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On O_n

By

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Abstract

We consider quasi-free states of type I on O_n, the C^*-algebras considered by Cuntz.

§1. Introduction

We consider some C^*-algebras which were shown to be simple by Cuntz in [5]. For separable Hilbert spaces H, these algebras $O(H)$ are constructed from full Fock space in a fashion similar to that for the CAR or CCR algebras on anti-symmetric or symmetric Fock spaces respectively. Borrowing terminology from those algebras, we define in Section 2 quasi-free automorphisms and quasi-free states on $O(H)$, and indicate how the work of [3, 11] fits into this framework. The main aim of this paper is to initiate a study of quasi-free states on $O(H)$, and in Section 2 we show how to construct primary and non-primary type I states in this class.

Throughout, H will denote a separable Hilbert space with $H \neq \mathbb{C}$, and $K(H)$ (respectively $T(H)$, $B(H)$) the compact (respectively trace class, bounded) operators on H.

§2.

Let $F(H)$ denote the full Fock space $\bigoplus_{r=0}^{\infty} (\otimes^r H)$, where $\otimes^0 H$ is a one dimensional Hilbert space spanned by a unit vector Ω, the vacuum. Define a linear map O_F: $H \to B(F(H))$ by

$$O_F(f) f_1 \otimes \cdots \otimes f_r = f \otimes f_1 \otimes \cdots \otimes f_r$$

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and

$$O_F(f)\Omega = f, \qquad f, f_i \in H.$$

Then

$$O_F(f)^* O_F(g) = \langle g, f \rangle 1, \qquad f, g \in H$$

and

$$\sum_{i=1}^{n} O_F(h_i) O_F(h_i)^* + \Omega \otimes \bar{\Omega} = 1$$

where $\Omega \otimes \bar{\Omega}$ is the projection on the vacuum, and $\{h_i\}_{i=1}^n$ is any complete orthonormal set in H. Let $O_F(H)$ denote the C^*-algebra generated by the range of O_F; which contains $K(F(H))$ when H is finite dimensional. If H is infinite dimensional, let $O(H) = O_F(H)$, whilst if H is finite dimensional let $O(H) = O_F(H)/K(F(H))$. Define a linear map $O: H \to O(H)$ by $O = O_F$ when H is infinite dimensional and $O = \pi \circ O_F$ when H is finite dimensional and where π is the natural projection $O_F(H) \to O(H)$. Then $O(H)$ is a C^*-algebra generated by the range of a linear map O which satisfies

$$O(f)^* O(g) = \langle g, f \rangle 1, \qquad f, g \in H \tag{2.1}$$

and

$$\sum_{i=1}^{n} O(h_i) O(h_i)^* \leqq 1 \tag{2.2}$$

for one, and hence all, complete orthonormal set $\{h_i\}_{i=1}^n$ in H, with equality in (2.2) should H be finite dimensional. Then $O(H)$ is isomorphic to O_n of [5], where n is the dimension of H. Moreover by [5] $O(H)$ is uniquely determined, up to isomorphism, as the C^*-algebra generated by the range of a (necessarily bounded linear) map O on H satisfying (2.1) and (2.2).

Note that if P_+ (respectively P_-) is the projection on anti-symmetric (respectively symmetric) Fock space, and $a_+(f)$ (respectively $a_-(f)$) is the anti-symmetric (respectively symmetric) annihilation operator, which determine the CAR (respectively CCR) C^*-algebras, and N is the number operator on $F(H)$ then

$$P_\pm N^{1/2} O(f) P_\pm = a_\pm^*(f), \qquad f \in H.$$

If $r \in \boldsymbol{N}$, the map

$$f_1 \otimes \cdots \otimes f_r \to O(f_1) \cdots O(f_r) \qquad f_i \in H$$

satisfies (2.1) on the algebraic tensor product $\odot^r H$. It thus extends to a map of the completion, $\otimes^r H$, into $O(H)$ such that (2.1) and (2.2) both hold. This embeds $O(\otimes^r H)$ in $O(H)$. The map $h\otimes\bar{k}\to O(h)O(k)^*$; $h, k\in H$, gives an algebraic isomorphism of the finite rank operators on H into $O(H)$. Hence using the previous embedding of $O(\otimes^r H)$ in $O(H)$, we can embed the compact operators on $\otimes^r H$ in $O(H)$.

Let $K(H)$ denote the compact operators on H, and $\tilde{K}(H)$ the C^*-algebra $K(H)+\boldsymbol{C}1$ on H. Let $\mathscr{F}(H)$ denote the C^*-subalgebra of $\bigotimes_{i=0}^{\infty}\tilde{K}(H)$ generated by $K(\otimes^r H)\otimes 1$, $r=0, 1,\dots$. Then $\mathscr{F}(H)$ has been embedded in $O(H)$ ([5]).

It follows from the preceding uniqueness statement on $O(H)$, that if U is a unitary between Hilbert spaces H and K, there is a unique *-isomorphism $O(U)$ between $O(H)$ and $O(K)$ such that $O(U)O(f)=O(Uf)$, $f\in H$. If H is infinite dimensional, then it is only necessary for U to be an isometry in which case $O(U)$ is a *-homomorphism. The map $U\to O(U)$ is continuous for the strong topologies because $\|O(f)\|=\|f\|$, $f\in H$. We call such maps quasi-free. One particular quasi-free automorphism, induced by the unitary $(z_1, z_2)\to(z_2, z_1)$ on $\boldsymbol{C}^2$, has been studied by Archbold [3] and shown to be outer on both $O(\boldsymbol{C}^2)$ and $\mathscr{F}(\boldsymbol{C}^2)$. His argument can easily be modified to show that if U is a unitary on a finite dimensional Hilbert space H; $U\neq 1$, then $O(U)$ is outer on $O(H)$. (Moreover $O(U)|_{\mathscr{F}(H)}=\otimes\operatorname{Ad}(U)$ and so is also clearly outer if $U\neq 1$.) In particular, the elements $\{O(t)\colon t\in\boldsymbol{T}, t\neq 1\}$ of the gauge group are outer, confirming suspicions raised by Remark 2.10 of [12] that the crossed product of $O(H)$ by the gauge group is simple. However let $\boldsymbol{T}^2$ act on $\boldsymbol{C}^2$ by $(t_1, t_2)\cdot(z_1, z_2)=(t_1z_1, t_2z_2)$, $t_i\in\boldsymbol{T}$, $z_i\in\boldsymbol{C}$. Then the crossed product of $O(\boldsymbol{C}^2)$ by $\boldsymbol{T}^2$ under the induced quasi-free action is stably isomorphic by [10] to the fixed point algebra, which is the GICAR algebra, and hence not simple. It would be interesting to know exactly when the crossed product of $O(H)$ by a quasi-free action is simple.

