Publ. RIMS, Kyoto Univ.
16 (1980), 875–888

On the Synthesis Problems of the Semimodular State Chart Theory

II. Semimodular Chart

By

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Introduction

In the preceding paper [1], we studied in detail the grounds of the distributive charts in the spaces spanned by cycles and developed a simple synthesis procedure. However the procedure can not be applied to the synthesis of semimodular charts. One of the difficulties with semimodular charts was that we failed to find a theorem which corresponds to Lemma 2.8 in [1]. The present paper undertakes the synthesis procedure for the semimodular state charts. Although a synthesis procedure for semimodular charts was first given in [2], it is indirect and impractical. Thus the purpose of this paper is to give a direct and clear one for semimodular charts. To help the explanation about the sequence of steps of the procedures, simple state charts are synthesized with figurative expressions for some intermediate steps.

Chapter I. Basic Tools and Theorems

§ 1. Minimal Cycles

Lemma 1.1. *Let* (V, h) *be a finite semimodular chart with at least one non-zero cycle. Then there exists the set of orthogonal cycles* $\{X(1),\ldots, X(m)\}$ *such that any cycle of* (V, h) *is written as a linear combination of them. Hereafter we call them the minimal cycles of* (V, h).

Proof. Since (V, h) is finite, there are at most finite similarity classes $T(1)$,

Communicated by S. Takasu, April 24, 1979.
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..., $T(n)$ in (V, h). Let $M^*=M(1)\vee\cdots\vee M(n)$, where $M(k)$ is one of the points of $T(k)$. Let us denote the cycles of M^* by $\{X(1),\dots, X(m)\}$. Then they are orthogonal by Lemma 1.22 in [1] and any cycle of (V, h) is written as a linear combination of them (1.14 in [1]).

Definition 1.2. Let (V, h) be a finite semimodular chart with nodes J and $\{X(1),\dots, X(m)\}$ be the minimal cycles of (V, h). Let $Q(q)$ denote the set of nodes spanned by $X(q)$ for each $q\in\{1,\dots, m\}$. Since minimal cycles are orthogonal, we have $Q(q)\cap Q(p)=\phi$ for distinct q and p. We define the non-negative integer $w(i)$ for each $i\in J$ as follows:

$$w(i)=\begin{cases} 0 & \text{if} \quad i\notin \cup Q(q), \\ X(q)_i & \text{if} \quad i\in Q(q). \end{cases}$$

We call the integer $w(i)$ as *the cyclic number on node i*. Since $X(q)_i\geqq 2$, we have $w(i)\geqq 2$ whenever $w(i)\neq 0$.

Lemma 1.3. *Let (V, h) be a finite semimodular chart with nodes J and $\{w(i)\}$ be its cyclic numbers. If M, N are points of V such that $M\sim N$ but $M_i\neq N_i$, then it follows that $w(i)\neq 0$ and $M_i\equiv N_i \pmod{w(i)}$ for the node i.*

Proof. It is no loss of generality that we assume $M_i>N_i$ for that particular i. We define sequences $\{M(k)\}$, $\{N(k)\}$ as follows: $M(0)=M$, $N(0)=N$, $M(k)=M\vee N(k-1)$ and $N(k)=M(k)-M+N$ for $k\geqq 1$. Trivially $M(k)>N(k-1)$. We prove that $M(k)\sim N(k)$, $M(k)\geqq M(k-1)$, $N(k)\geqq N(k-1)$ and $N(k)_i>M(k)_i=N(k-1)_i>M(k-1)_i$ for $k\geqq 1$ by the inductive method. Since $M(1)=M$, $N(0)=M$, $N=M+(M\vee N)-M$ and $N(1)=M(1)-M+N=N+(M\vee N)-M$ and $M\sim N$ hold, we have $M(1)\sim N(1)$ by 1.9 in [1].

Clearly, $M(1)\geqq M(0)$, $N(1)\geqq N(0)$ and $N(1)_i>M(1)_i=N(0)_i>M(0)_i$. Therefore the statement holds for $k=1$. Suppose that it holds for any non-negative integers not greater than k. Since $M(k+1)=M+(M\vee N(k))-M$, $N(k+1)=N+(M\vee N(k))-M$ and $M\sim N$, we have $M(k+1)\sim N(k+1)$ by 1.9 in [1]. It follows from the inductive assumption of $N(k)\geqq N(k-1)$, $N(k)_i>M_i$ that $M(k+1)\geqq M(k)$, $N(k+1)\geqq N(k)$ and $N(k+1)_i=M(k+1)_i-M_i+N_i>M(k+1)_i=(M\vee N(k))_i=N(K)_i=M(k)_i-M_i+N_i>M(k)_i$. Thus we complete the induction. Now we shall prove the lemma. Since (V, h) is finite and $\{M(k)\}$ is a sequence of infinite points, there exist two integers $k'>k$ such that $M(k')\sim M(k)$. Since $M(k')\geqq M(k)$, $M(k')\geqq N(k)$ and $M(k')\sim M(k)\sim N(k)$ hold, it follows from 1.13 in [1] that there exist $a(q)$, $b(q)$ of W such that $M(k')=M(k)+\sum a(q)Z(q)$

$=N(k)+\sum b(q)Z(q)$ where $Z(q)$'s are cycles of $M(k)$. Because $M(k')_i \geqq N(k)_i > M(k)_i$ and the $Z(q)$'s are orthogonal (1.22 in [1]), there exists q such that $M(k')_i = M(k)_i + a(q)Z(q)_i = N(k)_i + b(q)Z(q)_i$ and $a(q) > b(q)$. Hence we have $N_i - M_i = N(k)_i - M(k)_i = (a(q)-b(q))Z(q)_i$. Since $N_i - M_i \neq 0$, we have $Z(q)_i \neq 0$ and so $w(i) \neq 0$ and $N_i \equiv M_i \pmod{w(i)}$.

