Publ. RIMS, Kyoto Univ.
16 (1980), 693–720

Translationally Invariant Measure on the Infinite Dimensional Vector Space

By

Yasuo YAMASAKI*

Introduction

On an infinite dimensional vector space, any measure can not be invariant nor quasi-invariant under all translations ([1] [2]). However, for some dense subspace E_1, there exist such measures that are E_1-quasi-invariant, i.e. quasi-invariant under the translations: $x \to x+\xi$ where $\xi \in E_1$. For instance, Gaussian measure g on $\mathbb{R}^\infty$ is (l^2)-quasi-invariant. We have many other quasi-invariant measures ([3]), but the known examples including the measure g above have no equivalent invariant measure. Historically the translational quasi-invariance of probability measures has been discussed in detail ([4]), while the study of translational invariance has been neglected, perhaps because of difficulties of infinite measures which have less connection with the theory of probability.

In this paper, we shall construct directly a σ-finite invariant measure. First we consider the product of Lebesgue measure and uniform probability measures on $\mathbb{R}^n \times [-1/2, 1/2]^\infty$, and in the limit of $n \to \infty$ we obtain the requested measure μ on $\mathbb{R}^\infty$. We can easily show that μ is $\mathbb{R}_0^\infty$-invariant, and a detailed study shows that μ is (l^1)-invariant. We can modify μ to get an (l^2)-invariant measure, and other modifications can give invariant measures with respect to a larger subspace of $\mathbb{R}^\infty$.

The measure μ thus constructed is translationally ergodic, rotationally invariant, and rotationally ergodic. It is singular with Gaussian measures, so that the uniqueness of rotationally invariant measure ([1]) does not hold for infinite measures.

Finally we shall prove the existence of a measure which is invariant both under translations and homotheties. This fact, which is false for the finite

Received October 13, 1978.
* Research Institute for Mathematical Sciences, Kyoto University, Kyoto 606, Japan.

dimensional case, arises from the singularity of the infinite dimensional Lebesgue measure with respect to homotheties.

§ 1. Infiniteness of $\mathbf{R}_0^\infty$-Invariant Measure

Let X be a vector space, and μ be a measure on X (defined on a suitable σ-ring $\mathfrak{B}$). For $x \in X$, we define the translated measure $\tau_x\mu$ by

$$\tau_x\mu(B) = \mu(B - x), \qquad {}^\forall B \in \mathfrak{B}, \tag{1.1}$$

assuming that $\mathfrak{B}$ is invariant under translations.

Let Y be a subspace of X. We say that μ is Y-invariant if

$$^\forall y \in Y, \quad \tau_y\mu = \mu, \tag{1.2}$$

and that μ is Y-quasi-invariant if

$$^\forall y \in Y, \quad \tau_y\mu \sim \mu, \tag{1.3}$$

where $\mu_1 \sim \mu_2$ means that μ_1 and μ_2 are equivalent in the sense of absolute continuity.

If $\mu \sim \mu'$ and if μ is Y-quasi-invariant, then μ' is evidently Y-quasi-invariant. Especially, if $\mu \sim \mu'$ and if μ is Y-invariant, then μ' is Y-quasi-invariant. The converse is not always true. Namely, even if μ is Y-quasi-invariant, μ has not always an equivalent Y-invariant measure.

Let $\mathbf{R}^\infty$ be the vector space consisting of all real sequences. It contains $\mathbf{R}_0^\infty$ as a subspace where

$$\mathbf{R}_0^\infty = \{(x_n) \in \mathbf{R}^\infty;\ {}^\exists N,\ n \geqq N \Rightarrow x_n = 0\}. \tag{1.4}$$

Proposition 1.1. *On the space $\mathbf{R}^\infty$, no finite Borel measure (with respect to the weak topology) is $\mathbf{R}_0^\infty$-invariant.*

Proof. Let p_n be the projection of $\mathbf{R}^\infty$ to $\mathbf{R}^n$;

$$p_n\colon x = (x_1, x_2, \dots, x_n, \dots) \in \mathbf{R}^\infty \to (x_1, x_2, \dots, x_n) \in \mathbf{R}^n. \tag{1.5}$$

It is a measurable map from $\mathbf{R}^\infty$ onto $\mathbf{R}^n$, namely denoting the Borel fields of $\mathbf{R}^\infty$ and $\mathbf{R}^n$ with $\mathfrak{B}$ and $\mathfrak{B}_n$ respectively, we have $p_n^{-1}(\mathfrak{B}_n) \subset \mathfrak{B}$.

For a given Borel measure μ on $\mathbf{R}^\infty$, we define the measure μ_n on $\mathfrak{B}_n$ as

$$\mu(p_n^{-1}(B_n)) = \mu_n(B_n), \qquad {}^\forall B_n \in \mathfrak{B}_n. \tag{1.6}$$

If μ is a finite measure, then μ_n is evidently finite. On the other hand, for $x \in \mathbf{R}^\infty$

and $B_n \in \mathfrak{B}_n$, we have $p_n^{-1}(B_n) - x = p_n^{-1}(B_n - p_n(x))$, so that

$$\tau_x \mu(p_n^{-1}(B_n)) = \tau_{p_n(x)} \mu_n(B_n), \tag{1.7}$$

hence the $\mathbf{R}_0^\infty$-invariance of μ implies the $\mathbf{R}^n$-invariance of μ_n (because $p_n(\mathbf{R}_0^\infty) = \mathbf{R}^n$). However, it is well known that on the space $\mathbf{R}^n$, the Lebesgue measure is the only one which is σ-finite and $\mathbf{R}^n$-invariant, so no $\mathbf{R}^n$-invariant measure on $\mathbf{R}^n$ is finite. Therefore on the space $\mathbf{R}^\infty$, no $\mathbf{R}_0^\infty$-invariant measure is finite.

Q. E. D.

We can prove a more detailed result; for every σ-finite $\mathbf{R}_0^\infty$-invariant Borel measure μ on $\mathbf{R}^\infty$, its projection μ_n to $\mathbf{R}^n$ is necessarily $(0, \infty)$ type.

Proposition 1.2. *Let μ be a σ-finite $\mathbf{R}_0^\infty$-invariant Borel measure on $\mathbf{R}^\infty$. For the measure μ_n defined in* (1.6), *we have*

$$\begin{cases} \mu_n(B_n) = 0 & \text{if the Lebesgue measure of } B_n = 0, \\ \mu_n(B_n) = \infty & \text{if the Lebesgue measure of } B_n > 0. \end{cases} \tag{1.8}$$

Proof. First, let ν be a finite measure equivalent with μ. Since ν is $\mathbf{R}_0^\infty$-quasi-invariant, its projection ν_n is $\mathbf{R}^n$-quasi-invariant. It is well known that on the space $\mathbf{R}^n$, every $\mathbf{R}^n$-quasi-invariant σ-finite measure is equivalent with the Lebesgue measure, so ν_n, hence μ_n, is equivalent with the Lebesgue measure. Therefore $\mu_n(B_n) = 0$ if and only if the Lebesgue measure of B_n is zero.

Next, we shall derive a contradiction assuming that for some $B_n \in \mathfrak{B}_n$ we have $0 < \mu_n(B_n) < \infty$. For $x = (x_1, x_2, \ldots) \in \mathbf{R}^\infty$, put $q_n(x) = (x_{n+1}, x_{n+2}, \ldots) \in \mathbf{R}^\infty$, then by the correspondence $x \leftrightarrow (p_n(x), q_n(x))$, we have the isomorphism $\mathbf{R}^\infty \cong \mathbf{R}^n \times \mathbf{R}^\infty$. In this sense, μ can be regarded as a measure on $\mathbf{R}^n \times \mathbf{R}^\infty$. Putting

$$\nu(B) = \mu(B_n \times B), \qquad {}^\forall B \in \mathfrak{B}, \tag{1.9}$$

we define a Borel measure ν on $\mathbf{R}^\infty$. Since $\nu(\mathbf{R}^\infty) = \mu(B_n \times \mathbf{R}^\infty) = \mu_n(B_n) < \infty$, ν is a finite measure. On the other hand, for $x \in \mathbf{R}_0^\infty$ we have

$$\nu(B - x) = \mu(B_n \times (B - x)) = \mu(B_n \times B - (0, x)) = \mu(B_n \times B) = \nu(B),$$

so that ν is $\mathbf{R}_0^\infty$-invariant. This contradicts with Proposition 1.1, so we have proved the non-existence of such B_n that satisfies $0 < \mu_n(B_n) < \infty$. Q. E. D.

Thus, we have shown that even if there exists an $\mathbf{R}_0^\infty$-invariant σ-finite measure on $\mathbf{R}^\infty$, its projection on each $\mathbf{R}^n$ is $(0, \infty)$ type. This situation is just the case of a self-consistent sequence of $(0, \infty)$-type measures, and we can not define the limit measure in a definite sense (see [8]). So if we want to define an $\mathbf{R}_0^\infty$-invariant measure μ as a limit of some self-consistent sequence, we must rely on a somewhat indirect discussion. Nevertheless, we can construct an $\mathbf{R}_0^\infty$-invariant σ-finite measure rather easily, as shown later.

Before the construction of such a measure, we shall remark that among $\mathbf{R}_0^\infty$-quasi-invariant finite measures on $\mathbf{R}^\infty$, some one has an equivalent $\mathbf{R}_0^\infty$-invariant σ-finite measure, while another one has not. Many well known measures belong to the latter case, as shown in the next section.

§2. Gaussian Measure and Stationary Product

Proposition 2.1. *Let $f(x)$ be a measurable function on $\mathbf{R}^1$ which satisfies $f(x)>0$ and $\int_{-\infty}^{\infty} f(x)dx=1$. Let μ be the stationary product measure of f (i.e. $d\mu=\prod_{i=1}^{\infty} f(x_i)dx_i$), then μ is $\mathbf{R}_0^\infty$-quasi-invariant but μ has no equivalent $\mathbf{R}_0^\infty$-invariant σ-finite measure.*

Proposition 2.2. *Consider an inner product $(\ ,\ )_1$ on the space $\mathbf{R}_0^\infty$. If the topology defined by $(\ ,\ )_1$ is identical with the topology of (l^2) (on $\mathbf{R}_0^\infty$), then the Gaussian measure corresponding to $(\ ,\ )_1$ is $\mathbf{R}_0^\infty$-quasi-invariant, but it has no equivalent $\mathbf{R}_0^\infty$-invariant σ-finite measure.*

Proof of Proposition 2.1. As proved in [3], the stationary product measure μ is $\mathbf{R}_0^\infty$-ergodic. Let Σ be the permutation group on the set of all natural numbers $\mathbf{N}=\{1, 2,\dots\}$. Σ can be regarded as a transformation group on $\mathbf{R}^\infty$, and μ is Σ-invariant. Let Σ_0 be the subgroup of Σ generated by all transpositions (of two elements of $\mathbf{N}$). Σ_0 consists of such a permutation $\sigma\in\Sigma$ that satisfies $\sigma(i)=i$ except finite numbers of $i\in\mathbf{N}$. As shown in [3], the measure μ is Σ_0-ergodic.

Now, we shall derive a contradiction assuming that μ has an equivalent $\mathbf{R}_0^\infty$-invariant σ-finite measure ν. Since $\mu\sim\nu$ and μ is Σ_0-invariant and Σ_0-ergodic, if ν is Σ_0-invariant, then we have $\mu=c\nu$ for some constant $c>0$. Thus, the $\mathbf{R}_0^\infty$-invariance of ν implies that of μ, which is a contradiction.

Therefore, it is sufficient to prove that ν is Σ_0-invariant, namely $\forall\sigma\in\Sigma_0$, $\tau_\sigma\nu=\nu$, where

$$(2.1)\qquad \tau_\sigma\nu(B)=\nu(\sigma^{-1}(B)), \qquad \forall B\in\mathfrak{B}.$$

Since $\tau_\sigma\mu=\mu$, we have $\tau_\sigma\nu\sim\nu$. On the other hand, ν is $\mathbf{R}_0^\infty$-ergodic because μ is so. Therefore if $\tau_\sigma\nu$ is $\mathbf{R}_0^\infty$-invariant, then we have $\tau_\sigma\nu=c_\sigma\nu$ for some constant

$c_\sigma>0$. Especially for a transposition σ, $\sigma^2=I$ implies $c_\sigma^2=1$, hence $c_\sigma=1$. This means that ν is invariant under any transposition. Since Σ_0 is generated by the set of all transpositions, we have proved the Σ_0-invariance of ν.

