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A p-Adic Theory of Hyperfunctions, I

By

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Introduction

About 20 years ago, M. Sato constructed the theory of hyperfunctions in his papers [19] and [20]. But it is only from 10 years ago that his theory was used in many problems in mathematics. Anyway it was shown that the theory of hyperfunctions can be very effectively used (cf. e.g. Sato-Kawai-Kashiwara [21]). In this paper, we shall apply his idea to non-archimedean fields and construct a p-adic theory of hyperfunctions of one variable. Our main tool is Krasner's theory of p-adic analytic functions.

In Section 1, we shall axiomatize the results on Krasner's analytic functions which we need in this paper. The reason why we do so is: Though we use Krasner's theory in this paper, it seems likely that Tate's theory of rigid analytic spaces can be also used to construct hyperfunctions. Since Tate's theory can be applied to a wider class of spaces than Krasner's theory, we can not disregard this possibility.

In Section 2, we shall construct p-adic hyperfunctions. Let Ω be a locally closed subset of a compact subset K in $\boldsymbol{P}^1(k)$. Let V be a subset of $\boldsymbol{P}^1(k)$ such that (i) $V \supseteqq \Omega$ and (ii) V and $V \setminus \Omega$ are both completely regular quasi-connected sets. Let $\mathcal{O}$ be the presheaf of analytic functions. Then we shall show that

$$\mathcal{B}(\Omega) = \mathcal{O}(V \setminus \Omega) / \mathcal{O}(V)$$

is independent of a special choice of V and defines a flabby sheaf on any locally closed subset of K. We call an element of $\mathcal{B}(\Omega)$ a hyperfunction on Ω. We note that if L is a locally compact subfield of k,

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then $\boldsymbol{P}^1(L)$ is compact. Hence our theory can be applied to L or $\boldsymbol{P}^1(L)$.

In Section 3, we shall assume that Ω itself is compact, and show that $\mathcal{B}(\Omega)$ is the dual space of the space $\mathcal{A}(\Omega)$ of locally analytic functions on Ω. Of course, to do so, we must define the topologies of $\mathcal{B}(\Omega)$ and $\mathcal{A}(\Omega)$. But we have Tiel's theory of linear topological spaces over non-archimedean fields (cf. Tiel [25] and [26]). Hence we can construct a non-archimedean analogue of Komatsu's theory of projective and injective limits of weakly compact sequences of locally convex spaces (cf. Komatsu [7]). Hence we can define the topologies of $\mathcal{B}(\Omega)$ and $\mathcal{A}(\Omega)$ in a natural manner by making use of a non-archimedean analogue of Montel's theorem (cf. Lemma 3. 5).

In Section 4, examples of hyperfunctions will be given. In particular, we shall obtain an integral representation of the p-adic L-function. This result was also obtained in our former paper [16] and some applications of it were shown in that paper.

We note that Mazur and Swinnerton-Dyer obtained an integral representation of the p-adic L-function in their paper [11]. There they used the dual of the spaces of continuous functions instead of our spaces $\mathcal{B}(\Omega)$. Our integral is more convenient in analytic points but their integral has a merit of having a close relation to the Iwasawa theory.

It seems very interesting to generalize our results to a more general case. The author would like to treat it in a following paper.

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Notations and Terminology

Let $\boldsymbol{N}$, $\boldsymbol{Z}$, $\boldsymbol{Q}$, $\boldsymbol{R}$, $\boldsymbol{C}$, $\boldsymbol{Z}_p^\times$ and $\boldsymbol{Q}_p$ be the set of positive integers, the ring of rational integers, the rational number field, the real number field, the complex number field, the group of p-adic units and the p-adic number field, respectively. For any two integers a and b, let (a, b) be the greatest common divisor of a and b.

Let $\emptyset$ denote the empty set. For any two sets A and B, let $A \backslash B$ be the set consisting of all elements a of A such that $a \notin B$. We sometimes use the notation B^c if B is a subset of A and A is fixed and obvious (e.g. $A=\boldsymbol{P}^1(k)$ and B is a subset of it). Let f be a function on S, and let S' be a subset of S. Then the restriction of f to S' is denoted by $f|S'$.

Let $\overline{\boldsymbol{R}}$ be the union of $\boldsymbol{R}$ and two symbols $-\infty$ and $+\infty$. We define an order of $\overline{\boldsymbol{R}}$ by letting $-\infty<r<+\infty$ for any $r\in\boldsymbol{R}$. Of course, we assume that it is an extension of the natural order of $\boldsymbol{R}$. Let $\{-, 0, +\}$ be the set consisting of the three symbols $-, 0, +$. We define an order on $\{-, 0, +\}$ by $-<0<+$. Let $\overline{\boldsymbol{R}}\times\{-, 0, +\}$ be the product of these two ordered sets, and let $\leqq$ be the lexicographic order on it. Any element of $\overline{\boldsymbol{R}}$ (resp. $\overline{\boldsymbol{R}}\times\{-, 0, +\}$) is said to be a *real number* (resp. a *semi-real number*). Each element of $\overline{\boldsymbol{R}}\times\{-, 0, +\}$ can be written as $(r, -)$, $(r, 0)$ or $(r, +)$. We denote $(r, -)$, $(r, 0)$ and $(r, +)$ by r^-, r and r^+. We idetify $\overline{\boldsymbol{R}}$ and the subset $\{(r, 0)\,|\,r\in\overline{\boldsymbol{R}}\}$ of $\overline{\boldsymbol{R}}\times\{-, 0, +\}$ by the map $r\mapsto(r, 0)$.

§ 1. Results on Krasner's Analytic Function

In this section, we shall list all the results on Krasner's analytic functions which will be needed in the following sections.

Let k be an algebraically closed field with a non-trivial non-archimedean valuation $|\ |$. We assume that *k is maximally complete* with respect to $|\ |$. Namely, we assume that, for any decreasing sequence $k\supseteqq C_1\supseteqq C_2\supseteqq\cdots\supseteqq C_n\supseteqq\cdots$ of balls, the condition $\bigcap_{n=1}^{\infty} C_n\neq\emptyset$ is satisfied. We note that (i) "maximally complete" implies "complete"; (ii) it is known that, if the field k is not maximally complete, then there exists an algebraically closed field $k'\supseteqq k$ such that k' is maximally complete.

Let k be such a field. Then k has a metric defined by

$$d(x, y)=|x-y| \quad (x, y\in k). \tag{1.1}$$

Let $\boldsymbol{P}^1(k)=k\cup\{\infty\}$ be one dimensional projective space over k. Then the topology of k can be naturally extended to a metrizable topology on $\boldsymbol{P}^1(k)$ and the action of the linear fractional transformation

$$z\mapsto(az+b)/(cz+d) \tag{1.2}$$

is continuous with respect to this topology.

In our former paper [17], we defined the class of completely regular quasi-connected sets and the presheaf $\mathcal{O}$ of analytic functions on them.

Example 1. 1. Let

$$D=\{z\in \boldsymbol{P}^1(k)\,|\,|z-c_i|\geqq r_i \quad (i=1, \cdots, m)\} \tag{1. 3}$$

with $c_i\in k$, $r_i\in |k^{\times}|$. Then D is a completely regular quasi-connected set. Furthermore if none of the conditions $|z-c_i|\geqq r_i$ can be omitted without changing D, then any element f of $\mathcal{O}(D)$ can be expressed in the form

$$f(z)=a^{(\infty)}+\sum_{i=1}^{m}\sum_{n=1}^{\infty}a_{-n}^{(i)}(z-c_i)^{-n}. \tag{1. 4}$$

Since $r_i\in|k^{\times}|$, the right hand side of (1. 4) converges uniformly on D and $f(z)$ is bounded on D. Let

$$C_i=\{z\in \boldsymbol{P}^1(k)\,|\,|z-c_i|<r_i\}. \tag{1. 5}$$

Then the function f_{C_i} defined by

$$f_{C_i}(z)=\sum_{n=1}^{\infty}a_{-n}^{(i)}(z-c_i)^{-n} \tag{1. 6}$$

is an analytic function on C_i^c and $f(z)-f_{C_i}(z)$ is an analytic function on $D\cup C_i$.