The condition of finiteness of (V, h) is essential in the previous lemma.

Corollary 1.4. *Let (V, h) be a finite semimodular chart and let $\{w(i)\}$ be its cyclic numbers then it follows that $V|\{i\}=\{0, 1,..., k\}$ for some integer k or $V|\{i\}=W$ according as $w(i)=0$ or $w(i)\neq 0$.*

§2. Four Kinds of Extensions

Definition 1.5. (*Type-1 extension.*) Let (V, h) be a finite semimodular chart with nodes J, and let $w(i)=0$ for a node $i \in J$. By 1.4 there exists $k \in W$ such that $V|\{i\}=\{0, 1,..., k\}$ for the node $i \in J$. Let r be an integer such that $0<r<k$. Here we assume $k \geqq 2$, and the other case will be considered later. We define an extension (V', h') with nodes $\{i, \xi\}$, $\xi \notin J$, of $(V, h)|\{i\}$ as follows:

$$V'=\{(j, 0);\ 0 \leqq j \leqq r\} \cup \{(j, 1);\ r \leqq j \leqq k\}.$$

For each point $M' \in V'$, we define $h'(M')_\xi = M'_\xi$, and $h'(M')_i = h(M)_i$, where M is any point of V such that $M_i = M'_i$. Since $(V, h)|\{i\}=(V', h')|\{i\}$, we can make an amalgamation $(V, h)\otimes(V', h')$ with nodes $\{\xi, J\}$. We call the amalgamation *the type-1 extension with nodes $\{\xi, J\}$ of (V, h) with respect to $\{i, r\}$.* When $k=1$, let (V, h) itself be the type-1 extension with nodes J.

Lemma 1.6. *Let (V, h) be a finite semimodular chart with nodes J and $V|\{i\}=\{0, 1,..., k\}$ for a node $i \in J$. Let (V^e, h^e) be a type-1 extension with nodes $\{\xi, J\}$ of (V, h) with respect to (i, r) where $0<r<k$. If M^e, N^e are points of V^e such that $h^e(M^e)=h^e(N^e)$ and $M^e|J \sim N^e|J$ then $M^e \sim N^e$.*

Proof. Since $M^e|J \sim N^e|J$ in (V, h) and $w(i)=0$, we have $M^e_i = N^e_i$ from 1.3. Then $M^e = N^e$ also holds because of $h^e(M^e)=h^e(N^e)$. Hence we have $M^e|\{i, \xi\} = N^e|\{i, \xi\}$ and so $M^e \sim N^e$ by 1.26 in [1].

Definition 1.7. (*Type-2 extension.*) Let (V, h) be a finite semimodular chart with nodes J, and $w(i)\neq 0$ for a node $i \in J$. By 1.4 $V|\{i\}=W$ for the node i. Let a be an integer such that $0 \leqq a < w(i)$, b be any non-negative integer.

Let $a(k)$'s denote integers $a+kw(i)$ for $k=1,\dots, b$, and G be the Gray mapping for $2b+2$ with nodes $\Sigma=\{\xi(1),\dots, \xi(f)\}$ (see 2.11 in [1]). Let V' be the subset of $W^{\{i,\Sigma\}}$ defined as follows:

$$\begin{aligned}
&V'=V'(0)\cup V'(1)\cup\cdots\cup V'(2b+1)\\
&V'(0)=\{(j, G(0));\ 0\leqq j\leqq a(0)\}\\
&V'(2k-1)=\{(j, G(2k-1));\ a(k-1)\leqq j\leqq a(k-1)+1\}\\
&V'(2k)=\{(j, G(2k));\ a(k-1)+1\leqq j\leqq a(k)\}
\end{aligned}$$

for $b\geqq k\geqq 1$, and

$$V'(2b+1)=\{(j, G(2b+1));\ j\geqq a(b)\}\,.$$

For the point M' of V', we define $h'(M')$ as follows:

$h'(M')_j=M'_j$ (mod 2) for $j\in\Sigma$ and $h'(M')_i=h(M)_i$ for M of V such that $M'_i=M_i$.

Since $(V', h')\,|\,\{i\}=(V, h)\,|\,\{i\}$, we can make an amalgamation $(V, h)\otimes(V', h')$ with nodes $\{\Sigma, J\}$. We call the amalgamation the *type-2 extension with nodes* $\{\Sigma, J\}$ *of* (V, h) *with respect to* (i, a, b). (Refer to Fig. 3a.)

Lemma 1.8. *Let* (V, h) *be a finite semimodular chart with nodes* J *and* $w(i)\neq 0$ *for a node* $i\in J$. *Let* (V^e, h^e) *be a type-2 extension with nodes* $\{\Sigma, J\}$ *of* (V, h) *with respect to* (i, a, b) *where* $0\leqq a<w(i)$ *and* $b\geqq 0$. *Then* (V^e, h^e) *is a semimodular chart, and if* M^e *and* N^e *are points of* V^e *such that* $h^e(M^e)=h^e(N^e)$ *and* $M^e\,|\,J\sim N^e\,|\,J$, *then* $M^e\sim N^e$.