To complete the proof of Proposition, it remains only to prove that $\tau_\sigma\nu$ is $\mathbf{R}_0^\infty$-invariant. Since ν is $\mathbf{R}_0^\infty$-invariant, we have $\tau_x\nu=\nu$ for any $x\in\mathbf{R}_0^\infty$. Therefore

$$\forall x\in\mathbf{R}_0^\infty,\quad \tau_\sigma\tau_x\nu=\tau_\sigma\nu\,. \tag{2.2}$$

However, we can easily show $\tau_\sigma\tau_x\nu=\tau_{\sigma x}\tau_\sigma\nu$, so (2.2) implies that $\tau_\sigma\nu$ is $\sigma(\mathbf{R}_0^\infty)$-invariant. Since σ maps $\mathbf{R}_0^\infty$ onto $\mathbf{R}_0^\infty$, namely $\sigma(\mathbf{R}_0^\infty)=\mathbf{R}_0^\infty$, we have proved the $\mathbf{R}_0^\infty$-invariance of $\tau_\sigma\nu$. Q.E.D.

Next we shall prove Proposition 2.2. For the inner product (,) defining the topology of (l^2), the corresponding Gaussian measure is the stationary product of one-dimensional Gaussian measures, so Proposition 2.2 is a special case of Proposition 2.1. For a general inner product $(\ ,\ )_1$, the corresponding Gaussian measure is not always a product measure, so Proposition 2.2 is not included in Proposition 2.1.

For a Borel measure μ on $\mathbf{R}^\infty$, we shall denote the set of all admissible translations by T_μ, namely:

$$T_\mu=\{x\in\mathbf{R}^\infty;\ \tau_x\mu\sim\mu\}\,. \tag{2.3}$$

For the Gaussian measure μ corresponding to the inner product $(\ ,\ )_1$, we have $T_\mu=L$ where L is the topological dual of $\mathbf{R}_0^\infty$ with respect to $(\ ,\ )_1$, and μ is X-ergodic if and only if X is dense in L with the dual topology (c.f. [1] or [6]). If the topology defined by $(\ ,\ )_1$ is identical with that of (l^2), then $L=(l^2)$, so μ is $\mathbf{R}_0^\infty$-quasi-invariant and $\mathbf{R}_0^\infty$-ergodic.

As proved later, Proposition 2.2 is valid under a weaker assumption:

$$\mathbf{R}_0^\infty \text{ is contained densely in } L\,. \tag{*}$$

If $(\xi,\xi)_1=0$ implies $\xi=0$, there exists an algebraic isomorphism A from $\mathbf{R}_0^\infty$ onto $\mathbf{R}_0^\infty$ such that

$$(\xi,\eta)_1=(A\xi,A\eta)\,,\quad \forall\xi,\eta\in\mathbf{R}_0^\infty\,, \tag{2.4}$$

where (,) is the usual inner product of (l^2). In this case, since $L=A^*(l^2)$, the assumption (*) is equivalent with

$$A^{*-1}(\mathbf{R}_0^\infty) \text{ is contained densely in } (l^2)\,, \tag{**}$$

where A^* is the adjoint map of A defined on $\mathbf{R}^\infty$.

Let G be the rotation group of $\mathbf{R}_0^\infty$ with respect to the inner product $(\ ,\ )_1$, then the Gaussian measure μ is G-invariant. Let G_0 be the group of all finite dimensional rotations. ($U \in G$ is called finite dimensional, if there exists a finite dimensional subspace R of $\mathbf{R}_0^\infty$ such that $Ux=x$ on $R^\perp$.) Then G_0 acts transitively on the unit sphere of $\mathbf{R}_0^\infty$, so μ is G_0-ergodic (c.f. [1]). Note that G_0 is generated by the set of all two dimensional rotations.

Proof of Proposition 2.2. We shall derive a contradiction assuming that μ has an equivalent $\mathbf{R}_0^\infty$-invariant σ-finite measure ν. Since $\mu \sim \nu$ and μ is G_0-invariant and G_0-ergodic, if ν is G_0-invariant, then we have $\mu = c\nu$ for some constant $c>0$. Thus, the $\mathbf{R}_0^\infty$-invariance of ν implies that of μ, which is a contradiction.

Therefore, it is sufficient to prove that ν is G_0-invariant, namely $\forall U \in G_0$, $\tau_U \nu = \nu$, where

$$\tau_U \nu(B) = \nu(U^{*-1}(B)), \quad \forall B \in \mathfrak{B}. \tag{2.5}$$

Since $\tau_U \mu = \mu$, we have $\tau_U \nu \sim \nu$. On the other hand under the assumption (*), μ is $\mathbf{R}_0^\infty$-quasi-invariant and $\mathbf{R}_0^\infty$-ergodic, hence ν is so. Therefore if $\tau_U \nu$ is $\mathbf{R}_0^\infty$-invariant, then we have $\tau_U \nu = c_U \nu$ for some constant $c_U > 0$. We want to derive $c_U = 1$, but even if U is a two-dimensional rotation, we have not always $U^2 = I$, so the proof is not straightforward as in Proposition 2.1.

Moreover, the proof of $\mathbf{R}_0^\infty$-invariance of $\tau_U \nu$ is not so easy. Since ν is $\mathbf{R}_0^\infty$-invariant, we have $\tau_x \nu = \nu$ for any $x \in \mathbf{R}_0^\infty$. Therefore

$$\forall x \in \mathbf{R}_0^\infty, \quad \tau_U \tau_x \nu = \tau_U \nu. \tag{2.6}$$

However, we can easily show $\tau_U \tau_x U = \tau_{U^* x} \tau_U \nu$, so (2.6) implies that $\tau_U \nu$ is $U^*(\mathbf{R}_0^\infty)$-invariant. (U^* is the adjoint map of U defined on $\mathbf{R}^\infty$.) But U^* does not always map $\mathbf{R}_0^\infty$ onto $\mathbf{R}_0^\infty$, so we can not conclude the $\mathbf{R}_0^\infty$-invariance of $\tau_U \nu$.

Since U is a rotation with respect to the inner product $(\ ,\ )_1$, U^* maps L onto L (=the topological dual of $\mathbf{R}_0^\infty$ with respect to $(\ ,\ )_1$). Therefore, if ν is L-invariant, we have

$$\forall x \in L, \quad \tau_U \tau_x \nu = \tau_U \nu \tag{2.7}$$

instead of (2.6), thus $\tau_U \nu$ is $U^*(L) = L$-invariant, hence it is $\mathbf{R}_0^\infty$-invariant.

So, we want to prove the L-invariance of ν. It is L-quasi-invariant because μ is so, namely $\forall x \in L$, $\tau_x \nu \sim \nu$. Since $\tau_x \nu$ is $\mathbf{R}_0^\infty$-invariant, the $\mathbf{R}_0^\infty$-ergodicity of

ν implies that $\tau_x\nu=c_x\nu$ for some constant $c_x>0$. We want to derive $c_x=1$. In (2.6) if we put $U=-I$, then $U^*=-I$ maps $\mathbf{R}_0^\infty$ onto $\mathbf{R}_0^\infty$, so $\tau_{-I}\nu$ is $\mathbf{R}_0^\infty$-invariant. Moreover $U^2=(-I)^2=I$ implies $c_{-I}=1$, so that we have $\tau_{-I}\nu=\nu$. Now, applying τ_{-I} on the both hand sides of $\tau_x\nu=c_x\nu$ $(x\in L)$, we have $\tau_{-x}\tau_{-I}\nu=c_x\tau_{-I}\nu$, so $\tau_{-I}\nu=\nu$ implies $\tau_{-x}\nu=c_x\nu$ hence $c_{-x}=c_x$. Thus, we have $\nu=\tau_{-x}(\tau_x\nu)$ $=c_{-x}c_x\nu=c_x^2\nu$, so we get $c_x=1$.

Here, we have completed the proof of $\mathbf{R}_0^\infty$-invariance of $\tau_U\nu$ $(U\in G_0)$. So as mentioned before, we have

$$\forall U\in G_0,\quad \exists c_U>0,\quad \tau_U\nu=c_U\nu. \tag{2.8}$$

We want to derive $c_U=1$. It is sufficient to consider only the case that U is a two-dimensional rotation.

For any $x\in\mathbf{R}_0^\infty$ such that $\|x\|_1=1$, put

$$U_x\colon \mathbf{R}_0^\infty\ni y\to y-2(x,y)_1x, \tag{2.9}$$

then U_x is a rotation with respect to $(\ ,\ )_1$. Since $U_x^2=I$, we have $c_{U_x}=1$, hence $\tau_{U_x}\nu=\nu$.

Next, suppose that U is an arbitrary two-dimensional rotation, namely for some two-dimensional subspace R of $\mathbf{R}_0^\infty$, we have $U=I$ on $R^\perp$. Consider an orthonormal system $\{x, y\}$ of R with respect to $(\ ,\ )_1$, then U can be represented by an orthogonal matrix:

$$U\sim\begin{pmatrix}\cos\theta, & \sin\theta\\ -\sin\theta, & \cos\theta\end{pmatrix}. \tag{2.10}$$

The matrix corresponding to U_xUU_x changes the signs of the first column and the first row of that of U, so it changes θ to $-\theta$, hence we have $U_xUU_x=U^{-1}$. Thus the U_x-invariance of ν implies $c_U=c_{U^{-1}}$, hence $c_U^2=1$, so $c_U=1$. Q.E.D.

§3. Lebesgue Measure on $\mathbf{R}^\infty$

In this section, we shall construct concretely an $\mathbf{R}_0^\infty$-invariant σ-finite measure on $\mathbf{R}^\infty$.

First, consider the Lebesgue measure on $\mathbf{R}^n$ and denote it by λ_n. Evidently, for a Borel set B_n of $\mathbf{R}^n$ we have

$$\lambda_n(B_n)=\lambda_{n+1}\left(B_n\times\left[-\frac{1}{2},\frac{1}{2}\right]\right). \tag{3.1}$$

For $m>n$, put

$$L_{mn}=\left\{x=(x_k)\in\mathbf{R}^m;\ n<k\leqq m\Rightarrow|x_k|\leqq\frac{1}{2}\right\},\tag{3.2}$$

and restrict λ_m on L_{mn} (we shall denote the restricted measure by the same notation λ_m). Then, $\{\lambda_m\}_{m>n}$ becomes a self-consistent sequence on the measurable spaces $\{L_{mn}, \mathfrak{B}_m \cap L_{mn}\}_{m>n}$, where $\mathfrak{B}_m$ is the Borel field of $\mathbf{R}^m$. Since each λ_m is σ-finite, according to [8], $\{\lambda_m\}_{m>n}$ can be extended uniquely to the projective limit measure.

