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Complex Analytic Construction of the Kuranishi Family on a Normal Strongly Pseudo-Convex Manifold. II

By

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Introduction

This paper is a continuation of the previous paper [2] by the first named author. The deformation of partially complex structures on a compact strongly pseudo-convex manifold was investigated first by M. Kuranishi [4] in order to give a new insight into the problem of deformations of isolated singularities, and he constructed a family of partially complex structures on a compact strongly pseudo-convex real hypersurface of a complex analytic space with an isolated singularity parametrized by a finite number of parameters and inducing the versal family of deformations of the isolated singularity in some sense. If (V, x) is an analytic subset of a domain in a complex euclidean space with an isolated singular point x, we obtain a real submanifold M by cutting the analytic set by a sphere of sufficiently small radius centered at x, and V defines a subbundle ${}^{o}T''$ of $\mathbf{C}TM$, which is called the partially complex structure on M induced from V, consisting of all tangent vectors of type (0, 1) in $\mathbf{C}TV_{|M}$. Kuranishi represented an almost partially complex structure on M of finite distance from ${}^{o}T''$ by an element of $\Gamma(M, T'\otimes({}^{o}T'')^{*})$ and constructed the above-mentioned family of integrable almost partially complex structures making use of the harmonic theory for $\bar{\partial}_b$. He tided over the difficulties arising from the non-ellipticity of $\bar{\partial}_b$ by making use of Nash-Moser's inverse mapping theorem, thereby the family has only a differentiable structure of class C^{∞}. Then the problem as to when the family may have a complex analytic structure was left open. The first named author answered this problem affirmatively in

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[2] in the case that M is normal, $\dim_{\boldsymbol{R}} M \geqq 7$, and $H_{T'}^{(2)}=0$, depending on the following ideas: On a compact normal strongly pseudo-convex manifold M there exists a differential complex,

$$0 \to \Gamma(M, {}^{o}\bar{T}'') \xrightarrow{D_0} \Gamma(M, {}^{o}\bar{T}''\otimes({}^{o}T'')^*) \xrightarrow{D_1} \Gamma(M, {}^{o}\bar{T}''\otimes \wedge^2({}^{o}T'')^*) \to \cdots,$$

and the solutions of integrability condition $P(\phi)=0$ can be looked for in $\Gamma(M, {}^{o}\bar{T}''\otimes({}^{o}T'')^*)$ instead of in $\Gamma(M, T'\otimes({}^{o}T'')^*)$. He avoided the difficulties coming from the non-ellipticity of $\bar{\partial}_b$ by introducing the norms $\| \ \|'_{(m)}$ and $\| \ \|''_{(m)}$.

In this paper, following these ideas, we prove Main Theorem by the method used in [5].

Main Theorem. *If M is a compact normal strongly pseudo-convex real hypersurface of a complex manifold N with $\dim_{\boldsymbol{R}} M \geqq 7$, there exists a complex analytic family $\{\phi(t) \mid t \in T\}$ of partially complex structures on M of class C^2 such that $\phi(o)=0$, which is versal at o (cf. Definition 1.6), and the linear term of $\phi(t)$ with respect to the parameter t determines an injective map of the Zariski tangent space T'_oT of T at o into $H^1_{\bar{\partial}_b}(T')$, where $H^1_{\bar{\partial}_b}(T')$ denotes the first cohomology group of the complex $(\Gamma(M, T'\otimes \wedge^q({}^{o}T'')^*), \bar{\partial}_{T'}^{(q)})$.*

In Section 1 we recall some notations and results in [1] and [2] which is needed in this paper and give the formulation of Main Theorem. The proof of Main Theorem is given in Sections 2 and 3. In Section 2 we construct a family of partially complex structures on M. We prove that the family is the versal family in the sense of M. Kuranishi in Section 3. Throughout this paper, as a parameter space of a family of partially complex structures, we consider only a reduced complex analytic space.

§ 1. Preliminaries

In this section, we recall some formulations, notations and results in [1] and [2] which will be needed in Sections 2 and 3.

Let M be a differentiable manifold.

Definition 1.1. By an almost partially complex structure on M, we mean a complex subbundle E of $\boldsymbol{C}TM$ of class C^∞ such that $E \cap \bar{E}=\{0\}$ where $\bar{E}$ denotes the complex conjugate of E.

Further if it satisfies the condition that $[X, Y]$ is in $\Gamma(M, E)$ for any X, Y

in $\Gamma(M, E)$, we call E a partially complex structure on M, where $\Gamma(M, E)$ denotes the vector space of all sections of E over M of class C^∞. M with a partially complex structure E is called a partially complex manifold.

If M is a real hypersurface of a complex manifold N, N defines a partially complex structure ${}^oT''$ on M by

$${}^oT'' = T''N_{|M} \cap \mathbf{C}TM$$

where $T''N$ denotes the complex tangent bundle of N of type (0, 1). We call the partially complex structure on M the partially complex structure on M induced from the complex structure of N.

On the partially complex manifold $(M, {}^oT'')$, we have the (tangential) Cauchy-Riemann operator (cf. [1] § 3)

$$\bar{\partial}_b\colon \Gamma(U, \wedge^q({}^oT'')^*) \to \Gamma(U, \wedge^{q+1}({}^oT'')^*)$$

given by

$$\bar{\partial}_b\phi(X_1,\dots, X_{q+1}) = \sum_{i=1}^{q+1}(-1)^{i+1}X_i\phi(X_1,\dots, \hat{X}_i,\dots, X_{q+1}) + \sum_{i<j}(-1)^{i+j}\phi([X_i, X_j], X_1,\dots, \hat{X}_i,\dots, \hat{X}_j,\dots, X_{q+1})$$

for $\phi \in \Gamma(U, \wedge^q({}^oT'')^*)$ and $X_1,\dots, X_{q+1} \in \Gamma(U, {}^oT'')$, where U is an open domain of M.

We say that a partially complex manifold $(M, {}^oT'')$ with $\dim_{\mathbf{R}} M = 2n-1$ is strongly pseudo-convex if ${}^oT''$ is of rank $n-1$ and its Levi form is positive definite at each point of M (cf. [2], (1.2)).

Definition 1.2. A strongly pseudo-convex manifold $(M, {}^oT'')$ is called a normal strongly pseudo-convex manifold if there exists a non-vanishing global real C^∞-vector field ξ on M such that

$$[\xi, \Gamma(M, {}^oT'')] \subset \Gamma(M, {}^oT'')$$

and $\xi_p \notin \mathrm{Re}({}^oT''_p \oplus {}^o\bar{T}''_p)$ for any $p \in M$, where $\mathrm{Re}({}^oT''_p \oplus {}^o\bar{T}''_p)$ denotes the real part of ${}^oT''_p \oplus {}^o\bar{T}''_p$.

From now on, we assume that M is a normal strongly pseudo-convex real hypersurface of a complex manifold N with a fixed real C^∞-vector field ξ as above and $\dim_{\mathbf{R}} M = 2n-1$.

Then we have the following canonical C^∞-splitting of $\mathbf{C}TM$ as differentiable vector bundles,

$$\mathbf{C}TM = {}^oT'' \oplus {}^o\bar{T}'' \oplus F$$

where $F=\boldsymbol{C}\xi$. We set $T'={}^o\overline{T}''\oplus F$, then there exist the following differential complexes arising from the (tangential) Cauchy-Riemann complex for scalar valued forms,

$$0\to\Gamma(M,\ T')\xrightarrow{\bar\partial^{(0)}_{T'}}\Gamma(M,\ T'\otimes({}^oT'')^*)\xrightarrow{\bar\partial^{(1)}_{T'}}\Gamma(M,\ T'\otimes\wedge^2({}^oT'')^*)\to,$$
$$0\to\Gamma(M,{}^o\overline{T}'')\xrightarrow{D_0}\Gamma(M,\ {}^o\overline{T}''\otimes({}^oT'')^*)\xrightarrow{D_1}\Gamma(M,\ {}^o\overline{T}''\otimes\wedge^2({}^oT'')^*)\to,$$
$$0\to\Gamma(M,\ F)\xrightarrow{\bar\partial^{(0)}_{F}}\Gamma(M,\ F\otimes({}^oT'')^*)\xrightarrow{\bar\partial^{(1)}_{F}}\Gamma(M,\ F\otimes\wedge^2({}^oT'')^*)\to,$$

(cf. [1] and [2], for details).

We define operators which will be needed in Sections 2 and 3, and recall a lemma concerning with them (cf. [2], Lemma 4.2).

For each $q=0,\ 1,\dots,$

$$L_q\colon\Gamma(M,\ {}^o\overline{T}''\otimes\wedge^q({}^oT'')^*)\to\Gamma(M,\ F\otimes\wedge^{q+1}({}^oT'')^*)$$

is given by

$$L_q(\theta)(X_1,\dots,\ X_{q+1})=(\bar\partial^{(q)}_{T'}\theta(X_1,\dots,\ X_{q+1}))_F$$

for $\theta\in\Gamma(M,\ {}^o\overline{T}''\otimes\wedge^q({}^oT'')^*)$ and $X_1,\dots,\ X_{q+1}\in\Gamma(M,\ {}^oT'')$, where $(\ )_F$ denotes the projection to F according to the splitting $T'={}^o\overline{T}''\oplus F$.

We note that L_q are linear over $C^\infty(M)$, the ring of all C^∞-functions on M, and L_0 induces a C^∞-bundle isomorphism of ${}^o\overline{T}''$ to $F\otimes({}^oT'')^*$. We denote L_1 by L according to [2].

Proposition 1.1. (1) $LD_0=\bar\partial^{(1)}_F L_0$, (2) $L_2D_1=\bar\partial^{(2)}_F L$.

Next, we introduce an hermitian metric on M, then hermitian metrics along fibers on ${}^o\overline{T}''$, F and T' are introduced naturally, and we can speak of the harmonic theory on $\Gamma(M,\ {}^o\overline{T}''\otimes\wedge^q({}^oT'')^*)$, $\Gamma(M,\ F\otimes\wedge^q({}^oT'')^*)$ and $\Gamma(M,\ T'\otimes\wedge^q({}^oT'')^*)$ for $q=1,\ 2,\dots,\ n-2$, by J. J. Kohn (cf. [3] Ch. V and [6] § 6). In particular, if we denote by D_q^* (resp. $\square$) the adjoint operator of D_q with respect to the above hermitian metric (resp. the Laplacian operator $D_q^*D_q+D_{q-1}D_{q-1}^*$) and by $H^{(q)}_{{}^oT''}$ the vector space of all harmonic elements of $\Gamma(M,\ {}^o\overline{T}''\otimes\wedge^q({}^oT'')^*)$, $H^{(q)}_{{}^oT''}$ is finite dimensional and there exist the Neumann operator

$$N\colon\Gamma(M,\ {}^o\overline{T}''\otimes\wedge^q({}^oT'')^*)\to\Gamma(M,\ {}^o\overline{T}''\otimes\wedge^q({}^oT'')^*)$$

and the harmonic operator

$$\boldsymbol{H}^q_{{}^oT''}\colon\Gamma(M,\ {}^o\overline{T}''\otimes\wedge^q({}^oT'')^*)\to H^{(q)}_{{}^oT''}$$

with the relations

$$I = \mathbf{H}^q_{{}^oT''} + \Box N \quad \text{and} \quad \Box N = N \Box ,$$

where I denotes the identity operator of $\Gamma(M, {}^o\bar{T}'' \otimes \wedge^q({}^oT'')^*)$.

Further, on $\Gamma(M, F \otimes \wedge^q({}^oT'')^*)$ and $\Gamma(M, T' \otimes \wedge^q({}^oT'')^*)$, we employ similar notations and obtain similar results.