Proof. Since (V', h') is a finite semimodular chart, (V^e, h^e) is also finite semimodular chart (1.27 in [1]). Suppose that M^e, N^e are points in V such that $M^e\,|\,J\sim N^e\,|\,J$. Since $h^e(M^e)=h^e(N^e)$, $M^e\,|\,\Sigma=N^e\,|\,\Sigma$. If $M^e_i=N^e_i$ then $M^e|\{i, \Sigma\}=N^e|\{i, \Sigma\}$ and so $M^e\sim N^e$ by 1.26 in [1]. If $M^e_i\neq N^e_i$ then $|M^e_i-N^e_i|\geqq w(i)$ by 1.3 and so $M^e|\{i, \Sigma\}$ and $N^e|\{i, \Sigma\}$ should belong to $V'(2b+1)$ becase $M^e\,|\,\Sigma=N^e\,|\,\Sigma$. Then clearly, $M^e\,|\,\{i, \Sigma\}\sim N^e\,|\,\{i, \Sigma\}$ in (V', h') and so $M^e\sim N^e$ by 1.26 in [1].

Definition 1.9. (*Type-3 extension.*) Let (V, h) be a finite semimodular chart with nodes J and let $w(i)\geqq 3$ for a node $i\in J$. Let a be an integer such that $0\leqq a<w(i)$ and b be integer $\geqq 2$. Let $a(k)$'s denote integers $a+kw(i)$ for $k\in W$, and C be the periodical Gray mapping for b with nodes $\Sigma=\{\xi(1),\dots, \xi(f)\}$ (see 2.11 in [1]). Let V' be a subset of $W^{\{i,\Sigma\}}$ defined as follows:

$$V'=V'(0)\cup V'(1)\cup\cdots\cup V'(k)\cup\cdots$$

where

$$V'(0)=\{(j, C(0));\ 0\leqq j\leqq a(0)\}$$
$$V'(2k-1)=\{(j, C(2k-1));\ a(k-1)\leqq j\leqq a(k)-2\}$$
$$V'(2k)=\{(j, C(2k));\ a(k)-2\leqq j\leqq a(k)\} \qquad \text{for } k\geqq 1.$$

For each point M' of V', we define $h'(M')$ as follows:

$h'(M')_j=M'_j \pmod 2$ for $j\in\Sigma$, and $h'(M')_i=h(M)_i$ where M is any point of V such that $M_i=M'_i$. Since $(V', h')|\{i\}=(V, h)|\{i\}$, we can make an amalgamation $(V, h)\otimes(V', h')$ called as *the type-3 extension with nodes $\{\Sigma, J\}$ of (V, h) with respect to (i, a, b)*. (See Fig. 3b.)

Lemma 1.10. *Let V' be the subset of $W^{\{i,\Sigma\}}$ defined in 1.9. If M', N' are points of V' such that $M'_i\equiv N'_i \pmod{w(i)}$ and $h'(M')=h'(N')$, then $V'_{M'}=V'_{N'}$, and hence $M'\sim N'$.*

Proof. This directly follows from the definition.

Lemma 1.11. *Let (V, h) be a finite semimodular chart with nodes J and $w(i)\geqq 3$ for a node $i\in J$. Let (V^e, h^e) be the type-3 extension with nodes $\{\Sigma, J\}$ of (V, h) with respect to (i, a, b) where $0\leqq a<w(i)$ and $b\geqq 2$. Then (V^e, h^e) is a finite semimodular chart, and if M^e and N^e are points in V^e such that $h^e(M^e)=h^e(N^e)$ and $M^e|J\sim N^e|J$ then $M^e\sim N^e$.*

Proof. It follows from the similar argument as in 1.8 that (V^e, h^e) is a finite semimodular chart. Suppose that M^e, N^e are points of V^e such that $h^e(M^e)=h^e(N^e)$ and $M^e|J\sim N^e|J$. In the first place, we show $M^e|\{i, \Sigma\}\sim N^e|\{i, \Sigma\}$. For the sake of brevity, we denote $M^e|\{i, \Sigma\}$, $N^e|\{i, \Sigma\}$ by M', N' respectively. Since $M^e|J\sim N^e|J$ and $w(i)\neq 0$, we have $M^e_i\equiv N^e_i \pmod{w(i)}$. Then $M'\sim N'$ by 1.10. Thus $M^e|J\sim N^e|J$ and $M^e|\{i, \Sigma\}\sim N^e|\{i, \Sigma\}$, therefore $M^e\sim N^e$ by 1.26 in [1].

Definition 1.12. (*Type-4 extension.*) Let (V, h) be a finite semimodular chart with nodes J and $w(i)=2$ for a node $i\in J$. Let $b\geqq 2$ and C be the periodical Gray mapping for b with nodes $\Sigma=\{\xi(1),\dots, \xi(f)\}$. Let V' be the subset of $W^{\{i,\Sigma\}}$ defined as follows:

$$V'=\{(0, 0)\}\cup\{(j, C(j-1)), (j, C(j));\ j\geqq 1\}.$$

For each point M' of V', we define $h'(M')$ as follows:

$h'(M')=M'_j \pmod 2$ for $j\in\Sigma$, and $h'(M')_i=h(M)_i$ for M of V such that $M_i=M'_i$. Since $(V, h)|\{i\}=(V', h')|\{i\}$, we can make an amalgamation (V, h)

$\otimes(V', h')$ with nodes $\{\Sigma, j\}$. We call the amalgamation *the type-4 extension with nodes* $\{\Sigma, J\}$ *of* (V, h) *with respect to* (i, b).

The following two lemmas are proved like 1.10 and 1.11.