By the way, the projective limit measurable space of $\{L_{mn}, \mathfrak{B}_m \cap L_{mn}\}_{m>n}$ is measurably isomorphic with $\{L_n, \mathfrak{B}\cap L_n\}$, where

$$\begin{aligned}L_n&=\left\{x=(x_k)\in\mathbf{R}^\infty;\ n<k\Rightarrow|x_k|\leqq\frac{1}{2}\right\}\\&\cong\mathbf{R}^n\times\left[-\frac{1}{2},\frac{1}{2}\right]^\infty,\end{aligned}\tag{3.3}$$

and $\mathfrak{B}$ is the Borel field of $\mathbf{R}^\infty$. Therefore, the projective limit measure of $\{\lambda_m\}_{m>n}$ can be regarded as a measure on $\{L_n, \mathfrak{B}\cap L_n\}$. We shall denote it by μ_n. Namely we have

$$\begin{aligned}&m>n,\quad {}^\forall B_m\in\mathfrak{B}_m,\\&B_m\subset L_{mn}\Rightarrow\mu_n\left(B_m\times\left[-\frac{1}{2},\frac{1}{2}\right]^\infty\right)=\lambda_m(B_m).\end{aligned}\tag{3.4}$$

In other words, we get the following result. Imbedding L_n into $\mathbf{R}^\infty$, we regard μ_n as a measure on $\mathbf{R}^\infty$. Denoting the projection from $\mathbf{R}^\infty$ onto $\mathbf{R}^n$ by p_n, we have

$$m>n,\quad {}^\forall B_m\in\mathfrak{B}_m,\quad \mu_n(p_m^{-1}(B_m))=\lambda_m(B_m\cap L_{mn}).\tag{3.5}$$

Now, for $m>n$ we have evidently $L_n\subset L_m$. Here we shall prove that the restriction of μ_m on L_n is identical with μ_n. Namely we shall show;

$$m>n\Rightarrow{}^\forall B\in\mathfrak{B},\quad \mu_n(B)=\mu_m(B\cap L_n).\tag{3.6}$$

For this purpose, we shall define μ_n' by $\mu_n'(B)=\mu_m(B\cap L_n)$ and prove $\mu_n'=\mu_n$. Consider the projection of μ_n' onto $\mathbf{R}^j$ $(j>m)$, then we have

$$\begin{aligned}\mu_n'(p_j^{-1}(B_j))&=\mu_m(p_j^{-1}(B_j)\cap L_n)\\&=\mu_m(p_j^{-1}(B_j\cap L_{jn})\cap L_j).\end{aligned}\tag{3.7}$$

However, $j>m$ implies $\mu_m(L_j^C)=0$, so from (3.5) the right hand side of (3.7) is equal with

$$\lambda_j(B_j \cap L_{jn} \cap L_{jm}) = \lambda_j(B_j \cap L_{jn}) \\ = \mu_n(p_j^{-1}(B_j)).$$

This equality holds for every $j > m$ and every $B_j \in \mathfrak{B}_j$, so we get $\mu_n = \mu'_n$ because of the uniqueness of the extension of self-consistent σ-finite measures. Thus, we have proved (3.6).

Definition 3.1. Using the measure μ_n given in (3.5), we put

$$\mu(B) = \lim_{n\to\infty} \mu_n(B), \quad {}^\forall B \in \mathfrak{B}. \tag{3.8}$$

Then μ is called the *Lebesgue measure* on $\mathbf{R}^\infty$.

As shown later, we have a family of "Lebesgue measures" with parameters $\{a_k, b_k\}$, and the above one (given in (3.8)) is only an example of them. But here according to Definition 3.1, we adopt the special one (3.8) as the Lebesgue measure on $\mathbf{R}^\infty$.

Proposition 3.1. *The Lebesgue measure μ on $\mathbf{R}^\infty$ is an $\mathbf{R}_0^\infty$-invariant σ-finite Borel measure on $\mathbf{R}^\infty$.*

Proof. First we shall show that μ is a σ-additive measure on $\mathfrak{B}$. $\mu(\phi) = 0$ is evident, because $\mu_n(\phi) = 0$ for each n. Next, for a sequence of mutually disjoint Borel sets $\{B_k\}$ of $\mathbf{R}^\infty$, we shall show that

$$\mu(\bigcup_{k=1}^\infty B_k) = \sum_{k=1}^\infty \mu(B_k). \tag{3.9}$$

From (3.6), $\{\mu_n(B)\}$ is monotonically increasing with respect to n, so the limit in (3.8) exists certainly. Since each μ_n is a σ-additive measure, putting $\alpha_{nk} = \mu_n(B_k) \geqq 0$, the left side of (3.9) is equal with $\lim_{n\to\infty} \sum_{k=1}^\infty \alpha_{nk}$, while the right side is equal with $\sum_{k=1}^\infty \lim_{n\to\infty} \alpha_{nk}$. Since α_{nk} is monotonically increasing with respect to n, the sum and the limit commute mutually, according to Lebesgue's theorem (regarding the sum as the integral by an atomic measure on the set of natural numbers). Thus, (3.9) has been proved.

Second, we shall show that μ is σ-finite. Each μ_n is σ-finite, because it is the projective limit of σ-finite measures. From (3.6) and (3.8), we see that $B \subset L_n$ implies $\mu(B) = \mu_n(B)$. Therefore μ is σ-finite on L_n. Hence, μ is σ-finite on $\bigcup_{n=1}^\infty L_n$. On the other hand, from (3.6) we have $\mu_n((\bigcup_{k=1}^\infty L_k)^{\mathrm{C}}) = 0$ for each n, so we get $\mu((\bigcup_{k=1}^\infty L_k)^{\mathrm{C}}) = 0$. This means that μ is σ-finite on $\mathbf{R}^\infty$.

Lastly, we shall prove the $\mathbf{R}_0^\infty$-invariance of μ. Namely, we shall show that

$$(3.10) \qquad {}^{\forall}x \in \mathbf{R}_0^\infty, \quad {}^{\forall}B \in \mathfrak{B}, \quad \mu(B-x)=\mu(B).$$

First, note that $x=(x_k)\in \mathbf{R}_0^\infty$ implies ${}^{\exists}n$, $k>n \Rightarrow x_k=0$. For this n, we shall show that

$$(3.11) \qquad m>n \Rightarrow \mu_m(B-x)=\mu_m(B).$$

For this purpose we shall define μ'_m by $\mu'_m(B)=\mu_m(B-x)$, and prove $\mu'_m=\mu_m$. Consider the projection of μ'_m on $\mathbf{R}^j$ $(j>m)$, then we have

$$(3.12) \qquad \begin{aligned} \mu'_m(p_j^{-1}(B_j)) &= \mu_m(p_j^{-1}(B_j)-x) \\ &= \mu_m(p_j^{-1}(B_j - p_j(x))). \end{aligned}$$

However, using (3.5) as well as the relation $L_{jm}-p_j(x)=L_{jm}$ and the $\mathbf{R}^j$-invariance of λ_j, we can rewrite the right hand side of (3.12) as;

$$\begin{aligned} &\lambda_j((B_j-p_j(x))\cap L_{jm})=\lambda_j(B_j\cap L_{jm}-p_j(x)) \\ &=\lambda_j(B_j\cap L_{jm})=\mu_m(p_j^{-1}(B_j)). \end{aligned}$$

This equality holds for every $j>m$ and every $B_j\in\mathfrak{B}_j$, so we get $\mu_m=\mu'_m$ because of the uniqueness of the extension of self-consistent σ-finite measures. Thus, for any $m>n$ we have $\mu_m(B-x)=\mu_m(B)$, so letting $m\to\infty$ we get (3.10).

Q. E. D.

§4. Equivalent Measure in Product Type

The Lebesgue measure on $\mathbf{R}^\infty$ has equivalent finite measures in product type. In this section, we shall give an example of them.

Let $\{c_n\}$ be a sequence of positive numbers such that $0<c_n<1$. On the real line $\mathbf{R}^1$, for each n consider a function $f_n(x)$ which satisfies;

$$(4.1) \qquad \begin{aligned} &0<f_n(x)<1, \quad \int_{-\infty}^{\infty} f_n(x)dx=1, \\ &f_n(x)=c_n \quad \text{for} \quad -\tfrac{1}{2}\leqq x\leqq \tfrac{1}{2}. \end{aligned}$$

Such a function $f_n(x)$ exists certainly for any given constant $0<c_n<1$.

Proposition 4.1. *Define a product measure ν on $\mathbf{R}^\infty$ as follows:*

$$(4.2) \qquad d\nu=\prod_{n=1}^{\infty} f_n(x_n)dx_n.$$

If $\prod_{n=1}^{\infty} c_n>0$, then ν is equivalent with the Lebesgue measure μ on $\mathbf{R}^\infty$.

Remark. Of course, ν is $\mathbf{R}_0^\infty$-quasi-invariant. According to [3], every product measure on $\mathbf{R}^\infty$ is $\mathbf{R}_0^\infty$-ergodic. Therefore, ν, hence μ also, is $\mathbf{R}_0^\infty$-ergodic.

Proof. For $x=(x_n)\in\mathbf{R}^\infty$, define a function $f(x)$ by;

$$f(x)=\prod_{n=1}^{\infty} f_n(x_n)\,. \tag{4.3}$$

Since $0<f_n(x_n)<1$, the partial product decreases monotonically, so that the infinite product in (4.3) exists certainly. If $x\in L_n$, then $|x_k|\leqq 1/2$ for $k>n$, so we have

$$f(x)=\prod_{k=1}^{n} f_k(x_k)\prod_{k=n+1}^{\infty} c_k>0\,.$$

Thus $f(x)$ is positive on L_n, hence positive on $\bigcup_{n=1}^{\infty} L_n$. On the other hand, since $\mu((\bigcup_{n=1}^{\infty} L_n)^{\mathrm{C}})=0$, we see that $f(x)$ is positive for μ-almost all x.

Now, define a measure ν' on $\mathbf{R}^\infty$ by;

$$d\nu'=f\,d\mu\,. \tag{4.4}$$

Then, we have evidently $\nu'\sim\mu$. Therefore, it is sufficient to prove $\nu'=\nu$.

Consider the projection of ν' on $\mathbf{R}^m$;

$$\begin{aligned}\nu'(p_m^{-1}(B_m)) &= \lim_{n\to\infty}\nu'(p_m^{-1}(B_m)\cap L_n)\\ &= \lim_{n\to\infty}\int_{p_m^{-1}(B_m)\cap L_n} f(x)d\mu(x)\\ &= \lim_{n\to\infty}\int_{p_m^{-1}(B_m)} f(x)d\mu_n(x)\\ &= \lim_{n\to\infty}\lim_{N\to\infty}\int_{p_m^{-1}(B_m)}\prod_{k=1}^{N} f_k(x_k)d\mu_n(x)\\ &= \lim_{n\to\infty}\lim_{N\to\infty}\int_{p_{Nm}^{-1}(B_m)\cap L_{Nn}}\prod_{k=1}^{N} f_k(x_k)dx_1\cdots dx_N\,.\end{aligned} \tag{4.5}$$

The last equality is due to (3.5).

However, for $m<n<N$, we have

$$p_{Nm}^{-1}(B_m)\cap L_{Nn}=B_m\times\mathbf{R}^{n-m}\times\left[-\frac{1}{2},\frac{1}{2}\right]^{N-n},$$

so that the last side of (4.5) is equal with

$$\int_{B_m}\prod_{k=1}^{m} f_k(x_k)dx_1\cdots dx_m\cdot\lim_{n\to\infty}\left\{\prod_{k=m+1}^{n}\int_{-\infty}^{\infty} f_k(x_k)dx_k\right.$$

$$\times \lim_{N\to\infty} \prod_{k=n+1}^{N} \int_{-\frac{1}{2}}^{\frac{1}{2}} f_k(x_k)dx_k \Big\} .$$

Using (4.1) and (4.2), we see that it is equal with

$$\nu(p_m^{-1}(B_m)) \lim_{n\to\infty} \lim_{N\to\infty} \prod_{k=n+1}^{N} c_k .$$

But the assumption $\prod_{n=1}^{\infty} c_n > 0$ implies $\lim_{n\to\infty} \lim_{N\to\infty} \prod_{k=n+1}^{N} c_k = 1$, so we have proved the equality $\nu'(p_m^{-1}(B_m)) = \nu(p_m^{-1}(B_m))$.

This equality holds for every m and every $B_m \in \mathfrak{B}_m$, so we get $\nu = \nu'$ because of the uniqueness of the extension of self-consistent finite measures. Q. E. D.

§5. Determination of T_μ

As mentioned in Proposition 3.1, the Lebesgue measure μ on $\mathbf{R}^\infty$ is $\mathbf{R}_0^\infty$-invariant. In this section, we shall determine exactly the set of all admissible translations T_μ defined in (2.3).

