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On the Diameter of Compact Homogeneous Riemannian Manifolds

By

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Introduction

Let M be a compact Riemannian manifold. The diameter $d(M)$ of M is defined to be the maximum of $d(p, q)$ $p, q \in M$, where $d(\ ,\ )$ denotes the distance function on M induced by the Riemannian metric.

The main purpose of this paper is to find a positive constant d such that the diameter $d(M) \geqq d$ when the sectional curvature $K \leqq 1$.

In this paper we consider the case that the manifold M is homogeneous. In [3] the author proved that $d = \pi/2$ if the manifold has a big isotropy subgroup. It has been left to study the case that the isotropy subgroup is finite. Hence we shall mainly study invariant metrics on a Lie group and prove that the number $d > 0.23$ if the sectional curvature $K \not\equiv 0$ (Theorem 5.1).

§1. Fixed Points of Isometries

Let M be a compact C^∞ manifold with a Riemannian metric g. Let $d_g(\ ,\ )$ denote the distance function on M induced by g. Let $I(M, g)$ denote the group of isometries of (M, g). Let p be a point of M. We denote by $I_p(M, g)$ the isotropy subgroup of $I(M, g)$, i.e., $I_p(M, g) = \{a \in I(M, g); ap = p\}$. Let A be a connected subgroup of $I_p(M, g)$. Put $F(A) = \{x \in M; Ax = x\}$. Then it is easy to see that $F(A)$ is a disjoint union of closed totally geodesic submanifolds of M. For a curve c: $[0, 1] \to M$, we denote by $\mathrm{length}_g(c)$ the length of c with respect to the metric g.

Lemma 1.1. *Let A be a connected subgroup of $I_p(M, g)$ with* $\dim A \geqq 1$.

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Then $F(A)\subsetneqq M$. *Let* γ: $[0, 1]\to M$ *be a geodesic starting from a point of* $F(A)$ *in the normal direction to* $F(A)$. *Assume that the sectional curvature* $K_g\leqq k$ $(k>0)$ *and* $\operatorname{length}_g(\gamma)\leqq\pi/2\sqrt{k}$. *Then* $d_g(F(A), \gamma(1))=\operatorname{length}_g(\gamma)$, *i.e., the injectivity radius of* $F(A)$ *is not less than* $\pi/2\sqrt{k}$.

Proof. In case that $k=1$ and A is the identity component of $I_p(M, g)$, this is Proposition 4.2 in [3]. The proof in it is still valid for the case that A is a connected subgroup of $I_p(M, g)$ without any change. Hence we obtain

$$\begin{aligned}\operatorname{length}_g(\gamma)&=\frac{1}{\sqrt{k}}\operatorname{length}_{kg}(\gamma)\\&=\frac{1}{\sqrt{k}}d_{kg}(F(A), \gamma(1))\\&=d_g(F(A), \gamma(1)),\end{aligned}$$

since $K_{kg}=K_g/k\leqq 1$ and $\operatorname{length}_{kg}(\gamma)=\sqrt{k}\operatorname{length}_g(\gamma)\leqq\pi/2$. Q.E.D.

§2. The Length of a Killing Vector Field

Let (M, g) be a compact Riemannian manifold as in Section 1.

Theorem 2.1. *Let* ξ *be a non-trivial Killing vector field on the Riemannian manifold* (M, g). *Put* $\alpha=\max_{x\in M} g(\xi, \xi)_x$, *and* $\mathscr{F}=\{x\in M;\, g(\xi, \xi)_x=\alpha\}$. *Assume that the sectional curvature* $K_g\leqq 1$ *and* $\beta=\max_{x\in M} d_g(x, \mathscr{F})<\pi/2$. *Then for any point p of M we obtain*

(i) $g(\xi, \xi)_p\geqq\alpha\cos^2\beta$,

(ii) $\|(\operatorname{grad} g(\xi, \xi))_p\|\leqq 2\alpha\sin\beta$,

where $\|\ \|$ *denotes* $g(\ ,\ )^{1/2}$.

In order to prove the theorem, we provide the following propositions.

Proposition 2.2. *Let f be a positive differentiable function defined in the interval* (s_1, s_2) *such that* $-\pi/2<s_1<0<s_2<\pi/2$, $\max f=f(0)$ *and* $f''(s)\geqq -f(s)$. *Then* $f(s)\geqq f(0)\cos s$.