Lemma 1.13. *Let V' be the subset of $W^{\{i,\Sigma\}}$ defined in* 1.12. *If M', N' are points of V' such that $M'_i \equiv N'_i$ (mod 2) and $h'(M')=h'(N')$, then $V'_{M'}=V'_{N'}$ holds.*

Lemma 1.14. *Let (V, h) be a finite semimodular chart with nodes J and $w(i)=2$ for a node $i \in J$. Let (V^e, h^e) be the type-4 extension of (V, h) with nodes $\{\Sigma, J\}$ with respect to (i, b) where $b \geqq 2$. Then (V^e, h^e) is a semimodular chart and if M^e and N^e are points in V^e such that $h^e(M^e)=h^e(N^e)$ and $M^e \mid J \sim N^e \mid J$ then $M^e \sim N^e$.*

§3. Construction of κ-Extension

Lemma 1.15. *Let (V, h) be a finite semimodular chart with nodes J, and let $T(\alpha)$ and $T(\beta)$ be v-similarity classes such that $T(\beta)\mathscr{F}T(\alpha)$, i.e., $N \nleqq M$ for any $M \in T(\alpha)$, $N \in T(\beta)$. Then there exists a set Q of nodes satisfying the following*:

(a) *For each $N \in T(\beta)$, there exists a node $i \in Q$ such that $N_i > M_i$ holds for any $M \in T(\alpha)$.*

(b) *$M \mid Q$ is a constant vector for any $M \in T(\alpha)$.*

Proof. Take points $N \in T(\beta)$, $L \in T(\alpha)$ arbitrarily. $Q(0)$ denotes the set of nodes not spanned by any cycle of $Z(T(\alpha))$. If $N \mid Q(0) \leqq L \mid Q(0)$ holds, then we can find the point L' of $T(\alpha)$ which satisfies $L' \geqq N$ by adding some cycles of $Z(T(\alpha))$ to L. This is a contradiction. Hence there exists some node $i \in Q(0)$ such that $N_i > L_i$. Since $M_i = L_i$ hold for all M of $T(\alpha)$ by (3) of 1.30 in [1], $N_i > M_i$ hold for all $M \in T(\alpha)$. Since we can find such a node i for each $N \in T(\beta)$, we denote it by $i(N)$ for each $N \in T(\beta)$. We define a set Q as $Q=\{i(N); N \in T(\beta)\} \subset Q(0)$. Then evidently Q satisfies the conditions (a) and (b).

Definition 1.16. Let (V, h) be a finite semimodular chart with nodes J, and let K be its synthetic class. We divide the set K into "case-1" and "case-2" knots as follows:

(1) $\kappa=(T(\alpha), T(\beta))$ is a *case*-1 *knot* if either $T(\alpha)\bar{\mathscr{F}}T(\beta)$ or $T(\beta)\bar{\mathscr{F}}T(\alpha)$ holds.

(2) $\kappa=(T(\alpha), T(\beta))$ is a *case*-2 *knot* if both $T(\alpha)\mathscr{F}T(\beta)$ and $T(\beta)\mathscr{F}T(\alpha)$ hold.

Lemma 1.17. *Let (V, h) be a finite semimodular chart with nodes J, and let K be its synthetic class. Then there exists κ-extension for each case-*1 *knot.*

Proof. Let $\kappa=(T(\alpha), T(\beta))$ be a case-1 knot, and assume with no loss of generality that $T(\beta)\bar{\mathscr{F}}T(\alpha)$. Let Q be a set of nodes in 1.15 for these $T(\alpha)$, $T(\beta)$. Taking a node $i\in Q$, we define a set $T(\kappa, i)$ by $T(\kappa, i)=\{N\in T(\beta);\ N_i>M_i$ hold for all $M\in T(\alpha)$ for the $i\}$. Then we have $T(\beta)=\cup T(\beta, i)$, $i\in Q$. Now we shall construct a $(T(\alpha), T(\beta, i))$-extension $(V^{(i)}, h^{(i)})$ for each $i\in Q$. Then by 1.35 in [1] we get a κ-extension by forming the amalgamation $\otimes(V^{(i)}, h^{(i)})$, $i\in Q$. Consider a node i of Q. Since M_i is a constant integer for any $M\in T(\alpha)$, we denote the integer M_i+1 by r. Let $w(i)$ be the cyclic number on node i. If $w(i)=0$ then $(V^{(i)}, h^{(i)})$ denote the type-1 extension with nodes $\{\xi, J\}$ of (V, h) with respect to (i, r). Otherwise let $(V^{(i)}, h^{(i)})$ denote the type-2 extension with nodes $\{\Sigma, J\}$ of (V, h) with respect to (i, a, b), where a, b are the integers defined as $r=bw(i)+a$, $0\leqq a<w(i)$. We now show that $(V^{(i)}, h^{(i)})$ is $(T(\alpha), T(\beta, i))$-extension. The condition (1) of 1.32 in [1] has been already proved in 1.6 or 1.8 depending on the type of $(V^{(i)}, h^{(i)})$. To verify the condition (2) of 1.32 in [1], we note that $N_i\geqq M_i+2$ for any $N\in T(\beta, i)$, $M\in T(\alpha)$, because $h(T(\alpha))=h(T(\beta))$ and $N_i>M_i$.

First let us consider the case where $(V^{(i)}, h^{(i)})$ is a type-1 extension with respect to (i, r). Let M^e, N^e be points of $V^{(i)}$ such that $M^e\,|\,J\in T(\alpha)$ and $N^e\,|\,J\in T(\beta, i)$. Since $M^e_i=(M^e\,|\,J)_i=r-1$, we have $M^e_\xi=0$. On the other hand, since $N^e_i\geqq M^e_i+2=r+1$, $N^e_\xi=1$ follows from the definition of $(V^{(i)}, h^{(i)})$. Therefore $h^{(i)}(N^e)_\xi=1\neq 0=h^{(i)}(M^e)_\xi$. Finally, let M^e and N^e be points of $V^{(i)}$ such that $M^e\,|\,J\in T(\alpha)$ and $N^e\,|\,J\in T(\beta, i)$. Since $N^e_i>r=a+bw(i)$, we have $N^e\,|\,\Sigma=G(2b+1)$. On the other hand, $M^e\,|\,\Sigma$ equals either $G(2b)$ or $G(2b-1)$ because $a+(b-1)w(i)+1=r-w(i)+1<r-1=M^e_i<r=a+bw(i)$. Hence we have $h^{(i)}(M^e)_j\neq h^{(i)}(N^e)_j$ for some $j\in\Sigma$ ((2) of 2.12 in [1]), and so $h^{(i)}(N^e)\neq h^{(i)}(M^e)$.