First we shall remark that for every $x \in T_\mu$, we have $\tau_x\mu = \mu$. As seen from (3.5), μ_n is an even measure (i.e. $\mu_n(-B) = \mu_n(B)$), so that the limit measure μ is also an even measure. Therefore, by a similar discussion with the proof of L-invariance of ν below (2.7), we can conclude that $\tau_x\mu \sim \mu$ implies $\tau_x\mu = \mu$.

Since μ and $\tau_x\mu$ are $\mathbf{R}_0^\infty$-ergodic, if $x \notin T_\mu$, then we have $\tau_x\mu \perp \mu$ (singular with each other). Thus, we get the following conclusion:

For each $x \in \mathbf{R}^\infty$, we have either $\tau_x\mu = \mu$ or $\tau_x\mu \perp \mu$. Moreover $\tau_x\mu = \mu$ is equivalent with $x \in T_\mu$.

Proposition 5.1. $T_\mu = (l^1)$. *Namely, μ is exactly (l^1)-invariant.*

Proof. Put $L_0 = [-1/2, 1/2]^\infty$, then for each n we have $L_0 = p_n^{-1}([-1/2, 1/2]^n) \cap L_n$, so that from (3.5) we get $\mu_n(L_0) = \lambda_n([-1/2, 1/2]^n) = 1$. Therefore, letting $n \to \infty$, we have $\mu(L_0) = 1$.

Now, suppose that $\tau_x\mu = \mu$. Then, we must have $\mu(L_0 - x) = 1$. Here, we shall calculate $\mu_n(L_0 - x)$ to find a necessary condition for $\tau_x\mu = \mu$. For $x = (x_k) \in \mathbf{R}^\infty$, we have

$$(5.1) \quad L_0 - x \subset p_m^{-1}\left(\left[-\frac{1}{2}, \frac{1}{2}\right]^m - p_m(x)\right) = p_m^{-1}\left(\prod_{k=1}^{m}\left[-\frac{1}{2} - x_k, \frac{1}{2} - x_k\right]\right),$$

so supposing $n < m$, we have from (3.5)

$$(5.2)\qquad \mu_n(L_0-x)\leqq\lambda_m\Big(\prod_{k=1}^{n}\Big[-\frac{1}{2}-x_k,\ \frac{1}{2}-x_k\Big]\times\prod_{k=n+1}^{m}\Big(\Big[-\frac{1}{2}-x_k,\ \frac{1}{2}-x_k\Big]\cap\Big[-\frac{1}{2},\ \frac{1}{2}\Big]\Big)\Big)=\prod_{k=n+1}^{m}(1-|x_k|)_+,$$

where $r_+=\mathrm{Max}\,(0,\ r)$. Therefore, $\mu(L_0-x)=1$ implies that

$$(5.3)\qquad \lim_{n\to\infty}\lim_{m\to\infty}\prod_{k=n+1}^{m}(1-|x_k|)_+=1\,.$$

Thus, for a sufficiently large n, we have $\prod_{k=n+1}^{\infty}(1-|x_k|)_+>0$. This means $\sum_{k=1}^{\infty}|x_k|<\infty$, namely $x\in(l^1)$. So the fact: $\tau_x\mu=\mu\Rightarrow x\in(l^1)$ has been proved.

Conversely, assuming $x\in(l^1)$, we shall derive $x\in T_\mu$. For the measure ν defined by (4.2), we have $\nu\sim\mu$, so that $T_\mu=T_\nu$. Since ν is a product measure, $x=(x_n)$ belongs to T_ν if and only if

$$(5.4)\qquad \prod_{n=1}^{\infty}\int_{-\infty}^{\infty}\sqrt{f_n(y_n)f_n(y_n-x_n)}\,dy_n>0\,,$$

according to [3].

However, from (4.1) we have

$$(5.5)\qquad \int_{-\infty}^{\infty}\sqrt{f_n(y_n)f_n(y_n-x_n)}\,dy_n \geqq c_n\int_{[-\frac{1}{2},\frac{1}{2}]\cap[-\frac{1}{2}+x_n,\frac{1}{2}+x_n]}dy_n=c_n(1-|x_n|)_+\,.$$

From the assumption in Proposition 4.1, we have $\prod_{n=1}^{\infty}c_n>0$. Since $x\in(l^1)$ implies $\prod_{n=m}^{\infty}(1-|x_n|)_+>0$ for a sufficiently large m, we see that (5.4) is satisfied for every $x\in(l^1)$, hence we get $(l^1)\subset T_\nu$. Q.E.D.

Remark. As seen from (3.6), we have $\mu((\bigcup_{n=1}^{\infty}L_n)^c)=0$. Since $L_n\subset(l^\infty)$ for every n, we have $\mu((l^\infty)^c)=0$. Thus the Lebesgue measure μ on $\mathbf{R}^\infty$ lies on (l^∞). Therefore μ (also the equivalent measure ν in (4.2)) gives an example of an (l^1)-quasi-invariant measure lying on (l^∞).

§6. Linear Transformations of the Lebesgue Measure

Consider two sequences $a=(a_k)\in\mathbf{R}^\infty$ and $b=(b_k)\in\mathbf{R}^\infty$, and suppose that $a_k<b_k$ for every k.

For $n<m$, putting

$$(6.1)\qquad L_{mn}(a, b)=\{x=(x_k)\in \mathbf{R}^m;\ n<k\leqq m \Rightarrow x_k\in[a_k, b_k]\} \cong \mathbf{R}^n\times \prod_{k=n+1}^{m}[a_k, b_k],$$

the sequence of measurable spaces $\{L_{mn}(a, b), \mathfrak{B}_m\cap L_{mn}(a, b)\}_{m>n}$ forms a projective sequence (with respect to the usual projection). Furthermore, the sequence of measures $\{\lambda_m/\prod_{k=1}^{m}(b_k-a_k)\}_{m>n}$, where λ_m is the Lebesgue measure on $\mathbf{R}^m$, forms a self-consistent sequence on $\{L_{mn}(a, b), \mathfrak{B}_m\cap L_{mn}(a, b)\}$. Therefore, the sequence $\{\lambda_m/\prod_{k=1}^{m}(b_k-a_k)\}$ can be extended uniquely to a σ-additive measure $\mu_{n;a,b}$ on the projective limit measurable space:

$$(6.2)\qquad L_n(a, b)=\{x=(x_k)\in \mathbf{R}^\infty;\ n<k \Rightarrow x_k\in[a_k, b_k]\} \cong \mathbf{R}^n\times \prod_{k=n+1}^{\infty}[a_k, b_k].$$

Regarding $\mu_{n;a,b}$ as a measure on $\mathbf{R}^\infty$, we can define a measure $\mu_{a,b}$ as the limit measure of $\mu_{n;a,b}$. $\mu_{a,b}$ is a "Lebesgue measure" with parameters a and b, as remarked below Definition 3.1.

$\mu_{a,b}$ is a special case of translations and linear transformations of μ, as explained below. Let μ be the Lebesgue measure on $\mathbf{R}^\infty$ defined in Definition 3.1. Then μ is $\mathbf{R}_0^\infty$-invariant and $\mathbf{R}_0^\infty$-ergodic. For any $x\in\mathbf{R}^\infty$, the translated measure $\tau_x\mu$ is also $\mathbf{R}_0^\infty$-invariant and $\mathbf{R}_0^\infty$-ergodic, and $\tau_x\mu\perp\mu$ if $x\notin(l^1)$.

Next, for an algebraic isomorphism A from $\mathbf{R}_0^\infty$ onto $\mathbf{R}_0^\infty$, the transformed measure $\tau_A\mu$ is $A^*(\mathbf{R}_0^\infty)$-invariant and $A^*(\mathbf{R}_0^\infty)$-ergodic. ($\tau_A\mu$ is defined by $\tau_A\mu(B)=\mu(A^{*-1}(B))$ for every $B\in\mathfrak{B}$.) From Proposition 5.1, we can replace $\mathbf{R}_0^\infty$ by (l^1), so that whenever $A^{*-1}(\mathbf{R}_0^\infty)$ is contained densely in (l^1), $\tau_A\mu$ (hence $\tau_x\tau_A\mu$ also) is $\mathbf{R}_0^\infty$-invariant and $\mathbf{R}_0^\infty$-ergodic. (For ergodicity, see [6].)

Thus, starting from μ defined in Definition 3.1, by the translations and linear transformations we can construct infinitely many $\mathbf{R}_0^\infty$-invariant and $\mathbf{R}_0^\infty$-ergodic measures.

For given $a\in\mathbf{R}^\infty$ and $b\in\mathbf{R}^\infty$, putting $x=(a+b)/2$ and defining A by $y=(y_n)\to Ay=((b_n-a_n)y_n)$, we have $\mu_{a,b}=\tau_x\tau_A\mu$. In this sense, $\mu_{a,b}$ is a special example of translations and linear transformations of μ.

Especially for $a\in\mathbf{R}_+^\infty$ (namely $a_k>0$ for every k), $\mu_{-a,a}$ is denoted simply with μ_a. Defining A by $y=(y_n)\to Ay=(2a_ny_n)$, we have $\mu_a=\tau_A\mu$.

For a suitable $a\in\mathbf{R}_+^\infty$, μ_a becomes (l^2)-invariant. Generally μ_a is $A^*((l^1))$-invariant, however $A^*((l^1))$ is identical with the following $(l^1)_a$.

(6.3) $$(l^1)_a=\left\{y=(y_n)\in\mathbf{R}^\infty;\ \sum_{n=1}^{\infty}\frac{|y_n|}{a_n}<\infty\right\}.$$

If $\sum_{n=1}^{\infty}\frac{1}{a_n^2}<\infty$ (for instance if $a_n=n$), we have $(l^2)\subset(l^1)_a$, thus μ_a becomes an (l^2)-invariant measure.

Not insisting on (l^2), if we choose $a\in\mathbf{R}_+^\infty$ suitably, $T_{\mu_a}=(l^1)_a$ becomes a very large space. In this sense, even for a sufficiently large subspace X of $\mathbf{R}^\infty$, we can construct an X-invariant measure.

However the measure μ_a lies on $(l^\infty)_a=\{y=(y_n)\in\mathbf{R}^\infty;\ \sup|y_n|/a_n<\infty\}$, so if we choose $T_{\mu_a}=(l^1)_a$ to be a very large space, then $(l^\infty)_a$ becomes a larger space, thus the gap between T_{μ_a} and the carrier of μ_a is not narrowed.

§7. Mutual Equivalence of $\tau_A\mu$

In this section, we shall discuss the condition for $\mu\sim\tau_A\mu$. Here, we are interested in the norm of (l^1), so differently from the case of Gaussian measures (see [5] or [6]), it is difficult to find the necessary and sufficient condition for $\mu\sim\tau_A\mu$.

We shall denote with Λ_μ the set of algebraic isomorphisms A which satisfy $\mu\sim\tau_A\mu$.

Proposition 7.1. *For the Lebesgue measure μ on $\mathbf{R}^\infty$, we have $\tau_x\tau_A\mu\sim\tau_{x'}\tau_{A'}\mu$ if and only if $x-x'\in A^*((l^1))$ and $AA'^{-1}\in\Lambda_\mu$.*

Proof. Before the proof of sufficiency, we shall remark that $\tau_A(\tau_{A'}\mu)=\tau_{A'A}\mu$, because for every $B\in\mathfrak{B}$ we have $\tau_A(\tau_{A'}\mu)(B)=\tau_{A'}\mu(A^{*-1}(B))=\mu(A'^{*-1}A^{*-1}(B))=\mu((A'A)^{*-1}(B))=\tau_{A'A}\mu(B)$.

Now suppose that $AA'^{-1}\in\Lambda_\mu$, then we have $\mu\sim\tau_{AA'^{-1}}\mu$, so operating $\tau_{A'}$ on the both hand sides, we get $\tau_{A'}\mu\sim\tau_{A'}(\tau_{AA'^{-1}}\mu)=\tau_A\mu$. Since $\tau_A\mu$ is $A^*((l^1))$-invariant, if $x-x'\in A^*((l^1))$, then we have $\tau_{A'}\mu\sim\tau_{x-x'}\tau_A\mu$, so operating $\tau_{x'}$ on the both hand sides, we get $\tau_{x'}\tau_{A'}\mu\sim\tau_x\tau_A\mu$. This completes the proof of sufficiency.

