On Complete Kähler Domains with C^1-Boundary

By

Takeo OHSAWA[*]

Introduction

In the study of several complex variables the problem of characterizing Stein manifolds by their geometric properties has been one of the central topics since Oka showed that every pseudoconvex unramified domain over the complex number space C^n is a Stein manifold. In 1956 Grauert [4] showed that every unramified domain over C^n with real analytic boundary and complete Kähler metric is a Stein manifold. He proved the pseudoconvexity of the domain and applied Oka's theorem. The purpose of the present note is to prove the following theorem.

Theorem: *Let X be a domain with C^1-boundary and with a complete Kähler metric in a complex manifold M. Then X is pseudoconvex.*

To prove the theorem we apply the method of L^2 estimates of $\bar{\partial}$ which is summarized in Section 1.

§1. Differential Analysis on Complex Manifolds

Let X be a paracompact complex manifold of (complex) dimension n and ds^2 be a hermitian metric on X. The complex vector space $C_0^{p,q}(X)$ of infinitely differentiable (p, q) forms on X with compact support has a structure of a pre-Hilbert space with respect to the norm P defined by

$$(1) \qquad P(f) := \left(\int_X f \wedge \bar{*} f \right)^{1/2},$$

where $f \in C_0^{p,q}(X)$ and $\bar{*}$ denotes the conjugate of the star operator $*$ associated to ds^2. We define

(2) $(f, g) := \int_X f \wedge \bar{*}g$ for $f, g \in C_0^{p,q}(X)$.

We introduce another norm P_Φ defined by

(3) $P_\Phi(f) := \left(\int_X \exp(-\Phi)f \wedge \bar{*}f \right)^{1/2}$,

where $f \in C_0^{p,q}(X)$ and Φ denotes a differentiable real valued function on X. We define

(4) $(f, g)_\Phi := \int_X \exp(-\Phi)f \wedge \bar{*}g$,   for $f, g \in C_0^{p,q}(X)$.

We denote by $L_1^{p,q}(X)$ the square integrable (p, q) forms on X, i.e., the completion of $C_0^{p,q}(X)$ with respect to the norm P, and denote by $L_\Phi^{p,q}(X)$ the completion of $C_0^{p,q}(X)$ with respect to P_Φ. Note that if Φ is bounded on X, then the completion of $C_0^{p,q}(X)$ with respect to the norm P_Φ can be naturally identified with $L_1^{p,q}(X)$. A function $\Phi \colon X \to \mathbf{R}$ ($\mathbf{R}$ = real numbers) is called strictly plurisubharmonic if Φ is C^2, i.e., twice continuously differentiable, and the Levi form

$$\left(\frac{\partial^2 \Phi}{\partial z^\alpha \partial \bar{z}^\beta} \right)$$

is positive definite everywhere. Here $(z^1, \ldots, z^n)$ denotes a local coordinate system of X. Given a strictly plurisubharmonic function Φ on X,

(5) $d\sigma^2 := \sum_{\alpha, \beta} \frac{\partial^2 \Phi}{\partial z^\alpha \partial \bar{z}^\beta} dz^\alpha \cdot d\bar{z}^\beta$

defines a hermitian metric on X. Moreover the associated $(1, 1)$ form

(6) $\omega := \sqrt{-1} \sum_{\alpha, \beta} \frac{\partial^2 \Phi}{\partial z^\alpha \partial \bar{z}^\beta} dz^\alpha \wedge d\bar{z}^\beta$

is d-closed. As usual we say that a hermitian metric is a Kähler metric if the associated $(1, 1)$ form is d-closed. Let $L_2^{p,q}(X)$ be the square integrable (p, q) forms with respect to $d\sigma^2$, i.e., the completion of $C_0^{p,q}(X)$ with respect to the norm

(7) $Q(f) := \left(\int_X f \wedge \bar{\star}f \right)^{1/2}$,

where $\bar{\star}$ is the conjugate of the star operator $\star$ associated to $d\sigma^2$. Let $(\sigma_1, \ldots, \sigma_n)$ be a moving frame of the holomorphic cotangent bundle of X which is orthonormal with respect to $d\sigma^2$ in a neighbourhood of $x_0 \in X$. We let

$$(8) \qquad ds^2 = \sum_{i=1}^{n} \lambda_i^2 d\sigma_i \cdot d\bar{\sigma}_i,$$

where λ_i are positive valued functions in a neighbourhood of x_0 such that $\lambda_1 \leqq \cdots \leqq \lambda_n$. Note that λ_i are continuous functions and does not depend on the choice of $(\sigma_1, \ldots, \sigma_n)$, therefore λ_i are well defined on X.

