The Initial-Value Adjusting Method for Problems of the Least Squares Type of Ordinary Differential Equations

By

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Introduction

This work is concerned with the numerical processes to solve problems of the least squares type for ordinary differential equations. Consider the differential equation

$$\frac{dx}{dt} = X(t, x), \quad a < t < b$$

where x and $X(t, x)$ are real n-dimensional vectors and try to obtain a solution $x = x(t)$ of the above equation on the interval $[a, b]$ such that

$$\frac{1}{2} \sum_{j=1}^{N} {}^t[L_j x(t_j) - d_j][L_j x(t_j) - d_j] = \text{minimum},$$

where t_j are given points on $[a, b]$, $a = t_1 < t_2 < \cdots < t_N = b$, and L_j and d_j are given matrices and vectors, respectively.

Some authors have discussed these types of problems and proposed several numerical methods to solve them. Banks and Groome [1] have discussed the *quasilinearization* algorithm and established the quadratic convergence of it. Urabe [8] has shown that the problem can be reduced to the multipoint boundary value problem of nonlinear boundary condition, and proposed the application of the *Newton iterative process*. After Urabe's works, Fujii [2] has shown another reduction of the problem to a boundary value problem and discussed the Chebyshev-series-approximation with *a-posteriori* error bound.

In the present paper, we first propose a new method which is an applied version of the initial-value adjusting method given by Ojika and Kasue [6] for

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nonlinear boundary value problems of ordinary differential equations (on an
analysis of the method, see also [5]). In the next sections, we analyse the
process and prove the convergence of an approximate value to the exact one
under some conditions, roughly speaking, of sufficient smoothness of X and
suitable choice of the initial data of the iteration. Our method would be ex-
pected to have the advantages that they are easy to understand and program
the algorithm. Finally, in the last section, an illustrative numerical example
will be given.

§1. The Initial-Value Adjusting Algorithm

At first, we shall state the problem clearly. Consider a nonlinear differ-
ential equation

$$(1.1) \qquad \frac{dx}{dt} = X(t, x), \quad a < t < b.$$

Here x and $X(t, x)$ are n-dimensional vectors. Find a solution $x(t)$ of (1.1)
which minimizes the following value

$$(1.2) \qquad J = \frac{1}{2} \sum_{j=1}^{N} {}^{t}[L_j x(t_j) - d_j][L_j x(t_j) - d_j],$$

where t_j $(j = 1, ..., N)$ are the given points on the interval I, $a = t_1 < t_2 < \cdots < t_N$
$= b$, L_j and d_j $(j = 1, ..., N)$ are the given $n \times n$ matrices and n-dimensional
vectors, respectively. As far as the equation (1.1) is stable, the problem is
equivalent to find an initial value η such that the solution $x(t)$ starting with
$x(a) = \eta$ minimizes J.

Let us define a linear operator $L: C(I) \to \mathbf{R}^{nN}$ such that for $x \in C(I)$

$$(1.3) \qquad Lx = {}^{t}(L_1 x(t_1), ..., L_N x(t_N)).$$

Denote the nN-dimensional vector ${}^{t}(d_1, ..., d_N)$ by $\mathbf{d}$. Note that J is represented
by

$$(1.4) \qquad J = \frac{1}{2} {}^{t}\{Lx - d\}\{Lx - d\}.$$

Our *algorithm* is expressed in the following steps.

Step 0. Choose a suitable positive perturbation parameter ε and an initial
value $\eta_0 \in \mathbf{R}^n$, and set $k = 0$.

Step 1. Compute the numerical solution $x_k(t)$ of (1.1) with the initial con-

dition $x_k(a) = \eta_k$, and obtain the resulting values $L_k = Lx_k$ and $J_k = \frac{1}{2}{}^t\{L_k - d\}$ $\times \{L_k - d\}$.

Step 2. If the value J_k differs from J_{k-1} by less than a specified value, terminate the iteration. Otherwise, go to the next step. (If $k=0$, skip this step.)

Step 3. Set $j=1$.

Step 4. Compute the numerical solution $y_k^{(j)}(t)$ of (1.1) with the initial condition $y_k^{(j)}(a) = \eta_k + \varepsilon e_j$. Here e_j means the j-th unit vector of R^n.

Step 5. Replace j by $j+1$, and return to Step 4 until $j=n$.

Step 6. Determine the $nN \times n$ matrix $S(\varepsilon; x_k)$ (the *adjusting matrix*) such that

$$(1.5) \quad S(\varepsilon; x_k) = \left(\frac{1}{\varepsilon}\{Ly_k^{(1)} - L_k\}, \ldots, \frac{1}{\varepsilon}\{Ly_k^{(j)} - L_k\}, \ldots, \frac{1}{\varepsilon}\{Ly_k^{(n)} - L_k\}\right).$$

Step 7. Determine the initial value η_{k+1} for the next iteration by

$$(1.6) \quad \eta_{k+1} = \eta_k - \{{}^t S(\varepsilon; x_k)S(\varepsilon; x_k)\}^{-1}\, {}^t S(\varepsilon; x_k)\{L_k - d\}.$$

Then replace k by $k+1$, and return to Step 1.

Computational Remarks. In the above process, the numerical integration of the differential equation (1.1) are carried out by a suitable step-by-step method, for example, the Runge-Kutta method. Since the matrix ${}^t S(\varepsilon; x_k)S(\varepsilon; x_k)$ is an $n \times n$ symmetric, positive definite matrix, the square-root-free Cholesky's method is preferable for the solution of the linear equations in (1.6).

In what follows we shall denote the Euclidean norm of an n-dimensional vector x by $\|x\|$. $C(I)$ stands for the Banach space of vector-valued continuous functions on I, equipped with the norm

$$\|x\|_C = \max_{t \in I} \|x(t)\|.$$

$C^1(I)$ means a subset of $C(I)$ of continuously differentiable functions on I. The norms for matrices $R^m \to R^{m'}$ and other linear operators should be taken as the induced norms by the corresponding vector norms.

Let $\mathscr{D}$ be the domain of the tx-space bounded on x, intercepted by two hyperplanes $t=a$ and $t=b$. The boundary points of $\mathscr{D}$ on the hyperplanes $t=a$ and $t=b$ are supposed to be included in $\mathscr{D}$ and to make an open set on each hyperplane. Put

$$D = \{x \in C^1(I); (t, x(t)) \in \mathscr{D} \quad \text{for} \quad t \in I\},$$
$$D' = \{x \in C(I) \ ; (t, x(t)) \in \mathscr{D} \quad \text{for} \quad t \in I\}.$$

We shall need the following assumptions to the problem (1.1) and (1.2).

Assumption 1. (i) $X(t, x)$ is defined and three-times continuously differentiable with respect to x on $\mathscr{D}$. X and its derivatives up to the third order are continuous with respect to t on I.

(ii) In $\mathscr{D}$, there exist a positive number δ_0 and a mapping $Q(\xi; \cdot): \boldsymbol{R}^n \to \boldsymbol{R}^n$ such that the equation

$$(1.7) \qquad X(t, x+\xi) = X(t, x) + X_x(t, x)\xi + \frac{1}{2}X_{xx}(t, x)\xi\xi + Q(\xi; x)$$

holds whenever $\|\xi\| \leq \delta_0$ for each $(t, x) \in \mathscr{D}$.   Here $X_x(t, x)$ is the Jacobian matrix of $X(t, x)$ with respect to x and $X_{xx}(t, x)$ stands for the tensor of the third order with components $\partial^2 X_j(t, x)/\partial x_k \partial x_l$ ($j, k, l = 1, 2, ..., n$).

(iii) The constants $K_0 \sim K_3$ can be defined finitely as following:

$$K_0 = \sup_{\mathscr{D}} \|X(t, x)\|,$$

$$K_1 = \sup_{\mathscr{D}} \|X_x(t, x)\|,$$

$$K_2 = \sup_{\mathscr{D}} \|X_{xx}(t, x)\|,$$

$$\|Q(\xi; x)\| \leq K_3 \|\xi\|^3 \quad \text{for} \quad \|\xi\| \leq \delta_0 \quad \text{uniformly to} \quad (t, x) \in \mathscr{D}.$$

(iv) For arbitrary scalar number λ, the equality $Q(\lambda\xi; x) = \lambda^3 Q(\xi; x)$ holds.   If ξ_1, ξ_2, ξ_3 are such vectors that $\|\xi_1 + \xi_2\|, \|\xi_1\|, \|\xi_3\| \leq \delta_0$, the equation

$$(1.8) \qquad Q(\xi_1 + \xi_2; x) - Q(\xi_1; x) - Q(\xi_3; x)$$
$$= \xi_1 q_1(\xi_2) + \xi_2 q_2(\xi_1) + (\xi_2 - \xi_3)q_3(\xi_2, \xi_3)$$

holds for $(t, x) \in \mathscr{D}$.   Here $q_1(\xi)$ and $q_2(\xi)$ are of the behaviour of the quadratic form of ξ, i.e.

$$(1.9)_1 \qquad \lim_{\|\xi\| \to 0} q_1(\xi)/\|\xi\|^2, \quad \lim_{\|\xi\| \to 0} q_2(\xi)/\|\xi\|^2 < +\infty,$$

and q_3 has the estimation

$$(1.9)_2 \qquad\qquad \|q_3(\xi_2, \xi_3)\| \leq K_3'$$

for $\|\xi_2\|, \|\xi_3\| \leq \delta_0$.

Because of the equation (1.7), put

$$(1.10) \qquad X(t, x+\xi) = X(t, x) + X_x(t, x)\xi + R(\xi; x),$$

then it is evident that $R(\xi; x)$ has the estimation

$$(1.11) \qquad\qquad \|R(\xi; x)\| \leq K_2'\|\xi\|^2$$

and is Lipschitz-continuous

$$(1.12) \qquad \|R(\xi_1; x) - R(\xi_2; x)\| \le L_R \|\xi_1 - \xi_2\|$$

for $\|\xi_1\|$, $\|\xi_2\| \le \delta_0$.