Proof. Let ε be a positive number. Put $f_\varepsilon(s)=(f(0)+\varepsilon)\cos s/(f(s)+\varepsilon)$. Then we obtain

$$f'_\varepsilon(s)=\frac{-(f(0)+\varepsilon)(f(s)+\varepsilon)\sin s-(f(0)+\varepsilon)f'(s)\cos s}{(f(s)+\varepsilon)^2}$$

$$f''_\varepsilon(s)=-\frac{(f(0)+\varepsilon)(f(s)+\varepsilon+f''(s))\cos s}{(f(s)+\varepsilon)^2}-\frac{2f'_\varepsilon(s)f'(s)}{f(s)+\varepsilon}.$$

Since $f_\varepsilon'(0)=0$ and $f_\varepsilon''(0)<0$, f_ε is maximal at 0. If $f_\varepsilon'(s_0)=0$ for some s_0, then it follows that

$$f_\varepsilon''(s_0)=-\frac{(f(0)+\varepsilon)(f(s_0)+\varepsilon+f''(s_0))\cos s_0}{(f(s_0)+\varepsilon)^2}<0\,.$$

Hence every critical point of f_ε in the interval (s_1, s_2) is maximal, which implies that f_ε has no critical points in (s_1, s_2) except at 0. Therefore we obtain $f_\varepsilon(s) \leqq f_\varepsilon(0)=1$, i.e., $(f(0)+\varepsilon)\cos s \leqq f(s)+\varepsilon$. By ε passing to 0, the assertion is implied. Q.E.D.

Proposition 2.3. *Let $f: \boldsymbol{R}\to\boldsymbol{R}$ be a positive differentiable function such that $f''(s)\geqq -f(s)\geqq -\alpha$, where α is a positive number. Then $f'(s) \leqq\sqrt{2\alpha(\alpha-f(s))}$.*

Proof. Since $f''(s)\geqq -\alpha$, we have for any $t>s$

$$\begin{aligned}-\frac{1}{2}(t-s)^2\alpha &\leqq \int_s^t\left(\int_s^\tau f''(\sigma)d\sigma\right)d\tau\\ &=f(t)-f(s)-(t-s)f'(s)\,.\end{aligned}$$

It implies

$$f'(s)\leqq\frac{1}{2}(t-s)\alpha+\frac{\alpha-f(s)}{t-s}\,.$$

Putting $t-s=\sqrt{2(\alpha-f(s))/\alpha}$, we obtain

$$f'(s)\leqq\sqrt{2\alpha(\alpha-f(s))}\,. \qquad \text{Q.E.D.}$$

Proof of Theorem 2.1. Let $\gamma: \boldsymbol{R}\to M$ be a geodesic with $\|\dot\gamma\|=1$. Since ξ is a Killing vector field, it satisfies

$$\text{(2.1)} \qquad \frac{1}{2}\dot\gamma\dot\gamma\, g(\xi, \xi)_{\gamma(t)}=g(\nabla_{\dot\gamma}\xi, \nabla_{\dot\gamma}\xi)_{\gamma(t)}-g(R(\dot\gamma, \xi)\xi, \dot\gamma)_{\gamma(t)}\,,$$

where $\dot\gamma$ denotes the velocity vector of γ and R is the curvature tensor of the Riemannian connection of g.

(i) Put $f(s)=\|\xi_{\gamma(s)}\|$ and $F=\{s; f(s)=0\}$. We define $E_{\gamma(s)}=\xi_{\gamma(s)}/f(s)$ for $s\notin F$. Then from (2.1) we obtain

$$f(s)f''(s)=(f(s))^2 g(\nabla_{\dot\gamma}E, \nabla_{\dot\gamma}E)-(f(s))^2 g(R(\dot\gamma, E)E, \dot\gamma)\,.$$

Since the sectional curvature $K_g\leqq 1$, we obtain

$$\text{(2.2)} \qquad f''(s)\geqq -f(s) \qquad \text{for} \quad s\notin F\,.$$

There is a point q in $\mathscr{F}$ such that $d_g(p, q)=d_g(p, \mathscr{F})$. Let γ: $[0, s_0]\to M$ be a minimal geodesic from q to p, i.e., $d_g(q, p)=s_0<\pi/2$. First we show that $f(s)\neq 0$ $(s\in[0, s_0])$. Suppose that $F\cap[0, s_0]\neq\emptyset$. Put $\inf F\cap[0, s_0]=s_1$. Then $s_1>0$ and $s_1\in F$. From (2.2) and Proposition 2.2, we obtain $f(s)\geqq f(0)\cos s$ $(s\in[0, s_1])$. Hence it follows that

$$f(s_1)\geqq f(0)\cos s_1\geqq f(0)\cos s_0>0,$$

which contradicts $s_1\in F$. Therefore we obtain $F\cap[0, s_0]=\emptyset$. Hence (i) follows from Proposition 2.2.