Proposition 1.1. *If $\lambda_i \geqq 1$ for $i = 1, \ldots, n$, then*

$$(9) \qquad L_2^{n,q}(X) \subset L_1^{n,q}(X) \qquad q \geqq 0$$

and

$$(10) \qquad L_2^{p,0}(X) \supset L_1^{p,0}(X) \qquad p \geqq 0.$$

In particular $L_2^{n,0}(X) = L_1^{n,0}(X)$.

Proof. For the proof of $L_2^{n,q}(X) \subset L_1^{n,q}(X)$ it is sufficient to show that

$$(11) \qquad Q(f) \geqq P(f), \qquad \text{for} \quad f \in C_0^{n,q}(X).$$

Let us compute $f \wedge \overline{\underset{\cdot}{\star} f}$ at x_0. Let $(s_1, \ldots, s_n)$ and $(\sigma_1, \ldots, \sigma_n)$ be orthonormal moving frames of the holomorphic cotangent bundle of X around x_0 with respect to ds^2 and $d\sigma^2$ respectively, such that

$$(12) \qquad s_i = \lambda_i \sigma_i.$$

Letting

$$(13) \qquad f = \sum_{i_1 < \cdots < i_q} f_{i_1 \cdots i_q} \sigma_1 \wedge \cdots \wedge \sigma_n \wedge \bar{\sigma}_{i_1} \wedge \cdots \wedge \bar{\sigma}_{i_q}$$

we have

$$(14) \qquad f \wedge \overline{\underset{\cdot}{\star} f} = K \sum_{i_1 < \cdots < i_q} |f_{i_1 \cdots i_q}|^2 \sigma_1 \wedge \cdots \wedge \sigma_n \wedge \bar{\sigma}_1 \wedge \cdots \wedge \bar{\sigma}_n,$$

where

$$(15) \qquad K = (-1)^{n(n-1)/2} 2^{-q} (\sqrt{-1})^n.$$

On the other hand, since

$$(16) \qquad f = \sum_{i_1 < \cdots < i_q} f_{i_1 \cdots i_q} \frac{1}{\lambda_1 \cdots \lambda_n \lambda_{i_1} \cdots \lambda_{i_q}} s_1 \wedge \cdots \wedge s_n \wedge \bar{s}_{i_1} \wedge \cdots \wedge \bar{s}_{i_q}$$

we have

$$(17) \qquad f \wedge \bar{*} f = K \sum_{i_1 < \cdots < i_q} |f_{i_1 \cdots i_q}|^2 \frac{s_1 \wedge \cdots \wedge s_n \wedge \bar{s}_1 \wedge \cdots \wedge \bar{s}_n}{\lambda_1^2 \cdots \lambda_n^2 \lambda_{i_1}^2 \cdots \lambda_{i_q}^2}$$

$$= K \sum_{i_1 < \cdots < i_q} |f_{i_1 \cdots i_q}|^2 \frac{\sigma_1 \wedge \cdots \wedge \sigma_n \wedge \bar{\sigma}_1 \wedge \cdots \wedge \bar{\sigma}_n}{\lambda_{i_1}^2 \cdots \lambda_{i_q}^2}.$$

Hence we obtain $Q(f) \geq P(f)$.   The proof of $L_2^{p,0}(X) \supset L_1^{p,0}(X)$ is similar.

We define a first order differential operator

$$\bar{\partial}: C_0^{p,q}(X) \to C_0^{p,q+1}(X)$$

as follows;

$$\bar{\partial}f = (-1)^p \sum_{\substack{i_1 < \cdots < i_p \\ j_1 < \cdots < j_q}} \sum_{1 \leq j \leq n} \frac{\partial f_{i_1 \cdots i_p \bar{j}_1 \cdots \bar{j}_q}}{\partial \bar{z}^j} dz^{i_1} \wedge \cdots \wedge dz^{i_p} \wedge$$
$$dz^j \wedge d\bar{z}^{j_1} \wedge \cdots \wedge d\bar{z}^{j_q},$$

where

$$f = \sum_{\substack{i_1 < \cdots < i_p \\ j_1 < \cdots < j_q}} f_{i_1 \cdots i_p \bar{j}_1 \cdots \bar{j}_q} dz^{i_1} \wedge \cdots \wedge dz^{i_p} \wedge d\bar{z}^{j_1} \wedge \cdots \wedge d\bar{z}^{j_q}.$$