Let E be an open subset of $\boldsymbol{P}^1(k)$ such that there exists a linear fractional transformation $g: z\mapsto (az+b)/(cz+d)$ such that $g(E)$ has the form (1. 3). Then we say that E is a *connected open affine subset* of $\boldsymbol{P}^1(k)$. It is easy to see that a connected open affine subset of $\boldsymbol{P}^1(k)$ has the form (1. 3) iff it contains ∞.

Let D be an open subset of $\boldsymbol{P}^1(k)$. Then

Theorem 1. 2. *D is a completely regular set iff $D=\boldsymbol{P}^1(k)$ or there exists a sequence $\{D_n\}_{n=1}^{\infty}$ such that* (i) *the D_n are connected open affine subsets of $\boldsymbol{P}^1(k)$*, (ii) $D_1\subseteqq D_2\subseteqq\cdots\subseteqq D_n\subseteqq\cdots$ *and* (iii) $D=\bigcup_{n=1}^{\infty}D_n$. *Furthermore any two such sequences are cofinal.*

It is known that (i) the intersection of any finite number of completely regular quasi-connected sets is also a completely regular quasi-connected set, and that (ii) the image of a completely regular quasi-connected set by a linear fractional transformation is also a completely regular quasi-connected set. Let $\mathfrak{D}=\{D_\iota\}_{\iota\in I}$ be a family of completely regular quasi-connected sets. Then $\mathfrak{D}$ is a *chain* iff for any D, $D'\in\mathfrak{D}$, there exist a finite number of elements $D_1=D, D_2, \cdots, D_{n-1}, D_n=D'$ of $\mathfrak{D}$ such that $D_i\cap D_{i+1}\neq\emptyset$ for any $i=1,\cdots,n-1$. It is known that if $\mathfrak{D}$ is a chain, then $D=\bigcup_{\iota\in I} D_\iota$ is a completely regular quasi-connected set.

Theorem 1. 3. (i) *Let D be a completely regular quasi-connected set, and let g be a linear fractional transformation of $\mathbf{P}^1(k)$. Then a k-valued function f on D is an analytic function iff $f\circ g$ is an analytic function on $g^{-1}(D)$.*

(ii) *Let $\mathfrak{D}=\{D_\iota\}_{\iota\in I}$ be a chain of completely regular quasi-connected sets. Let $D=\bigcup_{\iota\in I} D_\iota$. Then a k-valued function f on D is an analytic function iff the restriction $f|D_\iota$ is an analytic function for any $\iota\in I$.*

(iii) *Let D be as in* (i), *and let $D=\bigcup_{n=1}^{\infty} D_n$ be as in Theorem* 1. 2. *We assume that D contains ∞. Then f is an analytic function on D iff the restriction $f|D_n$ can be expressed in the form* (1. 4) *for any $n\geqq 1$.*

Let D be a completely regular quasi-connected set. Then it is an easy consequence of this theorem that (a) if $f(z)$ is an analytic function on D, then $\left(\frac{d}{dz}\right)f(z)$ is also analytic on D; and (b) any rational function $f(z)\in k(z)$ is analytic on D if $f(z)$ has no pole in D.

Theorem B. *Let $\mathfrak{D}$ be a chain of a countable number of completely regular quasi-connected sets. Then the covering cohomology $H^p(\mathfrak{D},\mathcal{O})$ satisfies*

$$H^p(\mathfrak{D},\mathcal{O})=0$$

for any positive integer p.

In the following sections, we need only the facts that (i) $\mathcal{O}(D)$ is defined if D is a completely regular quasi-connected set and that (ii) $\mathcal{O}$ satisfies Theorem 1. 3 and Theorem B.

Example 1. 4. Let

$$D=\{z\in \boldsymbol{P}^1(k)\,|\,|z-c_i|\geqq r_i \quad (i=1,\cdots,m)\}, \tag{1.7}$$

where $c_i\in k$ but the r_i are semi-real numbers. Then D is a completely regular quasi-connected set. Furthermore if none of the conditions $|z-c_i|\geqq r_i$ can be omitted without changing D, then any analytic function f on D has the form (1. 4). But, since some of the r_i may not belong to $|k^\times|$, f may not converge uniformly and f may not be bounded. Of course, if we restrict f to

$$\{z\in \boldsymbol{P}^1(k)\,|\,|z-c_i|\geqq r'_i \quad (i=1,\cdots,m)\} \tag{1.8}$$

$(r'_i\geqq r_i,\ r'_i\in|k^\times|)$, then these conditions are satisfied.

Example 1. 5. Let K be a compact subset of $\boldsymbol{P}^1(k)$. Then we see easily that $D=\boldsymbol{P}^1(k)\setminus K$ is a completely regular quasi-connected set.

Example 1. 6. Let D be as in Example 1. 4. Let

$$D'=\{z\in \boldsymbol{P}^1(k)\,|\,|z-c'_j|\geqq r'_j \quad (j=1,\cdots,m)\} \tag{1.9}$$

be another such set containing D. Let

$$C_i=\{z\in \boldsymbol{P}^1(k)\,|\,|z-c_i|<r_i\} \tag{1.10}$$

and

$$C'_j=\{z\in \boldsymbol{P}^1(k)\,|\,|z-c'_j|<r'_j\}. \tag{1.11}$$

We define f_{C_i} $(f\in\mathcal{O}(D))$ and $g_{C'_j}$ $(g\in\mathcal{O}(D'))$ as in Example 1. 1. Then

$$(g|D)_{C_i}(z)=\sum_{C'_j\subseteqq C_i} g_{C'_j}(z) \tag{1.12}$$

holds for any element g of $\mathcal{O}(D')$.

§ 2. *P*-Adic Hyperfunctions

In this section, we shall define p-adic hyperfunctions of one variable and study fundamental properties of them. The arguments of this section will proceed essentially parallel to them of the archimedean case (cf. Sato [19] or Komatsu [8]).

Let k be as in Section 1, and let K be a compact subset of $\boldsymbol{P}^1(k)$.

For example, let $K=\boldsymbol{P}^1(L)=L\cup\{\infty\}$ for any locally compact subfield L of k.

Let Ω be a locally closed subset of K. Hence

$$\Omega=U\cap F \tag{2.1}$$

with an open subset U of K and a closed subset F of K. We note that F and $K\backslash U$ are compact. Let V be a subset of $\boldsymbol{P}^1(k)$ such that (i) $V\supseteqq\Omega$ and (ii) V and $V\backslash\Omega$ are completely regular quasi-connected sets. For example,

$$V_0=\boldsymbol{P}^1(k)\backslash(K\backslash U) \tag{2.2}$$

satisfies such conditions, because $K\backslash U$ and $(K\backslash U)\cup F$ are compact sets (cf. Example 1.5). Let

$$\mathcal{B}(\Omega)=\mathcal{O}(V\backslash\Omega)/\mathcal{O}(V) \tag{2.3}$$

and let

$$[\theta]_V=\theta \quad \text{modulo } \mathcal{O}(V) \tag{2.4}$$

for any element θ of $\mathcal{O}(V\backslash\Omega)$.

Let V_1 and V_2 be subsets of $\boldsymbol{P}^1(k)$ such that (i) $V_i\supseteqq\Omega$ and (ii) V_i and $V_i\backslash\Omega$ are completely regular quasi-connected sets for $i=1,2$. Then we claim that $\mathcal{O}(V_1\backslash\Omega)/\mathcal{O}(V_1)$ and $\mathcal{O}(V_2\backslash\Omega)/\mathcal{O}(V_2)$ are canonically isomorphic.