Let us consider an operator $\mathscr{F}$ mapping $\boldsymbol{R}^n$ to a function of $C^1(I)$ along the flow generated by the differential equation (1.1) such that

$$\frac{d}{dt}\mathscr{F}\eta = X[t, \mathscr{F}\eta(t)], \quad a < t < b,$$

$$\mathscr{F}\eta(a) = \eta.$$

If η^* is a vector such that $(a, \eta^*) \in \mathscr{D}$ and η^* is a minimal solution for

$$J(\eta) \equiv \frac{1}{2}{}^t\{L\mathscr{F}\eta - \boldsymbol{d}\}\{L\mathscr{F}\eta - \boldsymbol{d}\},$$

then grad $J(\eta)|_{\eta = \eta^*} = 0$ holds, that is

$${}^t\{L\Phi(\cdot\,; \mathscr{F}\eta^*)\}\{L\mathscr{F}\eta^* - \boldsymbol{d}\} = 0,$$

where $\Phi(t; x)$ denotes the *matrizant* of the linear homogeneous matrix differential equation

$$\frac{d\Phi}{dt} = X_x[t, x(t)]\Phi, \quad a < t < b.$$

Moreover, considering that J is a quadratic form of vector, we may take the following assumption.

Assumption 2.   There exists a vector η^* such that $(a, \eta^*) \in \mathscr{D}$, $\mathscr{F}\eta^* \in D$,

$$(1.13) \qquad {}^t\{L\Phi(\cdot\,; \mathscr{F}\eta^*)\}\{L\mathscr{F}\eta^* - \boldsymbol{d}\} = 0$$

and

$$(1.14) \qquad \det {}^t\{L\Phi(\cdot\,; \mathscr{F}\eta^*)\}L\Phi(\cdot\,; \mathscr{F}\eta^*) \ne 0$$

hold.

Furthermore, there exists a positive constant $\bar{\varepsilon}$ such that for $0 < \varepsilon \le \bar{\varepsilon}$ the initial value problem

$$\frac{dy}{dt} = X(t, y), \quad a < t < b$$

$$y(a) = \eta^* + \varepsilon e_j$$

has a unique solution in D for every j.

We shall call η^* the *exact (local) isolated minimal point* of $J(\eta)$.

Throughout the present paper we take the above two assumptions implicitly and will not mention in the statement of Theorems and Lemmas.

Then we shall make some preliminaries after [5].

The j-th column vector of $\Phi(t; x)$, denoted by $\varphi^{(j)}(t)$, is bounded by

$$(1.15) \qquad \|\varphi^{(j)}\|_C \leqq e^{2K_1(b-a)}$$

for $x \in D'$ and $j = 1, \ldots, n$. Therefore the estimation

$$(1.16) \qquad \|\Phi(\,\cdot\,; x)\| \leqq n e^{2K_1(b-a)} \equiv M_0$$

holds. We shall call the matrix $L\Phi(\,\cdot\,; \mathscr{F}\eta)$ the *G-matrix* and denote it by $G(\eta)$. Since D is open in $C^1(I)$, for a positive constant M_1 we can take a positive number $\varDelta$ such that in the $\varDelta$-neighbourhood $B_\varDelta$ of η^* defined by

$$B_\varDelta = \{\eta \in \mathbf{R}^n; \|\eta - \eta^*\| \leqq \varDelta\},$$

the inverse of *compound G-matrix* defined by $\mathfrak{G}(\eta) = {}^tG(\eta)G(\eta)$ exists and the estimation

$$(1.17) \qquad \|\mathfrak{G}(\eta)^{-1}\| \leqq M_1$$

holds. Replace $\varDelta$ by a suitable value less than $\varDelta$ if necessary, then it is possible that the inequality

$$\|\mathscr{F}\eta - \mathscr{F}\eta^*\|_C \leqq \delta_0$$

holds for $\eta \in B_\varDelta$. We shall fix the number $\varDelta$.

Corresponding to our iterative process, an operator $\mathscr{A}$ mapping $\mathbf{R}^n$ into itself is defined by the following:

$$(1.18) \quad \mathscr{A}\eta = \eta - \{{}^tS(\varepsilon; \mathscr{F}\eta)S(\varepsilon; \mathscr{F}\eta)\}^{-1}\, {}^tS(\varepsilon; \mathscr{F}\eta)\,\{L\mathscr{F}\eta - d\} \quad \text{for} \quad \eta \in B_\varDelta,$$

where the domain of $\mathscr{A}$ is identical to $B_\varDelta$. Then the iterative process (1.6) is simply represented by

$$(1.19) \qquad \eta_{k+1} = \mathscr{A}\eta_k, \quad k = 0, 1, 2, \ldots.$$

Lemmas $2 \sim 4$ in [5] assert the followings.

For $\eta \in B_\varDelta$, put $x(t) = \mathscr{F}\eta(t)$. $y^{(j)}(t)$ in Step 4 of the algorithm has the following expression.

$$(1.20) \qquad y^{(j)}(t) = x(t) + \varepsilon\{\varphi^{(j)}(t) + v^{(j)}(t)\} \quad \text{on} \quad I,$$
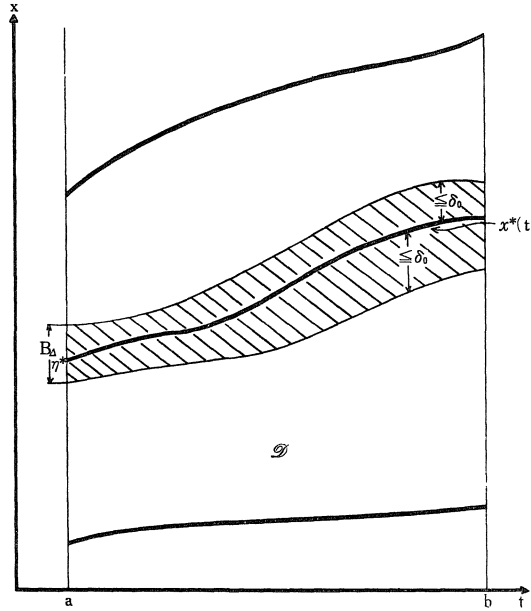
where $v^{(j)}(t)$ satisfies the differential equation

Figure 1.1.

$$(1.21) \qquad \frac{dv^{(j)}(t)}{dt} = \frac{1}{\varepsilon}\{X[t, x(t)+\varepsilon\{\varphi^{(j)}(t)+v^{(j)}(t)\}] - X[t, x(t)]\}$$
$$- X_x[t, x(t)]\varphi^{(j)}(t), \quad a<t<b$$

with the initial condition

$$(1.22) \qquad\qquad\qquad v^{(j)}(a)=0.$$

For arbitrary small positive number ε there exists a positive constant C^* such that $v^{(j)}(t)$ satisfies

$$(1.23) \qquad\qquad\qquad \|v^{(j)}\|_C \leqq C^*\varepsilon.$$

There exists a small positive number ε_0 such that for $0\leqq\varepsilon\leqq\varepsilon_0$ the estimation

$$(1.24) \qquad\qquad \|S(\varepsilon; \mathscr{F}\eta) - G(\eta)\| \leqq C^{**}\varepsilon \equiv \delta_1$$

holds in B_Δ. Moreover for $0\leqq\varepsilon\leqq\varepsilon_0$ and $\eta \in B_\Delta$ the inverse of the *compound adjusting matrix* ${}^tS(\varepsilon; \mathscr{F}\eta)S(\varepsilon; \mathscr{F}\eta)$ exists and the estimation

$$(1.25) \qquad \|\{{}^tS(\varepsilon; \mathscr{F}\eta)S(\varepsilon; \mathscr{F}\eta)\}^{-1}\| \leqq \frac{M_1}{1-\delta_1 M_1(2L_0M_0+\delta_1)}$$

holds, where L_0 is the bound of the operator norm of L on D.

§2. Fixed Point of the Operator $\mathscr{A}$

It is noteworthy that the exact minimal point η^* is *not* a fixed point of $\mathscr{A}$, because a fixed point of $\mathscr{A}$ must satisfy the equation

$$(2.1) \qquad \{{}^tS(\varepsilon;\ \mathscr{F}\eta)S(\varepsilon;\ \mathscr{F}\eta)\}^{-1}\,{}^tS(\varepsilon;\ \mathscr{F}\eta)\,\{L\mathscr{F}\eta-d\}=0,$$

i.e.

$$(2.1)' \qquad\qquad {}^tS(\varepsilon;\ \mathscr{F}\eta)\,\{L\mathscr{F}\eta-d\}=0.$$

On the other hand, from (1.13) η^* satisfies

$$ {}^tG(\eta^*)\,\{L\mathscr{F}\eta^*-d\}=0.$$

$S(\varepsilon;\ \mathscr{F}\eta)$ is surely an approximation of $G(\eta)$, but is not identical to that as far as $\varepsilon\neq 0$.

Therefore, we should answer the question whether a fixed point of $\mathscr{A}$ exists. Throughout this section we shall allow a *negative* value for ε for convenience' sake. In the case of negative ε, we can define the matrix $S(\varepsilon;\ \mathscr{F}\eta)$ and the operator $\mathscr{A}$ in the natural way. The estimations $(1.23)\sim(1.25)$ hold by replacing ε by $|\varepsilon|$.

We shall prove the existence of the fixed point of $\mathscr{A}$ in some neighbourhood of η^* by the implicit function theorem.

Define a function $\mathscr{T}(\eta,\ \varepsilon)$ by

$$(2.2) \qquad\qquad \mathscr{T}(\eta,\ \varepsilon)={}^tS(\varepsilon;\ \mathscr{F}\eta)\,\{L\mathscr{F}\eta-d\}$$

for $(\eta,\ \varepsilon)$ such that $\eta\in B_\Delta$ and $0<|\varepsilon|\leq\varepsilon_0$.

Lemma 1. *Define* $\mathscr{T}(\eta,\ 0)$ *by*

$$(2.3) \qquad\qquad \mathscr{T}(\eta,\ 0)=\lim_{\varepsilon\to 0}\mathscr{T}(\eta,\ \varepsilon),$$

then

$$(2.4) \qquad\qquad \mathscr{T}(\eta,\ 0)={}^tG(\eta)\,\{L\mathscr{F}\eta-d\}$$

holds.

Proof is evident.

Lemma 2. *For* $\eta\in B_\Delta$ *and* $|\varepsilon_1|,\ |\varepsilon_2|\leq\varepsilon_0,$ *the estimation*

$$(2.5) \qquad\qquad \|S(\varepsilon_1;\ \mathscr{F}\eta)-S(\varepsilon_2;\ \mathscr{F}\eta)\|\leq\text{const.}\ |\varepsilon_1-\varepsilon_2|$$

holds.   Here the constant does not depend on η, ε_1 and ε_2.