For each non-negative integer m and any open domain U of M, we define the Sobolev norm $\| \ \|_{(m)}$ and the norms $\| \ \|'_{(m)}$, $\| \ \|''_{(m)}$ on $C_0^\infty(U)$ and then on the spaces $\Gamma_c(U, {}^o\bar{T}'' \otimes \wedge^q({}^oT'')^*)$, $\Gamma_c(U, T' \otimes \wedge^q({}^oT'')^*)$, $\Gamma_c(U, F \otimes \wedge^q({}^oT'')^*)$, where $C_0^\infty(U)$ (resp. $\Gamma_c(U, E)$) denotes the vector space of all C^∞-functions (resp. all C^∞-sections of the vector bundle E) with compact supports in U. (Cf. [2], §3 for details.) We denote by $\Gamma_{(m)}(M, {}^o\bar{T}'' \otimes \wedge^q({}^oT'')^*)$, $\Gamma'_{(m)}(M, {}^o\bar{T}'' \otimes \wedge^q({}^oT'')^*)$ and $\Gamma''_{(m)}(M, {}^o\bar{T}'' \otimes \wedge^q({}^oT'')^*)$ the Hilbert spaces obtained by completing $\Gamma(M, {}^o\bar{T}'' \otimes \wedge^q({}^oT'')^*)$ with respect to the norms $\| \ \|_{(m)}$, $\| \ \|'_{(m)}$ and $\| \ \|''_{(m)}$ respectively, and so on.

We remark that these norms have the following properties: For any open set U of M and each $m \geqq n+1$, there exist constants c_m, c'_m and d_m such that

$$\|\phi\psi\|_{(m)} \leqq c_m \|\phi\|_{(m)} \|\psi\|_{(m)} \quad \text{for any} \quad \phi, \psi \in C_0^\infty(U),$$
$$\|\phi\psi\|'_{(m)} \leqq c'_m \|\phi\|'_{(m)} \|\psi\|'_{(m)} \quad \text{for any} \quad \phi, \psi \in C_0^\infty(U),$$
$$\|\bar{\partial}_b \phi\|_{(m)} \leqq d_m \|\phi\|'_{(m)} \quad \text{for any} \quad \phi \in C_0^\infty(U).$$

Moreover, we have the following results (cf. [2], Proposition 3.13 and (4.4)).

Proposition 1.2. *The following estimates hold.*

$$\|ND_1^* R_2(\phi)\|''_{(m)} \lesssim \|\phi\|''^2_{(m)} \quad \textit{uniformly for} \quad \phi \in \Gamma(M, {}^o\bar{T}'' \otimes ({}^oT'')^*),$$
$$\|D_0 L_0^{-1} \bar{\partial}_F^{(1)*} N_F \phi\|''_{(m)} \lesssim \|\phi\|''_{(m)} \quad \textit{uniformly for} \quad \phi \in \Gamma(M, F \otimes \wedge^2({}^oT'')^*),$$

where R_2 is a differential operator introduced in Proposition 1.4 *below, and see* [2] Section 3 *for the meaning of the symbol $\lesssim$.*

Further, since L is linear over $C^\infty(M)$, we have

$$\|L\phi\|''_{(m)} \lesssim \|\phi\|''_{(m)} \quad \text{uniformly for} \quad \phi \in \Gamma(M, {}^o\bar{T}'' \otimes ({}^oT'')^*).$$

Now we return to a family of partially complex structures on M.

First, we recall some propositions (cf. [1] Proposition 1.1, Theorem 2.1 and [2] Proposition 1.7.3).

Proposition 1.3. *Let A be the set of all almost partially complex struc-*

tures at a finite distance from ${}^oT''$. *Then there is a natural bijection*

$$b: \Gamma(M, T'\otimes({}^oT'')^*) \to A$$

such that if we write $b(\phi) = {}^{\phi}T''$ *for* $\phi \in \Gamma(M, T'\otimes({}^oT'')^*)$ *then* ${}^{\phi}T'' = \{X + \phi(X) \mid X \in {}^oT''\}$. *(See* [1], *for "at a finite distance from* ${}^oT''$".)

Proposition 1.4. *An almost partially complex structure* ${}^{\phi}T''$ *is a partially complex structure if and only if it satisfies the following integrability condition;*

$$P(\phi) = \bar{\partial}^{(1)}_{T''}\phi + R_2(\phi) + R_3(\phi) = 0$$

where $R_2(\phi)(X, Y) = [\phi(X), \phi(Y)]_{T'} - \phi([X, \phi(Y)]_{{}^oT''} + [\phi(X), Y]_{{}^oT''})$ *and* $R_3(\phi)(X, Y) = -\phi([\phi(X), \phi(Y)]_{{}^oT''})$ *for* $X, Y \in \Gamma(M, {}^oT'')$, $[\ ,\ ]_{T'}$ *(resp.* $[\ ,\ ]_{{}^oT''}$*) denoting the projection of* $[\ ,\ ]$ *to* T' *(resp.* ${}^oT''$*) according to the above splitting* $\boldsymbol{C}TM = T'\oplus {}^oT''$.

Proposition 1.5. *For any* $\phi \in \Gamma(M, {}^o\bar{T}''\otimes({}^oT'')^*)$, $P(\phi) = 0$ *if and only if* $D_1\phi + R_2(\phi) = 0$ *and* $L\phi = 0$.

Let E be a C^∞-vector bundle on M, T an analytic subset of a neighbourhood of the origin of $\boldsymbol{C}^r$.

Definition 1.3. A family $\{\phi(t) \mid t \in T\}$ of elements of $\Gamma(M, E)$ (resp. $C^2(M, E)$) is of class C^∞ (resp. C^2) if for each $p \in M$ there exists a neighbourhood U of p such that the map

$$\tilde{\phi}: U \times T \ni (q, t) \to \phi(t)(q) \in E$$

is of class C^∞ (resp. C^2), where $\phi(t)(q)$ denotes the value of $\phi(t)$ at q in E and $C^2(M, E)$ the vector space of all sections of E over M of class C^2.

Definition 1.4. A C^∞-family (resp. C^2-family) $\{\phi(t) \mid t \in T\}$ of elements of $\Gamma(M, E)$ (resp. $C^2(M, E)$) depends complex analytically on $t \in T$ if for each fixed $p \in M$, $\phi(t)(p)$ is complex analytic in t.

Using the notions in Definitions 1.3 and 1.4, we define a complex analytic family of partially complex structures on M.

Definition 1.5. By a complex analytic family of partially complex structures on M parametrized by T, we mean a C^∞-family $\{\phi(t) \mid t \in T\}$ of elements of $\Gamma(M, T'\otimes({}^oT'')^*)$ depending complex analytically on $t \in T$ and satisfying $P(\phi(t)) = 0$ for $t \in T$.

M. Kuranishi treated our family of partially complex structures on M as a family of isolated singularity in [4] and give the definition of the versal family as follows.

Definition 1.6. A complex analytic family $\{\phi(t) \mid t \in T\}$ of partially complex structures on M with $\phi(o)=0$ is versal at $o \in T$ if the family satisfies the following condition; For any neighbourhood $\tilde{N}$ of M in N and any complex analytic family $\pi\colon \mathscr{N} \to S$ of deformations of $\tilde{N}$ with $\mathscr{N}_o = \tilde{N}$ for $o \in S$, there are a neighbourhood S' of o in S, an analytic map $\tau\colon S' \to T$ and a differentiable embedding $F\colon M \times S' \to \mathscr{N}_{|S'}$ of class C^∞ depending complex analytically on $s \in S'$ (i.e. the map $S' \ni s \to F(p, s) \in \mathscr{N}$ is complex analytic for each fixed $p \in M$) such that

(1) $\tau(o)=o$,

(2) $\pi \circ F = p_2$ where p_2 denotes the projection of $M \times S'$ onto the second factor,

(3) $F_{|M \times o} = \mathrm{id}_M$,

(4) for any $s \in S'$, $F_{|M \times s}$ induces an isomorphism of the partially complex structure $\phi(\tau(s))$ on M to that induced on M from $\mathscr{N}_s$ via $F_{|M \times s}$.

If $\{\phi(t) \mid t \in T\}$ is a complex analytic family of partially complex structures on M,

$$\bar{\partial}_{T'}^{(1)} \phi(t) + R_2(\phi(t)) + R_3(\phi(t)) = 0 \quad \text{for} \quad t \in T,$$

by Proposition 1.5. Then for each $\partial/\partial t \in T'_o T$,

$$\bar{\partial}_{T'}^{(1)} \partial\phi/\partial t(o) = 0,$$

hence $\partial\phi/\partial t(o)$ determines an element of $H^1_{\bar{\partial}_b}(T')$. Then we have a linear map of $T'_o T$ to $H^1_{\bar{\partial}_b}(T')$ similar to the case of a family of compact complex manifolds (what is called the Kodaira-Spencer map).

Our purpose is to construct a complex analytic family $\{\phi(t) \mid t \in T\}$ of partially complex structures on M with $\phi(o)=0$ for $o \in T$ which is versal at $o \in T$ in the sense of M. Kuranishi and has the property that the linear map as above is an injective map of $T'_o T$ into $H^1_{\bar{\partial}_b}(T')$. In this paper, we construct only a complex analytic family of C^2-partially complex structures on M having the same properties as above. In fact, we can define a C^2-partially complex structure on M in the same way as Definition 1.1, see that it is represented by an element of $C^2(M, T' \otimes ({}^o T'')^*)$ satisfying $P(\phi)=0$ similarly to Propositions 1.3 and 1.4, and get the same result as Proposition 1.5. Then we define a complex analytic family of C^2-partially complex structures on M and the versality of a family in the same way as Definitions 1.5 and 1.6, and the linear map of T'_oT into $H^1_{\bar{\partial}_b}(T')$ as above is also defined for this family.

§2. Construction of a Versal Family

Let M be a compact normal strongly pseudo-convex real hypersurface of a complex manifold N of complex dimension $n \geqq 4$.

In this section, we construct a family $\phi(t)$ of C^2-sections of ${}^o\bar{T}'' \otimes ({}^oT'')^*$ over M depending complex analytically on t in a neighbourhood V of the origin of $H^{(1)}_{T'}$, and a finite number of holomorphic functions $h_1(t), \dots, h_l(t)$ on V satisfying the following relations:

(2.0) $\phi(0)=0$ and $h_\tau(0)=0$ for $\tau=1, \dots, l$,

(2.1) $P(\phi(t))=0$ for $t \in T$,

(2.2) the linear term of $\phi(t)$ is cohomologous to $\sum_{\sigma=1}^q \beta_\sigma t_\sigma$, where $q = \dim_{\boldsymbol{C}} H^{(1)}_{T'}$, $(\beta_1, \dots, \beta_q)$ is a base of $H^{(1)}_{T'}$, $(t_1, \dots, t_q)$ is the system of coordinates of $H^{(1)}_{T'}$ associated with the base, and T is an analytic subset of V defined by $h_1(t)= \cdots =h_l(t)=0$.

For the proof, we first introduce some notations.

For each $\psi \in \Gamma(M, T' \otimes ({}^oT'')^*)$, we set $\mathscr{L}\psi = \psi - \bar{\partial}^{(1)}_{T'}\theta_\psi$, where θ_ψ is a unique element of $\Gamma(M, {}^o\bar{T}'')$ such that

$$[X, \theta_\psi]_F = \psi(X)_F \qquad \text{for any} \quad X \in \Gamma(M, {}^oT'').$$

Then it is easily seen that $\mathscr{L}\psi$ is in $\Gamma(M, {}^o\bar{T}'' \otimes ({}^oT'')^*)$.