Let $T_1(H)$ denote the positive trace class operators K on H such that $\operatorname{tr}K=1$ if H is finite dimensional and $\operatorname{tr}K\leqq 1$ otherwise. If $K\in T_1(H)$, let ρ_K denote the normalized state on $\tilde{K}(H)$:

$$\rho_K(x+\lambda 1)=\operatorname{tr}(Kx)+\lambda, \qquad x\in K(H),\ \lambda\in\boldsymbol{C}.$$

If $\{K_i\}_{i=1}^{\infty}$ is a sequence in $T_1(H)$, let $\rho_{[K_i]}$ denote the restriction of the product state $\bigotimes_{i=1}^{\infty}\rho_{K_i}$ on $\otimes\tilde{K}(H)$ to $\mathscr{F}(H)$. Let P denote the canonical projection of $O(H)$ on $\mathscr{F}(H)$, which is the fixed point algebra of $O(H)$ under the gauge action.

We let $\omega_{[K_i]}$ denote the state $\rho_{[K_i]}\circ P$ on $O(H)$. Then for all $f_1,\dots,f_r, g_1,\dots, g_s \in H$:

$$\omega_{[K_i]}[O(f_1)\cdots O(f_r)O(g_s)^*\cdots O(g_1)^*]=\prod_{i=1}^{r}\langle K_i f_i, g_i\rangle\delta_{rs}.$$

We call such a state a quasi-free state on $O(H)$. If moreover K_i is a constant operator, K say, then we write ω_K for $\omega_{[K_i]}$. A state ω on $O(H)$ such that $\omega=\omega\circ P$ is said to be gauge invariant.

Proposition 2.1. *If H is infinite dimensional, ω_K is quasi-equivalent to ω_0 if and only if* $\operatorname{tr} K<1$.

Proof. Identify $O(H)$ with its (irreducible) representation on Fock space. Then the quasi-free state ω_0 is given by

$$\omega_0(x)=\langle x\Omega, \Omega\rangle, \qquad x\in O(H)$$

where Ω is the vacuum in $F(H)$. Suppose that ω_K is quasi-equivalent to ω_0, so that there exists a density operator ρ on $F(H)$ such that $\omega_K(x)=\operatorname{tr}(\rho x)$; $x\in O(H)$. Since ω_K is gauge invariant, there exist $\rho_r\in T(\otimes^r H)$ such that $\rho=\bigoplus_{r=0}^{\infty}\rho_r$. If H_1, H_2 are Hilbert spaces, and $\varphi\in T(H_1\otimes H_2)$, let $\operatorname{tr}_{H_2}(\varphi)$ denote the unique element of $T(H_1)$ such that

$$\operatorname{tr}(\operatorname{tr}_{H_2}(\varphi)x)=\operatorname{tr}(\varphi(x\otimes 1)), \qquad \text{for all} \quad x\in B(H_1).$$

For notational convenience, we write $H_i=H$, $i=1, 2,\dots$ and $F(H)=\bigoplus_{r=0}^{\infty}(\bigotimes_{i=1}^{r} H_i)$. Then straightforward computations show that for $f_1,\cdots, f_r\in H$:

$$\begin{aligned}&\operatorname{tr}(\rho O(f_1)\cdots O(f_r)O(f_r)^*\cdots O(f_1)^*)\\ &=\sum_{j=r}^{\infty}\langle \operatorname{tr}_{\bigotimes_{i=r+1}^{j} H_i}(\rho_j)f_1\otimes\cdots\otimes f_r, f_1\otimes\cdots\otimes f_r\rangle.\end{aligned}$$

But

$$\omega_K(O(f_1)\cdots O(f_r)O(f_r)^*\cdots O(f_1)^*)=\langle\otimes^r K(f_1\otimes\cdots\otimes f_r), f_1\otimes\cdots\otimes f_r\rangle.$$

Hence

$$\otimes^r K=\sum_{j=r}^{\infty}\operatorname{tr}_{\bigotimes_{i=r+1}^{j} H_i}(\rho_j). \tag{2.4}$$

Operating on this by tr_{H_r}, we see

$$(\operatorname{tr} K)\otimes^{r-1}K=\sum_{j=r}^{\infty}\operatorname{tr}_{\bigotimes_{i=r}^{j} H_i}(\rho_j). \tag{2.5}$$

Hence comparing (2.4) with (2.5), we have $\rho_r=(1-\mathrm{tr}\,K)\otimes^r K$, so that $\mathrm{tr}\,K=1$ would be absurd. Conversely, if $\mathrm{tr}\,K<1$, then $\rho=(1-\mathrm{tr}\,K)\bigoplus_{r=0}^{\infty}(\otimes^r K)$ defines a density operator such that

$$\omega_K(x)=\mathrm{tr}\,(\rho x), \qquad x\in O(H).$$

The following Proposition is essentially due to [11], who discussed the gauge group. This together with [9, Cor. 4.14] shows that if $K\in T_1(H)$ with $K>0$, then ω_K is primary.