Lemma 1.18. *Let (V, h) be a finite semimodular chart with nodes J, and let K be its synthetic class. Then there exists a κ-extension for each case-*2 *knot κ of K.*

Proof. Let $\kappa=(T(\alpha), T(\beta))$ be a case-2 knot. Since both $T(\alpha)\mathscr{F}T(\beta)$ and $T(\beta)\mathscr{F}T(\alpha)$ hold, they have the same cycles $\{Z(1),\dots, Z(m)\}$ by 1.15 in [1]. Let $Q(q)$ denote the nodes spanned by the cycle $Z(q)$ for each $q\in\{1,\dots, m\}$, and let $Q(0)$ denote the unspanned nodes i.e., $Q(0)=J-\cup Q(q)$. Since $M\geqq N$ for some $M\in T(\alpha)$, $N\in T(\beta)$, we have $M|Q(0)\leqq N|Q(0)$. Similarly we have $L|Q(0)\leqq P|Q(0)$ for some $L\in T(\beta)$, $P\in T(\alpha)$. On the other hand $M|Q(0)=P|Q(0)$ and $L|Q(0)=N|Q(0)$ by (3) of 1.30 in [1] and so $M|Q(0)=N|Q(0)$ holds for any $M\in T(\alpha)$, $N\in T(\beta)$. Hence also by (3) of 1.30 in [1], there exists a cycle $Z\in\{Z(1),\dots, Z(m)\}$ such that $M|Q\equiv N|Q \pmod{Z|Q}$ for points $M\in T(\alpha)$, $N\in T(\beta)$ where Q denotes the nodes spanned by the cycle Z. Then it follows from (2) of 1.30 in [1] that

$$T(\alpha)|Q=\{P^q(\alpha)+r(Z|Q);\ r\in W\}$$
$$T(\beta)|Q=\{P^q(\beta)+r(Z|Q);\ r\in W\}$$

where $P^q(\alpha)$ and $P^q(\beta)$ are the minimum points of $T(\alpha)|Q$ and $T(\beta)|Q$ respectively, and so $P^q(\alpha)\not\equiv P^q(\beta) \pmod{Z|Q}$. Now denote $Z|Q$, $T(\alpha)|Q$, $T(\beta)|Q$, $V|Q$ and $h|Q$ by Z^q, $T^q(\alpha)$, $T^q(\beta)$, V^q and h^q respectively. By 1.35 in [1] there exists an integer k as follows: represent $T^q(\alpha)$ and $T^q(\beta)$ as

$$T^q(\alpha)=T^q(\alpha, 0)\cup\cdots\cup T^q(\alpha, k-1)$$
$$T^q(\beta)=T^q(\beta, 0)\cup\cdots\cup T^q(\beta, k-1)$$

where

$$T^q(\alpha, t)=\{P^q(\alpha)+tZ^q+rkZ^q;\ r\in W\}$$
$$T^q(\beta, s)=\{P^q(\beta)+sZ^q+rkZ^q;\ r\in W\}$$

for our k. Then for each pair $(T^q(\alpha, t), T^q(\beta, s))$, there exists a node $i\in Q$ such that $M_i^q\not\equiv N_i^q \pmod{kZ_i}$ for all $M^q\in T^q(\alpha, t)$, $N^q\in T^q(\beta, s)$.

Define new symbols $\kappa(t, s)$ to be $\kappa(t, s)=(T(\alpha, t), T(\beta, s))$ for each (t, s), where $T(\alpha, t)$ and $T(\beta, s)$ are subsets of $T(\alpha)$ and $T(\beta)$ such that $T(\alpha, t)|Q=T^q(\alpha, t)$ and $T(\beta, s)|Q=T^q(\beta, s)$ respectively. Since $T(\alpha)=T(\alpha, 0)\cup\cdots\cup T(\alpha, k-1)$ and $T(\beta)=T(\beta, 0)\cup\cdots\cup T(\beta, k-1)$, we complete the proof by 1.34 in [1] if $\kappa(t, s)$-extension is constructed for each $\kappa(t, s)$. From now we shall construct $\kappa(t, s)$-extension for a given $\kappa(t, s)=(T(\alpha, t), T(\beta, s))$. Since $T(\alpha, t)|Q=T^q(\alpha, t)$ and $T(\beta, s)|Q=T^q(\beta, s)$, there exists a node $i\in Q$ such that $M_i\not\equiv N_i \pmod{kZ_i}$ for any choice of $M\in T(\alpha, t)$ and $N\in T(\beta, s)$. Let u and v be the minima of $T(\alpha, t)|\{i\}$ and $T(\beta, s)|\{i\}$ respectively, and with no loss of generality assume that $u\leqq v$. Since $u\equiv v \pmod{kZ_i}$, there exists an integer c such that $u+ckZ_i<v<u+(c+1)kZ_i$, and furthermore from the hypothesis $h(T(\alpha))=$

$h(T(\beta))$, we may strengthen the inequality to $u+ckZ_i+2\leqq v\leqq u+(c+1)kZ_i-2$. Hence we have $kZ_i\geqq 4$. We divide the case into two subcases as follows:

(1) Case 2.1; $w(i)=2$

(2) Case 2.2; $w(i)\geqq 3$.