Proposition 2.1 ([2] Proposition 5.1). *The map* $\mathscr{L}_{|H^{(1)}_{T'}}$ *is injective.*

Then if we set $\mathscr{H} = \mathscr{L}(H^{(1)}_{T'})$, $\mathscr{H}$ also represents the first cohomology group $H^1_{\bar{\partial}_b}(T')$.

Let $(\gamma'_1, \dots, \gamma'_p)$ be an orthonormal base of $\boldsymbol{H}^2_F L(H^{(1)}_{{}^oT''})$ with respect to the L^2-inner product (,) on $\Gamma(M, F \otimes \wedge^2({}^oT)^*)$ introduced in Section 1 and $(\gamma_1, \dots, \gamma_p)$ elements of $H^{(1)}_{{}^oT''}$ such that $\boldsymbol{H}^2_F L(\gamma_i) = \gamma'_i$ for $i=1, \dots, p$.

For $\eta \in \Gamma(M, F \otimes \wedge^2({}^oT'')^*)$, we set $\rho(\eta) = \sum_{i=1}^q (\eta, \gamma'_i)\gamma_i$.

Then we have the following lemma.

Lemma 2.2. (1) $\boldsymbol{H}^2_F L\rho \boldsymbol{H}^2_F L \boldsymbol{H}^1_{{}^oT''} = \boldsymbol{H}^2_F L \boldsymbol{H}^1_{{}^oT''}$,

(2) $(L\rho \boldsymbol{H}^2_F)^2 = L\rho \boldsymbol{H}^2_F$.

Proof. (1) Let $\eta \in H^{(1)}_{{}^oT''}$. Then

$$\begin{aligned} \boldsymbol{H}^2_F L\rho \boldsymbol{H}^2_F L(\eta) &= \boldsymbol{H}^2_F L\left(\textstyle\sum_{i=1}^p (\boldsymbol{H}^2_F L(\eta), \gamma'_i)\gamma_i\right) \\ &= \textstyle\sum_{i=1}^p (\boldsymbol{H}^2_F L(\eta), \gamma'_i)\gamma'_i \\ &= \boldsymbol{H}^2_F L(\eta). \end{aligned}$$

(2) Let $\psi \in H^{(2)}_F$. Then

$$\begin{aligned} L\rho \boldsymbol{H}^2_F L\rho(\psi) &= L\rho \boldsymbol{H}^2_F L\left(\textstyle\sum_{i=1}^p (\psi, \gamma'_i)\gamma_i\right) \\ &= L\rho\left(\textstyle\sum_{i=1}^p (\psi, \gamma'_i)\gamma'_i\right) \\ &= \textstyle\sum_{i=1}^p (\psi, \gamma'_i)L(\gamma_i) = L\rho(\psi). \end{aligned}$$ Q.E.D.

In view of Proposition 1.5 and Proposition 2.1, we shall construct $\phi(t)$ and $h_1(t),\dots, h_l(t)$ satisfying the following:

(2.0) $\phi(0)=0$ and $h_\tau(0)=0$ for $\tau=1,\dots, l$,

(2.1.1) $D_1\phi(t)+R_2(\phi(t))=0$ for $t\in T$,

(2.1.2) $L\phi(t)=0$ for $t\in T$,

(2.2) the linear term of $\phi(t)$ is equal to $\sum_{\sigma=1}^q (\mathscr{L}\beta_\sigma)t_\sigma$.

Before beginning the proof, it is convenient to introduce the following notation: Let U and D be open sets in M and $\boldsymbol{C}^q$ respectively, $h_1(t),\dots, h_l(t)$ holomorphic functions on D, and E a vector bundle on U. For two families $\phi(t)$ and $\psi(t)$ of C^∞-sections (resp. C^2-sections) of E on U which depend complex analytically on t in D, by

$$\phi(t)\equiv\psi(t) \mod (h_1(t),\dots, h_l(t), t^\mu),$$

we mean that there exist families $a_1(t),\dots, a_l(t)$ of C^∞-sections (resp. C^2-sections) of E on U which depend complex analytically on t in D such that $\phi(t)-\psi(t)-\sum_{\tau=1}^l a_\tau(t)h_\tau(t)$ contains no term of total degree less than μ as a power series in $(t_1,\dots, t_q)$.

(I) Construction of Formal Solution

In this paragraph, we shall construct formal power series $\phi(t)$, $h_1(t),\dots, h_l(t)$ in $t=(t_1,\dots, t_q)$ satisfying the formal versions of (2.0), (2.1.1), (2.1.2) and (2.2) below. To do this, we construct, for each $\mu\geqq 0$, a $\Gamma(M, {}^o\overline{T}''\otimes({}^oT'')^*)$-valued polynomial $\phi^\mu(t)$ in t of degree μ and an $H^{(2)}_{T'}$-valued polynomial $h^\mu(t)$ in t of the same degree, satisfying the following:

(2.0) $\phi^0=0$ and $h^0=0$,

$(2.1.1)_\mu$ $D_1\phi^\mu(t)+R_2(\phi^\mu(t))\equiv 0 \mod (h^\mu(t), t^{\mu+1})$,

$(2.1.2)_\mu$ $$L\phi^\mu(t) \equiv 0 \mod (h^\mu(t), t^{\mu+1}),$$

(2.2) $$\phi^1(t) = \sum_{\sigma=1}^q (\mathscr{L}\beta_\sigma)t_\sigma,$$

$(2.3)_\mu$ $$\phi^{\mu-1}(t) \equiv \phi^\mu(t) \mod (t^\mu),$$

and

$$h^{\mu-1}(t) \equiv h^\mu(t) \mod (t^\mu),$$

where $h^\mu(t) = h_1^\mu(t)e_1 + \cdots + h_l^\mu(t)e_l$ for some fixed base $(e_1, \dots, e_l)$ of $H_{T'}^{(2)}$ and $\mod (h^\mu(t), t^{\mu+1}) = \mod (h_1^\mu(t), \dots, h_l^\mu(t), t^{\mu+1})$.

We construct such $\phi^\mu(t)$ and $h^\mu(t)$ by induction on μ, while from technical reasons, we impose on them the following additional conditions for the induction:

$(2.4)_\mu$ $$D_1^*(P(\phi^\mu(t))_{{}^oT''}) \equiv 0 \mod (t^{\mu+1}),$$

$(2.5)_\mu$ $$\bar{\partial}_F^{(1)*}L\phi^\mu(t) = 0,$$

$(2.6)_\mu$ $$L\rho \boldsymbol{H}_F^2 L\phi^\mu(t) = 0.$$

In view of (2.0) and (2.2), we set

$$\phi^0 = 0, \quad h^0 = 0,$$

and

$$\phi^1(t) = \sum_{\sigma=1}^q (\mathscr{L}\beta_\sigma)t_\sigma, \quad h^1(t) = 0.$$

Then $\phi^1(t)$ is $\Gamma(M, {}^o\bar{T}'' \otimes ({}^oT'')^*)$-valued, $(2.3)_1$ is clearly satisfied, and since $\bar{\partial}_{T'}^{(1)}\phi^1(t) = 0$, $(2.1.1)_1$, $(2.1.2)_1$, $(2.5)_1$, and $(2.6)_1$ are satisfied. Moreover we have $P(\phi^1(t)) \equiv 0 \mod (t^2)$, so $(2.4)_1$ is also satisfied.

Suppose then that, for some $\mu \geqq 2$, $\phi^\nu(t)$ and $h^\nu(t)$ are already determined for all $\nu < \mu$ in such a way that $(2.1.1)_\nu$, $(2.1.2)_\nu$, $(2.3)_\nu$–$(2.6)_\nu$ are all satisfied.

Then we set

$$\phi_\mu(t) = -\kappa_t^\mu\{(I - D_0L_0^{-1}\bar{\partial}_F^{(1)*}N_FL)(I - \rho H_F^2 L)D_1^*N(P(\phi^{\mu-1}(t))_{{}^oT''})\}$$

and

$$\phi^\mu(t) = \phi^{\mu-1}(t) + \phi_\mu(t),$$

where κ_t^μ is the operator taking the homogeneous term of total degree μ in $(t_1, \dots, t_q)$. Since N and N_F are C^∞-operators, and $\phi^{\mu-1}(t)$ is $\Gamma(M, {}^o\bar{T}'' \otimes ({}^oT'')^*)$-valued by induction, $\phi^\mu(t)$ also is $\Gamma(M, {}^o\bar{T}'' \otimes ({}^oT'')^*)$-valued.

Next, we define $h^\mu(t)$ by the following congruence:

$$h^\mu(t) \equiv \boldsymbol{H}_{T'}^2(P(\phi^{\mu-1}(t))) \mod (t^{\mu+1}).$$

Then we shall show that these $\phi^{\mu}(t)$ and $h^{\mu}(t)$ satisfy the desired equalities. (We set $\phi_1(t)=\phi^1(t)$ in the below.)

$(2.4)_\mu$: $D_1^*(P(\phi^{\mu}(t))_{\circ T''})$

$$\begin{aligned}&\equiv D_1^*(P(\phi^{\mu-1}(t))_{\circ T''}+D_1\phi_\mu(t)) \quad \mathrm{mod}\,(t^{\mu+1})\\ &=D_1^*(P(\phi^{\mu-1}(t))_{\circ T''})-D_1^*D_1\kappa_t^{\mu}\{ND_1^*(P(\phi^{\mu-1}(t))_{\circ T''})\}\\ &=D_1^*(P(\phi^{\mu-1}(t))_{\circ T''})-\kappa_t^{\mu}D_1^*(P(\phi^{\mu-1}(t))_{\circ T''})\\ &\equiv 0 \quad \mathrm{mod}\,(t^{\mu+1}), \quad (\text{by } (2.4)_{\mu-1}).\end{aligned}$$

$(2.5)_\mu$: $\bar\partial_F^{(1)*}L\phi^{\mu}(t)$

$$\begin{aligned}&=-\kappa_t^{\mu}\{\bar\partial_F^{(1)*}(L-LD_0L_0^{-1}\bar\partial_F^{(1)*}N_FL)(I-\rho\boldsymbol{H}_F^2L)D_1^*N(P(\phi^{\mu-1}(t))_{\circ T''})\},\\ &\qquad\qquad (\text{by } (2.5)_{\mu-1}),\\ &=-\kappa_t^{\mu}\{\bar\partial_F^{(1)*}(I-\bar\partial_F^{(1)}\bar\partial_F^{(1)*}N_F)L(I-\rho\boldsymbol{H}_F^2L)D_1^*N(P(\phi^{\mu-1}(t))_{\circ T''})\},\\ &\qquad\qquad (\text{by Proposition 1.1}),\\ &=-\kappa_t^{\mu}\{\bar\partial_F^{(1)*}(\bar\partial_F^{(2)*}\bar\partial_F^{(2)}N_F+\boldsymbol{H}_F^2)L(I-\rho\boldsymbol{H}_F^2L)D_1^*N(P(\phi^{\mu-1}(t))_{\circ T''})\}\\ &=0.\end{aligned}$$

$(2.6)_\mu$: $L\rho\boldsymbol{H}_F^2L\phi^{\mu}(t)$

$$\begin{aligned}&=-\kappa_t^{\mu}\{L\rho\boldsymbol{H}_F^2(I-\bar\partial_F^{(1)}\bar\partial_F^{(1)*}N_F)L(I-\rho\boldsymbol{H}_F^2L)D_1^*N(P(\phi^{\mu-1}(t))_{\circ T''})\},\\ &\qquad\qquad (\text{by } (2.6)_{\mu-1} \text{ and Proposition 1.1}),\\ &=-\kappa_t^{\mu}\{L\rho\boldsymbol{H}_F^2(\bar\partial_F^{(2)*}\bar\partial_F^{(2)}N_F+\boldsymbol{H}_F^2)L(I-\rho\boldsymbol{H}_F^2L)D_1^*N(P(\phi^{\mu-1}(t))_{\circ T''})\}\\ &=-\kappa_t^{\mu}\{(L\rho\boldsymbol{H}_F^2-(L\rho\boldsymbol{H}_F^2)^2)LD_1^*N(P(\phi^{\mu-1}(t))_{\circ T''})\}\\ &=0, \quad (\text{by Lemma 2.2, (2)}).\end{aligned}$$

$(2.3)_\mu$: It is clear that

$$\begin{aligned}&\phi^{\mu-1}(t)\equiv\phi^{\mu}(t) \quad \mathrm{mod}\,(t^{\mu}),\\ &h^{\mu}(t)\equiv\boldsymbol{H}_{T'}^2(P(\phi^{\mu-1}(t))) \quad \mathrm{mod}\,(t^{\mu+1})\\ &\qquad\equiv\boldsymbol{H}_{T'}^2(P(\phi^{\mu-2}(s))+\bar\partial_{T'}^{(1)}\phi_{\mu-1}(t)) \quad \mathrm{mod}\,(t^{\mu})\\ &\qquad\equiv h^{\mu-1}(t) \quad \mathrm{mod}\,(t^{\mu}).\end{aligned}$$

To prove $(2.1.1)_\mu$, we first show the following lemma.