Conversely, suppose that $\tau_x\tau_A\mu\sim\tau_{x'}\tau_{A'}\mu$, then we have $\tau_{x-x'}\tau_A\mu\sim\tau_{A'}\mu$. Since both $\tau_A\mu$ and $\tau_{A'}\mu$ are even measures, operating τ_{-I} on the both hand sides, we get $\tau_{x'-x}\tau_A\mu\sim\tau_{A'}\mu$. Therefore we have $\tau_{x-x'}\tau_A\mu\sim\tau_{x'-x}\tau_A\mu$, hence $\tau_{2(x-x')}\tau_A\mu\sim\tau_A\mu$, which implies $x-x'\in A^*((l^1))$. Combining this with $\tau_{A'}\mu\sim\tau_{x-x'}\tau_A\mu$, we have $\tau_{A'}\mu\sim\tau_A\mu$, hence $\mu\sim\tau_{AA'^{-1}}\mu$. This implies $AA'^{-1}\in\Lambda_\mu$. Q. E. D.

Since it is difficult to determine the set Λ_μ concretely, we shall be content with a rough necessary condition: $A\in\Lambda_\mu \Rightarrow A^*((l^1))=(l^1)$. This condition comes from $T_\mu=(l^1)$ and $T_{\tau_A\mu}=A^*((l^1))$. We shall remark that from this condition, $AA'^{-1}\in\Lambda_\mu$ implies $A^*((l^1))=A'^*((l^1))$.

On the sufficient condition for $A\in\Lambda_\mu$, we shall discuss in a somewhat detailed manner.

Proposition 7.2.

1) *Let Σ be the permutation group of the set of all natural numbers* $\mathbf{N}$. *Regarding Σ as a transformation group on $\mathbf{R}_0^\infty$, we have $\Sigma\subset\Lambda_\mu$.*

2) *For an algebraic isomorphism A from $\mathbf{R}_0^\infty$ onto $\mathbf{R}_0^\infty$, suppose that the image of $I-A$ is finite dimensional, then we have $A\in\Lambda_\mu$ whenever $A^*((l^1))=(l^1)$.*

3) *For an algebraic isomorphism A such that $y=(y_n)\to Ay=(a_n y_n)$ (where $a_n>0$ for any n), we have $A\in\Lambda_\mu$ if and only if*

$$\sum_{n=1}^\infty |1-a_n|<\infty\,. \tag{7.1}$$

Corollary (of 3)). *For four elements a, b, a' and b' of $\mathbf{R}^\infty$, suppose that $a_n<b_n$ and $a'_n<b'_n$ for every n. Then, we have*

$$\mu_{a,b}\sim\mu_{a',b'} \tag{7.2}$$
$$\Leftrightarrow \sum_{n=1}^\infty \left|1-\frac{b_n-a_n}{b'_n-a'_n}\right|<\infty \quad \textit{and} \quad \sum_{n=1}^\infty \left|\frac{a_n+b_n-a'_n-b'_n}{b_n-a_n}\right|<\infty\,.$$

The corollary can be proved combining Proposition 7.1 with 3) of Proposition 7.2.

We shall prove Proposition 7.2 after the following Proposition 7.3.

Next, we shall inquire whether an $\mathbf{R}_0^\infty$-invariant and $\mathbf{R}_0^\infty$-ergodic measure exists besides the translations and linear transformations of the Lebesgue measure, namely besides measures of the form $\tau_x\tau_A\mu$. This question has not been answered up to the present. But under some additional condition, we can insist the uniqueness of the Lebesgue measure.

Proposition 7.3. *Let μ' be an $\mathbf{R}_0^\infty$-invariant and $\mathbf{R}_0^\infty$-ergodic, σ-finite Borel measure on $\mathbf{R}^\infty$. Putting $L_0=[-1/2, 1/2]^\infty$, if $0<\mu'(L_0)<\infty$, then we have $\mu'=c\mu$ for some constant $c>0$, where μ is the Lebesgue measure on $\mathbf{R}^\infty$.*

Proof. Put $L_n=\mathbf{R}^n\times[-1/2, 1/2]^\infty$, then we have $L_0\subset L_n$ and $\bigcup_{n=1}^\infty L_n$ is an $\mathbf{R}_0^\infty$-invariant set. Therefore from the $\mathbf{R}_0^\infty$-ergodicity of μ', we have

$$\mu'((\bigcup_{n=1}^\infty L_n)^{\mathrm{C}})=0\,. \tag{7.3}$$

From this and the σ-additivity of μ', we have

$$(7.4) \qquad \forall B \in \mathfrak{B}, \quad \mu'(B) = \lim_{n\to\infty} \mu'(B \cap L_n).$$

Therefore, putting $\mu'_n(B) = \mu'(B \cap L_n)$, if we can prove $\mu'_n(B) = c\mu_n(B)$ for every $B \in \mathfrak{B}$, then we get $\mu' = c\mu$ from (7.4).

Now, for each $x = (x_k) \in \mathbf{R}_0^\infty$ that satisfies $k > n \Rightarrow x_k = 0$, we have $\tau_x \mu'_n = \mu'_n$, because μ' is $\mathbf{R}_0^\infty$-invariant and $L_n = L_n - x$, hence $(B - x) \cap L_n = B \cap L_n - x$. Therefore, the projection of μ'_n onto $\mathbf{R}^n$ is $\mathbf{R}^n$-invariant. On the other hand, since $\mu'_n(p_n^{-1}([-1/2, 1/2]^n)) = \mu'(([-1/2, 1/2]^n \times \mathbf{R}^\infty) \cap L_n) = \mu'(L_0)$, the assumption $0 < \mu'(L_0) < \infty$ implies that the projection of μ'_n onto $\mathbf{R}^n$ is a σ-finite measure. Thus, from the uniqueness of an $\mathbf{R}^n$-invariant σ-finite measure on $\mathbf{R}^n$, there exists a constant $c > 0$ such that

$$(7.5) \qquad \forall B_n \in \mathfrak{B}_n, \quad \mu'_n(p_n^{-1}(B_n)) = c\lambda_n(B_n),$$

where λ_n is the Lebesgue measure on $\mathbf{R}^n$. Especially, substituting $B_n = [-1/2, 1/2]^n$ in (7.5), we have $c = \mu'(L_0)$, so that c does not depend on n.

Next, suppose that $m > n$. Since we have $\mu'_n(p_m^{-1}(B_m)) = \mu'(p_m^{-1}(B_m) \cap L_n) = \mu'(p_m^{-1}(B_m \cap L_{mn}) \cap L_m) = \mu'_m(p_m^{-1}(B_m \cap L_{mn})) = c\lambda_m(B_m \cap L_{mn}) = c\mu_n(p_m^{-1}(B_m))$, (the last equality comes from (3.5)), the projections of μ'_n and $c\mu_n$ onto $\mathbf{R}^m$ $(m > n)$ are identical. Therefore from the uniqueness of the extension of self-consistent σ-finite measures, we get $\mu'_n = c\mu_n$. Thus, in the limit of $n \to \infty$, we get $\mu' = c\mu$ where $c = \mu'(L_0)$. Q.E.D.

Corollary.

1) *Let μ' be an $\mathbf{R}_0^\infty$-invariant and $\mathbf{R}_0^\infty$-ergodic, σ-finite Borel measure on $\mathbf{R}^\infty$. For two elements a and b of $\mathbf{R}^\infty$ such that $a_n < b_n$ for every n, suppose that*

$$(7.6) \qquad 0 < \mu'\left(\prod_{n=1}^{\infty} [a_n, b_n]\right) < \infty,$$

then we have $\mu' = c\mu_{a,b}$ for some constant $c > 0$.

2) *For an algebraic isomorphism A from $\mathbf{R}_0^\infty$ onto $\mathbf{R}_0^\infty$, let μ' be an $A^*(\mathbf{R}_0^\infty)$-invariant and $A^*(\mathbf{R}_0^\infty)$-ergodic, σ-finite Borel measure on $\mathbf{R}^\infty$. Suppose that for some $x \in \mathbf{R}^\infty$ we have*

$$(7.7) \qquad 0 < \mu'(A^*(L_0) + x) < \infty,$$

then we have $\mu' = c\tau_x \tau_A \mu$ for some constant $c > 0$.

Proof. 1) is a special case of 2). To prove 2), it is sufficient to apply Proposition 7.3 to the measure $\tau_{A^{-1}} \tau_{-x} \mu'$.

Thus, the question whether an $\mathbf{R}_0^\infty$-invariant and $\mathbf{R}_0^\infty$-ergodic σ-finite measure exists besides the translations and linear transformations of the Lebesgue measure can be reduced to the question whether there exists a measure μ' which does not satisfy (7.6) for any a and b of $\mathbf{R}^\infty$, nor (7.7) for any x and A. If such a measure μ' exists, it becomes $(0, \infty)$-type on the set of infinite dimensional rectangles and their images by linear transformations.

Now, we shall prove Proposition 7.2, using Proposition 7.3. The proof of 1) is easy. For any $\sigma \in \Sigma$, we have evidently $\sigma(L_0)=L_0$, so that we have $\tau_\sigma\mu(L_0)=\mu(\sigma(L_0))=\mu(L_0)=1$. Therefore from Proposition 7.3, we get $\tau_\sigma\mu=\mu$.

Next, we shall prove 2). Suppose that the image of $I-A$ is finite dimensional, then for some n the image is contained in $\mathbf{R}^n\times\{0\}$. Therefore putting $e_k=(0, 0,\dots, 1, 0,\dots)$ (only the k-th coordinate is 1), A has the following form:

$$Ae_k=e_k+\sum_{j=1}^{n}\alpha_{kj}e_j\,. \tag{7.8}$$

Then for $y=(y_k)\in\mathbf{R}^\infty$, putting $A^*y=(z_k)$, we have

$$z_k=y_k+\sum_{j=1}^{n}\alpha_{kj}y_j\,. \tag{7.9}$$

On the other hand, from the assumption $A^*((l^1))=(l^1)$, $\tau_A\mu$ is $\mathbf{R}_0^\infty$-invariant and $\mathbf{R}_0^\infty$-ergodic, therefore from Proposition 7.3 it is sufficient to prove $0<\tau_A\mu(L_0)<\infty$. However from the definition of $\tau_A\mu$, we have $\tau_A\mu(L_0)=\mu(A^{*-1}(L_0))$. Since $y\in A^{*-1}(L_0)$ is equivalent with $A^*y\in L_0$, from (7.9) we have

$$A^{*-1}(L_0)=p_n^{-1}(M_n)\cap\bigcap_{k=n+1}^{\infty}p_k^{-1}(M_k), \tag{7.10}$$

where $M_n=\{y\in\mathbf{R}^n;\ |y_k+\sum_{j=1}^{n}\alpha_{kj}y_j|\leqq\frac{1}{2}\ \text{ for } 1\leqq k\leqq n\}$, and for $k>n$, $M_k=\left\{y\in\mathbf{R}^k;\ |y_k+\sum_{j=1}^{n}\alpha_{kj}y_j|\leqq\frac{1}{2}\right\}$.