(1) Case 2.1: Let b be the integer defined as $kZ_i=bw(i)=2b$. Since $kZ_i\geqq 4$, we have $b\geqq 2$. Let (V^e, h^e) be the type-4 extension with nodes $\{\Sigma, J\}$ of (V, h) with respect to (i, b). Now, we prove (V^e, h^e) is a $\kappa(t, s)$-extension for the $\kappa(t, s)$. The condition (1) of 1.32 in [1] has been already proved in 1.14. Let M^e, N^e be points of V^e such that $M^e|J\in T(\alpha, t)$, $N^e|J\in T(\beta, s)$. There exist non-negative integers x, y such that $M_i^e=u+xkZ_i$, $N_i^e=v+ykZ_i$. Then $M^e|\Sigma$ is either $C(u+xkZ_i-1)$ or $C(u+xkZ_i)$, here note that $u+xkZ_i=u+2bx\equiv u \pmod{2b}$. On the other hand, since $u+ckZ_i+2\leqq v\leqq u+(c+1)kZ_i-2$, we have $u+(c+y)kZ_i+2\leqq N_i^e\leqq u+(c+y+1)kZ_i-2$ and hence $C(u+2(c+y)b+1)\leqq N^e|\Sigma\leqq C(u+2(c+y)b+2b-2)$, utilizing the fact that $u+2(c+y)b+1\equiv u+1 \pmod{2b}$ and $u+2(c+y)b+2b-2\equiv u+2b-2 \pmod 2$. Then $M_j^e\not\equiv N_j^e \pmod 2$ for some $j\in\Sigma$ ((5) of 2.12 in [1]) because $b\geqq 2$. Therefore $h^e(M^e)\neq h^e(N^e)$.

(2) Case 2.2: Let a, b and q be the integers determined by $u+1=qw(i)+a$, $0\leqq a<w(i)$, and $kZ_i=bw(i)$. Let (V^e, h^e) be the type-3 extension with nodes $\{\Sigma, j\}$ of (V, h) with respect to (i, a, b). Now we shall prove that (V^e, h^e) is a $\kappa(t, s)$-extension for the $\kappa(t, s)$. The condition (1) of 1.32 in [1] has already been proved in 1.11. Let M^e, N^e be points in V^e such that $M^e|J\in T(\alpha, t)$, $N^e|J\in T(\beta, s)$. There exist non-negative integers x and y such that $M_i^e=u+xkZ_i$ and $N_i^e=v+ykZ_i$. Since $u+xkZ_i=a+qw(i)-1+xbw(i)=a+(q+xb)w(i)-1$, we have $M^e|\Sigma=C(2(q+xb))$ from 1.9, using $2(q+xb)\equiv 2q \pmod{2b}$. On the other hand, since $N_i^e=v+ykZ_i\geqq u+ckZ_i+2+ykZ_i=a+qw(i)-1+ckZ_i+2+ykZ_i=a+(q+(c+y)b)w(i)+1$ and $N_i^e=v+ykZ_i\leqq u+(c+1)kZ_i-2+ykZ_i=a+qw(i)-1+(c+1)kZ_i-2+ykZ_i=a+(q+(c+1+y)b)w(i)-3$, we have $a+(q+(c+y)b)w(i)+1\leqq N_i^e\leqq a+(q+(c+1+y)b)w(i)-3$. Since $bw(i)\geqq 4$, we have $C(2(q+(c+y)b+1)-1)\leqq N^e|\Sigma\leqq C(2(q+(c+y+1)b)-1)$ from 1.9, because $2(q+(c+y)b+1)\equiv 2q+1 \pmod{2b}$ and $2(q+(c+y+1)b)-1\equiv 2q+2b-1 \pmod{2b}$. Then since $b\geqq 2$, we have $M_j^e\not\equiv N_j^e \pmod 2$ for some $j\in\Sigma$ ((5) of 2.12 in [1]). Therefore $h^e(M^e)\neq h^e(N^e)$.

This completes the specification of the synthesis procedure for binary finite semimodular charts. The sequence of steps in this procedure goes as follows:

(1) Find the synthetic class K of (V, h).

(2) Construct a κ-extension (V^κ, h^κ) for each $\kappa \in K$.

(3) Make an amalgamation $\otimes(V^\kappa, h^\kappa)$, $\kappa \in K$.

Then $\otimes(V^\kappa, h^\kappa)$, $\kappa \in K$, is a digital extension.

Chapter II. Examples of Synthesis

We shall synthesize a simple distributive chart by using the procedure described in [1]. Figure 1a shows our distributive chart (V, h) with nodes $J=\{1, 2, 3\}$. This chart has one cycle $Z(1)=(220)$ which spans the nodes $Q(1)=\{1, 2\}$, and whose minimum point $L(1)=(001)$. Hence $Q(0)=\{3\}$. $Z^1=Z(1)\,|\,Q(1)=(22)$, $L^1=L(1)\,|\,Q(1)=(00)$. Since $\{M\,|\,Q(1);\ M \ngeq L(1) \text{ and } M \in V\}=\{(001), (000)\}$, $\hat{L}^1=(001)\,|\,Q(1)=(00)$. We first construct $\hat{V}^1 \otimes V$. Since $\Phi=\{M^1 \in V\,|\,Q(1);\ L^1 \leqq M^1 \ngeq L^1+Z^1\}=\{(00), (01), (10), (11), (12), (20), (21)\}$, the synthetic number $e=2$. Thus, $(eZ^1)_3/2=2$ and $L^1_3+1=1$, therefore the periodical