Lemma 2.3.

$$(2.7)_\mu \qquad \bar\partial_{T'}^{(2)}P(\phi^{\mu-1}(t))\equiv 0 \quad \mathrm{mod}\,(h^{\mu}(t),\ t^{\mu+1}),$$

$$(2.8)_\mu \qquad \boldsymbol{H}_{\circ T''}^2(P(\phi^{\mu-1}(t))_{\circ T''})\equiv 0 \quad \mathrm{mod}\,(h^{\mu}(t),\ t^{\mu+1}),$$

$$(2.9)_\mu \qquad D_2(P(\phi^{\mu-1}(t))_{\circ T''})\equiv 0 \quad \mathrm{mod}\,(h^{\mu}(t),\ t^{\mu+1}).$$

Proof. $(2.7)_\mu$: By Proposition 4.5 in [4], we have

$$\bar\partial_{T'}^{\phi^{\mu-1}(t)}P(\phi^{\mu-1}(t))=0.$$

Since $P(\phi^{\mu-1}(t))\equiv 0 \bmod (h^{\mu-1}(t), t^{\mu})$, we infer that $\bar\partial^{(2)}_{T'}P(\phi^{\mu-1}(t))\equiv 0 \bmod (th^{\mu-1}(t), t^{\mu+1})$. Since $th^{\mu-1}(t)\equiv 0 \bmod (h^{\mu}(t), t^{\mu+1})$, by $(2.3)_\mu$, $(2.7)_\mu$ follows.

$(2.8)_\mu$ and $(2.9)_\mu$: If we set $\psi(t)=-\bar\partial^{(1)*}_{T'}N_{T'}P(\phi^{\mu-1}(t))$,

$$\begin{aligned}\bar\partial^{(1)}_{T'}\psi(t)&=-\bar\partial^{(1)}_{T'}\bar\partial^{(1)*}_{T'}N_{T'}P(\phi^{\mu-1}(t))\\&=-P(\phi^{\mu-1}(t))+\bar\partial^{(2)*}_{T'}\bar\partial^{(2)}_{T'}N_{T'}P(\phi^{\mu-1}(t))+\boldsymbol{H}^2_{T'}P(\phi^{\mu-1}(t))\\&=-P(\phi^{\mu-1}(t))+N_{T'}\bar\partial^{(2)*}_{T'}\bar\partial^{(2)}_{T'}P(\phi^{\mu-1}(t))+\boldsymbol{H}^2_{T'}P(\phi^{\mu-1}(t))\\&\equiv -P(\phi^{\mu-1}(t)) \quad \bmod (h^{\mu}(t), t^{\mu+1}),\end{aligned}$$

(by $(2.7)_\mu$ and the definition of $h^\mu(t)$). Hence, by the definition of $\mathscr{L}$, we have

$$\bar\partial^{(1)}_{T'}\mathscr{L}\psi(t)+P(\phi^{\mu-1}(t))\equiv 0 \quad \bmod (h^{\mu}(t), t^{\mu+1})$$

and

$$D_1\mathscr{L}\psi(t)+P(\phi^{\mu-1}(t))_{\circ T''}\equiv 0 \quad \bmod (h^{\mu}(t), t^{\mu+1}).$$

From these, we infer $(2.8)_\mu$ and $(2.9)_\mu$.

$(2.1.1)_\mu$: By the definition of $\phi_\mu(t)$,

$$\begin{aligned}D_1\phi_\mu(t)&=-\kappa^\mu_t\{D_1ND_1^*(P(\phi^{\mu-1}(t))_{\circ T''})\}\\&\equiv -ND_1D_1^*(P(\phi^{\mu-1}(t))_{\circ T''}) \quad \bmod (t^{\mu+1}), \quad (\text{by } (2.4)_{\mu-1}).\end{aligned}$$

Then

$$\begin{aligned}&D_1\phi^\mu(t)+R_2(\phi^\mu(t))\\&\quad\equiv D_1\phi_\mu(t)+P(\phi^{\mu-1}(t))_{\circ T''} \quad \bmod (t^{\mu+1})\\&\quad\equiv ND_2^*D_2(P(\phi^{\mu-1}(t))_{\circ T''})+\boldsymbol{H}^2_{\circ T''}(P(\phi^{\mu-1}(t))_{\circ T''}) \quad \bmod (t^{\mu+1})\\&\quad\equiv 0 \quad \bmod (h^\mu(t), t^{\mu+1}), \quad (\text{by } (2.8)_\mu \text{ and } (2.9)_\mu).\end{aligned}$$

$(2.1.2)_\mu$: $L\phi^\mu(t)$

$$\equiv L\phi^{\mu-1}(t)-\{(I-\bar\partial^{(1)}_F\bar\partial^{(1)*}_F N_F)(I-L\rho\boldsymbol{H}^2_F)LD_1^*N(P(\phi^{\mu-1}(t))_{\circ T''})\} \bmod (t^{\mu+1}),$$

(by $(2.4)_{\mu-1}$ and Proposition 1.1). Now, if we set $\psi(t)=-\bar\partial^{(1)*}_{T'}N_{T'}P(\phi^{\mu-1}(t))$, as in the proof of $(2.8)_\mu$ and $(2.9)_\mu$, we have

$$D_1\mathscr{L}\psi(t)+P(\phi^{\mu-1}(t))_{\circ T''}=o(t)_{\circ T''}$$

and

$$L\mathscr{L}\psi(t)+L\phi^{\mu-1}(t)=o(t)_F$$

where $o(t)=\bar\partial^{(2)*}_{T'}\bar\partial^{(2)}_{T'}N_{T'}P(\phi^{\mu-1}(t))+\boldsymbol{H}^2_{T'}P(\phi^{\mu-1}(t))$. Then

$$\begin{aligned}&LD_1^*N(P(\phi^{\mu-1}(t))_{\circ T''})\\&\quad=-LD_1^*ND_1\mathscr{L}\psi(t)+LD_1^*N(o(t)_{\circ T''})\\&\quad=-L\mathscr{L}\psi(t)+LD_0D_0^*N\mathscr{L}\psi(t)+L\boldsymbol{H}^1_{\circ T''}\mathscr{L}\psi(t)+LD_1^*N(o(t)_{\circ T''})\end{aligned}$$

$$=L\phi^{\mu-1}(t)+LD_0D_0^*N\mathscr{L}\psi(t)+L\boldsymbol{H}^1_{\circ T''}\mathscr{L}\psi(t)+LD_1^*N(o(t)_{\circ T''})-o(t)_F.$$

Hence, since

$$L\phi^{\mu-1}(t)-(I-\bar\partial_F^{(1)}\bar\partial_F^{(1)*}N_F)(I-L\rho\boldsymbol{H}_F^2)L\phi^{\mu-1}(t)=0$$

by $(2.5)_{\mu-1}$ and $(2.6)_{\mu-1}$, we have

$$\begin{aligned}
L\phi^{\mu}(t)\equiv & -(I-\bar\partial_F^{(1)}\bar\partial_F^{(1)*}N_F)(I-L\rho\boldsymbol{H}_F^2)\{LD_0D_0^*N\mathscr{L}\psi(t)+L\boldsymbol{H}^1_{\circ T''}\mathscr{L}\psi(t)\} \\
& \bmod (h^\mu(t),\ t^{\mu+1}),\quad (\text{since } o(t)\equiv 0 \bmod (h^\mu(t),\ t^{\mu+1}) \text{ by } (2.7)_\mu \text{ and} \\
& \text{the definition of } h^\mu(t)), \\
= & -(I-\bar\partial_F^{(1)}\bar\partial_F^{(1)*}N_F)\bar\partial_F^{(1)}L_0D_0^*N\mathscr{L}\psi(t) \\
& -(\bar\partial_F^{(2)*}\bar\partial_F^{(2)}N_F+\boldsymbol{H}_F^2)(I-L\rho\boldsymbol{H}_F^2)L\boldsymbol{H}^1_{\circ T''}\mathscr{L}\psi(t),\quad (\text{by Proposition 1.1}), \\
= & -\bar\partial_F^{(2)*}\bar\partial_F^{(2)}N_F(I-L\rho\boldsymbol{H}_F^2)L\boldsymbol{H}^1_{\circ T''}\mathscr{L}\psi(t),\quad (\text{by Lemma 2.2, (1)}), \\
= & -N_F\bar\partial_F^{(2)*}\bar\partial_F^{(2)}L(I-\rho\boldsymbol{H}_F^2L)\boldsymbol{H}^1_{\circ T''}\mathscr{L}\psi(t) \\
= & -N_F\bar\partial_F^{(2)*}L_2D_1(I-\rho\boldsymbol{H}_F^2L)\boldsymbol{H}^1_{\circ T''}\mathscr{L}\psi(t),\quad (\text{by Proposition 1.1}), \\
= & \,0.
\end{aligned}$$

Thus $(2.1.2)_\mu$ is proved.

This completes the inductive constructions of $\phi^\mu(t)$ and $h^\mu(t)$.

(II) Proof of Convergence

In this paragraph, we shall prove that the formal power series $\phi(t)=\lim_{\mu\to\infty}\phi^\mu(t)$ and $h(t)=\lim_{\mu\to\infty}h^\mu(t)$ in $(t_1,\dots,t_q)$ are convergent with respect to $\|\ \|''_{(m)}$-norm and $|\ |$-norm respectively where $|\ |$ denotes the euclidean norm on the finite dimensional vector space $H^{(2)}_{T'}$. To prove this, we first show the existence of a convergent power series $A(t)$ in $(t_1,\dots,t_q)$ with real non-negative coefficients and with the following property:

$(2.10)_\mu$ $$\|\phi^\mu(t)\|''_{(m)}\ll A(t)\qquad \text{for all}\quad \mu\geqq 1.$$

(Cf. [2], for the meaning of the symbol $\ll$.)

First we fix $m\geqq n+2$, and set

$$A(t)=b/16c\sum_{\mu=1}^{\infty}(c^\mu/\mu^2)(t_1+\cdots+t_q)^\mu$$

where b and c are positive numbers to be determined later.

Then $A(t)$ satisfies the inequality $A(t)^2\ll(b/c)A(t)$.

Now it is clear that $(2.10)_1$ is satisfied, if we choose b sufficiently large.