Proof.   Let us denote the j-th column vector of $S(\varepsilon; \mathscr{F}\eta)$ by $s_j(\varepsilon; \mathscr{F}\eta)$. Put

$$\mathscr{F}(\eta + \varepsilon_1 e_j) = y_1^{(j)} = x + \varepsilon_1 \{\varphi^{(j)} + v_1^{(j)}\}$$

and

$$\mathscr{F}(\eta + \varepsilon_2 e_j) = y_2^{(j)} = x + \varepsilon_2 \{\varphi^{(j)} + v_2^{(j)}\}$$

(see (1.20)), then the equation

$$s_j(\varepsilon_1; \mathscr{F}\eta) - s_j(\varepsilon_2; \mathscr{F}\eta) = L(v_1^{(j)} - v_2^{(j)})$$

holds.   Define $w^{(j)}$ by $w^{(j)} = v_1^{(j)} - v_2^{(j)}$, then $w^{(j)}$ must satisfy the differential equation

$$(2.6) \qquad \frac{dw^{(j)}}{dt} = \frac{1}{\varepsilon_1}\{X[t, y_1^{(j)}(t)] - X[t, x(t)]\}$$
$$- \frac{1}{\varepsilon_2}\{X[t, y_2^{(j)}(t)] - X[t, x(t)]\}$$

with the initial condition

$$(2.7) \qquad\qquad\qquad w^{(j)}(a) = 0 .$$

By virtue of Assumption 1 and the equation (1.10),
    (the right-hand side of (2.6))

$$= \frac{1}{\varepsilon_1}\{X_x[t, x(t)]\varepsilon_1\{\varphi^{(j)}(t) + v_1^{(j)}(t)\} + R[y_1^{(j)} - x; x]\}$$
$$- \frac{1}{\varepsilon_2}\{X_x[t, x(t)]\varepsilon_2\{\varphi^{(j)}(t) + v_2^{(j)}(t)\} + R[y_2^{(j)} - x; x]\}$$
$$= X_x[t, x(t)]w^{(j)}(t) + \frac{1}{\varepsilon_1}R[y_1^{(j)} - x; x] - \frac{1}{\varepsilon_2}R[y_2^{(j)} - x; x]$$
$$= X_x[t, x(t)]w^{(j)}(t) + \varepsilon_1\{R[\varphi^{(j)} + v_1^{(j)}; x] - R[\varphi^{(j)} + v_2^{(j)}; x]\}$$
$$+ (\varepsilon_1 - \varepsilon_2)R[\varphi^{(j)} + v_2^{(j)}; x] .$$

Therefore, the integral form

$$(2.8) \quad w^{(j)}(t) = \int_a^t X_x[s, x(s)]w^{(j)}(s)ds$$
$$+ \varepsilon_1 \int_a^t \{R[\varphi^{(j)}(s) + v_1^{(j)}(s); x] - R[\varphi^{(j)}(s) + v_2^{(j)}(s); x]\}ds$$
$$+ (\varepsilon_1 - \varepsilon_2)\int_a^t R[\varphi^{(j)}(s) + v_2^{(j)}(s); x]ds$$

holds, which brings the inequality

(2.9) $\|w^{(j)}(t)\| \leqq K_1 \displaystyle\int_a^t \|w^{(j)}(s)\| ds$

$$+ |\varepsilon_1| \int_a^t \|R[\varphi^{(j)}(s) + v_1^{(j)}(s);\, x] - R[\varphi^{(j)}(s) + v_2^{(j)}(s);\, x]\| ds$$

$$+ |\varepsilon_1 - \varepsilon_2| \int_a^t \|R[\varphi^{(j)}(s) + v_2^{(j)}(s);\, x]\| ds .$$

Because of the estimations (1.11) and (1.12) we have

(2.10) $\|w^{(j)}(t)\| \leqq K_1 \displaystyle\int_a^t \|w^{(j)}(s)\| ds + |\varepsilon_1| L_R \int_a^t \|w^{(j)}(s)\| ds$

$$+ |\varepsilon_1 - \varepsilon_2| K_2' \delta_0^2 (b-a)$$

$$\leqq |\varepsilon_1 - \varepsilon_2| K_2' \delta_0^2 (b-a) + (K_1 + \varepsilon_0 L_R) \int_a^t \|w^{(j)}(s)\| ds .$$

Applying the Gronwall's lemma to (2.10), it holds that

$$\|w^{(j)}(t)\| \leqq |\varepsilon_1 - \varepsilon_2| K_2' \delta_0^2 (b-a) e^{(K_1 + \varepsilon_0 L_R)(t-a)} ,$$

which implies

(2.11) $\|w^{(j)}\|_C \leqq |\varepsilon_1 - \varepsilon_2| K_2' \delta_0^2 (b-a) e^{(K_1 + \varepsilon_0 L_R)(b-a)} .$ □

Lemma 3.   *For sufficiently small positive numbers $\tilde{\Delta}$ and $\tilde{\varepsilon}$, the relation*

(2.12) $S(\varepsilon;\, \mathscr{F}(\eta + \zeta)) - S(\varepsilon;\, \mathscr{F}\eta) = LP(\cdot;\, \mathscr{F}\eta, \varepsilon)\zeta + o(\|\zeta\|)$

holds if $\eta,\ \eta + \zeta \in B_{\tilde{\Delta}}$ and $|\varepsilon| \leqq \tilde{\varepsilon}$, where $P(t;\, \mathscr{F}\eta, \varepsilon)$ is a linear mapping which maps $\zeta \in \mathbf{R}^n$ to an $n \times n$ matrix $P(t;\, \mathscr{F}\eta, \varepsilon)\zeta$ with components of continuous functions on I.

Proof.   We shall put

$$\mathscr{F}(\eta + \zeta) = x_1, \quad \mathscr{F}\eta = x_2, \quad \mathscr{F}(\eta + \zeta + \varepsilon e_j) = y_1^{(j)} = x_1 + \varepsilon u_1^{(j)},$$
$$\mathscr{F}(\eta + \varepsilon e_j) = y_2^{(j)} = x_2 + \varepsilon u_2^{(j)} .$$

Then we have

(2.13) $s_j(\varepsilon;\, \mathscr{F}(\eta + \zeta)) - s_j(\varepsilon;\, \mathscr{F}\eta) = \dfrac{1}{\varepsilon}\{Ly_1^{(j)} - Lx_1 - Ly_2^{(j)} + Lx_2\}$

$$= L(u_1^{(j)} - u_2^{(j)}) .$$

Define $w^{(j)}$ by $w^{(j)} = u_1^{(j)} - u_2^{(j)}$, then $w^{(j)}$ must satisfy the differential equation

(2.14) $\dfrac{dw^{(j)}}{dt} = \dfrac{1}{\varepsilon}\{X[t,\, x_1(t) + \varepsilon u_1^{(j)}(t)]$

$$- X[t,\, x_1(t)] - X[t,\, x_2(t) + \varepsilon u_2^{(j)}(t)] + X[t,\, x_2(t)]\}, \quad a < t < b,$$

with the initial condition

(2.15) $$w^{(j)}(a) = 0.$$

Because of the equation (1.7), the right-hand side of (2.14) can be written as following:

$$-\frac{1}{\varepsilon}\{X[t, x_2(t)] + X_x[t, x_2(t)](x_1 - x_2 + \varepsilon u_1^{(j)})$$

$$+ \frac{1}{2}X_{xx}[t, x_2(t)](x_1 - x_2 + \varepsilon u_1^{(j)})(x_1 - x_2 + \varepsilon u_1^{(j)}) + Q(x_1 - x_2 + \varepsilon u_1^{(j)}; x_2)$$

$$- X[t, x_2(t)] - X_x[t, x_2(t)](x_1 - x_2) - \frac{1}{2}X_{xx}[t, x_2(t)](x_1 - x_2)(x_1 - x_2)$$

$$- Q(x_1 - x_2; x_2)$$

$$- X[t, x_2(t)] - X_x[t, x_2(t)]\varepsilon u_2^{(j)} - \frac{1}{2}X_{xx}[t, x_2(t)]\varepsilon u_2^{(j)}\varepsilon u_2^{(j)}$$

$$- Q(\varepsilon u_2^{(j)}; x_2) + X[t, x_2(t)]\}$$

$$= X_x[t, x_2(t)]w^{(j)} + \frac{1}{2}\varepsilon X_{xx}[t, x_2(t)](u_1^{(j)} + u_2^{(j)})w^{(j)}$$

$$+ X_{xx}[t, x_2(t)](x_1 - x_2)u_1^{(j)}$$

$$+ \frac{1}{\varepsilon}\{Q(x_1 - x_2 + \varepsilon u_1^{(j)}; x_2) - Q(x_1 - x_2; x_2) - Q(\varepsilon u_2^{(j)}; x_2)\}$$

$$= X_x[t, x_2(t)]w^{(j)} + \varepsilon X_{xx}[t, x_2(t)]u_2^{(j)}w^{(j)} + X_{xx}[t, x_2(t)](x_1 - x_2)u_2^{(j)}$$

$$+ X_{xx}[t, x_2(t)](x_1 - x_2)w^{(j)} + \frac{1}{2}\varepsilon X_{xx}[t, x_2(t)]w^{(j)}w^{(j)}$$

$$+ \frac{1}{\varepsilon}\{Q(x_1 - x_2 + \varepsilon u_1^{(j)}; x_2) - Q(x_1 - x_2; x_2) - Q(\varepsilon u_2^{(j)}; x_2)\}.$$

Utilizing (1.8) and (1.9), we have

$$\frac{1}{\varepsilon}\{Q(x_1 - x_2 + \varepsilon u_1^{(j)}; x_2) - Q(x_1 - x_2; x_2) - Q(\varepsilon u_2^{(j)}; x_2)\}$$

$$= \varepsilon(x_1 - x_2)q_1(u_2^{(j)}) + u_1^{(j)}q_2(x_1 - x_2) + \varepsilon^2 w^{(j)}q_3(u_1^{(j)}, u_2^{(j)}),$$

which brings the equation

(2.16) $$\frac{dw^{(j)}}{dt} = \{X_x[t, x_2(t)] + \varepsilon X_{xx}[t, x_2(t)]u_2^{(j)}(t)\}w^{(j)}$$

$$+ X_{xx}[t, x_2(t)](x_1(t) - x_2(t))u_2^{(j)}(t)$$

$$+ X_{xx}[t, x_2(t)]\left\{x_1(t) - x_2(t) + \frac{1}{2}\varepsilon w^{(j)}\right\}w^{(j)}$$

$$+ u_1^{(j)}(t)q(x_1 - x_2) + \varepsilon(x_1 - x_2)q_1(u_2^{(j)}) + \varepsilon^2 w^{(j)}q_3(u_1^{(j)}, u_2^{(j)}).$$