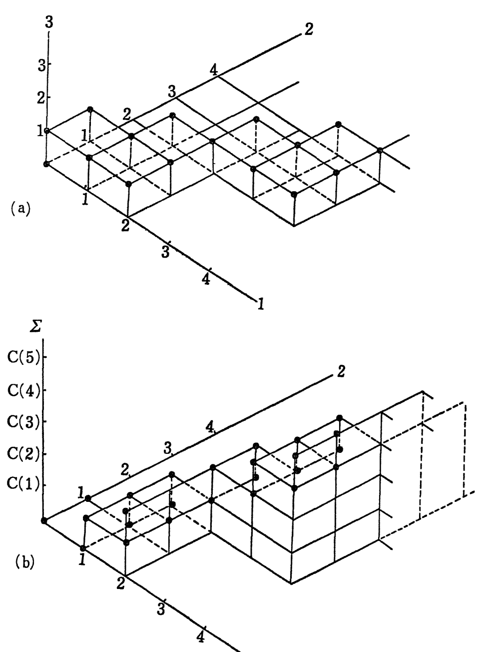


Figure 1. (a): A simple state chart (V, h) with $V=\{(000)\}\ \{(x, y, 1);\ n \leqq x, y \leqq n+1,\ n \in W\} \cup \{(x, y, 1);\ x=2(n+1),\ y=2n,\ n \in W\}$, $h(M)_i=M_i \pmod 2$ for M of V and i of J. (b): The extension of (V, h) with respect to the node 1.

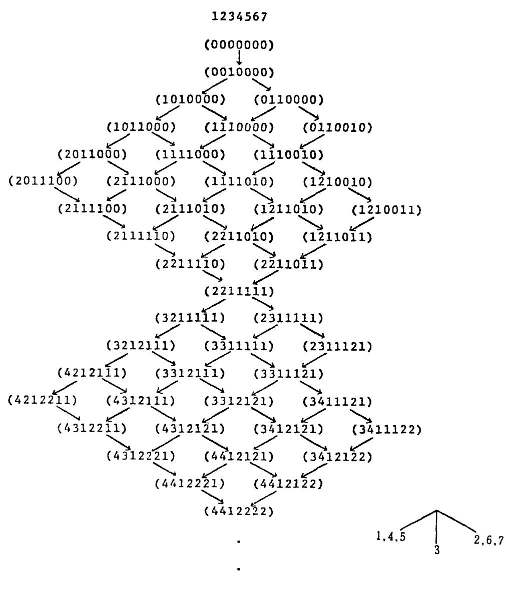


Figure 2. The binary digital extension of the (V, h) of Figure 3(a).

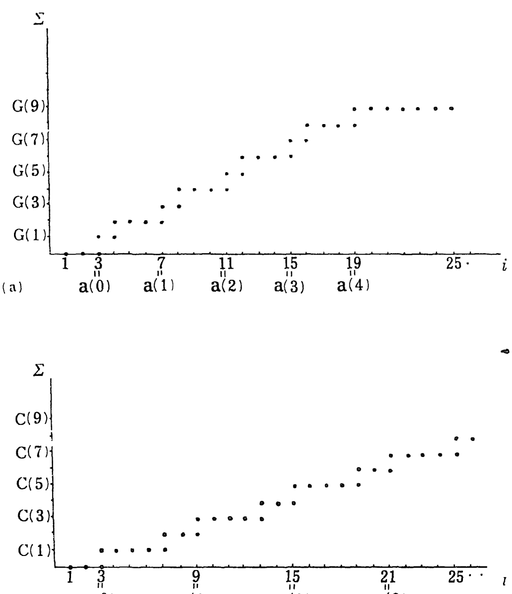
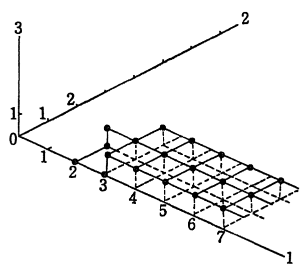


Figure 3. (a): The (V', h') of the type-2 extension in the case of $w(i)=4$, $a=3$ and $b=4$. (b): The (V', h') of the type-3 extension in the case of $w(i)=6$, $a=3$.

Gray mapping C^2 with nodes $\Sigma^1=\{4, 5\}$ is used to construct the simple extension V' of $V^1 \mid \{1\}$. Then the extension $V^{(1)}=V' \otimes V^1$ of V^1 with respect to the node 1 is as shown in Figure 1b. Similarly we have $V^{(2)}$ with new nodes $\Sigma^2=\{6, 7\}$. Then the amalgamation $\hat{V}^1=V^{(1)} \otimes V^{(2)}$ is the $\hat{1}$-extension of V^1.

On the other hand, $V^0=V \mid Q(0)=\{(0), (1)\}$ is itself $\hat{0}$-extension. Thus the digital extension $V^e=(\hat{V}^1 \otimes V) \otimes (\hat{V}^0 \otimes V)=(\hat{V}^1 \otimes V) \otimes (V^0 \otimes V)=(\hat{V}^1 \otimes V)$ is as shown in Figure 2.



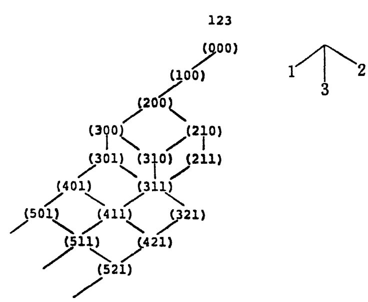


Figure 4. The state chart (V, h) to be synthesized.