Suppose that $(2.10)_{\mu-1}$ is satisfied for some b and c. By the definition of $\phi_\mu(t)$,

$$\|\phi_\mu(t)\|''_{(m)}\ll\|(I-D_0L_0^{-1}\bar\partial_F^{(1)*}N_FL)(I-\rho\boldsymbol{H}_F^2L)D_1^*NR_2(\phi^{\mu-1}(t))\|''_{(m)}.$$

By Proposition 1.2,

$$\|\phi_\mu(t)\|''_{(m)} \ll c_1 \|\phi^{\mu-1}(t)\|''^2_{(m)}$$

where c_1 is a constant independent of μ.

By our assumption $(2.10)_{\mu-1}$, we have

$$\|\phi_\mu(t)\|''_{(m)} \ll c_1 A(t)^2 \ll (bc_1/c)A(t).$$

Hence, if we choose b and c in such a way that $(bc_1/c) \leqq 1$ and $(2.10)_1$ are satisfied, we see that

$$\|\phi^\mu(t)\|''_{(m)} \ll A(t) \qquad \text{for all} \quad \mu \geqq 1,$$

by induction on μ.

Thus $\phi(t) = \lim_{\mu\to\infty} \phi^\mu(t)$ is convergent for t with $|t| < \varepsilon$, for some $\varepsilon > 0$, and it determines an element of $\Gamma''_{(m)}(M, {}^o\overline{T}'' \otimes ({}^oT'')^*)\{t_1,\dots, t_q\}$. By Sobolev's lemma, $\phi(t)$ is of class C^2 because $m \geqq n+2$.

Then if we set $h(t) = \boldsymbol{H}^2_{T'}(P(\phi(t)))$, $h(t)$ is in $H^{(2)}_{T'}\{t_1,\dots, t_q\}$ and it satisfies

$$\begin{aligned} h(t) &\equiv \boldsymbol{H}^2_{T'}(P(\phi^{\mu-1}(t))) \mod (t^{\mu+1}) \\ &\equiv h^\mu(t) \mod (t^{\mu+1}). \end{aligned}$$

Hence $\lim_{\mu\to\infty} h^\mu(t)$ coincides with $h(t)$.

From $(2.1.1)_\mu$ and $(2.1.2)_\mu$, we have

$(2.1.1)'_\mu$ $\qquad D_1\phi(t) + R_2(\phi(t)) \equiv 0 \mod (h(t), t^{\mu+1}),$

$(2.1.2)'_\mu$ $\qquad L\phi(t) \equiv 0 \mod (h(t), t^{\mu+1}),$

for any $\mu \geqq 1$.

Let T be the analytic subset of a smaller neighbourhood V of 0 in $D = \{t \mid |t| < \varepsilon\}$ defined by $h(t) = 0$. Let $\tau\colon \tilde{T} \to T$ be a resolution of the singularity of T and $(t'_1,\dots, t'_{q'})$ a system of local coordinates of $\tilde{T}$ at $0'$ for any $0' \in \tau^{-1}(0)$.

From $(2.1.1)'_\mu$ and $(2.1.2)'_\mu$, we have, for any $\mu \geqq 0$,

$(2.1.1)''_\mu$ $\qquad D_1\phi(\tau(t')) + R_2(\phi(\tau(t'))) \equiv 0 \mod (t'^{\mu+1}),$

$(2.1.2)''_\mu$ $\qquad L\phi(\tau(t')) \equiv 0 \mod (t'^{\mu+1}).$

Since the left hand sides of these equations are holomorphic in t', we infer that they all vanish. Hence (2.1.1) and (2.1.2) follow if we replace T by a smaller neighbourhood of 0 in T.

§ 3. Proof of Versality

The purpose of this section is to prove that the complex analytic family of C^2-partially complex structures on M constructed in Section 2 is the versal family.

Let $(\mathcal{N}, \pi, S)$ be an arbitrary family of deformations of a neighbourhood $\tilde{N}$ of M such that $\mathcal{N}_o=\tilde{N}$ for $o\in S$.

We may assume the following:

(3.i) o is the origin of $\boldsymbol{C}^r$ and S is an analytic subspace of a neighbourhood D of o in $\boldsymbol{C}^r$ defined by $b_1(s)=\cdots=b_l(s)=0$.

(3.ii) We find a finite system of open sets of $\mathcal{N}$, $\{\mathcal{U}_j\}_{j\in A}$, such that there exists an analytic embedding

$$\eta_j\colon \mathcal{U}_j\to W_j\times D \quad \text{with} \quad p_2\circ\eta_j=\pi \quad \text{for each} \quad j\in A,$$

and $M\subset\bigcup_{j\in A}\mathcal{U}_j$, where W_j is a neighbourhood of 0 in $\boldsymbol{C}^n$ and p_2 denotes the projection of $W_j\times D$ onto the second factor. We denote by $\zeta_j=(\zeta_j^1,\dots,\zeta_j^n)$ and $s=(s_1,\dots,s_r)$ the coordinates of W_j and D respectively, and set $z_j^\lambda=\zeta_j^\lambda\circ\eta_{j|\mathcal{N}_o}$ for $\lambda=1,\dots,n$ and $U_j=\mathcal{U}_j\cap M$ where we regard ζ_j^λ as a function on $W_j\times D$.

(3.iii) $\eta_j\circ\eta_k^{-1}$ is represented by

$$\begin{cases} \zeta_j^\lambda=f_{jk}^\lambda(\zeta_k, s) & \text{for} \quad \lambda=1,\dots,n,\\ s_\alpha=s_\alpha & \text{for} \quad \alpha=1,\dots,r,\end{cases}$$

and we set $\bar{f}_{jk}^\lambda(z_k)=f_{jk}^\lambda(z_k, 0)$ for $\lambda=1,\dots,n$.

From (3.iii), we infer that

$$\text{(3.iv)} \quad f_{ij}^\lambda(f_{jk}(\zeta_k, s), s)\equiv f_{ik}^\lambda(\zeta_k, s) \mod (b(s)) \quad \text{for} \quad \lambda=1,\dots,n.$$

To prove the versality of the family constructed in Section 2, it suffices to show the existence of a neighbourhood D' of 0 in D, of a $\boldsymbol{C}^2$-family $g_i(s)$ of sections of T' over U_i which depends complex analytically on s in D' for each $i\in A$, and of a $\mathcal{H}$-valued holomorphic function $\tau(s)$ on D' satisfying;

$$\text{(3.0)} \quad (g_i(0))^\lambda=z_i^\lambda \quad \text{for} \quad \lambda=1,\dots,n \quad \text{and} \quad \tau(0)=0,$$

$$\text{(3.1)} \quad (g_i(s))^\lambda-f_{ij}^\lambda(g_j(s), s)=0 \quad \text{for} \quad s\in S' \quad \text{and} \quad \lambda=1,\dots,n,$$

$$\text{(3.2)} \quad (\bar{\partial}_b+\phi(\tau(s)))(g_i(s))^\lambda=0 \quad \text{for} \quad s\in S' \quad \text{and} \quad \lambda=1,\dots,n,$$

$$\text{(3.3)} \quad h(\tau(s))=0 \quad \text{for} \quad s\in S',$$

where $S'=D'\cap S$, $g_i(s)$ has the expression $g_i(s)=\sum_{\lambda=1}^n (g_i(s))^\lambda\partial/\partial z_i^\lambda$ regarded as an element of $\Gamma(U_i, T'N_{|M})$ since T' is isomorphic to $T'N_{|M}$ as a C^∞-vector bundle, and $(\bar{\partial}_b+\phi(\tau(s)))(g_i(s))^\lambda$ denotes the element of $\Gamma(U_i, ({}^oT'')^*)$ defined by the congruence

$$(\bar{\partial}_b+\phi(\tau(s)))(g_i(s))^\lambda(X)=\bar{\partial}_b(g_i(s))^\lambda(X)+(\phi(\tau(s))(X))(g_i(s))^\lambda$$

for $X\in\Gamma(U_i, ({}^oT''))$.

(I) Construction of a Formal Solution

In this paragraph, we shall construct $\{g_i(s)\}_{i\in A}$ and $\tau(s)$ formally in s, that is to say, we construct sequences $\{g_i^\mu(s)\}_{i\in A}$ and $\tau^\mu(s)$ for $\mu=0, 1,\dots,$ satisfying the following:

(3.0) $(g_i^0)^\lambda=z_i^\lambda$ for $\lambda=1,\dots, n$ and $\tau^0=0$,

for any $\mu\geqq 0$,

$(3.1)_\mu$ $(g_i^\mu(s))^\lambda-f_{ij}^\lambda(g_j^\mu(s), s)\equiv 0 \mod (b(s), s^{\mu+1})$ for $\lambda=1,\dots, n$,

$(3.2)_\mu$ $(\bar{\partial}_b+\phi(\tau^\mu(s)))(g_i^\mu(s))^\lambda\equiv 0 \mod (b(s), s^{\mu+1})$ for $\lambda=1,\dots, n$,

$(3.3)_\mu$ $h(\tau^\mu(s))\equiv 0 \mod (b(s), s^{\mu+1})$,

$(3.4)_\mu$ $g_i^\mu(s)$ is a $\Gamma(U_i, T')$-valued polynomial in s of degree μ and $\tau^\mu(s)$ is a $\mathscr{H}$-valued polynomial in s of the same degree such that

$$g_i^\mu(s)\equiv g_i^{\mu-1}(s) \mod (s^\mu),$$

and

$$\tau^\mu(s)\equiv \tau^{\mu-1}(s) \mod (s^\mu).$$

We shall construct these $\{g_i^\mu(s)\}_{i\in A}$ and $\tau^\mu(s)$ by induction on μ.

For $\mu=0$, because of (3.0), we set

$$(g_i^0)^\lambda=z_i^\lambda \quad \text{for } \lambda=1,\dots, n, \qquad \text{and} \qquad \tau^0=0.$$

Suppose that $\{g_i^{\mu-1}(s)\}_{i\in A}$ and $\tau^{\mu-1}(s)$ are determined for some $\mu\geqq 1$. First we define a $\Gamma(U_i\cap U_j, T')$-valued polynomial in s of degree μ, $\sigma_{ij}^\mu(s)$, by

$$\sigma_{ij}^\mu(s)\equiv \textstyle\sum_{\lambda=1}^n \{(g_i^{\mu-1}(s))^\lambda-f_{ij}^\lambda(g_j^{\mu-1}(s), s)\}\partial/\partial z_i^\lambda \mod (s^{\mu+1}).$$

Then we set

$$g'_{i|\mu}(s)=\textstyle\sum_{k\in A}\rho_k\kappa_s^\mu(\sigma_{ki}^\mu(s)),$$

where $\{\rho_k\}_{k\in A}$ is a partition of unity subordinate to $\{U_k\}_{k\in A}$. Next we define $\Gamma(U_i, T'\otimes({}^oT'')^*)$-valued polynomials $\omega_i^\mu(s)$ and $\xi_i^\mu(s)$ of degree μ by

$$\omega_i^\mu(s)\equiv -\textstyle\sum_{\lambda=1}^n [\bar{\partial}_b\{(g_i^{\mu-1}(s))^\lambda+(g'_{i|\mu}(s))^\lambda\}$$

$$+\phi(\tau^{\mu-1}(s))\{(g_i^{\mu-1}(s))^\lambda-(g_i^0)^\lambda\}]\partial/\partial z_i^\lambda \quad \mathrm{mod}\,(s^{\mu+1}),$$

and

$$\xi_i^\mu(s)\equiv\omega_i^\mu(s)-\phi(\tau^{\mu-1}(s))_{|U_i} \quad \mathrm{mod}\,(s^{\mu+1}).$$

Then we set

$$\tau_\mu(s)=\sum_{\sigma=1}^q \tau_\mu^\sigma(s)(\mathscr{L}\beta_\sigma)=\mathscr{L}\boldsymbol{H}_{T'}^1(\sum_{i\in A}\rho_i\kappa_s^\mu(\xi_i^\mu(s)))$$

and

$$g'_\mu(s)=-\bar\partial_{T'}^{(0)*}N_{T'}(\sum_{i\in A}\rho_i\kappa_s^\mu(\xi_i^\mu(s))-\tau_\mu(s)).$$

Then we infer that $g'_\mu(s)$ is $\Gamma(M, T')$-valued, since $N_{T'}$ is a C^∞-operator. Finally we set

$$g_i^\mu(s)=g_i^{\mu-1}(s)+g'_{i|\mu}(s)+g'_\mu(s)$$

and

$$\tau^\mu(s)=\tau^{\mu-1}(s)+\tau_\mu(s).$$

It is clear that (3.0) and $(3.4)_\mu$ are satisfied for all $\mu\geqq 1$.