Proposition 2.2. *Let $\{e^{iht}: t\in \boldsymbol{R}\}$ be a strongly continuous one-parameter unitary group on H. Then:*

(a) *There exists a KMS state for $\{O(e^{iht}): t\in\boldsymbol{R}\}$ on $O(H)$ at a finite inverse temperature β if and only if $K=e^{-\beta h}\in T_1(H)$. In which case the KMS state is unique and is ω_K.*

(b) *There exists a ground state for $\{O(e^{iht}): t\in\boldsymbol{R}\}$ on $O(H)$ if and only if $h\geqq 0$, and $\mathrm{Ker}\,(h)\neq 0$ when H is finite dimensional. In which case there exists a unique gauge invariant ground state if and only if*

(i) $\mathrm{Ker}\,(h)=0$ *if H is infinite dimensional,*

(ii) $\mathrm{Ker}\,(h)$ *is one dimensional if H is finite dimensional.*

Proof. Let $\alpha_t=O(e^{iht})$, $t\in\boldsymbol{R}$, and let $\mathscr{D}(h)$ (respectively $\mathscr{E}(h)$) denote the domain (respectively entire vectors) of h.

(a) Suppose $K=e^{-\beta h}\in T_1(H)$. Then it is easy to check using (2.1) and (2.3) that $\omega_K(xy)=\omega_K(y\alpha_{\beta i}(x))$ for all x, y in the *-algebra generated by $\{O(f): f\in\mathscr{E}(h)\}$, which are clearly entire for $\alpha_{\boldsymbol{R}}$. Hence ω_K is KMS at inverse temperature β. Conversely, suppose there exists a KMS state ω at inverse temperature β. There exists $K\in B(H)_+$, such that $\omega(O(f)O(g)^*)=\langle Kf, g\rangle$, $f, g\in H$. In fact $K\in T_1(H)$ by (2.2). If f, $g\in\mathscr{E}(h)$ then

$$\begin{aligned}\langle Kf, g\rangle&=\omega(O(f)O(g)^*)=\omega(O(g)^*\alpha_{\beta i}(O(f))\\&=\omega(O(g)^*O(e^{-\beta h}f))=\langle e^{-\beta h}f, g\rangle.\end{aligned}$$

Hence $e^{-\beta h}$ is bounded and is equal to K. We claim that the linear span of $\{(e^{iht}\otimes\cdots\otimes e^{iht}-1)\eta: \eta\in\otimes^r H, t\in\boldsymbol{R}\}$ is dense in $\otimes^r H$. If not, by looking at the orthogonal complement, there exists a unit vector φ in $\otimes^r H$, such that $\otimes^r e^{iht}\varphi=\varphi$. Hence $\otimes^r K\varphi=\varphi$. Let ψ be a unit vector orthogonal to φ; then:

$$1\leqq\langle\otimes^r K\varphi, \varphi\rangle+\langle\otimes^r K\psi, \psi\rangle\leqq\mathrm{tr}\otimes^r K\leqq 1\,.$$

Thus $\otimes^r K\psi=0$, and so $\psi=0$ which is absurd.

Since ω is α_t invariant, we have for $f_1,\dots,f_r\in H$:

$$\omega[O(e^{iht}f_1)\cdots O(e^{iht}f_r)]=\omega[O(f_1)\cdots O(f_r)].$$

Hence

$$\omega[O(e^{iht}\otimes\cdots\otimes e^{iht}-1)(f_1\otimes\cdots\otimes f_r)]=0$$

using the embedding of $O(\otimes^r H)$ in $O(H)$. Thus by the proven density,

$$\omega[O(g_1)\cdots O(g_r)]=0\,,\qquad \text{for all}\quad g_1,\dots,g_r\in H\,. \tag{2.6}$$

Let $f_1,\dots,f_r,\ g_1,\dots,g_s\in\mathscr{E}(h)$. Then

$$\begin{aligned}&\omega[O(f_1)\cdots O(f_r)O(g_s)^*\cdots O(g_1)^*]\\&\quad=\omega[O(g_s)^*\cdots O(g_1)^*O(Kf_1)\cdots O(Kf_r)]\quad\text{by the KMS condition}\\&\quad=\prod_{i=1}^r\langle Kf_i,g_i\rangle\delta_{rs}\,,\quad\text{by (2.1) and (2.6)}.\end{aligned}$$

This means $\omega=\omega_K$.

(b) Let ω be a ground state for $\alpha_t\equiv e^{\delta t}$. Then by [13]

$$-i\omega(x^*\delta(x))\geqq 0\,,\qquad \forall x\in\mathscr{D}(\delta)\,. \tag{2.7}$$

Putting $x=O(f)$ for $f\in\mathscr{D}(h)$ we see that $h\geqq 0$. Conversely if $h\geqq 0$, let ρ be the projection on $\operatorname{Ker}(h)$; formally $\rho=e^{-\infty h}$. Let K_i be a sequence of operators in $T_1(H)$, $K_i\leqq\rho$. Then for x, y in the *-algebra generated by $\{O(f):f\in\mathscr{E}(h)\}$, it is easy to check that $t\to\omega_{[K_i]}(\alpha_t(x)y)$ has a bounded analytic extension to the upper half-plane, and so $\omega_{[K_i]}$ is a ground state for α_t. Thus if there exists a unique gauge invariant ground state $K_i=\rho$ always and so (i, ii) hold. Conversely, suppose (i, ii) hold, and let ω be a gauge invariant ground state. For $r\geqq 0$, let $R_r\in T_1(\otimes^r H)$ be given by

$$\langle R_r\varphi,\psi\rangle=\omega[O(\varphi)O(\psi)^*]\,,\qquad \varphi,\psi\in\otimes^r H\,.$$

Putting $x=O(\psi)^*$, where $\psi\in\odot^r\mathscr{D}(h)$ in (2.7), we see $R_rh_r\leqq 0$, where $e^{ih_rt}=\otimes^r e^{iht}$. But $h_r\geqq 0$, and so $R_r\leqq\otimes^r\rho$; hence $R_r=\otimes^r\rho$ by (i, ii), and so $\omega=\omega_\rho$.