(ii) Let $\gamma(0)=p$. Put $f(s)=\|\xi_{\gamma(s/\sqrt{2})}\|^2$. Then from (i) we obtain

$$f(s)\geqq\alpha\cos^2\beta>0.$$

On the other hand, from (2.1) and $K_g\leqq 1$, we obtain

$$f''(s)\geqq -f(s).$$

Hence it follows from Proposition 2.3 that

$$\begin{aligned}\dot{\gamma}g(\xi, \xi)_{\gamma(0)}&=\sqrt{2}f'(s)\\ &\leqq 2\sqrt{\alpha(\alpha-f(s))}\\ &\leqq 2\alpha\sin\beta.\end{aligned}$$

We note that

$$g(\dot{\gamma}, (\mathrm{grad}\, g(\xi, \xi))_p)=\dot{\gamma}g(\xi, \xi)_p.$$

Since we can choose the direction of γ at $\gamma(0)=p$ arbitrarily, our assertion is clear. Q.E.D.

§3. The Sectional Curvature of Invariant Metrics on a Lie Group

Let G be a compact connected Lie group with a left-invariant Riemannian metric g. We denote by $\mathfrak{g}$ the tangent space to G at the identity e. Let X be a tangent vector to G at e. We denote by X^L the left-invariant vector field on G such that the value X^L_e of X^L at e is X. We also define a right-invariant vector field X^R similarly. We denote by $\mathfrak{g}^L$ the Lie algebra of left-invariant vector fields on G.

A bi-linear form $U(g)$: $\mathfrak{g}^L\times\mathfrak{g}^L\to\mathfrak{g}^L$ is defined by

$$2g(U(g)(X^L, Y^L), Z^L)=g(X^L, [Z^L, Y^L])+g(Y^L, [Z^L, X^L])$$

$(X, Y, Z\in\mathfrak{g})$. We note that the Riemannian connection ∇ of g has an expression

$$\nabla_{X^L} Y^L = U(g)(X^L, Y^L) + \frac{1}{2}[X^L, Y^L] \qquad (X, Y \in \mathfrak{g})$$

and the curvature tensor $R(g)$ of ∇ satisfies

$$\begin{aligned} g(R(g)(X, Y)Y, X) = &\|U(g)(X^L, Y^L)_e\|^2 - g(U(g)(X^L, X^L)_e, U(g)(Y^L, Y^L)_e) \\ &-\frac{3}{4}\|[X^L, Y^L]_e\|^2 - \frac{1}{2}g([X^L, [X^L, Y^L]]_e, Y^L_e) \\ &-\frac{1}{2}g([Y^L, [Y^L, X^L]]_e, X^L_e). \end{aligned}$$

Lemma 3.1. $U(g)(X^L, Y^L)_e = -\frac{1}{2}(\text{grad}\, g(X^R, Y^R))_e \quad (X, Y \in \mathfrak{g})$.

Proof. For a vector $Z \in \mathfrak{g}$, we obtain

$$\begin{aligned} g(U(g)(X^L, Y^L)_e, Z) &= \frac{1}{2}\{g(\nabla_{X^L} Y^L, Z^L)_e + g(\nabla_{Y^L} X^L, Z^L)_e\} \\ &= -\frac{1}{2}\{g(Y^L, \nabla_{X^L} Z^L)_e + g(X^L, \nabla_{Y^L} Z^L)_e\} \\ &= -\frac{1}{2}\{g(Y^R, \nabla_{X^R} Z^L)_e + g(X^R, \nabla_{Y^R} Z^L)_e\} \\ &= -\frac{1}{2}\{g(Y^R, \nabla_{Z^L} X^R)_e + g(X^R, \nabla_{Z^L} Y^R)_e\} \\ &= -\frac{1}{2} Z g(Y^R, X^R). \end{aligned}$$

Q.E.D.

Let a be an element of G. We denote by R_a the right translation by a. Let dv be a bi-invariant volume element on G with $\int_G dv = 1$. We define a bi-invariant Riemannian metric $\tilde{g}$ on G by

$$\tilde{g} = \int_{a \in G} R_a^* g \, dv.$$

Let H be a finite subgroup of G such that g is invariant by the right action of H. Then there is a Riemannian metric on G/H such that the projection $(G, g) \to G/H$ is a Riemannian covering. We denote the metric also by g. We also define a Riemannian manifold $(G/H, \tilde{g})$ in like manner. The diameter of $(G/H, g)$ (resp. $(G/H, \tilde{g})$) is denoted by $d_g(G/H)$ (resp. $d_{\tilde{g}}(G/H)$). K_g denotes the sectional curvature of (G, g).