Since $V_1\cap V_2\supseteqq\Omega$, $V_1\cap V_2\neq\emptyset$. Hence $V_1\cap V_2$ is a completely regular quasi-connected set containing Ω. In particular, $V_1\cap V_2$ is an open subset of $\boldsymbol{P}^1(k)$. Since the residue field of k is infinite and discrete, $\Omega\subseteqq K$ does not contain any open subset of $\boldsymbol{P}^1(k)$. Hence

$$(V_1\cap V_2)\backslash\Omega=(V_1\backslash\Omega)\cap(V_2\backslash\Omega) \tag{2.5}$$

is not empty, and hence a completely regular quasi-connected set. Therefore, to prove the above claim, we may assume $V_2\supseteqq V_1$.

In this case, the restriction map

$$\rho:\mathcal{O}(V_2\backslash\Omega)\to\mathcal{O}(V_1\backslash\Omega) \tag{2.6}$$

induces the following commutative diagram:

$$
(2.7)\qquad
\begin{array}{ccccccccc}
0 \longrightarrow & \mathcal{O}(V_2) & \longrightarrow & \mathcal{O}(V_2\backslash\Omega) & \longrightarrow & \mathcal{O}(V_2\backslash\Omega)/\mathcal{O}(V_2) & \longrightarrow 0 \\
& \rho\big\downarrow & \curvearrowright & \rho\big\downarrow & \curvearrowright & \big\downarrow\eta & \\
0 \longrightarrow & \mathcal{O}(V_1) & \longrightarrow & \mathcal{O}(V_1\backslash\Omega) & \longrightarrow & \mathcal{O}(V_1\backslash\Omega)/\mathcal{O}(V_1) & \longrightarrow 0
\end{array}
$$

Here the horizontal two sequences are exact. Let η be the map $\eta:\mathcal{O}(V_2\backslash\Omega)/\mathcal{O}(V_2)\to\mathcal{O}(V_1\backslash\Omega)/\mathcal{O}(V_1)$ induced by ρ. Let θ be an element of $\mathcal{O}(V_2\backslash\Omega)\cap\mathcal{O}(V_1)$. Since $(V_2\backslash\Omega)\cap V_1\supseteqq(V_1\backslash\Omega)\neq\emptyset$, θ is an analytic function on $(V_2\backslash\Omega)\cup V_1=V_2$. Hence $\theta\in\mathcal{O}(V_2)$ and hence η is injective.

Let ψ be an element of $\mathcal{O}(V_1\backslash\Omega)$. Then, by Theorem B, there exist an element θ of $\mathcal{O}(V_2\backslash\Omega)$ and an element χ of $\mathcal{O}(V_1)$ such that

$$(2.8)\qquad \psi(z)=\theta(z)-\chi(z)$$

for any $z\in(V_2\backslash\Omega)\cap V_1=V_1\backslash\Omega$. Then

$$(2.9)\qquad \eta([\theta]_{V_2})=[\psi+\chi]_{V_1}=[\psi]_{V_1}+[\chi]_{V_1}=[\psi]_{V_1}.$$

Hence η is surjective. Therefore we have proved that the restriction map $\rho:\mathcal{O}(V_2\backslash\Omega)\to\mathcal{O}(V_1\backslash\Omega)$ induces an isomorphism $\eta:\mathcal{O}(V_2\backslash\Omega)/\mathcal{O}(V_2)\overset{\sim}{\to}\mathcal{O}(V_1\backslash\Omega)/\mathcal{O}(V_1)$.

Let Ω' be an open subset of Ω. Then there exists an open subset U' of U such that

$$(2.10)\qquad \Omega'=U'\cap F.$$

Let $V_0=\boldsymbol{P}^1(k)\backslash(K\backslash U)$ and $V_0'=\boldsymbol{P}^1(k)\backslash(K\backslash U')$ be as (2.2). Then $V_0'\subseteqq V_0$ and

$$(2.11)\qquad \begin{aligned} V_0'\backslash\Omega' &= \boldsymbol{P}^1(k)\backslash(K\backslash U')\backslash(K\backslash F^c) \\ &\subseteqq \boldsymbol{P}^1(k)\backslash(K\backslash U)\backslash(K\backslash F^c)=V_0\backslash\Omega. \end{aligned}$$

Hence the restriction map of $\mathcal{O}(V_0\backslash\Omega)$ to $\mathcal{O}(V_0'\backslash\Omega')$ induces the following commutative diagram:

$$
(2.12)\qquad
\begin{array}{ccccccccc}
0 \longrightarrow & \mathcal{O}(V_0) & \longrightarrow & \mathcal{O}(V_0\backslash\Omega) & \longrightarrow & \mathcal{B}(\Omega) & \longrightarrow 0 & \text{(exact)} \\
& \big\downarrow & \curvearrowright & \big\downarrow & \curvearrowright & \big\downarrow & & \\
0 \longrightarrow & \mathcal{O}(V_0') & \longrightarrow & \mathcal{O}(V_0'\backslash\Omega') & \longrightarrow & \mathcal{B}(\Omega') & \longrightarrow 0 & \text{(exact)}.
\end{array}
$$

Let $\rho^{\Omega}_{\Omega'}$ be the map $\mathcal{B}(\Omega)\to\mathcal{B}(\Omega')$ in this diagram. It is obvious that $\rho^{\Omega}_{\Omega}=\mathrm{id}$. Furthermore it follows that $\rho^{\Omega}_{\Omega''}=\rho^{\Omega'}_{\Omega''}\circ\rho^{\Omega}_{\Omega'}$ holds if $\Omega''\subseteqq\Omega'\subseteqq\Omega$. Therefore

Proposition 2. 1. *$\mathcal{B} = (\mathcal{B}(\Omega'), \rho^{\Omega'}_{\Omega''})$ gives a presheaf on any locally closed subset Ω of the comapct set K.*

Furthermore we obtain the following theorem:

Theorem 2. 2. *$\mathcal{B}$ is a flabby sheaf for any locally closed subset Ω of the compact set K.*

Proof. Since K is a compact metric space, K satisfies the second countability axiom. Hence any open covering of any open subset Ω' of Ω has a refinement by a countable number of open subset of Ω'. Therefore, to prove that $\mathcal{B}$ is a sheaf, it is sufficient to show that, for any open subset Ω' of Ω and for any open covering

(2. 13) $$\Omega' = \bigcup_i \Omega_i$$

of Ω' by a countable number of the Ω_i, $\mathcal{B}$ satisfies the conditions for the localization with respect to $\Omega' = \bigcup_i \Omega_i$.

Let $\Omega = U \cap F$ be as before. Let U_i be open subsets of U such that $\Omega_i = U_i \cap F$. Let $U' = \bigcup_i U_i$. Then $\Omega' = U' \cap F$. Let $V' = \boldsymbol{P}^1(k) \setminus (K \setminus U')$ and $V_i = \boldsymbol{P}^1(k) \setminus (K \setminus U_i)$. Then $V' = \bigcup_i V_i$.

Let $f = [\theta]_{V'} \in \mathcal{B}(\Omega')$. Then $f = 0$ iff $\theta \in \mathcal{O}(V')$, and $\rho^{\Omega'}_{\Omega_i}(f) = 0$ iff $\theta \in \mathcal{O}(V_i)$. Since $\{V_i\}$ is a chain of completely regular quasi-connected sets, $\theta \in \mathcal{O}(V')$ iff $\theta \in \mathcal{O}(V_i)$ for any i. Therefore $f = 0$ iff $\rho^{\Omega'}_{\Omega_i}(f) = 0$ for any i.