Let us denote the matrizant which corresponds to

$$X_x[t, x_2(t)] + \varepsilon X_{xx}[t, x_2(t)]u_2^{(j)}$$

by $\Psi_j(t; \mathscr{F}\eta, \varepsilon)$ (note that Ψ_j does not depend on ξ) and define a mapping $H_{j,\varepsilon}$ which maps a function $y(t) \in C(I)$ into $C^1(I)$ by

$$[H_{j,\varepsilon}y](t) = \Psi_j(t; \mathscr{F}\eta, \varepsilon) \int_a^t \Psi_j^{-1}(s; \mathscr{F}\eta, \varepsilon)y(s)ds,$$

then we can transform (2.16) and (2.15) into the integral equation

$$(2.17) \qquad w^{(j)}(t) = [H_{j,\varepsilon}Y\{\mathscr{F}(\eta+\xi) - \mathscr{F}\eta\}u_2^{(j)}](t)$$
$$+ \varepsilon[H_{j,\varepsilon}\{\mathscr{F}(\eta+\xi) - \mathscr{F}\eta\}q_1(u_2^{(j)})](t)$$
$$+ \varepsilon^2[H_{j,\varepsilon}\{w^{(j)}q_3(u_1^{(j)}, u_2^{(j)})\}](t)$$
$$+ [H_{j,\varepsilon}Y\{\mathscr{F}(\eta+\xi) - \mathscr{F}\eta + \frac{1}{2}\varepsilon w^{(j)}\}w^{(j)}](t)$$
$$+ [H_{j,\varepsilon}\{u_1^{(j)}q_2(\mathscr{F}(\eta+\xi) - \mathscr{F}\eta)\}](t).$$

Here $Y(t)$ stands for $X_{xx}[t, \mathscr{F}\eta(t)]$.

Since $\mathscr{F}(\eta+\xi) - \mathscr{F}\eta = \Phi(\cdot; \mathscr{F}\eta)\xi + o(\|\xi\|)$ and $u_2^{(j)}$ is equal to $\varphi^{(j)} + v_2^{(j)}$ which is bounded, the first term of the right-hand side of (2.17) is the order $O(\|\xi\|)$. Also, the second term is the order $O(\|\xi\|)$. Moreover, because of (1.9) in Assumption 1, the last term is the order $o(\|\xi\|)$. There is no factor which brings terms of the order lower than $O(\|\xi\|)$ in the other terms of the right-hand side of (2.17), thus we can assert that $w^{(j)}(t)$ is $O(\|\xi\|)$ *as a whole*.

Define an $n \times n$ matrix $h_1(t)$ with continuous components by

$$[H_{j,\varepsilon}\{\mathscr{F}(\eta+\xi) - \mathscr{F}\eta\}q_1(u_2^{(j)})](t) = h_1^{(j)}(t)\xi + o(\|\xi\|).$$

In the same manner we have

$$[H_{j,\varepsilon}\{w^{(j)}q_3(u_1^{(j)}, u_2^{(j)})\}](t) = h_2^{(j)}(t)\xi + o(\|\xi\|).$$

where $h_2(t)$ is an $n \times n$ matrix with continuous components. Put

$$w_0^{(j)}(t) = [H_{j,\varepsilon}Y\{\mathscr{F}(\eta+\xi) - \mathscr{F}\eta + \frac{1}{2}\varepsilon w^{(j)}\}w^{(j)}](t),$$

then $H_{j,\varepsilon}Y\{\mathscr{F}(\eta+\xi) - \mathscr{F}\eta\}w^{(j)}$ is the order $O(\|\xi\|^2)$. On the other hand $\frac{1}{2}\varepsilon H_{j,\varepsilon}Yw^{(j)}w^{(j)}$ must be the order $O(\|\xi\|^2)$. Therefore $w_0^{(j)} = o(\|\xi\|)$ holds. Summing up the above considerations, the equation (2.17) can be rewritten as

$$(2.18) \quad w^{(j)}(t) = [H_{j,\varepsilon}Y\Phi(\cdot; \mathscr{F}\eta)\xi u_2^{(j)}](t) + \varepsilon h_1^{(j)}(t)\xi + \varepsilon^2 h_2^{(j)}(t)\xi + o(\|\xi\|).$$

Hence, define the linear operator $P(t; \mathscr{F}\eta, \varepsilon)$ such that $P(t; \mathscr{F}\eta, \varepsilon)\xi$ is formed by

$$[H_{j,\varepsilon}Y\Phi(\cdot; \mathscr{F}\eta)\xi u_2^{(j)}](t) + \varepsilon h_1^{(j)}(t)\xi + \varepsilon^2 h_2^{(j)}(t)\xi$$

as the j-th column vector, then we have the conclusion. $\square$

Corollary. *For sufficiently small positive numbers $\tilde{\Delta}$ and $\tilde{\varepsilon}$, the estimation*

(2.19) $$\|S(\varepsilon; \mathscr{F}\eta_1) - S(\varepsilon; \mathscr{F}\eta_2)\| \leq const. \|\eta_1 - \eta_2\|$$

holds if $\|\eta_1 - \eta^*\|$, $\|\eta_2 - \eta^*\| \leq \tilde{\Delta}$ and $|\varepsilon| \leq \tilde{\varepsilon}$. Here the constant does not depend on η_1, η_2 and ε.

Now we are ready to state

Theorem 1. *There exist positive numbers Δ_1 and ε_1 such that, for any ε satisfying $|\varepsilon| \leq \varepsilon_1$, the equation*

$$\mathscr{T}(\eta, \varepsilon) = 0$$

has a unique solution $\hat{\eta} = \hat{\eta}(\varepsilon)$ in the ball $\|\eta - \eta^*\| \leq \Delta_1$, and $\hat{\eta} \to \eta^*$ as $\varepsilon \to 0$
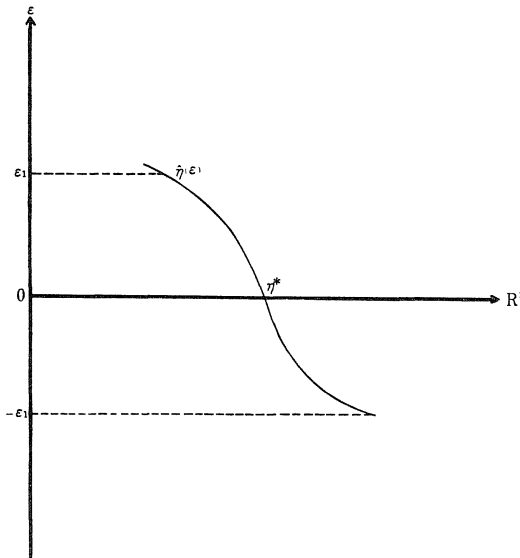

Figure 2.1.

Proof. We divide the proof into three parts.

First stage. Put the neighbourhood Nb of $(\eta^*, 0)$ as

$$Nb = \{(\eta, \varepsilon); \|\eta - \eta^*\| \leq \tilde{\Delta}, |\varepsilon| \leq \tilde{\varepsilon}\},$$

and assume that Nb is included in $B_\Delta \times \{\varepsilon; |\varepsilon| \leq \tilde{\varepsilon}\}$. For (η_1, ε_1), (η_2, ε_2) in Nb, the difference of $\mathscr{T}(\eta_1, \varepsilon_1)$ and $\mathscr{T}(\eta_2, \varepsilon_2)$ is written by

(2.20) $\mathscr{T}(\eta_1, \varepsilon_1) - \mathscr{T}(\eta_2, \varepsilon_2)$

$\qquad = {}^t S(\varepsilon_1; \mathscr{F}\eta_1)\{L\mathscr{F}\eta_1 - d\} - {}^t S(\varepsilon_2; \mathscr{F}\eta_2)\{L\mathscr{F}\eta_2 - d\}$

$\qquad = {}^t \{S(\varepsilon_1; \mathscr{F}\eta_1) - S(\varepsilon_2; \mathscr{F}\eta_1)\}\{L\mathscr{F}\eta_1 - d\}$

$\qquad\quad + {}^t \{S(\varepsilon_2; \mathscr{F}\eta_1) - S(\varepsilon_2; \mathscr{F}\eta_2)\}\{L\mathscr{F}\eta_1 - d\}$

$\qquad\quad + {}^t S(\varepsilon_2; \mathscr{F}\eta_2)L(\mathscr{F}\eta_1 - \mathscr{F}\eta_2).$

Because of Lemma 2, the first term of the right-hand side of (2.20) is estimated by const. $|\varepsilon_1 - \varepsilon_2|$. Besides, Corollary of Lemma 3 gives us the estimation

$$\|{}^t\{S(\varepsilon_2; \mathscr{F}\eta_1) - S(\varepsilon_2; \mathscr{F}\eta_2)\}\{L\mathscr{F}\eta_1 - d\}\| \leqq \text{const.}\ \|\eta_1 - \eta_2\|.$$

The inequality

$$\|\mathscr{F}\eta_1 - \mathscr{F}\eta_2\| \leqq \text{const.}\ \|\eta_1 - \eta_2\|$$

is easy to show.

Hence we have

$$\|\mathscr{T}(\eta_1, \varepsilon_1) - \mathscr{T}(\eta_2, \varepsilon_2)\| \leqq \text{const.}\ |\varepsilon_1 - \varepsilon_2| + \text{const.}\ \|\eta_1 - \eta_2\|.$$

This shows the joint continuity of $\mathscr{T}(\eta, \varepsilon)$ with respect to η and ε.