Lemma 3.2. *Assume that $K_g \leqq 1$ and $d_g(G/H) < \pi/2$. Then for any X* $(\in \mathfrak{g}, \neq 0)$

$$\cos^2 d_g(G/H) \leqq \frac{\tilde{g}(X, X)}{g(X, X)} \leqq (\cos d_g(G/H))^{-2}.$$

Proof. By definition we obtain

$$\tilde{g}(X, X)=\int_{a\in G} g(X^R, X^R)_a\, dv .$$

Since X^R is a Killing vector field on (G, g) and $g(X^R, X^R)$ is constant on each right orbit aH of H, it follows from Theorem 2.1 that

$$\begin{aligned} g(X, X)\cos^2 d_g(G/H) &= g(X^R, X^R)_e \cos^2 d_g(G/H) \\ &\leqq \max_{x\in G} g(X^R, X^R)_x \cos^2 d_g(G/H) \\ &\leqq g(X^R, X^R)_a \quad (\forall a\in G) \\ &\leqq \max_{x\in G} g(X^R, X^R)_x \\ &\leqq g(X^R, X^R)_e (\cos d_g(G/H))^{-2} \\ &= g(X, X)(\cos d_g(G/H))^{-2} . \end{aligned}$$

Hence the assertion is clear. Q.E.D.

Since both metrics g and $\tilde{g}$ on G are left-invariant, from Lemma 3.2 we obtain

Lemma 3.3. *If* $K_g \leqq 1$ *and* $d_g(G/H)<\pi/2$, *then*

$$\cos d_g(G/H) \leqq \frac{d_{\tilde{g}}(G/H)}{d_g(G/H)} \leqq (\cos d_g(G/H))^{-1} .$$

Lemma 3.4. *Let a be an element of G. Let $X, Y\ (\in \mathfrak{g})$ be linearly independent vectors such that $\tilde{g}(X, X)=\tilde{g}(Y, Y)=1$. Assume that $K_g \leqq 1$ and $d_g(G/H)<\pi/2$. Then*

(i)
$$\begin{aligned} &(R_a^* g)(U(R_a^* g)(X^L, X^L)_e, U(R_a^* g)(X^L, X^L)_e)^{1/2} \\ &\leqq (\cos d_g(G/H))^{-2} \sin d_g(G/H) , \end{aligned}$$

(ii)
$$(R_a^* g)(R(R_a^* g)(X, Y)Y, X) \leqq (\cos d_g(G/H))^{-4} .$$

Proof. Since we have

$$\tilde{\mathrm{g}}(X, X)=\int_{x\in G} g(X^R, X^R)_x\, dv=1 ,$$

there is a point p in G such that $g(X^R, X^R)_p=1$. Since X^R is a Killing vector field on (G, g) such that $g(X^R, X^R)$ is constant on each right orbit xH of H $(x\in G)$, it follows from Theorem 2.1 that

$$\max_{x\in G} g(X^R, X^R)_x \leqq g(X^R, X^R)_p (\cos d_g(G/H))^{-2} = (\cos d_g(G/H))^{-2} .$$

Similarly we obtain

$$\max_{x\in G} g(Y^R, Y^R)_x \leqq (\cos d_g(G/H))^{-2}.$$

(i) Since $(G/H, g)$ and $(G/aHa^{-1}, R_a^* g)$ are isometric, $K_{R_a^* g} \leqq 1$ and $d_{R_a^* g}(G/aHa^{-1}) = d_g(G/H) < \pi/2$. Since X^R is a Killing vector field also on $(G, R_a^* g)$ such that $(R_a^* g)(X^R, X^R)$ is constant on each right orbit of aHa^{-1}, it follows from Lemma 3.1 and Theorem 2.1 that

$$\begin{aligned}
&(R_a^* g)(U(R_a^* g)(X^L, X^L)_e, U(R_a^* g)(X^L, X^L)_e)^{1/2} \\
&= \frac{1}{2}(R_a^* g)((\mathrm{grad}_{R_a^* g}(R_a^* g)(X^R, X^R))_e, (\mathrm{grad}_{R_a^* g}(R_a^* g)(X^R, X^R))_e)^{1/2} \\
&\leqq \max_{x\in G} (R_a^* g)(X^R, X^R)_x \sin d_g(G/H) \\
&= \max_{x\in G} g(X^R, X^R)_x \sin d_g(G/H) \\
&\leqq (\cos d_g(G/H))^{-2} \sin d_g(G/H).
\end{aligned}$$

(ii) Since g and $R_a^* g$ are isometric, we obtain

$$1 \geqq K_{R_a^* g}(X, Y) = \frac{(R_a^* g)(R(R_a^* g)(X, Y)Y, X)}{(R_a^* g)(X, X)(R_a^* g)(Y, Y) - (R_a^* g)(X, Y)^2}.$$

Hence

$$\begin{aligned}
(R_a^* g)(R(R_a^* g)(X, Y)Y, X) &\leqq (R_a^* g)(X, X)(R_a^* g)(Y, Y) \\
&= g(X^R, X^R)_a\, g(Y^R, Y^R)_a \\
&\leqq (\cos d_g(G/H))^{-4}.
\end{aligned}$$

Q.E.D.