§3.

Let e be a rank one projection on H. Let A denote the infinite tensor product of $K(H)$ tailing off to 1 to the right and to e to the left [5, 6]. More precisely embed $\bigotimes_{-r}^{r}\tilde{K}(H)$ in $\bigotimes_{-r-1}^{r+1}\tilde{K}(H)$ by $x\to e\otimes x\otimes 1$, and let A be the C^*-subalgebra of the inductive limit of this sequence generated by $K(\overset{r}{\underset{-r}{\otimes}} H)$, $r=0, 1, 2, \ldots$. Let $\boldsymbol{Z}$ act on A induced by the shift Φ_0 to the right. Then the crossed product $C^*(A, \boldsymbol{Z})$ is isomorphic to $K \otimes O(H)$ where K denotes the compact operators on a separable infinite dimensional Hilbert space [5]. Let P_0 denote the canonical projection of $C^*(A, \boldsymbol{Z})$ on A. Let $\{K_i\}_{i=1}^{\infty}$ be a sequence in $T_1(H)$, and let $\theta_{[K_i]}$ denote the state on A obtained by taking the inductive limit of $(\overset{-1}{\underset{-r}{\otimes}} \rho_e) \otimes (\overset{r}{\underset{i=0}{\otimes}} \rho_{K_{i+1}})$ (on $\overset{r}{\underset{-r}{\otimes}} \tilde{K}(H)$) and restricting to A. We denote by $\varphi_{[K_i]}$ the state $\theta_{[K_i]} \circ P_0$ on $C^*(A, \boldsymbol{Z})$. With $\mathscr{F}(H)$ embedded in A, being generated by $(\otimes^{r-1} e) \otimes K(\overset{r}{\underset{0}{\otimes}} H) \subseteq K(\overset{r}{\underset{-r}{\otimes}} H)$ and if p is the identity of $\mathscr{F}(H)$, then $pC^*(A, \boldsymbol{Z})p \simeq O(H)$ ([5]), and $\varphi_{[K_i]}|_{O(H)}$ is the quasi-free state $\omega_{[K_i]}$ of Section 2. Suppose H is finite dimensional and p_i denotes the maximum eigenvalue of K_i. Let $\Omega_i \in H \otimes H$ satisfy $\rho_{K_i}(x) = \langle x \otimes 1 \Omega_i, \Omega_i \rangle$, $x \in B(H)$.

Theorem 3.1. *Suppose*

$$\sum_{i=1}^{\infty} (1-p_i) < \infty, \tag{3.1}$$

$$\sum_{i=1}^{\infty} (1-\langle \Omega_{i+1}, \Omega_i \rangle) < \infty. \tag{3.2}$$

Then $\varphi_{[K_i]}$ is type I *but not a factor state.*

Proof. Let $K_i = e$, and $\Omega_i = f \otimes f$ if $i < 0$, where f is a unit vector in the range of e. Let $H_i = H \otimes H$, $M_i = B(H) \otimes 1$, $i \in \boldsymbol{Z}$, and M be the ITPFI $R(H_i, M_i, \Omega_i, i \in \boldsymbol{Z})$ in the notation of [2], which is generated by the algebras $1 \otimes M_i \otimes 1$ on $\overset{\infty}{\underset{-\infty}{\otimes}} {}^{\Omega} H_i$, where $\Omega = \overset{\infty}{\underset{-\infty}{\otimes}} \Omega_i$. Because (3.2) holds, the shift to the right defines a unitary U on $\otimes^{\Omega} H_i$ which induces an automorphism, Φ say, of M as a shift to the right. Let π_0 denote the representation of A on $\otimes^{\Omega} H_i$ given by

$$\pi_0(\otimes x_i) = \underset{i \in \boldsymbol{Z}}{\otimes} (x_i \otimes 1).$$

Then (π_0, U) is a covariant representation of $(A, \boldsymbol{Z}, \Phi_0)$ on $\otimes^{\Omega} H_i$ such that $\pi_0(A)'' = M$. Let (z, W) be the covariant representation of $(M, \boldsymbol{Z}, \Phi)$ on $l^2(\boldsymbol{Z}, \otimes^{\Omega} H_i)$ by

$$z(m) = [\Phi^{-i}(m)]_{-\infty}^{\infty} \in l^{\infty}(\boldsymbol{Z}, M) \subseteq B(l^2(\boldsymbol{Z}, \otimes^{\Omega} H_i))$$

for $m \in M$, and W is the shift to the left on $l^2(\boldsymbol{Z}, \otimes^{\Omega} H_i)$. If $j \in \boldsymbol{Z}$, let δ_j denote the Dirac delta function at j. Then if n denotes the vector $\delta_0 \otimes \Omega$ in $l^2(\boldsymbol{Z}) \otimes (\otimes^{\Omega} H_i)$:

$$\langle z(\pi_0(a)) W^j n, n \rangle = \varphi_{[K_i]}(a \otimes \delta_j)$$

for $a \in A$, regarding $a \otimes \delta_j$ as an element of $L^1(\boldsymbol{Z}, A) \subseteq C^*(A, \boldsymbol{Z})$. Thus we can identify the GNS decomposition of $\varphi_{[K_i]}$ with the covariant representation $(z \circ \pi_0, W)$ of $(A, \boldsymbol{Z}, \Phi_0)$. In particular the von Neumann algebra generated by $C^*(A, \boldsymbol{Z})$ in the state $\varphi_{[K_i]}$ is that generated by $\{z\pi_0(A), W\}$, which is the crossed product of M by Φ. By (3.1) and [1, 4] M is type I. Hence by [14] the crossed product of M by Φ is isomorphic to $M \otimes L^\infty(\boldsymbol{T})$.