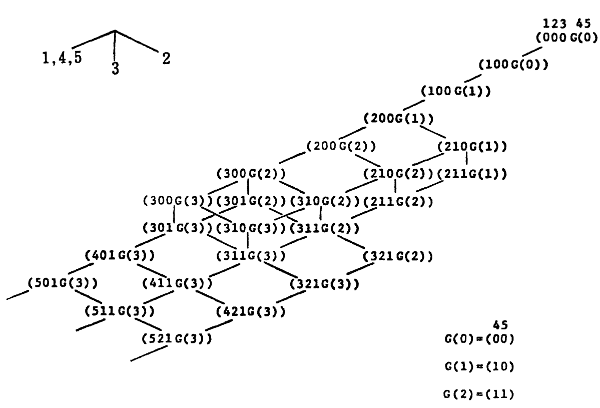


Figure 5. (T(211), T(411))-extension (V^1, h^1).

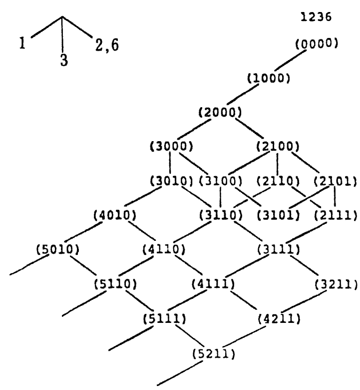


Figure 6. (T(301), T(321))-extension (V^2, h^2).

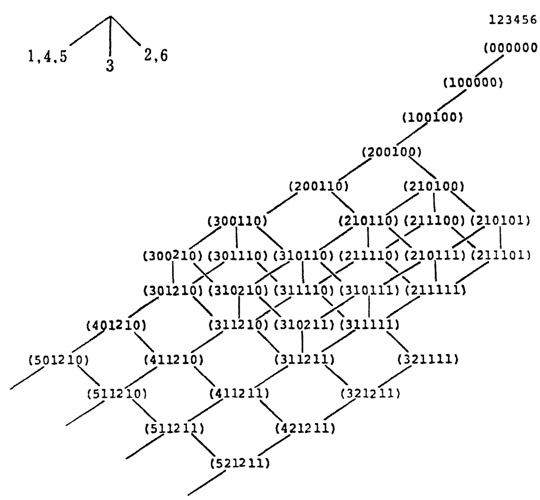


Figure 7. The binary digital extension $(V^e, h^e)=(V^1, h^1)\otimes(V^2, h^2)$ of the (V, h) of Figure 6.

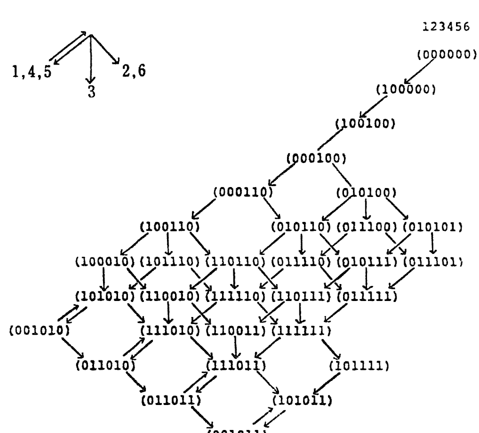


Figure 8. The binary digital graph obtained from the digital extension (V^e, h^e).

Secondly, let us synthesize a semimodular state chart (V, h) with nodes $J=\{1, 2, 3\}$ by using the procedure described in Chapter I where $V=\{(M_1, M_2, M_3);\ M_2=M_3=0$ and $0\leqq M_1\leqq 3\} \cup \{(M_1, M_2, M_3);\ M_2=0, M_3=1$ and $M_1\geqq 4\} \cup \{(M_1, M_2, M_3);\ M_2=1, M_3=0$ and $2\leqq M_1\leqq 3\} \cup \{(M_1, M_2, M_3);\ M_2=M_3=1$ and $M_1\geqq 2\} \cup \{(M_1, M_2, M_3);\ M_2=2, M_3=1$ and $M_1\geqq 3\}$ and $h(M)_i=M_i$ (mod 2) for a point M of V and i of J. (See Fig. 4.) The state chart (V, h) has one cycle $Z=(200)$. Taking minimum point from each similarity class of (V, h) as its representative, $V/\sim=\{T(000), T(100), T(200), T(300), T(210), T(310), T(310), T(301), T(211), T(311), T(401), T(411), T(321), T(421)\}$. In this state chart $\sim$ and $\overset{v}{\sim}$ are equal to each other. We have the synthetic class $K=\{(T(000), T(200)), (T(100), T(300)), (T(211), T(411)), (T(301), T(321)), (T(401), T(421))\}$. Let us first construct a $(T(211), T(411))$-extension. Since $T(211)\mathscr{F}T(411)$, this is a case-1 knot. To find a $(T(211), T(411, 1))$-extension, we construct the type-2 extension (V^1, h^1) with respect to $(1, 1, 1)$ because $V|\{1\}=W$ and $(2+1)=1\times 2+1$. (Refer to Fig. 5.) Fortunately, this (V^1, h^1) is also $(T(000), T(200)), (T(100), T(300)), (T(211), T(411))$-extension. Secondly let us construct a $(T(301), T(321, 2)$-extension. Since $V|\{2\}=\{0, 1, 2\}$, we construct the type-1 extension (V^2, h^2) with respect to $(2, 2)$. (Refer to Fig. 6.) Also in this case (V^2, h^2) is $(T(401), T(421)), (T(301), T(321))$-extension. Thus the amalgamation $(V^e, h^e)=(V^1, h^1)\otimes(V^2, h^2)$ is a digital extension of (V, h) (see Fig. 7). The graph of Figure 8 is the so-called "digital graph", or "state transition graph", obtained from (V^e, h^e), whose set of vertices is $\{h^e(M^e);\ M^e\in V^e\}$. Though V^e is a set of infinite points and *the digital graph* has only finite points, both demonstrate the same ordering of changes of the nodes.

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[2] Hattori, M., Existence of digital extensions of semimodular state charts, *J. Math. Soc. Japan*, **23** (1971), 417–434.