Let $f_i = [\theta_i]_{V_i} \in \mathcal{B}(\Omega)$. We assume that

(2. 14) $$\rho^{\Omega_i}_{\Omega_i \cap \Omega_j}(f_i) = \rho^{\Omega_j}_{\Omega_i \cap \Omega_j}(f_j)$$

holds for any i and j such that $\Omega_i \cap \Omega_j \neq \emptyset$. Since $\mathcal{B}(\Omega_i \cap \Omega_j) = \mathcal{O}((V_i \cap V_j) \setminus (\Omega_i \cap \Omega_j)) / \mathcal{O}(V_i \cap V_j)$,

(2. 15) $$\theta_{ij} = \theta_j - \theta_i \in \mathcal{O}(V_i \cap V_j),$$

holds for any i and j Furthermore $\{\theta_{ij}\}$ satisfies the cocycle condition. Hence it follows from Theorem B that there exists an element ψ_i of $\mathcal{O}(V_i)$ for any i such that

(2. 16) $$\theta_{ij} = \psi_j - \psi_i$$

for any i and j. Then

$$\theta_i - \psi_i = \theta_j - \psi_j \tag{2.17}$$

on $(V_i \backslash \Omega_i) \cap (V_j \backslash \Omega_j)$ for any i and j. Therefore there exists an element θ of $\mathcal{O}(\bigcup_i (V_i \backslash \Omega_i)) = \mathcal{O}(V' \backslash \Omega')$ such that

$$\theta(z) = \theta_i(z) - \psi_i(z) \tag{2.18}$$

for any $z \in V_i \backslash \Omega_i$. Then

$$f = [\theta]_{V'} \in \mathcal{B}(\Omega') \tag{2.19}$$

satisfies

$$\rho^{\Omega'}_{\Omega_i}(f) = [\theta]_{V_i} = [\theta_i - \psi_i]_{V_i} = [\theta_i]_{V_i} = f_i \,. \tag{2.20}$$

Therefore $\mathcal{B}$ satisfies the conditions for the localization. Hence $\mathcal{B}$ is a sheaf on Ω.

Let Ω' be an open subset of Ω. Let U' be as before and let $V = \boldsymbol{P}^1(k) \backslash (K \backslash U)$ and $V' = \boldsymbol{P}^1(k) \backslash (K \backslash U')$. Then

$$\begin{aligned} V' \cap (V \backslash \Omega) &= (\boldsymbol{P}^1(k) \backslash K) \cup (U' \cap U \cap F^c) \\ &= (\boldsymbol{P}^1(k) \backslash K) \cup (U' \cap F^c) = V' \backslash \Omega \,. \end{aligned} \tag{2.21}$$

Let $[\theta]_{V'}$ be any element of $\mathcal{B}(\Omega') = \mathcal{O}(V' \backslash \Omega') / \mathcal{O}(V')$. Then it follows from Theorem B that there exist an element ψ of $\mathcal{O}(V \backslash \Omega)$ and an element χ of $\mathcal{O}(V')$ such that

$$\theta = \psi - \chi \tag{2.22}$$

on $(V \backslash \Omega) \cap V' = V' \backslash \Omega$. Then $[\psi]_V \in \mathcal{B}(\Omega)$ satisfies

$$\begin{aligned} \rho^{\Omega}_{\Omega'}([\psi]_V) &= [\psi]_{V'} \\ &= [\psi - \chi]_{V'} = [\theta]_{V'}. \end{aligned} \tag{2.23}$$

Hence the restriction map $\rho^{\Omega}_{\Omega'}$ is surjective. Therefore $\mathcal{B}$ is flabby. Hence the theorem is proved.

Definition 2.3. Let Ω be a locally closed subset of the compact set K. Then the sheaf $\mathcal{B}$ is called the *sheaf of hyperfunctions* and an element of $\mathcal{B}(\Omega)$ is called a *hyperfunction on* Ω.

§ 3. Köthe's Theorem

In this section, we shall show that the space $\mathcal{B}(\Omega)$ of hyperfunctions on Ω is the dual space of the space $\mathcal{A}(\Omega)$ of locally analytic functions on Ω if Ω is compact. The corresponding fact for the archimedean case is well-known and due to Köthe.

3-1. Let k be a field with a non-trivial non-archimedean valuation. Let E be a linear topological Hausdorff k-vector space. We say that a subset S of E is *k-convex* if $x, y \in S$ and $\lambda, \mu \in k$ such that $|\lambda| \leqq 1$, $|\mu| \leqq 1$ implies $\lambda x + \mu y \in S$. If a filter on E has a base of the form $x + A$ with $x \in E$ and k-convex sets A, then the filter is said to be a *k-convex filter*. If any k-convex filter on S has at least one adherent point in S, we say that S is *c-compact*. It is known (cf. Springer [23]) that (i) a closed subset of a c-compact set is c-compact; (ii) a k-convex c-compact subset of E is closed in E; (iii) if f is a continuous linear map of locally k-convex Hausdorff linear topological spaces, then the image of a c-compact set is also c-compact; and (iv) the product of c-compact sets is c-compact. Furthermore it is known that the field k is c-compact if and only if k is maximally complete. Hereafter in 3-1, we assume that *k is maximally complete.*

In his papers [25] and [26], J. Van Tiel constructed a theory of linear topological k-vector spaces over such a field k. In particular, he showed that the notion of c-compact sets for k-vector spaces plays the role of the notion of compact sets for $\boldsymbol{R}$ or $\boldsymbol{C}$-vector spaces. Using this result of Tiel, we obtain the following analogue of H. Komatsu [7][1]. Since the proof of [7] can be translated in our case in the obvious man ner, we state only the results.

Let $u: X \to Y$ be a linear map of Banach k-vector spaces. Then we say that u is *weakly c-compact* if the image by u of the unit ball of X is relatively weakly c-compact.

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[1] The possibility of using the methods of Komatsu [7] in our case was pointed out by T. Kawai.

Lemma 3. 1. *Let $u: X \to Y$ be a continuous linear map of Banach spaces. Let $X', Y', u', \cdots$ be the dual objects of $X, Y, u, \cdots$. Then the following statements are equivalent*:

(i) *u is weakly c-compact*;

(ii) *u' is weakly c-compact*;

(iii) *u'' maps X'' into Y.*

A projective (resp. injective) sequence $\{X_i, u_{ij}\}$ of Banach k-vector spaces is said to be *weakly c-compact* if for each i there exists some j such that u_{ij} (resp. u_{ji}) is weakly c-compact. Furthermore if the image by $u_{i-1\,i}$ (resp. $u_{i\,i-1}$) of the unit ball is weakly c-compact for each i, then we say that $\{X_i, u_{ij}\}$ is *strictly weakly c-compact.*

Lemma 3. 2. *Any weakly c-compact projective* (*resp. injective*) *sequence of Banach k-vector spaces is equivalent to a strictly weakly c-compact projective* (*resp. injective*) *sequence of Banach k-vector spaces.*

Theorem 3. 3. (i) *The projective limit* $\operatorname{proj\,lim} X_i$ *of a weakly c-compact sequence of Banach k-vector spaces is a reflexive Fréchet space.*

(ii) *The injective limit* $Y = \operatorname{inj\,lim} Y_i$ *of a weakly c-compact sequence of Banach k-vector spaces is a Hausdorff complete reflexive and bornologic space. Furthermore the strong dual Y' is a Fréchet space, and for each bounded set B in Y there exists an index i such that B is the image $u_i(B_i)$ of a bounded set B_i in X_i.*

Theorem 3. 4. (i) *Let $\{X_i, u_{ij}\}$ be a weakly c-compact projective sequence of Banach k-vector spaces such that* $u_i(\operatorname{proj\,lim} X_j)$ *is dense in X_i for each i. Then the dual sequence $\{X'_i, u'_{ij}\}$ is a weakly c-compact injective system, and the strong dual space of the projective limit* $\operatorname{proj\,lim} X_i$ *is isomorphic to the injective limit* $\operatorname{inj\,lim} X'_i$.