Therefore we have

$$\begin{aligned}\mu(A^{*-1}(L_0))&=\lim_{m\to\infty}\mu_m(A^{*-1}(L_0))\\&=\lim_{m\to\infty}\lim_{N\to\infty}\mu_m(p_n^{-1}(M_n)\cap\bigcap_{k=n+1}^{N}p_k^{-1}(M_k))\,.\end{aligned} \tag{7.11}$$

Supposing $n<m<N$, we have from (3.5)

$$\begin{aligned}&\mu_m(p_n^{-1}(M_n)\cap\bigcap_{k=n+1}^{N}p_k^{-1}(M_k))\\&=\lambda_N((M_n\times\mathbf{R}^{N-n})\cap\bigcap_{k=n+1}^{N}(M_k\times\mathbf{R}^{N-k})\cap L_{Nm})\end{aligned}$$

$$=\int_{M_n}\lambda_{m-n}\Big(\prod_{k=n+1}^{m}\Big[-\frac{1}{2}-\sum_{j=1}^{n}\alpha_{kj}y_j,\ \frac{1}{2}-\sum_{j=1}^{n}\alpha_{kj}y_j\Big]\Big)$$

$$\times\lambda_{N-m}\Big(\prod_{k=m+1}^{N}\Big(\Big[-\frac{1}{2}-\sum_{j=1}^{n}\alpha_{kj}y_j,\ \frac{1}{2}-\sum_{j=1}^{n}\alpha_{kj}y_j\Big]\cap\Big[-\frac{1}{2},\frac{1}{2}\Big]\Big)\Big)dy_1dy_2\dots dy_n$$

$$=\int_{M_n}\prod_{k=m+1}^{N}(1-|\sum_{j=1}^{n}\alpha_{kj}y_j|)_+dy_1dy_2\dots dy_n\,.$$

We shall substitute this into (7.11). From the assumption $A^*((l^1))=(l^1)$, we have $A^*(\sum_{j=1}^{n}y_je_j)\in(l^1)$, hence we have $\sum_{k=1}^{\infty}|\sum_{j=1}^{n}\alpha_{kj}y_j|<\infty$. Therefore we get $\lim_{m\to\infty}\lim_{N\to\infty}\prod_{k=m+1}^{N}(1-|\sum_{j=1}^{n}\alpha_{kj}y_j|)_+=1$ for any $y=(y_j)\in\mathbf{R}^n$, so that Lebesgue's theorem assures that

$$\mu(A^{*-1}(L_0))=\lambda_n(M_n)=1/\det|(\delta_{kj}+\alpha_{kj})_{1\leq k,j\leq n}|\,.$$

This completes the proof of 2) of Proposition 7.2.

Lastly we shall prove 3). From the assumption for A, we have $A^*(\mathbf{R}_0^\infty)=\mathbf{R}_0^\infty$, so that $\tau_A\mu$ is $\mathbf{R}_0^\infty$-invariant and $\mathbf{R}_0^\infty$-ergodic. Therefore $A\in\Lambda_\mu$, namely $\tau_A\mu\sim\mu$, is equivalent with $0<\tau_A\mu(L_0)<\infty$.

However we have $\tau_A\mu(L_0)=\mu(A^{*-1}(L)_0)=\mu(\prod_{k=1}^{\infty}[-1/2a_k,1/2a_k])$. Since $\tau_A\mu\sim\mu$ is equivalent with $\mu\sim\tau_{A^{-1}}\mu$, for the convenience of calculation we shall consider A^{-1} instead of A. Then, the necessary and sufficient condition for $\tau_A\mu\sim\mu$ is as follows:

$$0<\mu\Big(\prod_{k=1}^{\infty}\Big[-\frac{a_k}{2},\frac{a_k}{2}\Big]\Big)<\infty\,. \tag{7.12}$$

In order to check the condition (7.12), we shall calculate as follows:

$$\mu\Big(\prod_{k=1}^{\infty}\Big[-\frac{a_k}{2},\frac{a_k}{2}\Big]\Big)=\lim_{n\to\infty}\mu_n\Big(\prod_{k=1}^{\infty}\Big[-\frac{a_k}{2},\frac{a_k}{2}\Big]\Big)$$
$$=\lim_{n\to\infty}\lim_{m\to\infty}\mu_n\Big(p_m^{-1}\Big(\prod_{k=1}^{m}\Big[-\frac{a_k}{2},\frac{a_k}{2}\Big]\Big)\Big)$$
$$=\lim_{n\to\infty}\lim_{m\to\infty}\lambda_m\Big(\prod_{k=1}^{m}\Big[-\frac{a_k}{2},\frac{a_k}{2}\Big]\cap L_{mn}\Big)$$
$$=\lim_{n\to\infty}\lim_{m\to\infty}\prod_{k=1}^{n}a_k\prod_{k=n+1}^{m}\mathrm{Min}(1,a_k)$$
$$=\lim_{n\to\infty}(\prod_{k=1}^{n}a_k\prod_{k=n+1}^{\infty}\mathrm{Min}(1,a_k))\,.$$

Therefore, (7.12) is satisfied if and only if $\prod_{k=1}^{\infty}\mathrm{Min}(1,a_k)>0$ and $\prod_{k=1}^{\infty}\mathrm{Max}(1,a_k)<\infty$. However, the former is equivalent with $\sum_{k=1}^{\infty}(1-a_k)_+<\infty$, while the latter is equivalent with $\sum_{k=1}^{\infty}(a_k-1)_+<\infty$. Combining them, (7.12) is equivalent with $\sum_{k=1}^{\infty}|1-a_k|<\infty$, namely with (7.1). In other words, (7.1) is a necessary and sufficient condition for $A\in\Lambda_\mu$. Q.E.D.

§8. Rotational Invariance and Ergodicity

The Lebesgue measure on $\mathbf{R}^\infty$, which is $\mathbf{R}_0^\infty$-invariant, is also invariant under finite dimensional rotations. We shall prove this below.

Let G be the rotation group of $\mathbf{R}_0^\infty$ with respect to the inner product (,) of (l^2), and let G_0 be a subgroup of G which consists of all finite dimensional rotations (namely such rotations which become the identity mapping on $R^\perp$ for some finite dimensional subspace R of $\mathbf{R}_0^\infty$).

Proposition 8.1. *Let μ be the Lebesgue measure on $\mathbf{R}^\infty$. For an element x of $\mathbf{R}^\infty$ and an algebraic isomorphism A from $\mathbf{R}_0^\infty$ onto $\mathbf{R}_0^\infty$, $\tau_x\tau_A\mu$ is G_0-invariant whenever it is $\mathbf{R}_0^\infty$-invariant and $\mathbf{R}_0^\infty$-ergodic.*

Proof. First we shall prove that $\tau_x\tau_A\mu$ is G_0-quasi-invariant. Namely we shall prove

$$\forall U\in G_0,\quad \tau_U\tau_x\tau_A\mu\sim\tau_x\tau_A\mu. \tag{8.1}$$

Since $\tau_U\tau_x\tau_A\mu=\tau_{U^*x}\tau_{AU}\mu$, from Proposition 7.1 it is sufficient to show

$$x-U^*x\in A^*((l^1)),\quad\text{and}\quad AU^{-1}A^{-1}\in\Lambda_\mu. \tag{8.2}$$

Now, for any $U\in G_0$ there exists n such that $Ue_k=e_k$ for $k>n$ (where $e_k=(0,0,\dots,1,0,\dots)$ (only the k-th coordinate is 1)). Then for each $x=(x_k)\in\mathbf{R}^\infty$, putting $U^*x=(y_k)$, we have $x_k=y_k$ for $k>n$, so that we get $x-U^*x\in\mathbf{R}_0^\infty$. On the other hand, from the assumption that $\tau_x\tau_A\mu$ is $\mathbf{R}_0^\infty$-invariant, we have $\mathbf{R}_0^\infty\subset A^*((l^1))$. Therefore we get $x-U^*x\in A^*((l^1))$ for every $x\in\mathbf{R}^\infty$, which is one of the desired relations.

Next, $U^{-1}\in G_0$ implies that the image of $I-U^{-1}$ is finite dimensional, hence the image of $A(I-U^{-1})A^{-1}=I-AU^{-1}A^{-1}$ is so. Therefore from 2) of Proposition 7.2, the relation $(AU^{-1}A^{-1})^*((l^1))=(l^1)$ implies $AU^{-1}A^{-1}\in\Lambda_\mu$. However, since $(AU^{-1}A^{-1})^*=A^{*-1}U^{*-1}A^*$, it is sufficient to show $A^{*-1}U^{*-1}A^*((l^1))=(l^1)$, namely $U^{*-1}A^*((l^1))=A^*((l^1))$.

Since for every $x\in\mathbf{R}^\infty$ we have $x-U^*x\in A^*((l^1))$ as proved above, $x\in A^*((l^1))$ is equivalent with $U^*x\in A^*((l^1))$, hence with $x\in U^{*-1}A^*((l^1))$. This means $A^*((l^1))=U^{*-1}A^*((l^1))$, which is the wanted relation.

Thus we have proved that $\tau_x\tau_A\mu$ is G_0-quasi-invariant. Then we have

$$\text{(8.3)} \qquad {}^\forall U\in G_0, \quad {}^\exists c_U>0, \quad \tau_U\tau_x\tau_A\mu=c_U\tau_x\tau_A\mu,$$

because both $\tau_x\tau_A\mu$ and $\tau_U\tau_x\tau_A\mu$ are $\mathbf{R}_0^\infty$-invariant and $\mathbf{R}_0^\infty$-ergodic. Using similar discussions with those below (2.8) (considering $\tau_x\tau_A\mu$ instead of ν), we can derive $c_U=1$. This means that $\tau_x\tau_A\mu$ is G_0-invariant. Q.E.D.

However, μ is not G-invariant. For instance, consider the following rotation U:

$$\text{(8.4)} \qquad \begin{aligned} Ue_{2k-1}&=\frac{1}{\sqrt{2}}(e_{2k-1}+e_{2k}),\\ Ue_{2k}&=\frac{1}{\sqrt{2}}(-e_{2k-1}+e_{2k}), \quad k=1,2,\dots. \end{aligned}$$

Then, we have

$$\text{(8.5)} \quad U^{*-1}(L_0)=\Big\{x=(x_k)\in\mathbf{R}^\infty;\ |x_{2k-1}+x_{2k}|\leqq\frac{\sqrt{2}}{2},\ |x_{2k-1}-x_{2k}|\leqq\frac{\sqrt{2}}{2} \text{ for } k=1,2,\dots\Big\}.$$

Therefore, putting

$$\text{(8.6)} \qquad B_2=\Big\{(x_1,x_2)\in\mathbf{R}^2;\ |x_1+x_2|\leqq\frac{\sqrt{2}}{2},\quad |x_1-x_2|\leqq\frac{\sqrt{2}}{2}\Big\},$$

we have $U^{*-1}(L_0)=B_2^\infty$, so that for $m>n$ we have

$$\begin{aligned} \mu_{2n}(U^{*-1}(L_0))&\leqq\mu_{2n}(B_2^m\times\mathbf{R}^\infty)\\ &=\lambda_{2m}(B_2^m\cap L_{2m,2n})\\ &=\Big\{\lambda_2\Big(B_2\cap\Big[-\frac{1}{2},\frac{1}{2}\Big]^2\Big)\Big\}^{m-n}=\{2(\sqrt{2}-1)\}^{m-n}. \end{aligned}$$

Letting $m\to\infty$, we get $\mu_{2n}(U^{*-1}(L_0))=0$, and letting $n\to\infty$, we get $\mu(U^{*-1}(L_0))=0$, namely $\tau_U\mu(L_0)=0$. This implies $\tau_U\mu\not\prec\mu$. (Furthermore we can conclude $\tau_U\mu\perp\mu$ because of $\mathbf{R}_0^\infty$-ergodicity of μ and $\tau_U\mu$.) Thus, μ is not quasi-invariant with respect to U, hence μ is not G-quasi-invariant.

We shall remark that G_0 acts transitively on the unit sphere of $\mathbf{R}_0^\infty$. Therefore, every G_0-invariant finite measure can be written as a superposition of Gaussian measures (see [1]). But Proposition 8.1 gives a counter-example of this fact for infinite G_0-invariant measures. (Since $\tau_x\tau_A\mu$ is $\mathbf{R}_0^\infty$-ergodic, if it is a superposition of Gaussian measures, it must be a single Gaussian measure.

This is impossible. Furthermore, $\tau_x \tau_A \mu$ is not equivalent with a superposition of Gaussian measures, because from Proposition 2.2 a Gaussian measure has no equivalent σ-finite $\mathbf{R}_0^\infty$-invariant measure.)