Suppose that K_i is a sequence of commuting rank one projections so that (3.1) holds, then (3.2) cannot hold if the sequence is aperiodic. (A sequence $K_1, K_2, \ldots$ is said to be aperiodic ([5]) if for any N, $K_N, K_{N+1}, \ldots$ is not periodic.) This situation will be studied further in Theorem 3.4 with the aid of the following lemma, which allows us to express $K \otimes O(H)$ as a transformation group C^*-algebra. Let $G_1 \underset{\lambda}{\times} G_2$ be the semidirect product of a locally compact group G_1 by another locally compact group G_2 under the continuous action λ. We omit proving the lemma in its greatest generality, it is enough for our purposes to assume that G_1, G_2 are unimodular and λ leaves Haar measure on G_1 invariant. Let $(A, G_1 \underset{\lambda}{\times} G_2, \alpha)$ be a C^*-dynamical system, and let $\alpha_0 = \alpha\,|_{G_1}$.

Lemma 3.2. *In the above situation, there exists a natural action β of G_2 on the crossed product $C^*(A, G_1)$ such that*

$$C^*(A, G_1 \underset{\lambda}{\times} G_2) \simeq C^*(C^*(A, G_1), G_2). \tag{3.3}$$

Proof. Let $C^c_{\alpha_0}(G_1, A)$ be the A-valued continuous functions on G_1, with compact support, with involution and multiplication given by:

$$x^*(g) = \alpha_0(g)\,[x(g^{-1})^*]$$
$$(xy)(g) = \int_{G_1} x(g)\alpha_0(h)\,[y(h^{-1}g)]dh$$

for $g \in G_1$, $x, y \in C^c_{\alpha_0}(G_1, A)$, and equipped with the L^1-norm. We write $\lambda(g_2)(g_1) = g_2^{-1} g_1 g_2$, $g_i \in G_i$. We can define an isometric action β of G_2 on $C^c_{\alpha_0}(G_1, A)$ by $(\beta(g_2)x)(g_1) = \alpha_0(g_2)x(g_2^{-1} g_1 g_2)$, $g_i \in G_i$. Then $\beta(g_2)$ gives a $*$-isomorphism of $C^*(A, G_1)$ because

$$\begin{aligned}(\beta(g_2)x^*)(g_1) &= \alpha(g_2)x^*(g_2^{-1}g_1g_2)\\ &= \alpha(g_2)\alpha(g_2^{-1}g_1g_2)x(g_2^{-1}g_1^{-1}g_2)^*\\ &= \alpha(g_1)\alpha(g_2)x(g_2^{-1}g_1^{-1}g_2)^*\\ &= \alpha(g_1)(\beta(g_2)x)(g_1^{-1})^*\\ &= (\beta(g_2)(x))^*(g_1),\end{aligned}$$

and

$$\begin{aligned}(\beta(g_2)xy)(g_1) &= \alpha(g_2)(xy)(g_2^{-1}g_1g_2) \\ &= \int_{G_1} \alpha(g_2)[x(h)\alpha(h)y(h^{-1}g_2^{-1}g_1g_2)]dh \\ &= \int_{G_1} \alpha(g_2)x(g_2^{-1}hg_2)\alpha(h)\alpha(g_2)y(g_2^{-1}h^{-1}g_1g_2)dh \\ &= \int_{G_1} (\beta(g_2)x)(h)\{\alpha(h)[\beta(g_2)(y)(h^{-1}g_1)]\}dh \\ &= [\beta(g_2)x\beta(g_2)y](g_1)\end{aligned}$$

for $x, y \in C^c_{\alpha_0}(G_1, A)$, $g_i \in G_i$. We can thus form $C^*(C^*(A, G_1), G_2))$ containing $C^c_\beta(G_2, C^c_{\alpha_0}(G_1, A))$ as a dense *-subalgebra. We can define a map i from this subalgebra into $C^c_\alpha(G_1 \times_\lambda G_2, A)$ by $(if)(g_1g_2) = f(g_2)(g_1)$, $g_i \in G_i$; which is isometric since Haar measure on $G_1 \times_\lambda G_2$ is the product of Haar measures on G_1 and G_2, using the invariance of Haar measure on G_1 under the action of G_2. In this way we see that (3.3) holds.

If n is finite, let $\boldsymbol{Z}_n$ denote the group of integers mod n, and let $\boldsymbol{Z}$ act on the restricted product $\coprod_{-\infty}^{\infty} \boldsymbol{Z}_n$ (equipped with the discrete topology) by a shift λ to the right. The semi-direct product $(\coprod_{-\infty}^{\infty} \boldsymbol{Z}_n) \times_\lambda \boldsymbol{Z} = G_n$, say, is amenable and acts on $\coprod_{-\infty}^{-1} \boldsymbol{Z}_n \times \prod_0^{\infty} \boldsymbol{Z}_n$ (equipped with the product topology) as follows. If $(x_i) \in \coprod_{-\infty}^{\infty} \boldsymbol{Z}_n$, $m \in \boldsymbol{Z}$; we let $((x_i), m)$ act on $\coprod \boldsymbol{Z}_n \times \prod \boldsymbol{Z}_n$, by first a translation m to the right, followed by pointwise addition:

$$(x_i)\cdot(z_i) = (x_i + z_i)\,, \qquad (z_i) \in \coprod \boldsymbol{Z}_n \times \prod \boldsymbol{Z}_n\,.$$