Second stage. We shall calculate the Fréchet derivative of $\mathscr{T}(\eta, \varepsilon)$ with respect to η in Nb. If $\varepsilon \neq 0$, the equation

(2.21) $\mathscr{T}(\eta + \xi, \varepsilon) - \mathscr{T}(\eta, \varepsilon)$

$\qquad = {}^t S(\varepsilon; \mathscr{F}(\eta + \xi))\{L\mathscr{F}(\eta + \xi) - d\} - {}^t S(\varepsilon; \mathscr{F}\eta)\{L\mathscr{F}\eta - d\}$

$\qquad = {}^t \{S(\varepsilon; \mathscr{F}(\eta + \xi)) - S(\varepsilon; \mathscr{F}\eta)\}\{L\mathscr{F}(\eta + \xi) - d\}$

$\qquad\quad + {}^t S(\varepsilon; \mathscr{F}\eta)L(\mathscr{F}(\eta + \xi) - \mathscr{F}\eta)$

holds. By virtue of Lemma 3, we have

$$\begin{aligned}
{}^t\{S(\varepsilon; \mathscr{F}(\eta + \xi)) &- S(\varepsilon; \mathscr{F}\eta)\}\{L\mathscr{F}(\eta + \xi) - d\}\\
&= {}^t(LP(\cdot; \mathscr{F}\eta, \varepsilon)\xi)\{L\mathscr{F}\eta - d\} + o(\|\xi\|).
\end{aligned}$$

On the other hand,

$${}^t S(\varepsilon; \mathscr{F}\eta)L(\mathscr{F}(\eta + \xi) - \mathscr{F}\eta) = {}^t S(\varepsilon; \mathscr{F}\eta)G(\eta)\xi + o(\|\xi\|)$$

holds. Therefore, from (2.21) we have

$$\begin{aligned}
\mathscr{T}(\eta + \xi, \varepsilon) &- \mathscr{T}(\eta, \varepsilon)\\
&= {}^t(LP(\cdot; \mathscr{F}\eta, \varepsilon)\xi)\{L\mathscr{F}\eta - d\} + {}^t S(\varepsilon; \mathscr{F}\eta)G(\eta)\xi + o(\|\xi\|),
\end{aligned}$$

which implies that the Fréchet derivative $\mathscr{T}_\eta(\eta, \varepsilon)$ exists in Nb and is given by

(2.22) $\qquad \mathscr{T}_\eta(\eta, \varepsilon)\xi = {}^t(LP(\cdot; \mathscr{F}\eta, \varepsilon)\xi)\{L\mathscr{F}\eta - d\} + {}^t S(\varepsilon; \mathscr{F}\eta)G(\eta)\xi.$

In the case $\varepsilon = 0$, we obtain

$$(2.23) \quad \mathcal{T}(\eta + \xi, 0) - \mathcal{T}(\eta, 0)$$
$$= {}^t G(\eta + \xi) \{L\mathcal{F}(\eta + \xi) - \boldsymbol{d}\} - {}^t G(\eta) \{L\mathcal{F}\eta - \boldsymbol{d}\}$$
$$= {}^t \{G(\eta + \xi) - G(\eta)\} \{L\mathcal{F}(\eta + \xi) - \boldsymbol{d}\} + {}^t G(\eta) L\{\mathcal{F}(\eta + \xi) - \mathcal{F}\eta\}.$$

As shown in [8], the Fréchet derivative $\Phi_\eta(t; \mathcal{F}\eta)$ of the matrizant $\Phi(t; \mathcal{F}\eta)$ with respect to η exists and is represented by

$$(2.24) \quad \Phi_\eta(t; \mathcal{F}\eta)\xi$$
$$= \Phi(t; \mathcal{F}\eta) \int_a^t \Phi^{-1}(s; \mathcal{F}\eta) X_{xx}[s, \mathcal{F}\eta(s)] \Phi(s; \mathcal{F}\eta)\xi \Phi(s; \mathcal{F}\eta) ds.$$

Hence the equation

$$(2.25) \quad \mathcal{T}_\eta(\eta, 0) = {}^t (L\Phi_\eta(\cdot; \mathcal{F}\eta)\xi) \{L\mathcal{F}\eta - \boldsymbol{d}\} + \mathfrak{G}(\eta)\xi$$

holds.

Let us consider $\lim_{\varepsilon \to 0} \mathcal{T}_\eta(\eta, \varepsilon)$ in (2.22). By (1.20) and (1.23) we have

$$u_2^{(j)}(t) = \frac{1}{\varepsilon} \{\mathcal{F}(\eta + \varepsilon e_j)(t) - \mathcal{F}\eta(t)\} = \varphi^{(j)}(t) + v^{(j)}(t)$$

and

$$\|v^{(j)}\|_C \leq C^* |\varepsilon|,$$

hence $u_2^{(j)} \to \varphi^{(j)}$ as $\varepsilon \to 0$. Moreover, as $\varepsilon \to 0$ the matrizant $\Psi_j(t; \mathcal{F}\eta, \varepsilon)$, which corresponds to

$$X_x[t, \mathcal{F}\eta(t)] + \varepsilon X_{xx}[t, \mathcal{F}\eta(t)] u_2^{(j)}(t),$$

has the limit $\Psi_j(t; \mathcal{F}\eta, 0)$ which is the matrizant corresponding to $X_x[t, \mathcal{F}\eta(t)]$. That is, $\Psi_j(t; \mathcal{F}\eta, 0)$ is identical to $\Phi(t; \mathcal{F}\eta)$ for every j. Therefore

$$[H_{j,\varepsilon} Y \Phi(\cdot; \mathcal{F}\eta)\xi u_2^{(j)}](t)$$
$$= \Psi_j(t; \mathcal{F}\eta, \varepsilon) \int_a^t \Psi_j^{-1}(s; \mathcal{F}\eta, \varepsilon) X_{xx}[s, \mathcal{F}\eta(s)] \Phi(s; \mathcal{F}\eta)\xi u_2^{(j)}(s) ds$$
$$\to \Phi(t; \mathcal{F}\eta) \int_a^t \Phi^{-1}(s; \mathcal{F}\eta) X_{xx}[s, \mathcal{F}\eta(s)] \Phi(s; \mathcal{F}\eta)\xi \varphi^{(j)}(s) ds$$

holds as $\varepsilon \to 0$. In such limitation,

$$\varepsilon h_1^{(j)}(t)\xi + \varepsilon^2 h_2^{(j)}(t)\xi \to 0,$$

which implies

$$P(t; \mathcal{F}\eta, \varepsilon)\xi \to \Phi_\eta(t; \mathcal{F}\eta)\xi$$

as $\varepsilon \to 0$.   Hence, we obtain the equation

$$\lim_{\varepsilon \to 0} \mathscr{T}_\eta(\eta, \varepsilon)\xi = {}^t(L\varPhi_\eta(\cdot\ ;\ \mathscr{F}\eta)\xi)\{L\mathscr{F}\eta - d\} + \mathfrak{G}(\eta)\xi$$
$$= \mathscr{T}_\eta(\eta, 0)\xi,$$

which implies the continuity of $\mathscr{T}_\eta(\eta, \varepsilon)$ at $\varepsilon = 0$.

Third stage.   For the isolated minimal point η^*, we have

(2.26) $\mathscr{T}_\eta(\eta^*, 0)\xi = \mathfrak{G}(\eta^*)\xi + {}^t(L\varPhi_\eta(\cdot\ ;\ \mathscr{F}\eta^*)\xi)\{L\mathscr{F}\eta^* - d\}$.

Then,

$${}^t(L\varPhi_\eta(\cdot\ ;\ \mathscr{F}\eta^*)\xi)\{L\mathscr{F}\eta^* - d\}$$
$$= {}^t(L\varPhi(\cdot\ ;\ \mathscr{F}\eta^*)\int_a^{\cdot} \varPhi^{-1}(s;\ \mathscr{F}\eta^*)X_{xx}[s,\ \mathscr{F}\eta^*(s)]\varPhi(s;\ \mathscr{F}\eta^*)\xi\varPhi(s;\ \mathscr{F}\eta^*)ds)$$
$$\times \{L\mathscr{F}\eta^* - d\}$$
$$= {}^t(p_N \int_a^{\cdot} \varPhi^{-1}(s;\ \mathscr{F}\eta^*)X_{xx}[s,\ \mathscr{F}\eta^*(s)]\varPhi(s;\ \mathscr{F}\eta^*)\xi\varPhi(s;\ \mathscr{F}\eta^*)ds)$$
$$\times {}^t G(\eta^*)\{L\mathscr{F}\eta^* - d\},$$

where p_N means an operation such that $p_N y = {}^t(y(t_1),\dots, y(t_N))$ for $y \in C(I)$. Because of (1.13) in Assumption 2, the above term is equal to 0.   Hence, from (2.26) the relation

$$\mathscr{T}_\eta(\eta^*, 0)\xi = \mathfrak{G}(\eta^*)\xi$$

holds, which implies the invertibility of $\mathscr{T}_\eta(\eta^*, 0)$ by virtue of (1.14) in Assumption 2.

By the implicit function theorem (Th. 1.7 in [3]), we have the desired conclusion.   □

§3. Convergence of the Iterative Process

In the preceding section we have shown the existence of a fixed point for the operator $\mathscr{A}$ in Nb. We shall denote the fixed point by $\hat{\eta} = \hat{\eta}(\varepsilon)$ and call it the *approximate minimal point* of $J(\eta)$ because it is an approximation of η^* and tends to η^* as $\varepsilon \to 0$.

In this section we shall show that in some neighbourhood of $\hat{\eta}$ the iterative process (1.19) converges to $\hat{\eta}$ while ε is fixed as a positive number less than ε_1.

Since $\hat{\eta}$ is defined by $\mathscr{T}(\hat{\eta}, \varepsilon) = 0$, the equation

(3.1) $\{{}^t S(\varepsilon;\ \mathscr{F}\hat{\eta})S(\varepsilon;\ \mathscr{F}\hat{\eta})\}^{-1}\ {}^t S(\varepsilon;\ \mathscr{F}\hat{\eta})\{L\mathscr{F}\hat{\eta} - d\} = 0$

holds as far as $\|\hat{\eta} - \eta^*\| \leqq \Delta$.

On the convergence we have the following theorem.

Theorem 2. *There exists a positive number Δ_2 such that the iterative process (1.19) starting with any $\eta_0 = \eta$ in the Δ_2-neighbourhood of $\hat{\eta}$ converges to $\hat{\eta}$.*

Proof. Let Δ_2 be sufficiently small such that the Δ_2-neighbourhood of $\hat{\eta}$ is included in B_Δ. We shall show that $\mathscr{A}$ is a contraction mapping on the Δ_2-neighbourhood of $\hat{\eta}$.