Example. Defining A by $y=(y_n) \to Ay=(2a_n y_n)$, the measure $\mu_a = \tau_A \mu$ is $(l^1)_a$-invariant and lies on $(l^\infty)_a$, where $(l^1)_a$ is given by (6.3) and $(l^\infty)_a$ is explained below (6.3). If $a \in (l^2)$, then we have $(l^\infty)_a \subset (l^2)$, so that we get the following result: "On the space (l^2), there exists a σ-finite Borel measure (other than Dirac measure) which is invariant under all finite dimensional rotations". For any given $b \in (l^1) \cap \mathbf{R}_+^\infty$, this measure can be chosen to be $(l^2)_b$-invariant, because putting $a_n = \sqrt{b_n}$, μ_a is $(l^2)_b$-invariant and lies on (l^2). ($b \in (l^1)$ implies $a \in (l^2)$, so μ_a lies on (l^2). On the other hand, $(l^2)_b \subset (l^1)_a$ comes from $\sum_{n=1}^\infty |x_n|/a_n = \sum_{n=1}^\infty \sqrt{b_n}|x_n|/b_n \leqq \{\sum_{n=1}^\infty b_n \cdot \sum_{n=1}^\infty x_n^2/b_n^2\}^{1/2}$.)

Proposition 8.2. *The Lebesgue measure μ on $\mathbf{R}^\infty$ is G_0-ergodic, where G_0 is the group of all finite dimensional rotations.*

Proof. Assuming $\mu(B \ominus U^*(B))=0$ for every $U \in G_0$, we shall prove $\mu(B)=0$ or $\mu(B^{\mathrm{C}})=0$.

From (3.5), the restriction of μ on L_0 becomes the infinite product of uniform measures on $[-1/2, 1/2]$ (namely, restrictions of one-dimensional Lebesgue measures on $[-1/2, 1/2]$). Therefore, as shown in [3], the restriction of μ on L_0 is Σ_0-ergodic. (Σ_0 denotes the group of permutations generated by all transpositions of two elements of $\mathbf{N}$.)

Now, we shall assume

$$\forall U \in G_0, \quad \mu(B \ominus U^*(B))=0. \tag{8.7}$$

Then, since $\Sigma_0 \subset G_0$, we have $\mu(B \ominus \sigma(B))=0$ for every $\sigma \in \Sigma_0$. On the other hand, $\sigma(L_0)=L_0$ implies $\sigma(B \cap L_0)=\sigma(B) \cap L_0$, therefore we have

$$\mu(B \cap L_0 \ominus \sigma(B \cap L_0)) = \mu((B \ominus \sigma(B)) \cap L_0)=0. \tag{8.8}$$

Since the restriction of μ on L_0 is Σ_0-ergodic, from (8.8) we get $\mu(B \cap L_0)=0$ or $\mu(B^{\mathrm{C}} \cap L_0)=0$. So, considering B^{C} instead of B if necessary, the proof will be completed if we can derive $\mu(B)=0$ from $\mu(B \cap L_0)=0$ under (8.7).

Under (8.7), $\mu(B \cap L_0)=0$ implies $\mu(U^*(B) \cap L_0)=0$ for every $U \in G_0$, so we get $\mu(B \cap U^{*-1}(L_0))=0$ from the G_0-invariance of μ. Furthermore, considering a countable union with respect to U, we get

(8.9) $$U_k \in G_0, \quad k=1, 2,\dots, \quad \Rightarrow \mu(B \cap \bigcup_{k=1}^{\infty} U_k^{*-1}(L_0))=0.$$

Therefore, the proof will be completed, if we can show

(8.10) $$\exists U_k \in G_0, \quad k=1, 2,\dots, \quad \mu((\bigcup_{k=1}^{\infty} U_k^*(L_0))^C)=0.$$

Let G_{0n} be a subgroup of G_0 which consists of all rotations U such that $Ue_k=e_k$ for $k>n$. G_{0n} can be identified with the n-dimensional rotation group $O(n)$, which is separable with respect to the natural topology (the induced topology from $\mathbf{R}^{n^2}$). Namely, there exists a dense countable subset A_n of $O(n)$. Identifying $O(n)$ with G_{0n}, the set A_n is a countable subset of G_{0n}, hence of G_0. If we put $A=\bigcup_{n=1}^{\infty} A_n$, A is a countable subset of G_0. Now, we shall show that $A=\{U_k\}$ satisfies (8.10).

On the set L_n, we have $|x_k| \leqq 1/2$ for $k>n$, and $(x_k)_{k>n}$ is mutually independent with respect to μ. On the other hand, the mean of x_k^2 with respect to μ is

(8.11) $$\int_{-\frac{1}{2}}^{\frac{1}{2}} x^2 dx = \frac{2}{3}\left(\frac{1}{2}\right)^3 = \frac{1}{12}.$$

Therefore, from the law of large numbers, for almost all $x \in L_n$ we have

(8.12) $$\lim_{N\to\infty} \frac{1}{N} \sum_{k=1}^{N} x_k^2 = \frac{1}{12}.$$

This holds for each n, so that combining with $\mu((\bigcup_{n=1}^{\infty} L_n)^C)=0$, we have (8.12) for almost all $x \in \mathbf{R}^\infty$. In other words, putting

(8.13) $$D=\left\{x \in \mathbf{R}^\infty; \lim_{N\to\infty} \frac{1}{N} \sum_{k=1}^{N} x_k^2 = \frac{1}{12}\right\},$$

we have $\mu(D^C)=0$. Thus, we have also $\mu(D^C \cup (\bigcup_{n=1}^{\infty} L_n)^C)=0$.

Therefore, we can conclude (8.10) if we can show

(8.14) $$D \cap (\bigcup_{n=1}^{\infty} L_n) \subset \bigcup_{k=1}^{\infty} U_k^*(L_0).$$

Suppose that $x \in D \cap (\bigcup_{n=1}^{\infty} L_n)$, then we have

(8.15) $$\exists N, \quad \frac{1}{N} \sum_{k=1}^{N} x_k^2 < \frac{1}{9}, \quad \text{and} \quad |x_k| \leqq \frac{1}{2} \quad \text{for} \quad k>N.$$

Then, for a suitable rotation U of $\mathbf{R}^N$ (namely for some $U \in O(N)$), $(x_1, x_2,\dots, x_N)$

can be mapped to $(c, c, \ldots, c)$, where $c=\sqrt{\sum_{k=1}^{N} x_k^2/N}$. From (8.15), we have $c<1/3$. Since A_N is a dense countable subset of $O(N)$, we have

$$\exists U_k \in A_N \subset O(N), \quad \max_{\|x\|\leqq 1} \|Ux-U_k x\| \leqq \sqrt{\frac{1}{4N}}, \tag{8.16}$$

where $\|\cdot\|$ is the Euclid norm in $\mathbf{R}^N$. Putting $U_k(x_1, x_2, \ldots, x_N)=(y_1, y_2, \ldots, y_N)$, we get $|y_k-c|\leqq\sqrt{1/4N}\cdot\sqrt{N/9}=1/6$, hence $|y_k|\leqq c+1/6<1/3+1/6=1/2$. Thus, we get $U_k(x_1, x_2, \ldots, x_N)\in[-1/2, 1/2]^N$.

Imbedding $O(N)$ into G_0, U_k becomes a rotation of $\mathbf{R}_0^\infty$ which keeps e_j invariant for $j>N$. Then, the mapping U_k^{*-1} is defined on $\mathbf{R}^\infty$. It is identical with U_k on $\mathbf{R}_0^\infty$, and keeps x_j invariant for $j>N$. Therefore we have $U_k^{*-1}x \in[-1/2, 1/2]^\infty=L_0$, hence we have $x\in U_k^*(L_0)$. This completes the proof of (8.14). Q.E.D.

Remark. Generally, not only the proof, but also Proposition 8.2 itself is invalid for the measure μ_a. As shown in the example before Proposition 8.2, if $a\in(l^2)$, the measure μ_a lies on (l^2), so that denoting the unit ball of (l^2) with B, we have $\mu_a(nB)>0$ for some $n>0$. Since nB is a G_0-invariant set, combining with $\mu_a((nB)^C)>0$, we see that μ_a is not G_0-ergodic. (Putting $A=\{x\in\mathbf{R}^\infty;\ |x_1|>n\}$, we have $A\subset(nB)^C$ and $\mu_a(A)=\infty$, which comes from $\mathbf{R}_0^\infty$-invariance of μ_a.)

§9. Invariance under Homotheties

In this section, we shall discuss about the invariance under homotheties. Denoting the Lebesgue measure on $\mathbf{R}^\infty$ with μ, we shall define a countably additive measure $\bar{\mu}$ on $\mathfrak{B}$ by

$$\bar{\mu}(B)=\int_0^\infty \frac{1}{c}\tau_{cI}\mu(B)dc=\int_0^\infty \frac{1}{c}\mu\left(\frac{1}{c}B\right)dc. \tag{9.1}$$

The measure $\bar{\mu}$ is $\mathbf{R}_0^\infty$-invariant (actually (l^1)-invariant), G_0-invariant, and invariant also under homotheties. Namely we have

$$\forall c_0>0, \quad \forall B\in\mathfrak{B}, \quad \bar{\mu}(c_0B)=\bar{\mu}(B). \tag{9.2}$$

In the case of finite dimensional space $\mathbf{R}^n$, the Lebesgue measure is a unique $\mathbf{R}^n$-invariant σ-finite measure, and it is not invariant under homotheties. (In (9.1), if we adopt the Lebesgue measure on $\mathbf{R}^n$ as μ, the corresponding measure $\bar{\mu}$ becomes $(0, \infty)$-type, so it is not σ-finite.) In contrast with this, we have:

Proposition 9.1. *The measure $\bar{\mu}$ defined in* (9.1) *is a σ-finite Borel measure* *on* $\mathbf{R}^\infty$. *Therefore, on the space* $\mathbf{R}^\infty$, *there exists a* σ-*finite Borel measure which is invariant simultaneously under translations* $\mathbf{R}_0^\infty$, *finite dimensional rotations* G_0, *and homotheties.*

Proof. Let D be the set defined in (8.13). Then, for $c>0$ we have

$$cD=\left\{x\in\mathbf{R}^\infty;\ \lim_{N\to\infty}\frac{1}{N}\sum_{k=1}^{N}x_k^2=\frac{c^2}{12}\right\}, \tag{9.3}$$

so that $\{cD\}_{c>0}$ is mutually disjoint. Since $\mu(D^{\mathrm{C}})=0$, we have $\tau_{cI}\mu((cD)^{\mathrm{C}})=0$.

On the other hand, since μ is a σ-finite measure, we have

$$^{\exists}E_k\in\mathfrak{B},\quad k=1,2,\dots,\quad \mu(E_k)<\infty,\quad \mu((\bigcup_{k=1}^{\infty}E_k)^{\mathrm{C}})=0. \tag{9.4}$$

Here, we can suppose that $E_k\subset D$ for every k, because $\mu(D^{\mathrm{C}})=0$ implies $\mu(E_k)=\mu(E_k\cap D)$. Furthermore, we can suppose that $\{E_k\}$ is monotonically increasing.

The mapping $(x, c)\to cx$ maps continuously $\mathbf{R}^\infty\times(0,\infty)$ onto $\mathbf{R}^\infty$, and one-to-one on $E_k\times[1/k, k]$ because $\{cE_k\}_{c>0}$ is mutually disjoint. Therefore, the image $F_k=\bigcup_{c\in[1/k,k]}(cE_k)$ becomes a continuous one-to-one image of a Borel subset of $\mathbf{R}^\infty\times(0,\infty)$, so F_k itself is a Borel subset of $\mathbf{R}^\infty$. (For instance, see [7].)