Now $C^*(\coprod_{-\infty}^{-1} \boldsymbol{Z}_n \times \prod_0^{\infty} \boldsymbol{Z}_n, \coprod_{-\infty}^{\infty} \boldsymbol{Z}_n)$ is isomorphic to $A(\boldsymbol{C}^n)$, (which is defined at the beginning of § 3) and the action of $\boldsymbol{Z}$ on $C^*(\coprod \boldsymbol{Z}_n \times \prod \boldsymbol{Z}_n, \coprod \boldsymbol{Z}_n)$ given by Lemma 3.2 is the same as that of the shift Φ_0 on $A(\boldsymbol{C}^n)$. Hence by Lemma 3.2 we have

$$K \otimes O_n \simeq C^*(\coprod \boldsymbol{Z}_n \times \prod \boldsymbol{Z}_n;\ \ (\coprod \boldsymbol{Z}_n) \times_\lambda \boldsymbol{Z})\,.$$

Let G_∞ denote the semi-direct product of $\coprod_{-\infty}^{\infty} \boldsymbol{Z}$ by a shift λ to the right. Let $\boldsymbol{Z}^*$ denote the one-point compactification of the integers, and let G_∞ act on $\prod_{-\infty}^{\infty} \boldsymbol{Z}^*$ in a similar fashion to the action of G_n on $\coprod \boldsymbol{Z}_n \times \prod \boldsymbol{Z}_n$. $\boldsymbol{Z}$ acts by a shift to the right, and $(g_i) \in \coprod \boldsymbol{Z}$ by:

$$(g_i)(x_i) = (g_i + x_i) \qquad x_i \in \prod \boldsymbol{Z}^*$$

with the convention $n+\infty=\infty$, $n\in\boldsymbol{Z}$. For $i\geqq 0$, we embed $C(\prod_{-i}^{i}\boldsymbol{Z}^*)$ in $C(\prod_{-\infty}^{\infty}\boldsymbol{Z}^*)$ by an injection $f\to\tilde{f}$:

$$(\tilde{f})(x_j)_{-\infty}^{\infty}=\begin{cases} f(x_{-i},\dots,x_i) & \text{if } x_j=0,\ \forall j<-i,\\ 0 & \text{otherwise}.\end{cases}$$

Let C_∞ denote the C^*-subalgebra of $C(\prod_{-\infty}^{\infty}\boldsymbol{Z}^*)$ generated by $C_0(\prod_{-i}^{i}\boldsymbol{Z})$ $(\subseteq C(\prod_{-i}^{i}\boldsymbol{Z}^*)$, $i=0, 1, 2,\dots)$. Then C_∞ is invariant under the action of G_∞ on $C(\prod\boldsymbol{Z}^*)$. Then we see as before that $C^*(C_\infty, \coprod_{-\infty}^{\infty}\boldsymbol{Z})\simeq A$, and $K\otimes O_\infty\simeq C^*(C_\infty, G_\infty)$. We summarize this as:

Proposition 3.3.

$$K\otimes O_n\simeq C^*(\coprod_{-\infty}^{-1}\boldsymbol{Z}_n\times\prod_{0}^{\infty}\boldsymbol{Z}_n,\ (\coprod_{-\infty}^{\infty}\boldsymbol{Z}_n)\times_\lambda\boldsymbol{Z}) \quad \textit{if} \quad 2\leqq n<\infty,$$

$$K\otimes O_\infty\simeq C^*(C_\infty,\ (\coprod_{-\infty}^{\infty}\boldsymbol{Z})\times_\lambda\boldsymbol{Z}).$$

Let $\boldsymbol{Z}_\infty=\boldsymbol{Z}$; and $\{e_i: i\in\boldsymbol{Z}_n\}$ be a sequence of orthogonal minimal projections on H with $\sum e_i=1$, where e_0 is the fixed projection e. For each $(i, j)\in\boldsymbol{N}\times\boldsymbol{Z}_n$ let k_{ij} be a positive real number with $\sum_{j\in\boldsymbol{Z}_n}k_{ij}=1$ if n is finite, and $\sum_j k_{ij}\leqq 1$ otherwise. Let K_i denote the operator $\sum_{j\in\boldsymbol{Z}_n}k_{ij}e_j$ on H. If $n<\infty$, let μ_i denote the probability measure on $\boldsymbol{Z}_n$ given by $\mu_i(j)=k_{ij}$. If $n=\infty$, let μ_i be the probability measure on $\boldsymbol{Z}^*$ given by $\mu_i(j)=k_{ij}$, $j\neq\infty$, and $\mu_i(\infty)=1-\sum_j k_{ij}$. Let μ denote the product measure $\prod_{-\infty}^{\infty}\mu_i$ on $\prod_{-\infty}^{\infty}\boldsymbol{Z}_n$ (if $n<\infty$, otherwise on $\prod_{-\infty}^{\infty}\boldsymbol{Z}^*$), where μ_i is the Dirac point measure at 0 if $i<0$. Let Q denote the canonical projection of $K\otimes O(H)$ on $C_0(\coprod_{-\infty}^{-1}\boldsymbol{Z}_n\times\prod_{0}^{\infty}\boldsymbol{Z}_n)$ (if $n<\infty$, otherwise on C_∞). Then the state $\mu\circ Q$ is precisely the state $\varphi_{[K_i]}$ on $K\otimes O(H)$.

Now let $n<\infty$. If each $K_i\in\{e_0,\dots, e_{n-1}\}$ then $((e_0)_{-\infty}^{-1}, (K_i)_{i=0}^{\infty})$ corresponds to a point x say of $\coprod\boldsymbol{Z}_n\times\prod\boldsymbol{Z}_n$, and μ is the Dirac point measure at x. We write $G=G_n$, and if $y=(y_i)_{-\infty}^{\infty}\in\coprod\boldsymbol{Z}_n\times\prod\boldsymbol{Z}_n$, let G_y denote the stabilizer at y, i.e. $G_y=\{g\in G: gy=y\}$. Then G_y is either trivial, $\{1\}$, or isomorphic to $\boldsymbol{Z}$, depending on whether the sequence $y_1, y_2,\dots$ is aperiodic or not.