(ii) *Let $\{Y_i, u_{ji}\}$ be a weakly c-compact injective sequence of Banach k-vector spaces. Then the dual sequence $\{Y'_i, u'_{ji}\}$ is a* *weakly c-compact projective sequence of Banach k-vector spaces, and the strong dual of the injective limit* $\operatorname{injlim} Y_i$ *is isomorphic to the projective limit* $\operatorname{projlim} Y'_i$.

3-2. Let k be as in Section 1, r a positive real number, α an element of k,

$$V_{\alpha,r}=\{z\in k \mid |z-\alpha|\leqq r^-\}. \tag{3.1}$$

Let $\mathcal{O}_b(V_{\alpha,r})$ be the space of bounded analytic functions on $V_{\alpha,r}$. It is obvious that $\mathcal{O}_b(V_{\alpha,r})\subseteqq\mathcal{O}(V_{\alpha,r})$ and any element $f\in\mathcal{O}(V_{\alpha,r})$ has the form

$$f(z)=\sum_{m\geqq 0} a_m(z-\alpha)^m \quad (a_m\in k). \tag{3.2}$$

Then it is known (cf. e.g. Krasner [9]) that

$$\begin{aligned}\|f\|_{V_{\alpha,r}} &= \sup_{z\in V_{\alpha,r}} |f(z)| \\ &= \sup_{m\geqq 0} |a_m| r^m\end{aligned} \tag{3.3}$$

holds. Since

$$\mathcal{O}_b(V_{\alpha,r})=\{f(z)=\sum_{m\geqq 0} a_m(z-\alpha)^m \mid \sup_{m\geqq 0} |a_m|r^m<+\infty\}, \tag{3.4}$$

it follows that $\mathcal{O}_b(V_{\alpha,r})$ is a Banach k-vector space with the norm $\| \ \|_{V_{\alpha,r}}$.

Lemma 3.5. *Let* r, r' *be real numbers such that* $0\lneqq r\lneqq r'$. *Let* α *be an element of* k, *let*

$$\rho:\mathcal{O}_b(V_{\alpha,r'}) \to \mathcal{O}_b(V_{\alpha,r})$$

be the restriction map. Then the image by ρ *of the unit ball* B *of* $\mathcal{O}_b(V_{\alpha,r'})$ *is c-compact.*

Proof. By the definition,

$$B=\{f(z)=\sum_{m\geqq 0} a_m(z-\alpha)^m \mid |a_m|r'^m\leqq 1\} \tag{3.5}$$

and the induced topology on $\rho(B)$ is given by

$$\|f\|_{V_{\alpha,r}}=\sup_{m\geqq 0} |a_m|r^m. \tag{3.6}$$

Since $r' \gneqq r$,

(3.7) $$|a_m| r^m \leqq (r/r')^m \to 0$$

for $m \to \infty$. Let

(3.8) $$k^\infty = \{(a_0, a_1, \cdots, a_m, \cdots) \mid a_m \in k\}$$

be the direct product of a countable number of copies of k. Then the product topology of k^∞ can be defined by

(3.9) $$d(a, b) = \sum_{m=0}^{\infty} \frac{1}{2^m} \frac{|a_m - b_m| r^m}{1 + |a_m - b_m| r^m}$$

for any $a = (a_0, \cdots, a_m, \cdots)$ and $b = (b_0, \cdots, b_m, \cdots)$. Since k is maximally complete, k^∞ is a c-compact set. Since $\| \ \|_{V_{\alpha,r}}$ and $d(a, b)$ are equivalent on $\rho(B)$,

(3.10) $$\rho(B) \ni f(z) = \sum_{m \geqq 0} a_m (z - \alpha)^m \to (a_0, \cdots, a_m, \cdots) \in k^\infty$$

is a k-linear (injective) homeomorphism. Since the image

(3.11) $$\{(a_0, \cdots, a_m, \cdots) \in k^\infty \mid |a_m| \leqq (1/r')^m\}$$

is a closed subset of the c-compact space k^∞, $\rho(B)$ is a c-compact subset of $\mathcal{O}_b(V_{\alpha,r})$.

3-3. For any quasi-connected set D, let $\mathcal{O}_b(D)$ be the set of all bounded analytic functions on D. Let r be a finite positive real number, α an element of k,

(3.12) $$C = \{z \in k \mid |z - \alpha| \leqq r\}.$$

Let $V = \boldsymbol{P}^1(k)$, $\mathcal{O}(V) = k$. Then, by Theorem 1.3 and Example 1.4,

(3.13) $$\mathcal{O}(V \backslash C) = \operatorname*{proj\,lim}_{r \lneqq \rho} \mathcal{O}_b(\{z \in V \mid |z - \alpha| \geqq \rho^+\}).$$

Hence we define on $\mathcal{O}(V \backslash C)$ the projective limit topology of the Banach spaces $\mathcal{O}_b(\{z \in V \mid |z - \alpha| \geqq \rho^+\})$. Let

(3.14) $$\mathcal{O}^*(C) = \operatorname*{inj\,lim}_{r \lneqq \rho} \mathcal{O}_b(\{z \in k \mid |z - \alpha| \leqq \rho^-\})$$

be the injective limit of the Banach spaces $\mathcal{O}_b(\{z \in k \mid |z - \alpha| \leqq \rho^-\})$. By Theorem 3.3 and Lemma 3.5, $\mathcal{O}(V \backslash C)$ is a reflexive Fréchet space and $\mathcal{O}^*(C)$ is a complete Hausdorff bornologic space and the strong dual of a reflexive Fréchet space. Furthermore we observe that $\mathcal{O}(V)$ is a closed subspace of $\mathcal{O}_b(\{z\in V \mid |z-\alpha|\geqq\rho^+\})$ and

$$(3.15)\qquad \mathcal{O}(V\backslash C)/\mathcal{O}(V)=\operatorname{proj}\lim_{r\lneqq\rho}\mathcal{O}_b(\{z\in V \mid |z-\alpha|\geqq\rho^+\})/\mathcal{O}(V)$$

is a reflexive Fréchet space.

Let

$$(3.16)\qquad f(z)=\sum_{m=0}^{\infty}a_m(z-\alpha)^{-m}$$

be an element of $\mathcal{O}(V\backslash C)$ and let

$$(3.17)\qquad g(z)=\sum_{n=0}^{\infty}b_n(z-\alpha)^n$$

be an element of $\mathcal{O}_b(\{z\in k \mid |z-\alpha|\leqq\rho^-\})$ $(r\lneqq\rho)$. Let

$$(3.18)\qquad \langle f(z),g(z)\rangle_C=\operatorname*{Res}_{|z-\alpha|\leqq r} f(z)g(z)dz = \sum_{-m+n=-1}a_mb_n\,.$$

Since

$$(3.19)\qquad \sum_{-m+n=-1}a_mb_n=\sum_{n\geqq 0}(a_{n+1}\rho^{-n-1})(b_n\rho^n)\rho\,,$$

$\langle\ ,\ \rangle_C$ is a continuous bilinear form on $\mathcal{O}(V\backslash C)\times\mathcal{O}_b(\{z\in k \mid |z-\alpha|\leqq\rho^-\})$. Hence $\langle\ ,\ \rangle_C$ is a continuous bilinear form on $\mathcal{O}(V\backslash C)\times\mathcal{O}^*(C)$. Since $\langle f(z),g(z)\rangle_C=0$ if $f(z)\in\mathcal{O}(V)$, $\langle\ ,\ \rangle_C$ is a continuous bilinear form on $(\mathcal{O}(V\backslash C)/\mathcal{O}(V))\times\mathcal{O}^*(C)$. If $a_m\neq0$ for $m\geqq1$, then $g(z)=(z-\alpha)^{m-1}\in\mathcal{O}^*(C)$ satisfies $\langle f(z),g(z)\rangle_C=a_m\neq0$. Similarly if $b_n\neq0$ for $n\geqq0$, then $f(z)=(z-\alpha)^{-n-1}\in\mathcal{O}(V\backslash C)$ satisfies $\langle f(z),g(z)\rangle_C=b_n\neq0$. Therefore $\langle\ ,\ \rangle_C$ is a non-degenerate continuous bilinear form on $(\mathcal{O}(V\backslash C)/\mathcal{O}(V))\times\mathcal{O}^*(C)$.

