{nB} \sum_{k=1}^{\infty} a_k |x_k|^p d\mu(x), \tag{6.7}$$

but the integrand being $\leqq n^p$, we get $\sum_{k=1}^{\infty} a_k \leqq C^p n^p < \infty$. This completes the proof of Proposition 6.2. Q.E.D.

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Added in proof: Note for the last statement in Section 4: Under the assumption that τ_μ is locally convex, $T_\mu = E_\mu^*$ occurs only if (E, τ_μ) is a Hilbert space. This result was obtained recently by H. Shimomura, and will be published in the near future.