Theorem 3.4. *In the above situation $\varphi_{[K_i]}$ is always type* I. *Moreover the following conditions are equivalent*:

(i) $\varphi_{[K_i]}$ *is pure*.

(ii) $G_x=\{1\}$.

(iii) *The sequence* $K_1, K_2, \ldots$ *is aperiodic*.

Proof. If $y \in \coprod \boldsymbol{Z}_n \times \prod \boldsymbol{Z}_n$, let χ_y denote the character $f \to f(y)$ on $C_0(\coprod \boldsymbol{Z}_n \times \prod \boldsymbol{Z}_n)$, so that $\mu = \chi_x$. Then from [8, Lemma 2.3] we can identify the covariant representation (π, u) of $(\coprod \boldsymbol{Z}_n \times \prod \boldsymbol{Z}_n, G)$ arising in the GNS representation $(\pi \times u)$ of $\mu \circ Q$ as that induced on $l^2(G)$ from the covariant representation (μ, ι) of $(\coprod \boldsymbol{Z}_n \times \prod \boldsymbol{Z}_n, \{1\})$. That is

$$(\pi(f)\varphi)(g) = f(gx)\varphi(g)$$
$$(u(h)\varphi)(g) = \varphi(h^{-1}g)$$

for $h, g \in G$, $\varphi \in l^2(G)$, $f \in C_0(\coprod \boldsymbol{Z}_n \times \prod \boldsymbol{Z}_n)$. Let G/G_x denote the space of cosets $\{gG_x : g \in G\}$ and let $c: G/G_x \to G$ be a cross section so that $c(G_x) = 1$. Then let $\varphi(g) = c(gG_x)$, $\psi(g) = \varphi(g)^{-1}g$, $g \in G$; so that $g \to (\varphi(g), \psi(g))$ identifies G with the cartesian product $G/G_x \times G_x$. Here, and in what follows, we use the cross section to confuse G/G_x with a subset of G, in order to simplify the notation. Then

$$\pi = \bigoplus_{g \in G} \chi_{gx} = \bigoplus_{g \in G} \chi_{\varphi(g)\psi(g)x} = \bigoplus_{a \in G/G_x} \Big(\bigoplus_{G_x} \chi_{ax} \Big).$$

Note that if $a, a' \in G/G_x$, with $a \neq a'$, then $ax \neq a'x$, so that if $b \in (\pi \times u)(C^*(\coprod \boldsymbol{Z}_n \times \prod \boldsymbol{Z}_n, G_n))'$, (where $\pi \times u$ is the representation of $C^*(\coprod \boldsymbol{Z}_n \times \prod \boldsymbol{Z}_n, G_n)$ obtained from (π, u)), then $b = \bigoplus_{a \in G/G_x} b_a$ where $b_a \in B(l^2(G_x))$. Let $\delta_{a,m}$ be the canonical basis for $l^2(G/G_x \times G_x)$, $a \in G/G_x$, $m \in G_x$. Then

$$b_a(\delta_{am}) = \sum_{n \in G_x} b^a_{mn} \delta_{an}$$

for some $\{b^a_{m,n}\}_{n \in G_x}$ in $l^2(G_x)$, for each $(a, m) \in G/G_x \times G_x$. Then for $h \in G$:

$$\begin{aligned} bu(h^{-1})\delta_{am} &= b\delta_{ham} = b\delta_{\varphi(ha)\psi(ha)m} \\ &= \sum_{n \in G_x} b^{\varphi(ha)}_{\psi(ha)m,n} \delta_{\varphi(ha)n} \\ &= \sum_{n \in G_x} b^{\varphi(ha)}_{\psi(ha)m,\psi(ha)n} \delta_{han}, \end{aligned}$$

and

$$\begin{aligned} u(h^{-1})b\delta_{am} &= u(h^{-1}) \sum b^a_{mn} \delta_{an} \\ &= \sum b^a_{mn} \delta_{han}. \end{aligned}$$

Therefore, since $b \in u(G)'$, $b^{\varphi(ha)}_{\psi(ha)m,\psi(ha)n} = b^a_{mn}$ for all $a \in G/G_x$, $m, n \in G_x$. Taking $h = a^{-1}$, we have

$$b^1_{mn} = b^a_{mn} = b^0_{mn} \quad \text{say}.$$

Putting $a=1$, $h\in G_x$, $b^0_{hm,hn}=b^0_{m,n}$; i.e. (under the Fourier transform) $b^0\in L^\infty(\hat{G}_x)$, and $b=1\otimes b^0$. Hence

$$(\pi\times u)(C^*(\coprod \boldsymbol{Z}_n\times\prod \boldsymbol{Z}_n,\ G_n))'\simeq 1_{l^2(G/G_x)}\otimes L^\infty(\hat{G}_x),$$

and

$$(\pi\times u)(C^*(\coprod \boldsymbol{Z}_n\times\prod \boldsymbol{Z}_n,\ G_n))''\simeq B(l^2(G/G_x))\otimes L^\infty(\hat{G}_x).$$

The theorem now follows from this. (It also follows from the above that if F is the set $\{a\in G/G_x\colon ax\in\prod_{-\infty}^{-1}\{0\}\times\prod_0^\infty \boldsymbol{Z}_n\}$ then the von Neumann algebra generated in the quasi-free state $\omega_{[K_i]}$ on $O(H)$ is $B(l^2(F))\otimes L^\infty(\hat{G}_x)$.)