Now we have

Theorem 3.6. *Let the notation and assumptions be as above. Then $\mathcal{O}(V\backslash C)/\mathcal{O}(V)$ and $\mathcal{O}^*(C)$ are mutually dual linear topological k-vector spaces with respect to $\langle\ ,\ \rangle_C$.*

Proof. Since $\mathcal{O}^*(C)$ is reflexive, it is sufficient to prove that

$$(3.20)\qquad \mathcal{O}(V\backslash C)/\mathcal{O}(V)\ni f(z) \text{ modulo } \mathcal{O}(V)\to\langle f(z),\ \rangle_C\in\mathcal{O}^*(C)'$$

is an isomorphism. Since $\langle\ ,\ \rangle_C$ is a continuous non-degenerate bilinear form, this map is continuous and injective.

Let Φ be an element of $\mathcal{O}^*(C)'$. Let

$$(3.21)\qquad \theta(z)=\Phi\left(\frac{1}{z-t}\right)$$

for any element z of $V\backslash C$. Since $\frac{1}{z-t}\in\mathcal{O}^*(C)$ as a rational function in t, this is well-defined. Furthermore, since

$$(3.22)\qquad \frac{1}{z-t}=\sum_{m=0}^{\infty}\frac{(t-\alpha)^m}{(z-\alpha)^{m+1}}$$

is an expansion in $\mathcal{O}^*(C)$ for any fixed $z\in V\backslash C$,

$$(3.23)\qquad \theta(z)=\sum_{m=0}^{\infty}\Phi((t-\alpha)^m)\frac{1}{(z-\alpha)^{m+1}}.$$

Since this expansion converges for any $z\in V\backslash C$, $\theta(z)$ is an element of $\mathcal{O}(V\backslash C)$. Let $g(z)=\sum_{n=0}^{\infty}b_n(z-\alpha)^m$ be as before. Then

$$\begin{aligned}(3.24)\qquad \langle\theta(z),g(z)\rangle_C&=\langle\sum_{m=0}^{\infty}\Phi((t-\alpha)^m)\frac{1}{(z-\alpha)^{m+1}},\ \sum_{n=0}^{\infty}b_n(z-\alpha)^n\rangle_C\\ &=\sum_{m=0}^{\infty}\Phi((t-\alpha)^m)b_m\\ &=\Phi(\sum_{m=0}^{\infty}b_m(t-\alpha)^m)\\ &=\Phi(g(t)).\end{aligned}$$

Therefore (3.20) is surjective. Since $\mathcal{O}(V\backslash C)/\mathcal{O}(V)$ and $\mathcal{O}^*(C)'$ are both Fréchet spaces, it follows from the open mapping theorem (cf. Bourbaki [3]) that this continuous bijective linear map is an isomorphism. Hence the theorem is proved.

Let V be as before. Let $C_1,\cdots,C_t$ be mutually disjoint closed balls Let $\mathcal{O}^*(C_i)$ and $\operatorname{Res}_{C_i} f(z)g(z)dz$ be as before. Let

$$(3.25)\qquad \mathcal{O}^*(C_1\cup\cdots\cup C_t)=\bigoplus_{i=1}^{t}\mathcal{O}^*(C_i).$$

Namely, we take small open balls $V_1, \cdots, V_t$ such that $V_i \supseteqq C_i$ and $V_1, \cdots, V_t$ are mutually disjoint, and define

$$\mathcal{O}^*(C_1 \cup \cdots \cup C_t) = \operatorname*{inj\,lim}_{V_i} \bigoplus_{i=1}^{t} \mathcal{O}_b(V_i). \tag{3.26}$$

On the other hand, it follows from Example 1.4 that

$$\mathcal{O}(V \backslash C_1 \backslash \cdots \backslash C_t)/\mathcal{O}(V) \cong \bigoplus_{i=1}^{t} \mathcal{O}(V \backslash C_i)/\mathcal{O}(V) \tag{3.27}$$

(a canonical topological isomorphism). Hence

$$\mathcal{O}(V \backslash C_1 \backslash \cdots \backslash C_t) \times \mathcal{O}^*(C_1 \cup \cdots \cup C_t) \to k \tag{3.28}$$

$$(f(z), g(z)) \to \langle f(z), g(z) \rangle = \sum_{i=1}^{t} \operatorname*{Res}_{C_i} f(z) g(z) dz$$

induces

Corollary 3.7. *$\mathcal{O}(V \backslash C_1 \backslash \cdots \backslash C_t)/\mathcal{O}(V)$ and $\mathcal{O}^*(C_1 \cup \cdots \cup C_t)$ are mutually dual linear topological spaces.*

3-4. Let K be a compact set in $\boldsymbol{P}^1(k)$. For the sake of simplicity we assume that $K \not\ni \infty$. Let $\mathcal{A}(K)$ be the set consisting of all k-valued functions g on K such that, for any $\alpha \in K$, there exists a neighbourhood U_α of α in K such that $g|U_\alpha$ is given by a convergent power series. Let $V = \boldsymbol{P}^1(k)$. Then V and $V \backslash K$ are completely regular quasi-connected sets. Hence

$$\mathcal{B}(K) = \mathcal{O}(V \backslash K)/\mathcal{O}(V). \tag{3.29}$$

Let $\{r_n\}_{n=1}^{\infty}$ be a strictly decreasing sequence of real numbers satisfying $r_n \in |k|$ and $\lim r_n = 0$. For any positive integer n and for any element α of K, let $C_\alpha^{(n)}$ be the closed ball in V containing α of diameter r_n. Since K is compact, K is covered by a finite number of them. Let $C_1^{(n)}, \cdots, C_{t_n}^{(n)}$ be closed balls of diameter r_n such that (i) $\bigcup_{i=1}^{t_n} C_i^{(n)} \supseteqq K$, (ii) $C_i^{(n)} \cap K \neq \emptyset$ and (iii) $C_i^{(n)} \cap C_j^{(n)} = \emptyset$ $(i \neq j)$. Let

$$K_n = C_1^{(n)} \cup \cdots \cup C_{t_n}^{(n)}. \tag{3.30}$$

Then $\{K_n\}_{n=1}^{\infty}$ is a strictly decreasing sequence and

(3. 31) $$K=\bigcap_{n=1}^{\infty} K_n .$$

Furthermore $V\backslash K=\bigcup_{n=1}^{\infty}(V\backslash K_n)$ is a completely regular quasi-connected set. Similarly, let $C_{\alpha}^{(n)0}$ be the open ball in V containing α of diameter r_n^-, and let

(3. 32) $$K_n^0=C_1^{(n)0}\cup\cdots\cup C_{t_n'}^{(n)0}$$

be the corresponding covering of K. Let

(3. 33) $$\mathcal{O}_b(K_n^0)=\bigoplus_{i=1}^{t_n'}\mathcal{O}_b(C_i^{(n)0}).$$

Then $\{K_n^0\}_{n=1}^{\infty}$ satisfies similar conditions. Furthermore, by Example 1.4,

(3. 34) $$\mathcal{A}(K)=\bigcup_{n=1}^{\infty}\mathcal{O}_b(K_n^0),$$

(3. 35) $$\mathcal{B}(K)=\bigcap_{n=1}^{\infty}\mathcal{O}_b(V\backslash K_n)/\mathcal{O}(V),$$

Hence we define topologies on $\mathcal{A}(K)$ and $\mathcal{B}(K)$ by

(3. 36) $$\mathcal{A}(K)=\operatorname{inj\,lim}\mathcal{O}_b(K_n^0)$$

and

(3. 37) $$\mathcal{B}(K)=\operatorname{proj\,lim}\mathcal{O}_b(V\backslash K_n)/\mathcal{O}(V).$$

Here the injective sequence and the projective sequence are constructed by the natural restriction maps. Hence, by Theorem 3. 3 and Lemma 3. 5, $\mathcal{A}(K)$ is a Hausdorff complete bornologic space and the strong dual of a reflexive Fréchet space, and $\mathcal{B}(K)$ is a reflexive Fréchet space.