Proposition 3.1. *For any* $\mu\geqq 0$,

$(1)_\mu$ $(g_i^\mu(s))^\lambda-f_{ij}^\lambda(g_j^\mu(s), s)\equiv 0 \quad \mathrm{mod}\,(b(s), s^{\mu+1})$ *for* $\lambda=1,\dots,n$,

$(2)_\mu$ $\theta_i^\mu(s)-\phi(\tau^\mu(s))_{|U_i}\equiv 0 \quad \mathrm{mod}\,(b(s), s^{\mu+1})$

where $\theta_i^\mu(s)$ *is a* $\Gamma(U_i, T'\otimes({}^oT'')^*)$*-valued polynomial in* s *of degree* μ *defined by*

$$(\bar\partial_b+\theta_i^\mu(s))(g_i^\mu(s))^\lambda\equiv 0 \quad \mathrm{mod}\,(s^{\mu+1}) \qquad \textit{for} \quad \lambda=1,\dots,n,$$

and $(\bar\partial_b+\theta_i^\mu(s))(g_i^\mu(s))^\lambda$ *is defined by the same congruence of* $(\bar\partial_b+\phi(\tau(s)))\times(g_i(s))^\lambda$.

$(3)_\mu$ $h(\tau^\mu(s))\equiv 0 \quad \mathrm{mod}\,(b(s), s^{\mu+1})$,

$(4)_\mu$ $\sum_{k\in A}\rho_k\{\sigma_{kj}^{\mu+1}(s)-\sigma_{ki}^{\mu+1}(s)\}\equiv 0 \quad \mathrm{mod}\,(s^{\mu+1})$,

$(5)_\mu$ $\bar\partial_{T'}^{(0)}\bar\partial_{T'}^{(0)*}N_{T'}\{\sum_{i\in A}\rho_i\omega_i^{\mu+1}-\phi(\tau^\mu(s))\}\equiv 0 \quad \mathrm{mod}\,(s^{\mu+1})$,

$(6)_\mu$ $\boldsymbol{H}_{T'}^1\{\sum_{i\in A}\rho_i\omega_i^{\mu+1}(s)-\phi(\tau^\mu(s))\}\equiv 0 \quad \mathrm{mod}\,(s^{\mu+1})$.

Proof. For $\mu=0$, it is clear that $(1)_0-(3)_0$ are satisfied. Since

$$\sigma_{ij}^1(s)\equiv 0 \quad \mathrm{mod}\,(s),$$
$$\omega_i^1(s)\equiv 0 \quad \mathrm{mod}\,(s),$$

and

$$\phi(\tau^0(s))=0,$$

$(4)_0-(6)_0$ are also satisfied.

We suppose that $(1)_{\mu-1}-(6)_{\mu-1}$ are satisfied for some $\mu\geqq 1$. To prove $(1)_\mu$, we show the following lemma.

Lemma 3.2. $\sigma^\mu_{ki}(s)-\sigma^\mu_{kj}(s)+\sigma^\mu_{ij}(s)\equiv 0 \mod (b(s), s^{\mu+1})$.

Proof. Omitting the index subordinate to the local base $(\partial/\partial z_i^1,\dots,\partial/\partial z_i^n)$ of T',

$$\begin{aligned}\sigma^\mu_{ki}(s)&\equiv g_k^{\mu-1}(s)-f_{ki}(g_i^{\mu-1}(s), s) \mod (s^{\mu+1})\\ &\equiv g_k^{\mu-1}(s)-f_{ki}(f_{ij}(g_j^{\mu-1}(s), s), s)\\ &\quad -\textstyle\sum_{\lambda=1}^n(\partial f_{ki}/\partial\zeta_i^\lambda)(f_{ij}(g_j^{\mu-1}(s), s), s)(\sigma^\mu_{ij}(s))^\lambda \mod (s^{\mu+1}, (\sigma^\mu_{ij}(s))^2)\\ &\equiv \sigma^\mu_{kj}(s)-\textstyle\sum_{\lambda=1}^n(\partial\bar f_{ki}/\partial z_i^\lambda)(z_i)(\sigma^\mu_{ij}(s))^\lambda \mod (s^{\mu+1}, (\sigma^\mu_{ij}(s))^2, s\sigma^\mu_{ij}(s), b(s)).\end{aligned}$$

From $(1)_{\mu-1}$, $\sigma^\mu_{ij}(s)\equiv 0 \mod (b(s), s^\mu)$. Then Lemma 3.2 follows.

Proof of $(1)_\mu$: $g_i^\mu(s)-f_{ij}(g_j^\mu(s), s)$

$$\begin{aligned}&\equiv g_i^{\mu-1}(s)+g'_{i|\mu}(s)+g'_\mu(s)-f_{ij}(g_j^{\mu-1}(s), s)\\ &\quad -\textstyle\sum_{\lambda=1}^n(\partial\bar f_{ij}/\partial z_j^\lambda)(z_j)\{(g'_{j|\mu}(s))^\lambda+(g'_\mu(s))^\lambda\} \mod (s^{\mu+1})\\ &\equiv \sigma^\mu_{ij}(s)-\textstyle\sum_{k\in\Lambda}\rho_k\kappa_s^\mu(\sigma^\mu_{kj}(s)-\sigma^\mu_{ki}(s)) \mod (s^{\mu+1}),\\ &\qquad\qquad\text{(by the definition of } g'_{i|\mu}(s) \text{ and } g'_\mu(s)),\\ &\equiv \sigma^\mu_{ij}(s)-\textstyle\sum_{k\in\Lambda}\rho_k(\sigma^\mu_{kj}(s)-\sigma^\mu_{ki}(s)) \mod (s^{\mu+1}),\quad \text{(by } (4)_{\mu-1}),\\ &\equiv 0 \mod (b(s), s^{\mu+1}),\quad \text{(by Lemma 3.2)}.\end{aligned}$$

Proof of $(4)_\mu$: As we have proved above,

$$\sigma^{\mu+1}_{kj}(s)\equiv\sigma^\mu_{kj}(s)-\textstyle\sum_{l\in\Lambda}\rho_l(\sigma^\mu_{lj}(s)-\sigma^\mu_{lk}(s)) \mod (s^{\mu+1}).$$

Hence

$$\sigma^{\mu+1}_{kj}(s)-\sigma^{\mu+1}_{ki}(s)\equiv\sigma^\mu_{kj}(s)-\sigma^\mu_{ki}(s)-\textstyle\sum_{l\in\Lambda}\rho_l(\sigma^\mu_{lj}(s)-\sigma^\mu_{li}(s)) \mod (s^{\mu+1})$$

Then,

$$\textstyle\sum_{k\in\Lambda}\rho_k(\sigma^{\mu+1}_{kj}(s)-\sigma^{\mu+1}_{ki}(s))\equiv 0 \mod (s^{\mu+1}).$$

To prove $(2)_\mu$, $(3)_\mu$, we prove some lemmas.

Lemma 3.3. $\theta_i^\mu(s)\equiv\omega_i^\mu(s)-\bar\partial^{(0)}_{T'}g'_\mu(s)_{|U_i} \mod (b(s), s^{\mu+1})$.

Proof. From the definition of $\theta_i^\mu(s)$ and $\omega_i^\mu(s)$,

$$\begin{aligned}&\bar\partial_b\{(g_i^{\mu-1}(s))^\lambda+(g'_{i|\mu}(s))^\lambda\}+\bar\partial_b(g'_\mu(s))^\lambda{}_{|U_i}+(\theta_i^\mu(s))^\lambda\\ &\qquad+\theta_i^\mu(s)\{(g_i^{\mu-1}(s))^\lambda-(g_i^0)^\lambda\}\equiv 0 \mod (s^{\mu+1})\end{aligned}$$

and

$$\bar{\partial}_b\{(g_i^{\mu-1}(s))^\lambda+(g'_{i|\mu}(s))^\lambda\}+(\omega_i^\mu(s))^\lambda$$
$$+\phi(\tau^{\mu-1}(s))\{(g_i^{\mu-1}(s))^\lambda-(g_i^0)^\lambda\}\equiv 0 \mod (s^{\mu+1}),$$

for $\lambda=1,\dots, n$. From $(2)_{\mu-1}$, we infer that

$$(\theta_i^\mu(s))^\lambda\equiv(\omega_i^\mu(s))^\lambda-\bar{\partial}_b(g'_\mu(s)_{|U_i})^\lambda \mod (b(s),\ s^{\mu+1}) \qquad \text{for} \quad \lambda=1,\dots, n.$$

This yields Lemma 3.3.

Corollary 3.4. $P(\omega_i^\mu(s))\equiv 0 \mod (b(s),\ s^{\mu+1})$.

Proof. Let $\theta_i(s)$ be a complex analytic family of elements of $\Gamma(U_\iota, T'\otimes({}^oT'')^*)$ parametrized by s in a sufficiently small neighbourhood of 0 in D, defined by

$$(\bar{\partial}_b+\theta_i(s))(g_i^\mu(s))^\lambda=0 \qquad \text{for} \quad \lambda=1,\dots, n,$$

where $(\bar{\partial}_b+\theta_i(s))(g_i^\mu(s))^\lambda$ is defined by the same congruence of $(\bar{\partial}_b+\phi(\tau(s)))\times(g_i(s))^\lambda$.

Since, for each s, $\theta_i(s)$ represents the partially complex structure on U_i induced from the complex structure on $\mathbf{C}^n$ via a C^∞-embedding $g_i^\mu(s)$, it is clear that

$$P(\theta_i(s))=0.$$

Since $\theta_i^\mu(s)\equiv\theta_i(s) \mod (s^{\mu+1})$, (by the definition of $\theta_i^\mu(s)$), and $\omega_i^\mu(s)\equiv\theta_i^\mu(s)+\bar{\partial}_{T'}^{(0)}g'_\mu(s)_{|U_i} \mod (b(s),\ s^{\mu+1})$, (by Lemma 3.3),

$$\begin{aligned} P(\omega_i^\mu(s)) &\equiv P(\theta_i^\mu(s)+\bar{\partial}_{T'}^{(0)}g'_\mu(s)_{|U_i}) \mod (b(s),\ s^{\mu+1}) \\ &\equiv P(\theta_i^\mu(s)) \mod (s^{\mu+1}) \\ &\equiv P(\theta_i(s)) \mod (s^{\mu+1}) \\ &=0. \end{aligned}$$

Q.E.D.

Lemma 3.5. $\theta_i^\mu(s)\equiv\theta_j^\mu(s) \mod (b(s),\ s^{\mu+1})$ *on* $U_i\cap U_j$.