We have

$$(3.2) \quad \begin{aligned} \mathscr{A}\eta - \hat{\eta} &= \eta - \hat{\eta} - \{{}^t S(\varepsilon; \mathscr{F}\eta) S(\varepsilon; \mathscr{F}\eta)\}^{-1} \mathscr{T}(\eta, \varepsilon) \\ &= \eta - \hat{\eta} - \{{}^t S(\varepsilon; \mathscr{F}\eta) S(\varepsilon; \mathscr{F}\eta)\}^{-1} \{\mathscr{T}(\eta, \varepsilon) - \mathscr{T}(\hat{\eta}, \varepsilon)\} \\ &= \{{}^t S(\varepsilon; \mathscr{F}\eta) S(\varepsilon; \mathscr{F}\eta)\}^{-1} \{{}^t S(\varepsilon; \mathscr{F}\eta) S(\varepsilon; \mathscr{F}\eta)(\eta - \hat{\eta}) \\ &\quad - \mathscr{T}_\eta(\hat{\eta}, \varepsilon)(\eta - \hat{\eta})\} + o(\|\eta - \hat{\eta}\|). \end{aligned}$$

Because of (2.22), the equation

$$(3.3) \quad \begin{aligned} &{}^t S(\varepsilon; \mathscr{F}\eta) S(\varepsilon; \mathscr{F}\eta)(\eta - \hat{\eta}) - \mathscr{T}_\eta(\hat{\eta}, \varepsilon)(\eta - \hat{\eta}) \\ &= {}^t S(\varepsilon; \mathscr{F}\eta) S(\varepsilon; \mathscr{F}\eta)(\eta - \hat{\eta}) - {}^t S(\varepsilon; \mathscr{F}\hat{\eta}) G(\hat{\eta})(\eta - \hat{\eta}) \\ &\quad - {}^t \{\boldsymbol{L} P(\cdot; \mathscr{F}\hat{\eta}, \varepsilon)(\eta - \hat{\eta})\} \{\boldsymbol{L}\mathscr{F}\hat{\eta} - \boldsymbol{d}\} + o(\|\eta - \hat{\eta}\|) \end{aligned}$$

holds. The first term of the right-hand side of the above equation is written by

$$(3.4) \quad \begin{aligned} &\{{}^t S(\varepsilon; \mathscr{F}\eta) S(\varepsilon; \mathscr{F}\eta) - {}^t S(\varepsilon; \mathscr{F}\hat{\eta}) G(\hat{\eta})\}(\eta - \hat{\eta}) \\ &= \{{}^t S(\varepsilon; \mathscr{F}\eta) S(\varepsilon; \mathscr{F}\eta) - {}^t S(\varepsilon; \mathscr{F}\hat{\eta}) S(\varepsilon; \mathscr{F}\hat{\eta})\}(\eta - \hat{\eta}) \\ &\quad + {}^t S(\varepsilon; \mathscr{F}\hat{\eta}) \{S(\varepsilon; \mathscr{F}\hat{\eta}) - G(\hat{\eta})\}(\eta - \hat{\eta}). \end{aligned}$$

By virtue of (2.12), the estimation for the compound adjusting matrices

$$\|{}^t S(\varepsilon; \mathscr{F}\eta) S(\varepsilon; \mathscr{F}\eta) - {}^t S(\varepsilon; \mathscr{F}\hat{\eta}) S(\varepsilon; \mathscr{F}\hat{\eta})\| \leqq \text{const.} \|\eta - \hat{\eta}\|$$

holds, which brings

$$(3.5) \quad \{{}^t S(\varepsilon; \mathscr{F}\eta) S(\varepsilon; \mathscr{F}\eta) - {}^t S(\varepsilon; \mathscr{F}\hat{\eta}) S(\varepsilon; \mathscr{F}\hat{\eta})\}(\eta - \hat{\eta}) = o(\|\eta - \hat{\eta}\|).$$

Utilizing the inequality (1.24), we have

$$(3.6) \quad \|{}^t S(\varepsilon; \mathscr{F}\hat{\eta}) \{S(\varepsilon; \mathscr{F}\hat{\eta}) - G(\hat{\eta})\}\| \leqq (L_0 M_0 + \delta_1)\delta_1.$$

On the other hand, the principal part of the second term of (3.3) has the following j-th column vector

$$ {}^t (\boldsymbol{L}\Psi_j(\cdot; \mathscr{F}\hat{\eta}, \varepsilon)) \int_a^{\cdot} \Psi_j^{-1}(s; \mathscr{F}\hat{\eta}, \varepsilon) X_{xx}[s, \mathscr{F}\hat{\eta}(s)] \Phi(s; \mathscr{F}\hat{\eta})(\eta - \hat{\eta}) u_2^{(j)}(s) ds) $$

$$\times \{ \boldsymbol{L}\mathscr{F}\hat{\eta} - \boldsymbol{d} \}$$

$$= {}^{t}(p_{N}\int_{a}^{\cdot} \Psi_{j}^{-1}(s;\, \mathscr{F}\hat{\eta},\, \varepsilon)X_{xx}[s,\, \mathscr{F}\hat{\eta}(s)]\Phi(s;\, \mathscr{F}\hat{\eta})(\eta-\hat{\eta})u_{2}^{(j)}(s)ds)$$

$$\times {}^{t}(\boldsymbol{L}\Psi_{j}(\cdot\,;\, \mathscr{F}\hat{\eta},\, \varepsilon))\{ \boldsymbol{L}\mathscr{F}\hat{\eta} - \boldsymbol{d} \},$$

where p_N means the same operation in Section 2. Then we have the estimation

$$\boldsymbol{L}\Psi_{j}(\cdot\,;\, \mathscr{F}\hat{\eta},\, \varepsilon) - S(\varepsilon;\, \mathscr{F}\hat{\eta})$$
$$= \boldsymbol{L}\Psi_{j}(\cdot\,;\, \mathscr{F}\hat{\eta},\, \varepsilon) - G(\hat{\eta}) + G(\hat{\eta}) - S(\varepsilon;\, \mathscr{F}\hat{\eta})$$
$$= o(\varepsilon)$$

because of (1.24) and the same assertion in the proof of Theorem 1.

Therefore, we have

(the j-th column vector of ${}^{t}\{ \boldsymbol{L}P(\cdot\,;\, \mathscr{F}\hat{\eta},\, \varepsilon)(\eta-\hat{\eta})\}\{ \boldsymbol{L}\mathscr{F}\hat{\eta} - \boldsymbol{d} \})$

$$= {}^{t}(p_{N}\int_{a}^{\cdot} \Psi_{j}^{-1}(s;\, \mathscr{F}\hat{\eta},\, \varepsilon)X_{xx}[s,\, \mathscr{F}\hat{\eta}(s)]\Phi(s;\, \mathscr{F}\hat{\eta})(\eta-\hat{\eta})u_{2}^{(j)}(s)ds)$$

$$\times {}^{t}S(\varepsilon;\, \mathscr{F}\hat{\eta})\{ \boldsymbol{L}\mathscr{F}\hat{\eta} - \boldsymbol{d} \} + o(\varepsilon)(\eta-\hat{\eta}),$$

which implies the estimation

$$(3.7) \qquad \| {}^{t}\{ \boldsymbol{L}P(\cdot\,;\, \mathscr{F}\hat{\eta},\, \varepsilon)(\eta-\hat{\eta})\}\{ \boldsymbol{L}\mathscr{F}\hat{\eta} - \boldsymbol{d} \} \| \leqq o(\varepsilon)\|\eta-\hat{\eta}\|$$

because of (3.1).

Remembering the estimation (1.25) and summing up (3.4), (3.5) and (3.6), we have

$$\| \eta-\hat{\eta} - \{ {}^{t}S(\varepsilon;\, \mathscr{F}\eta)S(\varepsilon;\, \mathscr{F}\hat{\eta})\}^{-1}\, {}^{t}S(\varepsilon;\, \mathscr{F}\hat{\eta})G(\hat{\eta})(\eta-\hat{\eta}) \|$$
$$\leqq \frac{\delta_1 M_1(L_0 M_0 + \delta_1)}{1 - \delta_1 M_1(2L_0 M_0 + \delta_1)} \|\eta-\hat{\eta}\|\,.$$

Thus choose ε and $\|\eta-\hat{\eta}\|$ sufficiently small such that the inequality

$$\| \mathscr{A}\eta-\hat{\eta} \| \leqq \left(\frac{\delta_1 M_1(L_0 M_0 + \delta_1)}{1 - \delta_1 M_1(2L_0 M_0 + \delta_1)} + \alpha \right)\|\eta-\hat{\eta}\|$$

holds, where α is a positive number satisfying the inequality

$$\alpha \leqq \frac{1 - \delta_1 L_0 M_0 M_1}{1 - \delta_1 M_1(2L_0 M_0 + \delta_1)}(1 - \kappa)$$

for an arbitrary positive $\kappa < 1$. Therefore, we have

$$\| \mathscr{A}\eta-\hat{\eta} \| \leqq (1 - \kappa)\|\eta-\hat{\eta}\|,$$

which shows the contraction property of $\mathscr{A}$. $\qquad\square$

From the above considerations we have the assertion as in [5] that the

convergence of the iteration would not be expected to be quadratic in the neighbourhood of $\hat{\eta}$ while a fixed ε is chosen to be apart from zero. The reason is the following. The equation

$$\mathscr{A}(\hat{\eta}+\xi)-\mathscr{A}\hat{\eta}$$
$$=\mathscr{A}(\hat{\eta}+\xi)-\hat{\eta}$$
$$=\xi-\{{}^t S(\varepsilon;\ \mathscr{F}(\hat{\eta}+\xi))S(\varepsilon;\ \mathscr{F}(\hat{\eta}+\xi))\}^{-1}\{\mathscr{T}(\hat{\eta}+\xi,\ \varepsilon)-\mathscr{T}(\hat{\eta},\ \varepsilon)\},$$

implies that the Fréchet derivative of $\mathscr{A}$ at $\hat{\eta}$ is given by

$$(3.8)\quad \mathscr{A}'(\hat{\eta})\xi = \xi - \{{}^t S(\varepsilon;\ \mathscr{F}\hat{\eta})S(\varepsilon;\ \mathscr{F}\hat{\eta})\}^{-1}[{}^t(\boldsymbol{L}P(\cdot\ ;\ \mathscr{F}\hat{\eta},\ \varepsilon)\xi)\{\boldsymbol{L}\mathscr{F}\hat{\eta}-\boldsymbol{d}\}$$
$$+{}^t S(\varepsilon;\ \mathscr{F}\hat{\eta})G(\hat{\eta})\xi]$$
$$=\{{}^t S(\varepsilon;\ \mathscr{F}\hat{\eta})S(\varepsilon;\ \mathscr{F}\hat{\eta})\}^{-1}[{}^t S(\varepsilon;\ \mathscr{F}\hat{\eta})\{S(\varepsilon;\ \mathscr{F}\hat{\eta})-G(\hat{\eta})\}\xi$$
$$-{}^t(\boldsymbol{L}P(\cdot\ ;\ \mathscr{F}\hat{\eta},\ \varepsilon)\xi)\{\boldsymbol{L}\mathscr{F}\hat{\eta}-\boldsymbol{d}\}].$$

As in the derivation of (3.7), we have

$$ {}^t(\boldsymbol{L}P(\cdot\ ;\ \mathscr{F}\hat{\eta},\ \varepsilon)\xi)\{\boldsymbol{L}\mathscr{F}\hat{\eta}-\boldsymbol{d}\}=o(\varepsilon)\xi. $$

On the other hand, $S(\varepsilon;\mathscr{F}\hat{\eta})-G(\hat{\eta})$ does not vanish if $\varepsilon\neq 0$. Then, we can not expect that $\mathscr{A}'(\hat{\eta})\xi$ vanishes for any $\xi\neq 0$. Hence the convergence is not quadratical.