Theorem 3.5. *Assume that the sectional curvature $K_g \leqq 1$ and the diameter $d_g(G/H) < \pi/2$. Then the sectional curvature $K_{\tilde g}$ of $\tilde g$ satisfies*

$$K_{\tilde g} \leqq (\cos d_g(G/H))^{-4}(1 + \sin^2 d_g(G/H)).$$

Proof. Since $\tilde g$ is bi-invariant, $U(\tilde g) \equiv 0$ (cf. Lemma 3.1). We take vectors $X, Y (\in \mathfrak{g})$ such that $\tilde g(X, X) = \tilde g(Y, Y) = 1$ and $\tilde g(X, Y) = 0$. Then from Lemma 3.4 we obtain

$$\begin{aligned}
K_{\tilde g}(X, Y) &= \frac{\tilde g(R(\tilde g)(X, Y)Y, X)}{\tilde g(X, X)\tilde g(Y, Y) - \tilde g(X, Y)^2} \\
&= -\frac{3}{4}\tilde g([X^L, Y^L]_e, [X^L, Y^L]_e) \\
&\quad -\frac{1}{2}\tilde g([X^L, [X^L, Y^L]]_e, Y_e^L) - \frac{1}{2}\tilde g([Y^L, [Y^L, X^L]]_e, X_e^L) \\
&= \int_{a\in G} \Big\{ -\frac{3}{4}(R_a^* g)([X^L, Y^L]_e, [X^L, Y^L]_e) \\
&\quad -\frac{1}{2}(R_a^* g)([X^L, [X^L, Y^L]]_e, Y_e^L)
\end{aligned}$$

$$-\frac{1}{2}(R_a^*g)([Y^L, [Y^L, X^L]]_e, X_e^L)\Big\}dv$$

$$=\int_{a\in G}\Big\{(R_a^*g)(R(R_a^*g)(X, Y)Y, X)$$

$$-(R_a^*g)(U(R_a^*g)(X^L, Y^L)_e, U(R_a^*g)(X^L, Y^L)_e)$$

$$+(R_a^*g)(U(R_a^*g)(X^L, X^L)_e, U(R_a^*g)(Y^L, Y^L)_e)\Big\}dv$$

$$\leqq (\cos d_g(G/H))^{-4} + \sin^2 d_g(G/H)(\cos d_g(G/H))^{-4}. \qquad \text{Q.E.D.}$$

§ 4. Bi-invariant Metrics and Finite Subgroups of a Lie Group

Let G be a compact connected Lie group as in Section 3. Let exp denote the usual exponential mapping from $\mathfrak{g}$ to G, i.e., for a tangent vector $X\in\mathfrak{g}$ $\gamma(t)=\exp tX$ $(t\in\boldsymbol{R})$ is a one-parameter subgroup of G such that $\dot\gamma(0)=X$. The usual bracket operation is defined by

$$[X, Y]=\frac{d}{dt}(\mathrm{Ad}(\exp tX)Y)|_{t=0}$$

$X, Y\in\mathfrak{g}$. Let $\tilde g$ be a bi-invariant Riemannian metric on G with sectional curvature $K_{\tilde g}\leqq k$ $(k>0)$. We note the mapping $\exp:\mathfrak{g}\to G$ coincides with the usual exponential mapping of the Riemannian manifold $(G, \tilde g)$ because the metric $\tilde g$ is bi-invariant. For non-zero vectors X and Y of $\mathfrak{g}$ we denote by $\measuredangle(X, Y)$ the angle which X and Y make. $\|\ \|$ denotes $\tilde g(\ ,\ )^{1/2}$.

Lemma 4.1. *Let X and Y be non-zero vectors of $\mathfrak{g}$. We have*

$$\measuredangle(\mathrm{Ad}(\exp Y)X, X)\leqq\frac{\|[Y, X]\|}{\|X\|}.$$

Proof. Since the metric $\tilde g$ is bi-invariant and $\frac{d}{dt}\mathrm{Ad}(\exp tY)X|_{t=0}=[Y, X]$, we see $\|\mathrm{Ad}(\exp tY)X\|=\|X\|$ and $\left\|\frac{d}{dt}\mathrm{Ad}(\exp tY)X\right\|=\|[Y, X]\|$. Hence it follows that

$$\measuredangle(\mathrm{Ad}(\exp Y)X, X)\leqq\frac{1}{\|X\|}\int_0^1\left\|\frac{d}{dt}\mathrm{Ad}(\exp tY)X\right\|dt$$

$$=\frac{\|[Y, X]\|}{\|X\|}. \qquad \text{Q.E.D.}$$

Lemma 4.2. *Let X and Y be non-zero vectors of $\mathfrak{g}$ such that* $\exp tX$ $(0\leqq t\leqq 1)$ *and* $\exp tY$ $(0\leqq t\leqq 1)$ *are minimal geodesics. Suppose that* $\|X\|=d_{\tilde g}(e, \exp X)<\pi/\sqrt{k}$ and $\|[X, Y]\|/\|X\|<\pi/3$. *Then*

$$d_{\tilde{g}}(e, \exp X) > d_{\tilde{g}}(\exp X, \exp Y \exp X \exp Y^{-1}).$$

Remark. The similar estimate is found in [1].