$\{F_k\}$ is also monotonically increasing, and for $c\in[1/k, k]$, we have $F_k\supset cE_k$. So that, for any given $c>0$, choosing k such that $1/k\leqq c\leqq k$, we have

$$\mu\left(\frac{1}{c}\left(\bigcup_{j=1}^{\infty}F_j\right)^{\mathrm{C}}\right)=\mu\left(\frac{1}{c}\left(\bigcup_{j=k}^{\infty}F_j\right)^{\mathrm{C}}\right)\leqq\mu\left(\left(\bigcup_{j=k}^{\infty}E_j\right)^{\mathrm{C}}\right)=0. \tag{9.5}$$

Since this holds for any given $c>0$, from (9.1) we have

$$\bar{\mu}\left(\left(\bigcup_{j=1}^{\infty}F_j\right)^{\mathrm{C}}\right)=0. \tag{9.6}$$

On the other hand, since we have

$$\mu\left(\frac{1}{c}F_j\right)=\mu\left(\frac{1}{c}F_j\cap D\right)=\begin{cases}\mu(E_j), & \text{if } c\in\left[\frac{1}{j}, j\right],\\ 0, & \text{if } c\notin\left[\frac{1}{j}, j\right],\end{cases}$$

we get

$$\bar{\mu}(F_j)=\mu(E_j)\int_{\frac{1}{j}}^{j}\frac{dc}{c}=2\log j\cdot\mu(E_j)<\infty. \tag{9.7}$$

Combining this with (9.6), we see that $\bar{\mu}$ is a σ-finite Borel measure on $\mathbf{R}^\infty$.

Q. E. D.

It seems to be somewhat curious that there exists a σ-finite measure on $\mathbf{R}^\infty$ which is invariant simultaneously under translations and homotheties. This comes from the fact that the Lebesgue measure μ on $\mathbf{R}^\infty$ is singular with respect to homotheties, so that we can find suitable sets F_j in the proof of Proposition 9.1 with finite $\bar{\mu}$-measures. However, for a "more natural" set, $\bar{\mu}$ is $(0, \infty)$-type. For instance, on the family of rectangles in $\mathbf{R}^\infty$, $\bar{\mu}$ is $(0, \infty)$-type as shown below.

For an algebraic isomorphism A from $\mathbf{R}_0^\infty$ onto $\mathbf{R}_0^\infty$, whenever $A^*(\mathbf{R}_0^\infty)$ is contained densely in (l^1), $\bar{\mu}(A^*(L_0)+x)$ is 0 or ∞ for any $x \in \mathbf{R}^\infty$. Because if we assume $0<\bar{\mu}(A^*(L_0)+x)<\infty$, the set of $c>0$ such that $0<\tau_{cI}\mu(A^*(L_0)+x)<\infty$ has a positive measure, so especially for two different values c and c', we have $0<\tau_{cI}\mu(A^*(L_0)+x)<\infty$ and $0<\tau_{c'I}\mu(A^*(L_0)+x)<\infty$. Then from the corollary of Proposition 7.3, both $\tau_{cI}\mu$ and $\tau_{c'I}\mu$ are equivalent with $\tau_x\tau_A\mu$, hence we have $\tau_{cI}\mu \sim \tau_{c'I}\mu$. This is a contradiction to $c \neq c'$.

For an algebraic isomorphism A from $\mathbf{R}_0^\infty$ onto $\mathbf{R}_0^\infty$, from (9.1) we have

$$\tau_A\bar{\mu}(B)=\int_0^\infty \frac{1}{c}\tau_A\tau_{cI}\mu(B)dc\,. \tag{9.8}$$

From now on, we shall assume that $A^{*-1}(\mathbf{R}_0^\infty)$ is contained densely in (l^1).

Then, $\tau_A\mu$ is $\mathbf{R}_0^\infty$-invariant and G_0-invariant, so that $\tau_A\bar{\mu}$ is $\mathbf{R}_0^\infty$-invariant, G_0-invariant, and invariant also under homotheties. Since $\bar{\mu}$ is σ-finite, $\tau_A\bar{\mu}$ is also σ-finite. Therefore, we conclude that on the space $\mathbf{R}^\infty$ there exist infinitely many σ-finite Borel measures which are invariant simultaneously under translations $\mathbf{R}_0^\infty$, finite dimensional rotations G_0, and homotheties.

Proposition 9.2.

1) *Let H be a transformation group of $\mathbf{R}^\infty$ which is generated by homotheties and translations by elements of $\mathbf{R}_0^\infty$. Then, $\tau_A\bar{\mu}$ is H-ergodic.*

2) *If $\mu \sim \tau_{c_0A}\mu$ for some $c_0>0$, then we have $\bar{\mu} \sim \tau_A\bar{\mu}$ (more exactly we have $\tau_A\bar{\mu}=\alpha\bar{\mu}$ for some $\alpha>0$). Otherwise, we have $\bar{\mu} \perp \tau_A\bar{\mu}$.*

Proof. Assume that a Borel subset B of $\mathbf{R}^\infty$ satisfies:

$$\begin{aligned} &{}^\forall x \in \mathbf{R}_0^\infty, \quad \tau_A\bar{\mu}(B \ominus (B-x))=0\,, \\ &{}^\forall c>0\,, \quad \tau_A\bar{\mu}(B \ominus (cB))=0\,. \end{aligned} \tag{9.9}$$

The proof of 1) will be completed if we can derive $\tau_A\bar{\mu}(B)=0$ or $\tau_A\bar{\mu}(B^C)=0$ from (9.9).

On the space $\mathbf{R}_0^\infty$, consider the inductive limit topology of Euclid topologies on $\mathbf{R}^n\times\{0\}$. Then, $\mathbf{R}_0^\infty$ is separable, so that there exists a countable dense subgroup $X=\{x_n\}$. According to [6], the measure $\tau_A\mu$ is X-ergodic.

Now, we shall assume the first line of (9.9). Then, we have ${}^\forall n$, $\tau_A\bar{\mu}(B\ominus(B-x_n))=0$, hence for almost all $c>0$ we have ${}^\forall n$, $\tau_A\mu(c^{-1}B\ominus(c^{-1}B-c^{-1}x_n))=0$, which implies $\tau_A\mu(c^{-1}B)=0$ or $\tau_A\mu(c^{-1}B^C)=0$ because of $c^{-1}X$-ergodicity of $\tau_A\mu$. We shall put

$$(9.10)\qquad N=\{c>0;\ \tau_A\mu\left(\frac{1}{c}B\right)=0\}.$$

Denoting the Haar measure on $(0, \infty)$ with λ, if $\lambda(N^C)=0$, then $\tau_A\mu(c^{-1}B)=0$ for almost all $c>0$ so that $\tau_A\bar{\mu}(B)=0$. Similarly, $\lambda(N)=0$ implies $\tau_A\bar{\mu}(B^C)=0$. Therefore, the proof will be completed if we can exclude the case: $\lambda(N)>0$ and $\lambda(N^C)>0$. In this case, we must have $\lambda(N\cap c_0N^C)>0$ for some $c_0>0$. (For instance, see [9].) If $c\in N\cap c_0N^C$, then we have $\tau_A\mu(c^{-1}B)=0$ and $\tau_A\mu(c_0c^{-1}B)>0$, hence we have $\tau_A\mu(c^{-1}(B\ominus(c_0B)))>0$. Combining with $\lambda(N\cap c_0N^C)>0$, this implies $\tau_A\bar{\mu}(B\ominus(c_0B))>0$, which contradicts with the second line of (9.9).

Thus, the proof of 1) has been completed.

Since both $\bar{\mu}$ and $\tau_A\bar{\mu}$ are H-ergodic, we have either $\bar{\mu}\sim\tau_A\bar{\mu}$ or $\bar{\mu}\perp\tau_A\bar{\mu}$. Therefore, for the proof of 2), it is sufficient to show

$$(9.11)\qquad {}^\exists c_0>0,\quad \mu\sim\tau_{c_0A}\mu\Leftrightarrow\bar{\mu}\sim\tau_A\bar{\mu}$$
$$\Leftrightarrow {}^\exists\alpha>0,\quad \tau_A\bar{\mu}=\alpha\bar{\mu}.$$

If we assume $\mu\sim\tau_{c_0A}\mu$, then we have $\tau_{c_0A}\mu=\alpha\mu$ for some $\alpha>0$, so from (9.8) we have $\tau_{c_0A}\bar{\mu}=\alpha\bar{\mu}$. Since $\tau_A\bar{\mu}$ is invariant under homotheties, we have $\tau_{c_0A}\bar{\mu}=\tau_A\bar{\mu}$, thus we get $\tau_A\bar{\mu}=\alpha\bar{\mu}$, so especially we get $\tau_A\bar{\mu}\sim\bar{\mu}$.

Conversely, assuming $\bar{\mu}\sim\tau_A\bar{\mu}$, we shall derive $\mu\sim\tau_{c_0A}\mu$. Since both μ and $\tau_{c_0A}\mu$ are $\mathbf{R}_0^\infty$-ergodic, $\mu\lesssim\tau_{c_0A}\mu$ implies $\tau_{c_0A}\mu\sim\mu$. Therefore, it is sufficient to show

$$(9.12)\qquad {}^\exists c_0>0,\quad {}^\forall E\in\mathfrak{B},\quad \mu(E)>0\Rightarrow\tau_{c_0A}\mu(E)>0.$$

Consider the set D defined in (8.13). Since $\mu(D^C)=0$, we have $\mu(E)=\mu(E\cap D)$, so that it is sufficient to prove (9.12) under the assumption $E\subset D$. Then, as explained below (9.4), $F=\bigcup_{0<c<\infty}(cE)$ is a Borel subset of $\mathbf{R}^\infty$, and satisfies

$$\bar{\mu}(F)=\int_0^\infty \frac{1}{c}\,\mu\Big(\frac{1}{c}F\Big)dc=\mu(E)\int_0^\infty \frac{dc}{c}=\infty\,.$$

Thus, from the assumption $\tau_A\bar{\mu}\sim\bar{\mu}$, we must have $\tau_A\bar{\mu}(F)>0$. However, since $c^{-1}F=F$ for every $c>0$, we get $\tau_A\mu(F)>0$ from (9.8).

Especially, choosing $E=D$, we have $\tau_A\mu(\bigcup_{0<c<\infty}(cD))>0$. Since $\tau_A\mu$ is $\mathbf{R}_0^\infty$-ergodic and each cD is an $\mathbf{R}_0^\infty$-invariant set, this implies $\tau_A\mu((c_0^{-1}D)^{\mathbf{C}})=0$ for some $c_0>0$, which is equivalent with $\tau_{c_0A}\mu(D^{\mathbf{C}})=0$.

In the general case of $E\subset D$, $\mu(E)>0$ implies $\tau_A\mu(F)>0$ as seen above. Since $F=c_0^{-1}F$, we have $\tau_A\mu(F)=\tau_{c_0A}\mu(F)$, so that we have $0<\tau_{c_0A}\mu(F)=\tau_{c_0A}\mu(F\cap D)=\tau_{c_0A}\mu(E)$. This completes the proof of (9.12). Q.E.D.

References

[1] Umemura, Y., Measures on infinite dimensional vector spaces, *Publ. RIMS, Kyoto Univ.,* **1** (1965), 1–47.

[2] Sudakov, V. N., Linear sets with quasi-invariant measure, (in Russian), *Doklady Akad. Nauk,* **127** (1959), 524–525.

[3] Shimomura, H., An aspect of quasi-invariant measures on $\mathbf{R}^\infty$, *Publ. RIMS, Kyoto Univ.,* **11** (1976), 749–773.

[4] Xia, Dao-Xing, *Measure and integration on infinite-dimensional spaces,* Academic Press, New York, 1972.

[5] Rosanov, Yu, A., On the density of one Gaussian measure with respect to another, *Theory of Prob. and Appl.,* **7** (1962), 62–86.

[6] Shimomura, H., Linear transformation of quasi-invariant measures, *Publ. RIMS, Kyoto Univ.,* **12** (1977), 777–800.

[7] Bourbaki, N., *Eléments de mathématique, topologie générale,* Hermann, 1958, Chap. 9, §6, $n^\circ 7$.

[8] Yamasaki, Y., Kolmogorov's extension theorem for infinite measures, *Publ. RIMS, Kyoto Univ.,* **10** (1975), 381–411.

[9] Halmos, P. R., *Measure Theory,* Springer, 1974, §59, Theorem E.