If we assume that $X_{xx}[t,\ x]$ is identically zero on $\mathscr{D}$ (almost linear case for differential equation), then the convergence is quadratic because in such case $v^{(j)}(t)$ in (1.20) is identically zero and $S(\varepsilon;\ \mathscr{F}\eta)=G(\eta)$ holds, which makes our iterative process very simple and guarantees the quadratic convergence.

Remark. In [8], Urabe mentioned the convergence rate of the several iterative processes, and wrote that the quasilinearization algorithm by Banks and Groome is not quadratically convergent because it is not the Newton iterative process (p. 182 of [8]). But, the sufficient condition of the quadratic convergence is the zero spectral radius of the Fréchet derivative of the nonlinear operator representing the iterative process at the limit point (fixed point of the operator), and the quasilinearization algorithm has such property as shown in [1].

Concluding this section, we investigate the approximation order of $\hat{\eta}$ for η^*. We shall improve the statement of Lemma 2.

Lemma 4. *For $\eta\in B_A$ and $|\varepsilon|,\ |\varepsilon+v|\leqq\varepsilon_0$, the equation*

$$(3.9)\qquad S(\varepsilon+v;\ \mathscr{F}\eta)-S(\varepsilon;\ \mathscr{F}\eta)=\boldsymbol{L}p(\cdot\ ;\ \mathscr{F}\eta,\ \varepsilon)v+o(|v|)$$

holds, where $p(t; \mathscr{F}\eta, \varepsilon)$ is an $n \times n$ matrix with components of continuous functions on I.

   Proof.   After the notations in Lemma 2, let us put

$$\mathscr{F}(\eta + (\varepsilon + v)e_j) = y_1^{(j)} = x + (\varepsilon + v)\{\varphi^{(j)} + v_1^{(j)}\},$$
$$\mathscr{F}(\eta + \varepsilon e_j) = y_2^{(j)} = x + \varepsilon\{\varphi^{(j)} + v_2^{(j)}\}$$

and

$$w^{(j)} = v_1^{(j)} - v_2^{(j)}.$$

Then we have

$$s_j(\varepsilon + v; \mathscr{F}\eta) - s_j(\varepsilon; \mathscr{F}\eta) = Lw^{(j)},$$

and $w^{(j)}$ must satisfy the differential equation

(3.10) $$\frac{dw^{(j)}}{dt} = \frac{1}{\varepsilon + v}\{X[t, y_1^{(j)}(t)] - X[t, x(t)]\}$$
$$- \frac{1}{\varepsilon}\{X[t, y_2^{(j)}(t)] - X[t, x(t)]\}, \quad a < t < b$$

and the initial condition

(3.11) $$w^{(j)}(a) = 0$$

just as (2.6) and (2.7).   Utilizing the equations (1.7) and (1.8), we have
   (the right-hand side of (3.10))

$$= \frac{1}{\varepsilon + v}\Big\{ X_x[t, x(t)](\varepsilon + v)(\varphi^{(j)} + v_1^{(j)})$$
$$+ \frac{1}{2}X_{xx}[t, x(t)](\varepsilon + v)^2(\varphi^{(j)} + v_1^{(j)})(\varphi^{(j)} + v_1^{(j)})$$
$$+ Q[(\varepsilon + v)(\varphi^{(j)} + v_1^{(j)}); x]\Big\} - \frac{1}{\varepsilon}\Big\{ X_x[t, x(t)]\varepsilon(\varphi^{(j)} + v_2^{(j)})$$
$$+ \frac{1}{2}X_{xx}[t, x(t)]\varepsilon^2(\varphi^{(j)} + v_2^{(j)})(\varphi^{(j)} + v_2^{(j)}) + Q[\varepsilon(\varphi^{(j)} + v_2^{(j)}); x]\Big\}$$
$$= X_x[t, x(t)]w^{(j)} + \frac{1}{2}(\varepsilon + v)X_{xx}[t, x(t)](\varphi^{(j)} + v_1^{(j)})(\varphi^{(j)} + v_1^{(j)})$$
$$- \frac{1}{2}\varepsilon X_{xx}[t, x(t)](\varphi^{(j)} + v_2^{(j)})(\varphi^{(j)} + v_2^{(j)}) + (\varepsilon + v)^2 Q(\varphi^{(j)} + v_1^{(j)}; x)$$
$$- \varepsilon^2 Q(\varphi^{(j)} + v_2^{(j)}; x)$$
$$= \{X_x[t, x(t)] + \varepsilon X_{xx}[t, x(t)](\varphi^{(j)} + v_2^{(j)}) + \varepsilon^2 q_2(\varphi^{(j)} + v_2^{(j)})\}w^{(j)}$$
$$+ \frac{1}{2}v X_{xx}[t, x(t)](\varphi^{(j)} + v_2^{(j)})(\varphi^{(j)} + v_2^{(j)}) + 2v\varepsilon Q(\varphi^{(j)} + v_2^{(j)}; x)$$
$$+ v X_{xx}[t, x(t)] (\varphi^{(j)} + v_2^{(j)})w^{(j)} + \frac{1}{2}(\varepsilon + v)X_{xx}[t, x(t)]w^{(j)}w^{(j)}$$
$$+ \varepsilon^2\{(\varphi^{(j)} + v_2^{(j)})q_1(w^{(j)}) + w^{(j)}q_3(w^{(j)})\},$$

where $q_3(w^{(j)}, 0)$ is abbreviated by $q_3(w^{(j)})$.

Let $\Omega_j(t; \mathscr{F}\eta, \varepsilon)$ be the matrizant which corresponds to

$$X_x[t, x(t)] + \varepsilon X_{xx}[t, x(t)](\varphi^{(j)} + v_2^{(j)}) + \varepsilon^2 q_2(\varphi^{(j)} + v_2^{(j)})$$

and transform (3.10) and (3.11) into the integral equation

$$(3.12) \qquad w^{(j)}(t) = v\Omega_j(t; \mathscr{F}\eta, \varepsilon) \int_a^t \Omega_j^{-1}(s; \mathscr{F}\eta, \varepsilon) \Xi_j(s; \mathscr{F}\eta, \varepsilon) ds$$
$$+ \Omega_j(t; \mathscr{F}\eta, \varepsilon) \int_a^t \Omega_j^{-1}(s; \mathscr{F}\eta, \varepsilon) W_j(s; w^{(j)}) ds.$$

Here

$$(3.13) \quad \Xi_j(t; \mathscr{F}\eta, \varepsilon) = \frac{1}{2} X_{xx}[t, x(t)] \{\varphi^{(j)}(t) + v_2^{(j)}(t)\} \{\varphi^{(j)}(t) + v_2^{(j)}(t)\}$$
$$+ 2\varepsilon Q(\varphi^{(j)}(t) + v_2^{(j)}(t); x)$$

and

$$(3.14) \qquad W_j(t; w^{(j)}) = v X_{xx}[t, x(t)] \{\varphi^{(j)}(t) + v_2^{(j)}(t)\} w^{(j)}$$
$$+ \frac{1}{2} (\varepsilon + v) X_{xx}[t, x(t)] w^{(j)} w^{(j)}$$
$$+ \varepsilon^2 \{(\varphi^{(j)}(t) + v_2^{(j)}(t)) q_1(w^{(j)}) + w^{(j)} q_3(w^{(j)})\}.$$

Then, the first term of (3.12) is the order of $O(|v|)$ and the second term has no factors of the order lower than $O(|v|)$. Hence $w^{(j)}(t)$ is the order of $O(|v|)$ as a whole. Considering this fact, each term of W_j is the order of $O(|v|^2)$.

Therefore we have the equation

$$(3.15) \qquad w^{(j)}(t) = v\Omega_j(t; \mathscr{F}\eta, \varepsilon) \int_a^t \Omega_j^{-1}(s; \mathscr{F}\eta, \varepsilon) \Xi_j(s; \mathscr{F}\eta, \varepsilon) ds + o(|v|).$$

Put $p(t; \mathscr{F}\eta, \varepsilon)$ such that p has the j-th column vector equal to $w^{(j)}/v$, then we have (3.9). $\qquad \square$

Corollary. *For $\eta \in B_4$ and $|v| \le \varepsilon_0$, the equation*

$$(3.16) \qquad S(v; \mathscr{F}\eta) - G(\eta) = Lp(\cdot; \mathscr{F}\eta)v + o(|v|)$$

holds. Here $p(t; \mathscr{F}\eta)$ is formed by

$$\frac{1}{2} \Phi(t; \mathscr{F}\eta) \int_a^t \Phi^{-1}(s; \mathscr{F}\eta) X_{xx}[\mathscr{F}\eta(s), s] \varphi^{(j)}(s) \varphi^{(j)}(s) ds$$

as the j-th column vector.

By virtue of Lemma 4 and its Corollary we obtain informations about the derivatives of $\mathscr{T}(\eta, \varepsilon)$ with respect to ε at some neighbourhood of $(\eta^*, 0)$, which

implies the following

Theorem 3. *In the neighbourhood*

$$Nb(\Delta_3, \varepsilon_3) = \{(\eta, \varepsilon); \|\eta - \eta^*\| \leqq \Delta_3 \quad and \quad |\varepsilon| \leqq \varepsilon_3\},$$

$\hat{\eta}(\varepsilon)$ *whose existence is guaranteed in Theorem 1 has the estimation such as*

$$(3.17) \qquad\qquad \|\hat{\eta}(\varepsilon) - \eta^*\| \leqq const. |\varepsilon| \quad as \quad \varepsilon \to 0.$$

Proof. We have shown in the third stage of the proof of Theorem 1 that $\mathcal{T}_\eta(\eta^*, 0)$ is identical to $\mathfrak{G}(\eta^*)$ and is invertible. Taking the continuity of the derivative $\mathcal{T}_\eta(\eta, \varepsilon)$ at $\varepsilon = 0$ into consideration, we can choose the numbers Δ_3 and ε_3 such that $\mathcal{T}_\eta(\hat{\eta}, \varepsilon)$ has the bounded inverse in $Nb(\Delta_3, \varepsilon_3)$.