It is obvious that

(3. 38) $$\mathcal{O}_b(K_n^0)\supseteqq\mathcal{O}^*(K_n)\supseteqq\mathcal{O}_b(K_{n-1}^0),$$

(3. 39) $$\mathcal{O}_b(V\backslash K_n)\subseteqq\mathcal{O}(V\backslash K_n)\subseteqq\mathcal{O}_b(V\backslash K_{n-1}).$$

Since all inclusion maps in (3. 38) and (3. 39) are continuous, we obtain

(3. 40) $$\mathcal{A}(K)=\operatorname{inj\,lim}\mathcal{O}^*(K_n)$$

and

(3. 41) $$\mathcal{B}(K)=\operatorname{proj\,lim}\mathcal{O}(V\backslash K_n)/\mathcal{O}(V).$$

For any $f(z)\in\mathcal{O}(V\backslash K)$ and for any $g(z)\in\mathcal{O}^*(K_n)$, let

$$(3.42)\qquad \langle f(z), g(z)\rangle_{K_n}=\sum_{i=1}^{t_n} \operatorname*{Res}_{C_i^{(n)}} f(z)g(z)\,dz\,,$$

where we define $\operatorname*{Res}_{C_i^{(n)}} f(z)g(z)\,dz=\langle f(z), g(z)\rangle_{C_i^{(n)}}$ by (3.18). Then this bilinear form is continuous on $(\mathcal{O}(V\backslash K)/\mathcal{O}(V))\times\mathcal{O}^*(K_n)$. Since this pairing does not depend on a choice of $\mathcal{O}^*(K_n)\ni g(z)$ by Example 1.6, it induces a continuous bilinear form $\langle\ ,\ \rangle_K$ on $\mathcal{B}(K)\times\mathcal{A}(K)$. Moreover, by Example 1.6, the restriction of $\langle\ ,\ \rangle_K$ to a closed subset K' of K coinsides with $\langle\ ,\ \rangle_{K'}$. Furthermore it follows from the proof of the non-degeneracy of $\langle\ ,\ \rangle_C$ that $\langle\ ,\ \rangle_K$ is non-degenerate (because $(z-\alpha)^{-n-1}\in\mathcal{O}(V\backslash K)$ if $\alpha\in K$). Hence we have a continuous injection

$$(3.43)\qquad \begin{array}{ccc} \mathcal{O}(V\backslash K)/\mathcal{O}(V) & \to & \mathcal{A}(K)' \\ \Downarrow & & \Downarrow \\ f(z) \text{ modulo } \mathcal{O}(V) & \mapsto & \langle f(z),\ \rangle_K. \end{array}$$

Now we have

Theorem 3.7. *$\mathcal{A}(K)$ and $\mathcal{B}(K)$ are mutually dual linear topological k-vector spaces with respect to the bilinear form $\langle\ ,\ \rangle_K$.*

Proof. Since the both sides of (3.43) are reflexive Fréchet space, it is sufficient to prove the map (3.43) is surjective. But the surjectivity follows from

$$\begin{aligned}(3.44)\qquad \mathcal{A}(K)' &= (\operatorname{inj\,lim} \mathcal{O}_b(K_n^o))' \\ &= \operatorname{proj\,lim} \mathcal{O}_b(K_n^o)' \\ &= \operatorname{proj\,lim} \mathcal{O}^*(K_n)' \\ &\cong \operatorname{proj\,lim} \mathcal{O}(V\backslash K_n)/\mathcal{O}(V) \\ &= \mathcal{B}(K).\end{aligned}$$

Remark 3.8. Theorem 3.7 can be generalized to the case $K\ni\infty$ by checking that $\mathcal{A}(K)$, $\mathcal{B}(K)$, $\langle\ ,\ \rangle_K$ are invariant by the action of linear fractional transformations (i.e., they are transformed by a linear fractional transformation to the corresponding objects of the image of K).

3-5. Let $f(z)$ be an element of $\mathcal{O}(V\backslash K)$. Then $\left(\frac{d}{dz}\right)f(z)$ also belongs to $\mathcal{O}(V\backslash K)$. Hence we define the derivation $\frac{d}{dz}$ on $\mathcal{B}(K)$ by

$$\mathcal{B}(K)\ni f(z) \text{ modulo } \mathcal{O}(V) \longrightarrow \left(\frac{d}{dz}\right) f(z) \text{ modulo } \mathcal{O}(V)\in\mathcal{B}(K). \tag{3.45}$$

Since $\left(\frac{d}{dz}\right)\mathcal{O}(V)\subseteqq\mathcal{O}(V)$, this definition is well-defined.

Proposition 3.9. *Let* $f(z)$ *and* $g(z)$ *be an element of* $\mathcal{O}(V\backslash K)$ *and an element of* $\mathcal{A}(K)$. *Then*

$$\left\langle\left(\frac{d}{dz}\right)f(z), g(z)\right\rangle_K=\left\langle f(z), -\left(\frac{d}{dz}\right)g(z)\right\rangle_K.$$

Proof. Since $\mathcal{A}(K)=\operatorname{inj\,lim}\mathcal{O}^*(K_n)$, $\mathcal{O}^*(K_n)=\bigoplus_{i=1}^{t_n}C_i^{(n)}$ and

$$\langle f(z), g(z)\rangle=\sum_{i=1}^{t_n}\langle f(z), g(z)\rangle_{C_i^{(n)}} \tag{3.46}$$

for any element $g(z)$ of $\mathcal{O}^*(K_n)$, it is sufficient to prove the corresponding formula for $\langle\ ,\ \rangle_{C_i^{(n)}}$. Let

$$C=C_i^{(n)}=\{z\in V\,|\,|z-\alpha|\leqq r\}. \tag{3.47}$$

Let $f(z)=\sum_{m=0}^{\infty}a_m(z-\alpha)^{-m}\in\mathcal{O}(V\backslash C)$, $g(z)=\sum_{n=0}^{\infty}b_n(z-\alpha)^n\in\mathcal{O}^*(C)$. Then

$$\left(\frac{d}{dz}\right)f(z)=\sum_{m=0}^{\infty}(-m)\,a_m(z-\alpha)^{-m-1} \tag{3.48}$$

and

$$\left(\frac{d}{dz}\right)g(z)=\sum_{n=0}^{\infty}nb_n(z-\alpha)^{n-1}. \tag{3.49}$$

Hence

$$\begin{aligned}\left\langle\left(\frac{d}{dz}\right)f(z), g(z)\right\rangle_C &=\sum_{m=n}(-m)\,a_m b_n\\ &=\sum_{m=n}a_n(-n)\,b_n\\ &=\left\langle f(z), -\left(\frac{d}{dz}\right)g(z)\right\rangle_C.\end{aligned} \tag{3.50}$$