Proof. $(\bar{\partial}_b+\theta_j^\mu(s))(g_i^\mu(s))$

$$\equiv\sum_{\lambda=1}^n(\partial f_{ij}/\partial\zeta_j^\lambda)(g_j^\mu(s),\ s)(\bar{\partial}_b+\theta_j^\mu(s))(g_j^\mu(s))^\lambda \mod (b(s),\ s^{\mu+1}),$$

(from $(1)_\mu$),

$\equiv 0 \mod (b(s),\ s^{\mu+1})$, (from the definition of $\theta_j^\mu(s)$).

Since $\theta_i^\mu(s)$ is determined uniquely by

$$(\bar{\partial}_b+\theta_i^\mu(s))(g_i^\mu(s))^\lambda\equiv 0 \mod (s^{\mu+1}) \qquad \text{for} \quad \lambda=1,\dots, n,$$

Lemma 3.5 follows.

From Lemma 3.3 and Lemma 3.5, we have the following corollary.

Corollary 3.6. $\omega_i^\mu(s) \equiv \omega_j^\mu(s) \mod (b(s), s^{\mu+1})$ *on* $U_i \cap U_j$.

Lemma 3.7. $h(\tau^{\mu-1}(s)) \equiv 0 \mod (b(s), s^{\mu+1})$.

Proof. By the definition of $h(t)$ in Section 2 (II), we have

$$h(\tau^{\mu-1}(s)) = \boldsymbol{H}_{T'}^2 P(\phi(\tau^{\mu-1}(s))).$$

Since

$$\begin{aligned} P(\phi(\tau^{\mu-1}(s))) &= \textstyle\sum_{i \in A} \rho_i P(\phi(\tau^{\mu-1}(s))_{|U_i}) \\ &\equiv \textstyle\sum_{i \in A} \rho_i P(\omega_i^\mu(s) - \xi_i^\mu(s)) \mod (s^{\mu+1}), \end{aligned}$$

$$P(\omega_i^\mu(s) - \xi_i^\mu(s)) \equiv P(\omega_i^\mu(s)) - \bar{\partial}_{T'}^{(1)} \xi_i^\mu(s) \mod (b(s), s^{\mu+1}),$$

(by Lemma 3.3 and $(2)_{\mu-1}$), and

$$\begin{aligned} &P(\omega_i^\mu(s)) \equiv 0 \mod (b(s), s^{\mu+1}), \quad \text{(by Corollary 3.4)}, \\ &h(\tau^{\mu-1}(s)) \equiv -\boldsymbol{H}_{T'}^2 \{\bar{\partial}_{T'}^{(1)} (\textstyle\sum_{i \in A} \rho_i \xi_i^\mu(s))\} \mod (b(s), s^{\mu+1}) \\ &\qquad = 0. \end{aligned}$$

Q.E.D.

Lemma 3.8. $\bar{\partial}_{T'}^{(1)} (\sum_{i \in A} \rho_i \xi_i^\mu(s)) \equiv 0 \mod (b(s), s^{\mu+1})$.

Proof. By the definition of $\xi_i^\mu(s)$, we have

$$\bar{\partial}_{T'}^{(1)} \xi_i^\mu(s) \equiv \bar{\partial}_{T'}^{(1)} \omega_i^\mu(s) - \bar{\partial}_{T'}^{(1)} \phi(\tau^{\mu-1}(s))_{|U_i} \mod (s^{\mu+1}).$$

Since

$$P(\omega_i^\mu(s)) \equiv 0 \mod (b(s), s^{\mu+1})$$

and

$$P(\phi(\tau^{\mu-1}(s))) \equiv 0 \mod (h(\tau^{\mu-1}(s)), s^{\mu+1}),$$

$$\begin{aligned} \bar{\partial}_{T'}^{(1)} \xi_i^\mu(s) &\equiv P(\omega_i^\mu(s)) - P(\omega_i^\mu(s) - \xi_i^\mu(s)) \mod (b(s), s^{\mu+1}) \\ &\equiv P(\omega_i^\mu(s)) - P(\phi(\tau^{\mu-1}(s)))_{|U_i} \mod (b(s), s^{\mu+1}), \quad \text{(by the definition of } \xi_i^\mu(s)), \\ &\equiv 0 \mod (b(s), s^{\mu+1}, h(\tau^{\mu-1}(s))) \\ &\equiv 0 \mod (b(s), s^{\mu+1}), \quad \text{(by Lemma 3.7)}. \end{aligned}$$

Hence, taking account of Corollary 3.6,

$$\begin{aligned} \bar{\partial}_{T'}^{(1)} (\textstyle\sum_{i \in A} \rho_i \xi_i^\mu(s)) &\equiv \textstyle\sum_{i \in A} \rho_i \bar{\partial}_{T'}^{(1)} \xi_i^\mu(s) \mod (b(s), s^{\mu+1}) \\ &\equiv 0 \mod (b(s), s^{\mu+1}). \end{aligned}$$

Q.E.D.

Lemma 3.9. $\sum_{i \in A} \rho_i \theta_i^\mu(s) - \phi(\tau^\mu(s)) \equiv 0 \mod (b(s), s^{\mu+1})$.

Proof. By Lemma 3.3, we have

$\sum_{i\in\Lambda}\rho_i\theta_i^\mu(s)-\phi(\tau^\mu(s))$

$\equiv\sum_{i\in\Lambda}\rho_i\omega_i^\mu(s)+\bar\partial_{T'}^{(0)}g'_\mu(s)-\phi(\tau^{\mu-1}(s))-\phi_1\tau_\mu(s) \mod (b(s),\ s^{\mu+1})$

$\equiv\sum_{i\in\Lambda}\rho_i\xi_i^\mu(s)-\bar\partial_{T'}^{(0)}\bar\partial_{T'}^{(0)*}N_{T'}\{\sum_{i\in\Lambda}\rho_i\kappa_s^\mu(\xi_i^\mu(s))$
$-\mathscr{L}\boldsymbol{H}_{T'}^1(\sum_{i\in\Lambda}\rho_i\kappa_s^\mu(\xi_i^\mu(s)))\}-\mathscr{L}\boldsymbol{H}_{T'}^1(\sum_{i\in\Lambda}\rho_i\kappa_s^\mu(\xi_i^\mu(s)))$

$\mod (b(s),\ s^{\mu+1})$, (from the definition of $g'_\mu(s)$ and $\tau_\mu(s)$),

$=\sum_{i\in\Lambda}\rho_i\xi_i^\mu(s)-\bar\partial_{T'}^{(0)}\bar\partial_{T'}^{(0)*}N_{T'}(\sum_{i\in\Lambda}\rho_i\kappa_s^\mu(\xi_i^\mu(s)))$
$-\bar\partial_{T'}^{(0)}\bar\partial_{T'}^{(0)*}N_{T'}\bar\partial_{T'}^{(0)}\theta_{\boldsymbol{H}_{T'}^1(\sum_{i\in\Lambda}\rho_i\kappa_s^\mu(\xi_i^\mu(s)))}$
$-\boldsymbol{H}_{T'}^1(\sum_{i\in\Lambda}\rho_i\kappa_s^\mu(\xi_i^\mu(s)))+\bar\partial_{T'}^{(0)}\theta_{\boldsymbol{H}_{T'}^1(\sum_{i\in\Lambda}\rho_i\kappa_s^\mu(\xi_i^\mu(s)))}$,

(where θ_ψ denotes the element of $\Gamma(M,\ {}^o\bar T'')$ introduced in Section 2 for $\psi\in\Gamma(M,\ T'\otimes({}^o\bar T'')^*)$),

$\equiv\sum_{i\in\Lambda}\rho_i\xi_i^\mu(s)-\bar\partial_{T'}^{(0)}\bar\partial_{T'}^{(0)*}N_{T'}(\sum_{i\in\Lambda}\rho_i\xi_i^\mu(s))-\boldsymbol{H}_{T'}^1(\sum_{i\in\Lambda}\rho_i\xi_i^\mu(s))$

$\mod (s^{\mu+1})$, (by $(5)_{\mu-1}$ and $(6)_{\mu-1}$),

$=\bar\partial_{T'}^{(1)*}\bar\partial_{T'}^{(1)}N_{T'}(\sum_{i\in\Lambda}\rho_i\xi_i^\mu(s))$

$\equiv 0 \mod (b(s),\ s^{\mu+1})$, (by Lemma 3.8). Q.E.D.

Proof of $(2)_\mu$: From Lemma 3.5, we infer that

$$\{\sum_{j\in\Lambda}\rho_j\theta_j^\mu(s)\}_{|U_i}\equiv\theta_i^\mu(s) \mod (b(s),\ s^{\mu+1}).$$

Then from Lemma 3.9, we have

$$\theta_i^\mu(s)\equiv\phi(\tau^\mu(s))_{|U_i} \mod (b(s),\ s^{\mu+1}).$$

Proof of $(3)_\mu$: Since the linear term of $h(t)$ is null, we have $h(\tau^\mu(s))\equiv h(\tau^{\mu-1}(s)) \mod (s^{\mu+1})\equiv 0 \mod (b(s),\ s^{\mu+1})$, (by Lemma 3.7).

Proof of $(5)_\mu$: $\bar\partial_{T'}^{(0)}\bar\partial_{T'}^{(0)*}N_{T'}\{\sum_{i\in\Lambda}\rho_i\omega_i^{\mu+1}(s)-\phi(\tau^\mu(s))\}$

$\equiv\bar\partial_{T'}^{(0)}\bar\partial_{T'}^{(0)*}N_{T'}\{\sum_{i\in\Lambda}\rho_i\omega_i^\mu(s)+\bar\partial_{T'}^{(0)}g'_\mu(s)-\phi(\tau^{\mu-1}(s))-\phi_1\tau_\mu(s)\}$

$\mod (s^{\mu+1})$, (by the definition of $\omega_i^{\mu+1}(s)$),

$\equiv\bar\partial_{T'}^{(0)}\bar\partial_{T'}^{(0)*}N_{T'}(\sum_{i\in\Lambda}\rho_i\xi_i^\mu(s))-\bar\partial_{T'}^{(0)}\bar\partial_{T'}^{(0)*}N_{T'}\{\sum_{i\in\Lambda}\rho_i\kappa_s^\mu(\xi_i^\mu(s))-\phi_1\tau_\mu(s)\}$
$-\bar\partial_{T'}^{(0)}\bar\partial_{T'}^{(0)*}N_{T'}\phi_1\tau_\mu(s) \mod (s^{\mu+1})$, (by the definition of $g'_\mu(s)$),

$\equiv 0 \mod (s^{\mu+1})$, (by $(5)_{\mu-1}$).

Proof of $(6)_\mu$: $\boldsymbol{H}_{T'}^1\{\sum_{i\in\Lambda}\rho_i\omega_i^{\mu+1}(s)-\phi(\tau^\mu(s))\}$

$\equiv\boldsymbol{H}_{T'}^1\{\sum_{i\in\Lambda}\rho_i\xi_i^\mu(s)-\bar\partial_{T'}^{(0)}\bar\partial_{T'}^{(0)*}N_{T'}(\sum_{i\in\Lambda}\rho_i\kappa_s^\mu(\xi_i^\mu(s))-\phi_1\tau_\mu(s))$
$-\mathscr{L}\boldsymbol{H}_{T'}^1(\sum_{i\in\Lambda}\rho_i\kappa_s^\mu(\xi_i^\mu(s)))\} \mod (s^{\mu+1})$

$=\boldsymbol{H}_{T'}^1(\sum_{i\in\Lambda}\rho_i\xi_i^\mu(s))-\boldsymbol{H}_{T'}^1(\sum_{i\in\Lambda}\rho_i\kappa_s^\mu(\xi_i^\mu(s)))$, (since $\boldsymbol{H}_{T'}^1\mathscr{L}=\boldsymbol{H}_{T'}^1$),

$\equiv 0 \mod (s^{\mu+1})$, (by $(6)_{\mu-1}$). Q.E.D.