On the other hand the equation

$$\begin{aligned}
\mathcal{T}(\hat{\eta}, &\varepsilon + v) - \mathcal{T}(\hat{\eta}, \varepsilon) \\
&= {}^t S(\varepsilon + v; \mathscr{F}\hat{\eta})\{\boldsymbol{L}\mathscr{F}\hat{\eta} - \boldsymbol{d}\} - {}^t S(\varepsilon; \mathscr{F}\hat{\eta})\{\boldsymbol{L}\mathscr{F}\hat{\eta} - \boldsymbol{d}\} \\
&= {}^t\{S(\varepsilon + v; \mathscr{F}\hat{\eta}) - S(\varepsilon; \mathscr{F}\hat{\eta})\}\{\boldsymbol{L}\mathscr{F}\hat{\eta} - \boldsymbol{d}\}
\end{aligned}$$

holds, and by Lemma 4 we obtain the assertion that $\mathcal{T}(\eta, \varepsilon)$ has the derivative with respect to ε at $(\hat{\eta}(\varepsilon), \varepsilon)$ such as

$$(3.18) \qquad\qquad \mathcal{T}_\varepsilon(\hat{\eta}, \varepsilon)v = {}^t\{\boldsymbol{L}p(\,\cdot\,; \mathscr{F}\hat{\eta}, \varepsilon)v\}\{\boldsymbol{L}\mathscr{F}\hat{\eta} - \boldsymbol{d}\}.$$

Then, replacing Δ_3 and ε_3 by suitable numbers less than them if necessary, Theorem 20.3 in [3] implies the existence of the derivatives of $\hat{\eta}(\varepsilon)$ in $Nb(\Delta_3, \varepsilon_3)$, which is defined by

$$(3.19) \qquad\qquad \hat{\eta}_\varepsilon(\varepsilon)v = -(\mathcal{T}_\eta(\hat{\eta}, \varepsilon))^{-1}\mathcal{T}_\varepsilon(\hat{\eta}, \varepsilon)v.$$

Thus $\hat{\eta}(\varepsilon)$ is the order of $O(|\varepsilon|)$ as $\varepsilon \to 0$ in $Nb(\Delta_3, \varepsilon_3)$. $\square$

Remark. Let ε tend to 0 in (3.18), then we have

$$\mathcal{T}_\varepsilon(\eta^*, 0)v = {}^t\{\boldsymbol{L}p(\,\cdot\,; \mathscr{F}\eta^*)v\}\{\boldsymbol{L}\mathscr{F}\eta^* - \boldsymbol{d}\}$$

by virtue of Corollary of Lemma 4.   By the fact

$$\mathcal{T}_\eta(\eta^*, 0) = \mathfrak{G}(\eta^*),$$

the equation

$$\hat{\eta}_\varepsilon(0)v = -\mathfrak{G}^{-1}(\xi^*){}^t\{\boldsymbol{L}p(\,\cdot\,; \mathscr{F}\eta^*)v\}\{\boldsymbol{L}\mathscr{F}\eta^* - \boldsymbol{d}\}$$

holds.   However, we obtain

$${}^t\{\boldsymbol{L}p(\,\cdot\,; \mathscr{F}\eta^*)v\}$$

$$=\,^t\left\{\frac{1}{2}vL\Phi(\cdot\,;\,\mathscr{F}\eta^*)\int_a^{\cdot}\Phi^{-1}(s;\,\mathscr{F}\eta^*)Z(s;\,\mathscr{F}\eta^*)ds\right\}$$

$$=\,^t\left\{\frac{1}{2}vp_N\int_a^{\cdot}\Phi^{-1}(s;\,\mathscr{F}\eta^*)Z(s;\,\mathscr{F}\eta^*)ds\right\}{}^tG(\eta^*),$$

where $Z(t;\,\mathscr{F}\eta)$ stands for the matrix which is composed by $X_{xx}[t,\,\mathscr{F}\eta(t)]\varphi^{(j)}(t)$ $\times\varphi^{(j)}(t)$ as the j-th column vector. Remembering that ${}^tG(\eta^*)\{L\mathscr{F}\eta^*-d\}=0$, we obtain the statement that the derivative $\dot{\eta}_\varepsilon(0)$ vanishes, which implies that we can expect the approximate minimal point $\hat{\eta}(\varepsilon)$ to be the order of $o(|\varepsilon|)$ extremely near to η^*.

§4. An Illustrative Example

Let us consider the following problem originally mentioned in [4], which occurs relating to the tubular flow chemical reactor with axial mixing.

The differential equation is

(4.1) $$\frac{d^2x}{d\tau^2}-6\frac{dx}{d\tau}-12x^2=0\,,\quad 0<\tau<1\,.$$

By the transformation

(4.2) $$t=1-2\tau\,,$$

the equation (4.1) can be reduced to the following differential equation:

(4.3) $$\frac{d^2x}{dt^2}+3\frac{dx}{dt}-3x^2=0\,,\quad -1<t<1\,.$$

The constraining condition of least squares type is the following:

(4.4) $$\sum_{j=0}^{10}\{x(t_j)-y_j\}^2=\text{minimum}\,,$$

where

(4.5) $$t_j=0.2j-1.0\quad(j=0,\,1,...,\,10)$$

and y_j $(j=0,\,1,...,\,10)$ are given in Table 4.1.

Table 4.1.

j	0	1	2	3	4	5
y_j	0.38727	0.39476	0.41305	0.43862	0.47017	0.50764
	6	7	8	9	10	
	0.55172	0.60372	0.66559	0.74012	0.83129	

This problem is equivalent to the example in [2].

We shall rewrite the problem into the vector form as (1.1) and (1.2). The equation (4.3) is transformed into

(4.6)
$$\begin{cases} \dfrac{dx_1}{dt} = x_2\,, \\[2mm] \dfrac{dx_2}{dt} = 3x_1^2 - 3x_2\,, \end{cases} \qquad -1 < t < 1.$$

Let x be the vector $x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$, then (4.6) is equivalent to

(4.6)′
$$\frac{dx}{dt} = \begin{bmatrix} x_2 \\ 3x_1^2 - 3x_2 \end{bmatrix}, \qquad -1 < t < 1$$

and the functional value J to be minimized is

(4.7)
$$J = \frac{1}{2} \sum_{j=0}^{10} {}^t[L_j x(t_j) - d_j][L_j x(t_j) - d_j]\,.$$

Here t_j ($j = 0, 1, \ldots, 10$) are the same as in (4.5), the matrices L_j ($j = 0, 1, \ldots, 10$) are

$$L_0 = L_1 = \cdots = L_{10} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix},$$

and the vectors d_j ($j = 0, 1, \ldots, 10$) are given by

$$d_j = \begin{bmatrix} y_j \\ 0.0 \end{bmatrix}.$$

The results of the numerical computation carried out by FACOM M–190 in the Data Processing Center, Kyoto University are shown in Table 4.2. For the numerical integration of ordinary differential equations the Runge-Kutta-Gill method programed by T. Ojika was used. All the calculations were carried out in the double precision arithmetic.

Table 4.2.

iteration times	0		1	
	x_1	x_2	x_1	x_2
$t_0 = -1.0$	0.5	0.0	0.37806769956	0.10806510612
$t_1 = -0.8$	0.51252090985	0.11507113359	0.40171204036	0.12821189883
$t_2 = -0.6$	0.54334736769	0.18909213769	0.42937707807	0.14869156970
$t_3 = -0.4$	0.58721750119	0.24863472740	0.46135283076	0.17162147360
$t_4 = -0.2$	0.64281390289	0.30821629103	0.49830337268	0.19871943583
$t_5 = 0.0$	0.71115159268	0.37748477983	0.54123626991	0.23176626442
$t_6 = 0.2$	0.79503126629	0.46516837677	0.59155143668	0.27296261250
$t_7 = 0.4$	0.89914650517	0.58192807415	0.65116185255	0.32531012072
$t_8 = 0.6$	1.0307690460	0.74350525298	0.72270288155	0.39313865121
$t_9 = 0.8$	1.2012163950	0.97571861039	0.80987497209	0.48294851916
$t_{10} = 1.0$	1.4287209566	1.3241632086	0.91800946931	0.60486691312
J_k	4.93×10^{-1}		1.16×10^{-2}	

iteration times	2		3	
	x_1	x_2	x_1	x_2
$t_0 = -1.0$	0.38702086627	0.00261251861	0.38727177735	−0.00004288586
$t_1 = -0.8$	0.39490532424	0.07015010707	0.39476040385	0.06869739433
$t_2 = -0.6$	0.41341909766	0.11229585764	0.41304637216	0.11139645273
$t_3 = -0.4$	0.43913778607	0.14393243601	0.43861177501	0.14326111462
$t_4 = -0.2$	0.47081948324	0.17287829974	0.47016701758	0.17226427328
$t_5 = 0.0$	0.50841638291	0.20369262002	0.50763820223	0.20303615170
$t_6 = 0.2$	0.55263785706	0.23962085391	0.55171808024	0.23885031164
$t_7 = 0.4$	0.60480645049	0.28369352839	0.60371555428	0.28274087195
$t_8 = 0.6$	0.66689448238	0.33949736231	0.66558820885	0.33828049978
$t_9 = 0.8$	0.74170948834	0.41193878825	0.74012428084	0.41034340656
$t_{10} = 1.0$	0.83325492973	0.50828488609	0.83129911340	0.50613875295
J_k	5.81×10^{-6}		1.12×10^{-10}	

iteration times	4	
	x_1	x_2
$t_0 = -1.0$	0.38727191330	−0.00004431630
$t_1 = -0.8$	0.39476032659	0.06869661205
$t_2 = -0.6$	0.41304617227	0.11139596872
$t_3 = -0.4$	0.43861149266	0.14326075367
$t_4 = -0.2$	0.47016666726	0.17226394342
$t_5 = 0.0$	0.50763778440	0.20303579926
$t_6 = 0.2$	0.55171758641	0.23884989812
$t_7 = 0.4$	0.60371496863	0.28274036083
$t_8 = 0.6$	0.66558750766	0.33827984706
$t_9 = 0.8$	0.74012343005	0.41034255099
$t_{10} = 1.0$	0.83129806389	0.50613760232
J_k	1.10×10^{-10}	

convergence criterion 1.0×10^{-7}

step size for the Runge-Kutta method 0.003125

$\varepsilon = 1.0 \times 10^{-8}$

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