Proof. From Lemma 4.1, we obtain $\measuredangle(\mathrm{Ad}(\exp Y)X, X) < \pi/3$. It implies $\|X\| > \|X - \mathrm{Ad}(\exp Y)X\|$. Let us define a curve $\gamma: [0, 1] \to \mathfrak{g}$ by $\gamma(t) = tX + (1-t)\,\mathrm{Ad}(\exp Y)X$. We have the sectional curvature $K_{\tilde{g}} \geqq 0$ since the metric $\tilde{g}$ is bi-invariant. From Rauch's comparison theorem we easily see that

$$\begin{aligned} d_{\tilde{g}}(e, \exp X) &= \|X\| \\ &\geqq \|X - \mathrm{Ad}(\exp Y)X\| \\ &= \mathrm{length}(\gamma) \\ &\geqq \mathrm{length}(\exp \circ \gamma) \\ &\geqq d_{\tilde{g}}(\exp X, \exp Y \exp X \exp Y^{-1}). \end{aligned}$$ Q. E. D.

Lemma 4.3. *Let $c(\neq e)$ be an element of the center of G. If G is semi-simple, then $d_{\tilde{g}}(e, c) \geqq \pi/\sqrt{k}$.*

Proof. Let $\gamma: [0, 1] \to G$ be a minimal geodesic from e to c. Then γ is expressed as $\gamma(t) = \exp tX$ for some $X \in \mathfrak{g}$. Since G is semi-simple, the orbit $\mathrm{Ad}(G)X$ ($\subset \mathfrak{g}$) of X by the adjoint action of G is at least of one dimension. Since $\exp \mathrm{Ad}(G)X = c$, c is conjugate to e along γ. Hence the assertion follows from the Morse-Shoenberg theorem. Q. E. D.

Let x be a point of G. $C(x)$ denotes the cut locus of x with respect to the metric $\tilde{g}$.

Lemma 4.4. *If G is semi-simple, then $d_{\tilde{g}}(e, C(e)) \geqq \pi/2\sqrt{k}$.*

Proof. Since the metric $\tilde{g}$ is bi-invariant, the isotropy subgroup $I_e(G, \tilde{g})$ at e contains the inner automorphisms $\mathrm{Ad}(G)$ of G. Since the fixed points $F(\mathrm{Ad}(G))$ is the center of G and since the center consists of finite points, the assertion follows from Lemma 1.1. Q. E. D.

Let H be a finite subgroup of G. Let h be an element of $H \backslash \{e\}$ which is the closest to e. Put $Z(h) = \{x \in G;\ xh = hx\}$.

Lemma 4.5. *If G is semi-simple and $d_{\tilde{g}}(e, h) < \pi/2\sqrt{k}$, then $Z(h) \subsetneqq G$ and $\max_{x \in G} d_{\tilde{g}}(x, Z(h)) \geqq \pi/2\sqrt{k}$.*

Proof. From Lemma 4.4, we see that the minimal geodesic from e to h is unique. We denote the geodesic by $\exp tX$ ($0 \leqq t \leqq 1$), where $X \in \mathfrak{g}$. Since for any $x \in Z(h)$ the inner automorphism by x fixes the endpoints of the geodesic and since the minimal geodesic connecting e and h is unique, we obtain $x(\exp tX)x^{-1} = \exp tX$ $(0 \leqq t \leqq 1)$, which implies that

$$Z(h) = \{x \in G;\ (\exp tX)x(\exp tX)^{-1} = x \quad (t \in \boldsymbol{R})\}$$
$$= F(\{L_{\exp tX}(R_{\exp tX})^{-1};\ t \in \boldsymbol{R}\}).$$

Since G is semi-simple, the assertion follows from Lemma 1.1. Q. E. D.

Lemma 4.6. *Suppose that* $H \not\subseteq Z(h)$. *Then for any* $a \in H \backslash Z(h)$ $d_{\tilde{g}}(e, a) \geqq \pi/6\sqrt{k}$.