By Proposition 3.1, we have $(3.1)_\mu$ and $(3.3)_\mu$ for any $\mu\geqq 0$. From $(2)_\mu$ in

Proposition 3.1, we infer that, for any $\mu\geqq 0$,

$$(3.2)_\mu \quad (\bar{\partial}_b+\phi(\tau^\mu(s)))(g_i^\mu(s))^\lambda\equiv 0 \mod (b(s), s^{\mu+1}) \qquad \text{for } \lambda=1,\dots, n.$$

This completes the inductive construction of $g_i^\mu(s)$ and $\tau^\mu(s)$.

(II) Proof of Convergence

In this paragraph, we prove that the formal power series $g_i(s)=\lim_{\mu\to\infty} g_i^\mu(s)$ and $\tau(s)=\lim_{\mu\to\infty}\tau^\mu(s)$ are convergent with respect to $\| \ \|'_{(m)}$-norm and $| \ |$-norm respectively where $m\geqq n+2$ and $| \ |$ denotes the euclidean norm on the finite dimensional vector space $\mathscr{H}$.

To prove that $\{g_i^\mu(s)\}_{i\in A}$ and $\tau^\mu(s)$ are convergent, it suffices to show the following estimates: for all $\mu\geqq 1$,

$$(3.5)_\mu \qquad \|g_i^\mu(s)-g_i^0\|'_{(m)}\ll A(s),$$
$$(3.6)_\mu \qquad |\tau^\mu(s)|\ll A(s),$$

for some convergent power series $A(s)$.

Similarly to Section 2 (II), we set

$$A(s)=(b/16c)\sum_{\mu=1}^{\infty}(c^\mu/\mu^2)(s_1+\cdots+s_r)^\mu,$$

and show that $(3.5)_\mu$ and $(3.6)_\mu$ are satisfied for suitable b and c.

Since $\phi(t)$ is holomorphic in t and $f_{ij}(\zeta_j, s)$ is holomorphic in (ζ_j, s), we may assume the following:

$$(3.\text{v}) \quad \|\phi(t)\|_{(m)}\ll(b_0/c_0)\sum_{\mu=1}^{\infty}c_0^\mu(t_1+\cdots+t_q)^\mu,$$
$$(3.\text{vi}) \quad \|f_{ij}^\lambda(z_j+x, s)-\bar{f}_{ij}^\lambda(z_j)-\sum_{\nu=1}^{n}(\partial\bar{f}_{ij}^\lambda/\partial z_j^\nu)(z_j)x^\nu - \sum_{\alpha=1}^{r}(\partial f_{ij}^\lambda/\partial s_\alpha)(z_j, 0)s_\alpha\|'_{(m)} \ll(b_0/c_0)\sum_{\mu=2}^{\infty}c_0^\mu(x^1+\cdots+x^n+s_1+\cdots+s_r)^\mu \qquad \text{for } \lambda=1,\dots, n.$$

For $\mu=1$, we can choose b so large that $(3.5)_1$ and $(3.6)_1$ are satisfied.

We suppose that $(3.5)_{\mu-1}$ and $(3.6)_{\mu-1}$ are both satisfied for some $\mu\geqq 2$. Under this situation we claim the following lemma.

Lemma 3.10. *For sufficiently large* c, *the following estimates are satisfied.*

(1) $\|\kappa_s^\mu(\sigma_{ij}^\mu(s))\|'_{(m)}\ll(b/c)K_1A(s)$,

(2) $\|g'_{i|\mu}(s)\|'_{(m)}\ll(b/c)K_2A(s)$,

(3) $\|\kappa_s^\mu(\omega_i^\mu(s))\|_{(m)}\ll(b/c)K_3A(s)$,

(4) $\|\kappa_s^\mu(\xi_i^\mu(s))\|_{(m)}\ll(b/c)K_4A(s)$,

where K_1–K_4 *are constants independent of* μ.

Proof. (1) By the definition of $\sigma_{ij}^{\mu}(s)$,

$$\begin{aligned}\kappa_s^{\mu}(\sigma_{ij}^{\mu}(s)) = \textstyle\sum_{\lambda=1}^{n} -\kappa_s^{\mu}\{f_{ij}^{\lambda}(g_j^{\mu-1}(s), s) - \bar{f}_{ij}^{\lambda}(z_j) \\ - \textstyle\sum_{\gamma=1}^{n} (\partial\bar{f}_{ij}^{\lambda}/\partial z_j^{\gamma})(z_j)((g_j^{\mu-1}(s))^{\gamma} - (g_j^0)^{\gamma}) \\ - \textstyle\sum_{\alpha=1}^{r} (\partial f_{ij}^{\lambda}/\partial s_{\alpha})(z_j, 0)s_{\alpha}\}\partial/\partial z_i^{\lambda}.\end{aligned}$$

From (3.vi) and $(3.5)_{\mu-1}$, we have

$$\begin{aligned}\|\kappa_s^{\mu}(\sigma_{ij}^{\mu}(s))\|'_{(m)} &\ll (b_0/c_0)\textstyle\sum_{\nu=2}^{\infty} c_0^{\nu}(c'_m)^{\nu}(n+r)^{\nu}A(s)^{\nu} \\ &\ll b_0c'_m(n+r)\textstyle\sum_{\nu=1}^{\infty} \{c_0c'_m(n+r)b/c\}^{\nu}A(s)\end{aligned}$$

where c'_m is the constant introduced in Section 1. Hence if $c_0c'_m(n+r)b/c<1/2$ is satisfied,

$$b_0c'_m(n+r)\textstyle\sum_{\nu=1}^{\infty} \{c_0c'_m(n+r)b/c\}^{\nu}A(s) \ll 2b_0c_0(c'_m)^2(n+r)^2(b/c)A(s).$$

(2) From the definition of $g'_{i|\mu}(s)$ and (1), we infer (2).

(3) From the definition of $\omega_i^{\mu}(s)$,

$$\bar{\partial}_b(g'_{i|\mu}(s))^{\lambda} + \kappa_s^{\mu}(\omega_i^{\mu}(s))^{\lambda} - \kappa_s^{\mu}\{\phi(\tau^{\mu-1}(s))(g_i^{\mu-1}(s) - g_i^0)^{\lambda}\} = 0.$$

Then, from $(3.5)_{\mu-1}$, $(3.6)_{\mu-1}$ and (2), we have

$$\|\kappa_s^{\mu}(\omega_i^{\mu}(s))\|_{(m)} \ll (nb_0c_m/c_0)\{\textstyle\sum_{\nu=1}^{\infty} c_0^{\nu}q^{\nu}c_m^{\nu-1}A(s)^{\nu}\}A(s) + (d_mb/c)K_2A(s),$$

where c_m and d_m are constants introduced in Section 1. Hence, if $c_0qc_mb/c<1/2$, we have

$$\|\kappa_s^{\mu}(\omega_i^{\mu}(s))\|_{(m)} \ll (b/c)(2nb_0c_mq + d_mK_2)A(s).$$

(4) Since $\kappa_s^{\mu}(\phi(\tau^{\mu-1}(s))) = \kappa_s^{\mu}(\phi(\tau^{\mu-1}(s)) - \phi_1\tau^{\mu-1}(s))$,

$$\begin{aligned}\|\kappa_s^{\mu}(\phi(\tau^{\mu-1}(s)))\|_{(m)} &\ll (b_0/c_0)\textstyle\sum_{\nu=2}^{\infty} c_0^{\nu}q^{\nu}A(s)^{\nu}, \quad \text{(by (3.v) and } (3.6)_{\mu-1}), \\ &\ll b_0q\textstyle\sum_{\nu=1}^{\infty} (c_0qb/c)^{\nu}A(s).\end{aligned}$$

Hence, if $qc_0b/c<1/2$ is satisfied, we have

$$\|\kappa_s^{\mu}(\phi(\tau^{\mu-1}(s)))\|_{(m)} \ll 2b_0c_0q^2(b/c)A(s).$$

Since $\xi_i^{\mu}(s) \equiv \omega_i^{\mu}(s) - \phi(\tau^{\mu-1}(s))_{|U_i} \bmod (s^{\mu+1})$, we infer (4) from (3) and this estimate. Q.E.D.

Using this lemma we shall show $(3.5)_{\mu}$ and $(3.6)_{\mu}$.

$$\begin{aligned}|\tau_{\mu}(s)| &= |\mathscr{L}\boldsymbol{H}_{T'}^1(\textstyle\sum_{i\in A} \rho_i\kappa_s^{\mu}(\xi_i^{\mu}(s)))| \\ &\ll c_2c_m \textstyle\sum_{i\in A} \|\rho_i\|_{(m)}\|\kappa_s^{\mu}(\xi_i^{\mu}(s))\|_{(m)} \\ &\ll (b/c)c_2c_mK_4 \textstyle\sum_{i\in A} \|\rho_i\|_{(m)}A(s), \quad \text{(by Lemma 3.10 (4))},\end{aligned}$$

where c_2 is a constant satisfying $|\mathscr{L}\boldsymbol{H}^1_{T'}\phi| \leqq c_2\|\phi\|_{(m)}$ for any $\phi \in \Gamma(M, T'\otimes({}^oT'')^*)$.

$$\begin{aligned}\|g'_\mu(s)\|'_{(m)} &= \|\bar{\partial}^{(0)*}_{T'}N_{T'}(\textstyle\sum_{i\in A}\rho_i\kappa^\mu_s(\xi^\mu_i(s)) - \phi_1\tau_\mu(s))\|'_{(m)} \\ &\ll c_3\|\textstyle\sum_{i\in A}\rho_i\kappa^\mu_s(\xi^\mu_i(s))\|_{(m)} + c_3c_4|\tau_\mu(s)| \\ &\ll (b/c)(1+c_2c_4)c_3c_mK_4\textstyle\sum_{i\in A}\|\rho_i\|_{(m)}A(s),\end{aligned}$$

where c_3 and c_4 are constants satisfying

$$\|\bar{\partial}^{(0)*}_{T'}N_{T'}\phi\|'_{(m)} \leqq c_3\|\phi\|_{(m)} \qquad \text{for} \quad \phi\in\Gamma(M, T'\otimes({}^oT'')^*)$$

(cf. [2], Proposition 3.11) and

$$\|\eta\|_{(m)} \leqq c_4|\eta| \qquad \text{for any} \quad \eta\in\mathscr{H}$$

(since $\mathscr{H}$ is finite dimensional), respectively.

Hence if we choose b and c in such a way that $(3.5)_1$, $(3.6)_1$, (1)–(4) in Lemma 3.10, $(b/c)c_2c_mK_4\sum_{i\in A}\|\rho_i\|_{(m)}<1$, $(b/c)K_2<1/2$, $(b/c)(1+c_2c_4)c_3c_mK_4 \times \sum_{i\in A}\|\rho_i\|_{(m)}<1/2$ are all satisfied, $(3.5)_\mu$ and $(3.6)_\mu$ follow.

From these arguments, we infer that $g_i(s)$ and $\tau(s)$ are a $\Gamma'_{(m)}(U_i, T')$-valued and a $\mathscr{H}$-valued holomorphic functions on some neighbourhood D' of 0 in D, respectively. Since $m\geqq n+2$, by the Sobolev's lemma, $\{g_i(s)\}_{i\in A}$ are of class C^2.

From $(3.1)_\mu$–$(3.3)_\mu$, using the same argument at the end of Section 2, we infer that $F=(\{g_i(s)\}_{i\in A}, \tau(s))$ satisfies (3.1), (3.2) and (3.3) for some neighbourhood D' of 0 in D.

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