Remarks. (i) If $x, x'\in\coprod \boldsymbol{Z}_n\times\prod \boldsymbol{Z}_n$, then the states $\varphi_x=\chi_x\circ Q$, $\varphi_{x'}=\chi_{x'}\circ Q$ are equivalent on $K\otimes O(\boldsymbol{C}^n)$ if and only if x, x' lie on the same orbit under G_n ([7]) (see also [8]). Now consider the unitary $u=\begin{bmatrix}0&1\\1&0\end{bmatrix}$ on $\boldsymbol{C}^2$, and let e_i be orthogonal projections in $\boldsymbol{C}^2$ such that $u^*e_1u=e_2$. Let x be the point $(O)_{-\infty}^{\infty}$ in $\coprod\boldsymbol{Z}_2\times\prod\boldsymbol{Z}_2$ (corresponding to $(e_1)_{-\infty}^{\infty}$) and x' the point $((O)_{-\infty}^{-1}, (1)_0^\infty)$ (corresponding to $((e_1)_{-\infty}^{-1}, (e_2)_0^\infty)$). Then clearly x, x' lie on different orbits under G_2 so that φ_x, $\varphi_{x'}$ are inequivalent. Now A (see [5] and the beginning of this section) is an inductive limit as $j\to-\infty$ of a sequence A_j, each isomorphic to $\mathscr{F}(\boldsymbol{C}^2)=\otimes B(\boldsymbol{C}^2)$, with embeddings $x\to e\otimes x$ of A_j in A_{j-1}. Thus A can be identified with $K\otimes\mathscr{F}(\boldsymbol{C}^2)$ (which is a restriction of the identification of $C^*(A, \mathbf{Z})$ with $K\otimes O(\boldsymbol{C}^2)$), in such a way that the identity of $A_0\subseteq A$ corresponds to $q\otimes 1$ in $K\otimes\mathscr{F}(\boldsymbol{C}^2)$, where q is a minimal projection in K. This identifies φ_x on $C^*(A, \boldsymbol{Z})$ with the state $\rho_q\otimes\omega_{e_1}$ on $K\otimes O(H)$, and $\varphi_{x'}$ with $\rho_q\otimes\omega_{e_2}$. Hence ω_{e_1}, ω_{e_2} are inequivalent on $O(H)$; but $\omega_{e_1}\circ O(u)=\omega_{e_2}$. Hence $O(u)$ is outer on $O(\boldsymbol{C}^2)$ (c.f. [3] and § 2).

With the same unitary u as above, let f_1, f_2 be orthogonal non zero projections with $u^*f_iu=f_i$. Let $K_1, K_2\cdots$ be an aperiodic sequence, where $K_i\in\{f_1, f_2\}$. Then the quasi-free state $\omega_{[K_i]}$ is pure on $O(H)$ and $\omega_{[K_i]}\circ O(u)=\omega_{[K_i]}$. Hence $O(u)$ is weakly inner in the GNS representation of $\omega_{[K_i]}$ (c.f. [3]).

These remarks clearly generalize to $O(\boldsymbol{C}^n)$.

(ii) It is not necessary for a sequence $K_1, K_2,\dots$ in $T_1(H)$ to be aperiodic before $\omega_{[K_i]}$ becomes factorial, e.g. $\omega_{1/n}$ is factorial of type $\mathrm{III}_{1/n}$ if $2\leqq n<\infty$. See also remarks before Proposition 2.2.

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Note: After the completion of this work, I have learnt that (iii)$\Rightarrow$(i) of Theorem 3.4 has been shown by Joachim Cuntz; (announced in December 1978 at the Beilefeld encounters in Mathematics and Physics II, and to appear in the proceedings of that conference).

References

[1] Araki, H., A lattice of von Neumann algebras associated with the quantum theory of a free Bose field, *J. Math. Phys.*, **4** (1963), 1343–1362.

[2] Araki, H. and Woods, E. J., A classification of factors, *Publ. RIMS, Kyoto Univ.*, **4** (1968), 51–130.

[3] Archbold, R. J., On the 'flip-flop' automorphism of $C^*(S_1, S_2)$, *Quart. J. Math.*, (Oxford) (2), **30** (1979), 129–132.

[4] Bures, D., Certain factors constructed as infinite tensor products, *Comp. Math.*, **15** (1963), 169–191.

[5] Cuntz, J., Simple C^*-algebras generated by isometries, *Commun. Math. Phys.*, **57** (1977), 173–185.

[6] Elliott, G. A., On totally ordered groups and K_0, Ring theory, Conference at Waterloo, 1978, *Lecture Notes in Math.*, **734**, Springer, 1979.

[7] Glimm, J., Families of induced representations, *Pac. J. Math.*, **12** (1962), 885–911.

[8] Gootman, E., The type of some C^*- and W^*-algebras associated with transformation groups, *Pacific J. Math.*, **48** (1973), 93–106.

[9] Hugenholtz, N. M., *States and representations in statistical mechanics*; *in Mathematics of Contemporary Physics*, ed., R. F. Streater, Academic Press, London, 1972.

[10] Kishimoto, A. and Takai, H., Some remarks on C^*-dynamical systems with a compact abelian group, *Publ. RIMS, Kyoto Univ.*, **14** (1978), 383–397.

[11] Olesen, D. and Pedersen, G. K., Some C^*-dynamical systesm with a single KMS state, *Math. Scand.*, **42** (1978), 111–118.

[12] Pedersen, G. K. and Takai, H., Crossed products of C^*-algebras by approximately uniformly continuous actions, *Preprint*, Copenhagen, March 1979.

[13] Powers, R. T. and Sakai, S., Existence of ground states and KMS states for approximately inner dynamics, *Comm. Math. Phys.*, **39** (1975), 273–288.

[14] van Daele, A., *Continuous crossed products and type* III *von Neumann algebras*, Cambridge University Press., Cambridge, 1978.