Proof. Let γ, δ: $[0, 1] \to G$ be minimal geodesics from e to h and a respectively. Then there are X and Y in $\mathfrak{g}$ such that $\gamma(t) = \exp tX$ and $\delta(t) = \exp tY$. If $\|[X, Y]\|/\|X\| < \pi/3$, Lemma 4.2 implies that

$$d_{\tilde{g}}(e, h) > d_{\tilde{g}}(h, aha^{-1}) = d_{\tilde{g}}(e, aha^{-1}h^{-1}),$$

which contradicts the choice of h. Hence we obtain $\|[X, Y]\|/\|X\| \geqq \pi/3$ On the other hand, we have

$$k \geqq K_{\tilde{g}}(X, Y) = \frac{\frac{1}{4}\|[X, Y]\|^2}{\|X\|^2\|Y\|^2 - \tilde{g}(X, Y)^2}.$$

Hence it follows that

$$k \geqq \frac{\frac{1}{4}\|[X, Y]\|^2}{\|X\|^2\|Y\|^2} \geqq \frac{\pi^2}{36\|Y\|^2},$$

which implies

$$d_{\tilde{g}}(e, a) = \|Y\| \geqq \frac{\pi}{6\sqrt{k}}.$$ Q. E. D.

Theorem 4.7. *Assume that the group G is not abelian. Then the following* (i), (ii) *and* (iii) *hold.*

(i) $d_{\tilde{g}}(G) \geqq \dfrac{\pi}{2\sqrt{k}}$.

(ii) *If G is simply connected, then* $d_{\tilde{g}}(G) \geqq \dfrac{\pi}{\sqrt{k}}$.

(iii) $\max_{x \in G} d_{\tilde{g}}(x, H) = d_{\tilde{g}}(G/H) \geqq \dfrac{\pi}{12\sqrt{k}}$.

Proof. Let Z be the identity component of the center of G. We put $G' = G/Z$. Then G' is semi-simple and if G is simply connected, so is G'. The metric $\tilde{g}$ on G induces a Riemannian metric $\tilde{g}'$ on G' so that the projection

$\pi: G \to G'$ is a Riemannian submersion. Let $\mathfrak{z}$ denote the tangent space to Z at e. We take orthonormal vectors X and Y in $\mathfrak{g}$ such that $X, Y \perp \mathfrak{z}$. Then from O'Neil's theorem for Riemannian submersion we obtain

$$K_{\tilde{g}'}(\pi_* X, \pi_* Y) = K_{\tilde{g}}(X, Y) + \frac{3}{4} \|[X, Y]_{\mathfrak{z}}\|^2,$$

where $[X, Y]_{\mathfrak{z}}$ denotes the orthogonal projection of $[X, Y]$ to $\mathfrak{z}$. Since $[X, Y] \perp \mathfrak{z}$, we obtain $K_{\tilde{g}'} \leqq k$. On the other hand it is clear that, for any point x and y of G, $d_{\tilde{g}}(x, y) \geqq d_{\tilde{g}'}(\pi(x), \pi(y))$. Hence we have only to prove the theorem with the assumption that G is semi-simple. So we suppose G is semi-simple.

(i) From Lemma 4.4, we obtain

$$d_{\tilde{g}}(G) \geqq d_{\tilde{g}}(e, C(e)) \geqq \frac{\pi}{2\sqrt{k}}.$$

(ii) Corollary 5.12 in [2] states that the cut locus and first conjugate locus coincide. The assertion follows easily from the Morse-Shoenberg theorem.

(iii) If $H = \{e\}$, the inequality follows from (i). Hence we assume that $H \supsetneqq \{e\}$. Let h be an element of $H \backslash \{e\}$ which is the closest to e. Let m be the middle point of a minimal geodesic from e to h. Then it is easy to see that $d_{\tilde{g}}(m, H) = d_{\tilde{g}}(m, e) = \frac{1}{2} d_{\tilde{g}}(e, h)$. If $d_{\tilde{g}}(e, h) \geqq \pi/2\sqrt{k}$, then $\max_{x \in G} d_{\tilde{g}}(x, H) \geqq d_{\tilde{g}}(m, H) \geqq \pi/4\sqrt{k}$. Hence we may assume that $d_{\tilde{g}}(e, h) < \pi/2\sqrt{k}$. If $H \subset Z(h)$, the inequality follows from Lemma 4.5. Therefore we suppose that $H \not\subseteq Z(h)$. Let a be an element of $H \backslash Z(h)$ which is the closest to e. Let m' be the middle point of a minimal geodesic from e to a. Then we have $d_{\tilde{g}}(m', H) = d_{\tilde{g}}(m', e)$. (In fact, suppose that there is an element $b \in H$ with $d_{\tilde{g}}(m', b) < d_{\tilde{g}}(m', e)$. Then it follows from $d_{\tilde{g}}(e, b) \leqq d_{\tilde{g}}(e, m') + d_{\tilde{g}}(m', b) < d_{\tilde{g}}(e, a)$ that $b \in Z(h)$. Hence we obtain $ab^{-1} \in H \backslash Z(h)$ and $d_{\tilde{g}}(e, ab^{-1}) = d_{\tilde{g}}(b, a) \leqq d_{\tilde{g}}(b, m') + d_{\tilde{g}}(m', a) < d_{\tilde{g}}(e, a)$, which means that a is not the closest to e of $H \backslash Z(h)$.) Therefore the inequality follows from Lemma 4.6. Q. E. D.