3-6. Let Φ be an element of $\mathcal{A}(K)'$. Let z be a parame'er that moves over $V\backslash K$. Then

$$\frac{1}{t-z}=-\sum_{n=0}^{\infty}\frac{(t-\alpha)^{n}}{(z-\alpha)^{n+1}} \quad (\alpha\in K) \tag{3.51}$$

converges for $|t-\alpha|<|z-\alpha|$. Hence $\frac{1}{t-z}$ is a locally analytic function on K as a function of t. Let

$$\theta(z)=\Phi\left(\frac{1}{t-z}\right). \tag{3.52}$$

Then $\theta(z)$ is a function on $V\backslash K$. Since

$$V\backslash K=\bigcup_{n=1}^{\infty}(V\backslash K_n), \tag{3.53}$$

any element z of $V\backslash K$ belongs to some $V\backslash K_n$. Since

$$V\backslash K_n=\bigcap_{n=1}^{t_n}(V\backslash C_i^{(n)}), \tag{3.54}$$

z belongs to any $V\backslash C_i^{(n)}$. By Theorem 3.7, (3.41) and the proof of Corollary 3.7,

$$\theta(z)=\Phi\left(\frac{1}{t-z}\right) = \sum_{i=1}^{t_n}\{\Phi|\mathcal{O}(C_i^{(n)})\}\left(\frac{1}{t-z}\right). \tag{3.55}$$

Hence, by the proof of Theorem 3.6 (cf. (3.23)),

$$\theta(z)=-\sum_{i=1}^{t_n}\sum_{m=1}^{\infty}\Phi((t-c_i^{(n)})^{m})\frac{1}{(z-c_i^{(n)})^{m+1}} \quad (c_i^{(n)}\in C_i^{(n)}) \tag{3.56}$$

for any $z\in V\backslash K_n$. Hecne $\theta(z)\in\mathcal{O}(V\backslash K_n)$. Furthermore it follows from (3.24) that

$$\Phi(g(t))=\langle\theta(z),g(z)\rangle_{K_n} \tag{3.57}$$

for any element $g(z)$ of $\mathcal{O}^*(K_n)$. Therefore it follows from (3.53) that $\theta(z)\in\mathcal{O}(V\backslash K)$ and

$$\Phi(g(t))=\langle\theta(z),g(z)\rangle_{K} \tag{3.58}$$

holds for any element g of $\mathcal{A}(K)$. We note

$$\theta(\infty)=0. \tag{3.59}$$

§ 4. Examples

4-1. Let k, V be as in 3-4. Let α be an element of k. Then

$$f(z)=\frac{(-1)^m m!}{(z-\alpha)^{m+1}} \tag{4.1}$$

is an element of $\mathcal{O}(V\setminus\{\alpha\})$. Let

$$g(z)=\sum_{n=0}^{\infty} b_n(z-\alpha)^n \tag{4.2}$$

be an element of $\mathcal{A}(\{\alpha\})$. Then

$$\langle f(z), g(z)\rangle_{\{\alpha\}}=(-1)^m m! b_m. \tag{4.3}$$

Hence $f(z)$ modulo $\mathcal{O}(V)\in\mathcal{B}(\{\alpha\})$ is the m-th derivative $\delta_\alpha^{(m)}$ of the delta function δ_α at α.

4-2. Now we assume that k contains the p-adic number field $\boldsymbol{Q}_p$. For simplicity, we assume that $p\neq 2$. Let s_1 be an element of k such that $|s_1|<|p^{-1}p^{1/(p-1)}|$. Let s_2 be an element of $\boldsymbol{Z}/(p-1)\boldsymbol{Z}$, $s=(s_1, s_2)$.

If z is an element of

$$\{z\in k \mid d(z, \boldsymbol{Z}_p^\times)<|p^{1/(p-1)}|\}, \tag{4.4}$$

then

$$\omega(z)=\lim_{n\to\infty} z^{p^n} \tag{4.5}$$

is well-defined and gives a $(p-1)$th root of unity such that $|z-\omega(z)|<1$. Let

$$\langle z\rangle=\omega(z)^{-1}z, \tag{4.6}$$

$$\langle z\rangle^{s_1}=\exp\{s_1\log\langle z\rangle\}, \tag{4.7}$$

$$z^s=\omega(z)^{s_2}\langle z\rangle^{s_1}. \tag{4.8}$$

Then z^s is an analytic function on

$$\{z\in k \mid |z-\zeta|<\min(|s_1^{-1}p^{1/(p-1)}|, |p^{1/(p-1)}|)\} \tag{4.9}$$

for each $(p-1)$th root ζ of unity. In particular z^s is a locally analytic function on $\boldsymbol{Z}_p^\times$.

Let χ be a k-valued Dirichlet character modulo f. Then we proved in [14] that the right hand side of

$$(4.10)\qquad \left(\frac{d}{dz}\right)^2 \log \Gamma_{p,\chi}(-z+1) = \lim_{n\to\infty} \frac{1}{fp^n} \sum_{\substack{1\leq x\leq fp^n \\ (x,p)=1}} \frac{\chi(x)}{-z+x}$$

converges in the topology of $\mathcal{O}(V\backslash Z_p^\times)$. Hence

$$(4.11)\qquad -\left\langle \left(\frac{d}{dz}\right)^2 \log \Gamma_{p,\chi}(-z+1),\, z^{-s+1} \right\rangle_{Z_p^\times}$$

$$= \lim_{n\to\infty} \frac{1}{fp^n} \sum_{\substack{1\leq x\leq fp^n \\ (x,p)=1}} \chi(x) \left\langle \frac{1}{z-x},\, z^{-s+1} \right\rangle_{Z_p^\times}$$

$$= \lim_{n\to\infty} \frac{1}{fp^n} \sum_{\substack{1\leq x\leq fp^n \\ (x,p)=1}} \chi(x) \langle \delta_x,\, z^{-s+1} \rangle_{Z_p^\times}$$

$$= \lim_{n\to\infty} \frac{1}{fp^n} \sum_{\substack{1\leq x\leq fp^n \\ (x,p)=1}} \chi(x)\, \omega^{-s_2+1}(x) \langle x \rangle^{-s_1+1}$$

$$= \lim_{n\to\infty} \frac{1}{fp^n} \sum_{\substack{1\leq x\leq fp^n \\ (x,p)=1}} \chi\omega^{-s_2+1}(x) \langle x \rangle^{-s_1+1},$$

which is equal to $(s_1-1)L_p(s_1:\chi\omega^{-s_2+1})$ by the definition of the p-adic L-function $L_p(s:\chi)$ (cf. Kubota-Leopoldt [10]). Hence we have obtained the following theorem.

Theorem 4. 1. *Let the notation and assumptions be as above. Then*

$$(s_1-1)L_p(s_1:\chi\omega^{-s_2+1}) = -\left\langle \left(\frac{d}{dz}\right)^2 \log \Gamma_{p,\chi}(-z+1),\, z^{-s+1} \right\rangle_{Z_p^\times}.$$

Remark 4. 2. This theorem exactly implies the result stated in the introduction of Morita [16]. But it does not cover the main theorem of that paper.

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(2) After writing the present paper, the author proved that the Krasner theory and the Tate theory give the same results if the field k is maximally complete and the domain D is a completely regular quasi-connected set. Furthermore, he proved that we can obtain the results of Section 1 without assuming that k is maximally complete if we use the Tate theory instead of the Krasner theory. Hence [17] will not appear.

For the references and the details, see Morita, Y., Analytic functions on an open subset of $\boldsymbol{P}^1(k)$, *J. reine u. angew. Math.*, **311/312** (1979), 361-383.