§ 5. Diameter Estimate

Theorem 5.1. *Let (M, g) be a compact homogeneous Riemannian manifold with the sectional curvature $K_g \leqq 1$ and $K_g \not\equiv 0$. Then the diameter $d_g(M)$ of (M, g) is not less than a positive constant d (>0.23).*

Proof. Let p be a point of M. If $\dim I_p(M, g) \geqq 1$, then from Lemma 1.1 we obtain

$$d_g(M) \geqq \max_{x \in M} d_g(x, F(I_p^\circ(M, g))) \geqq \frac{\pi}{2},$$

where $I_p^\circ(M, g)$ denotes the identity component of $I_p(M, g)$ (see also [3]).

Hence we assume that $\dim I_p(M, g)=0$. We denote by G the identity component of $I(M, g)$ and put $H=G \cap I_p(M, g)$. Since the projection $G \to G/H=M$ is a covering, g induces a left-invariant Riemannian metric on G such that the projection is a Riemannian covering. We denote the metric also by g. It is invariant by the inner automorphism by H. We define a Riemannian metric $\tilde{g}$ as in Section 3. Suppose that $d_g(G/H)<\pi/2$. Then from Theorem 3.5 we obtain

$$K_{\tilde{g}} \leqq (\cos d_g(G/H))^{-4}(1+\sin^2 d_g(G/H)).$$

Since $K_g \not\equiv 0$, it is easily seen that G is not abelian. Hence it follows from Lemma 3.3 and Theorem 4.7 that

$$(5.1) \quad \frac{\pi}{12\sqrt{(\cos d_g(G/H))^{-4}(1+\sin^2 d_g(G/H))}} \leqq d_{\tilde{g}}(G/H) \leqq d_g(G/H)(\cos d_g(G/H))^{-1}.$$

We put

$$d=\inf\left\{t \geqq 0;\ \frac{\pi}{12} \leqq t(\cos t)^{-3}(1+\sin^2 t)^{1/2}\right\}.$$

Then $\pi/2>d>0.23$ and $d_g(M) \geqq d$. Q.E.D.

Theorem 5.2. *Let (M, g) be a simply connected compact homogeneous Riemannian manifold with sectional curvature $K_g \leqq 1$ and $K_g \not\equiv 0$. Then the diameter $d_g(M)$ of (M, g) is not less than a positive constant d_0 (>0.81).*

Proof. As in the proof of Theorem 5.1, we may assume that $\dim I_p(M, g)=0$. We define G and H as in the proof of Theorem 5.1. Since M is simply connected, $H=\{e\}$ and $G=M$. Hence it follows from Lemma 3.3 and (i) of Theorem 4.7 that we can replace (5.1) by

$$(5.2) \quad \frac{\pi}{\sqrt{(\cos d_g(G))^{-4}(1+\sin^2 d_g(G))}} \leqq d_g(G)(\cos d_g(G))^{-1}.$$

We put

$$d_0=\inf\{t \geqq 0;\ \pi \leqq t(\cos t)^{-3}(1+\sin^2 t)^{1/2}\}.$$

Then $\pi/2>d_0>0.81$ and $d_g(M)=d_g(G) \geqq d_0$. Q.E.D.

Theorem 5.3. *Let G be a compact connected Lie group with a left-*

invariant metric g. Assume that the sectional curvature $K_g \leqq 1$ and G is not abelian. Then the diameter $d_g(G)$ of (G, g) is not less than a positive constant d_1 (>0.66).

Proof. We define a metric $\tilde{g}$ as in Section 3. We may assume that $d_g(G) < \pi/2$. From Theorem 3.5, we obtain

$$K_{\tilde{g}} \leqq (\cos d_g(G))^{-4}(1+\sin^2 d_g(G)).$$

Hence it follows from Lemma 3.3 and Theorem 4.7 that

$$(5.3) \qquad \frac{\pi}{2\sqrt{(\cos d_g(G))^{-4}(1+\sin^2 d_g(G))}} \leqq d_{\tilde{g}}(G) \leqq d_g(G)(\cos d_g(G))^{-1}.$$

We put

$$d_1 = \inf\left\{t>0;\ \frac{\pi}{2} \leqq t(\cos t)^{-3}(1+\sin^2 t)^{1/2}\right\}.$$

Then

$$\frac{\pi}{2} > d_1 > 0.66 \quad \text{and} \quad d_g(G) \geqq d_1. \qquad \text{Q.E.D.}$$

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