The domain of $\bar{\partial}$ which we denote by $D^{p,q}$ is defined as the set of $f \in L_1^{p,q}(X)$ such that the current $\bar{\partial}f$ belongs to $L_1^{p,q+1}(X)$.   We regard $\bar{\partial}$ as a closed linear operator from $L_1^{p,q}(X)$ to $L_1^{p,q+1}(X)$ with dense domain $D^{p,q}$.   Since $D^{p,q}$ is dense in $L_1^{p,q}(X)$ the adjoint $\bar{\partial}^*$ of $\bar{\partial}$ is well defined in the following sense. f belongs to the domain $D_*^{p,q+1}$ of $\bar{\partial}^*$ if and only if $f \in L_1^{p,q}(X)$ and there exists $h \in L_1^{p,q-1}(X)$ such that

$$(h, u) = (f, \bar{\partial}u) \qquad \text{for any} \quad u \in D^{p,q-1}.$$

Such h is uniquely determined by f and we define $\bar{\partial}^* f = h$.

Note that $C_0^{p,q}(X) \subset D_*^{p,q}$ by the theorem of Stokes.

We define the adjoint of $\bar{\partial}$ with respect to the norm P_Φ similarly and denote it by $\bar{\partial}_\Phi^*$.   The domain of $\bar{\partial}$ in $L_\Phi^{p,q}$ and the domain of $\bar{\partial}_\Phi^*$ is denoted by $D_\Phi^{p,q}$ and $D_{*,\Phi}^{p,q}$, respectively.

We say that a hermitian metric ds^2 is complete if for any sequence $\{x_\nu\}$, $\nu = 1, 2, \ldots$ of points in X which does not have an accumulation point, the distance between x_1 and x_ν tends to infinity as $\nu \to \infty$.

Proposition 1.2.   *Let ds^2 be a complete C^2 hermitian metric on X and Φ be a C^2 function on X with values in $\mathbf{R}$.   Then $C_0^{p,q}(X)$ is dense in $D_\Phi^{p,q}$, $D_{*,\Phi}^{p,q}$, and $D_\Phi^{p,q} \cap D_{*,\Phi}^{p,q}$, with respect to the graph norms $P_\Phi(f) + P_\Phi(\bar{\partial}f)$, $P_\Phi(f) + P_\Phi(\bar{\partial}_\Phi^* f)$, and $P_\Phi(f) + P_\Phi(\bar{\partial}f) + P_\Phi(\bar{\partial}_\Phi^* f)$, respectively.*

Proof.   See Andreotti-Vesentini [1], Proposition 5.

Since $\bar{\partial}$ and $\bar{\partial}_\Phi^*$ are closed operators we obtain

Corollary 1.3. *Under the same assumptions as above, $D_\Phi^{p,q}$, $D_{*\Phi}^{p,q}$, and $D_\Phi^{p,q} \cap D_{*\Phi}^{p,q}$ are the completions of $C_0^{p,q}(X)$ with respect to the graph norms.*

Corollary 1.3 is not used in this paper.

From now on we assume that X has a complete Kähler metric ds^2 of class C^2, there exists a bounded strictly plurisubharmonic function Φ on X, and λ_i defined above with respect to ds^2 and Φ are $\geqq 1$. Then P and P_Φ are epuivalent norms on $C_0^{p,q}(X)$ so that the completions of $C_0^{p,q}(X)$ with respect to P and P_Φ agree.

Proposition 1.4. *Let X, ds^2, and Φ be as above. Then we have the following inequality:*

(†) $$C_f\{P_\Phi(\bar\partial u) + P_\Phi(\bar\partial_\Phi^* u)\} \geqq |(f, u)_\Phi|^2.$$

Here $f \in L_2^{n,q}(X)$, $q \geqq 1$, $u \in D_\Phi^{n,q} \cap D_{*\Phi}^{n,q}$, and C_f is a continuous function of $Q(f)$.

Proof. In virtue of Proposition 1.2, it is sufficient to show (†) assuming $f, u \in C_0^{n,q}(X)$. Let Λ be the operator defined by

(16) $$(\Omega \wedge u, v) = (u, \Lambda v),$$

where $u \in C_0^{p,q}(X)$, $v \in C_0^{p+1,q+1}(X)$ and Ω denotes the $(1, 1)$ form associated to ds^2. The following formula is first due to Calabi-Vesentini [2];

(17) $$(\bar\partial\bar\partial_\Phi^* + \bar\partial_\Phi^*\bar\partial)u - *^{-1}(\bar\partial\bar\partial_\Phi^* + \bar\partial_\Phi^*\bar\partial)*u = (\Theta\Lambda - \Lambda\Theta)u \quad \text{for} \quad u \in C_0^{p,q}(X)$$

where Θ denotes the left multiplication by ω. From this equality it follows that

(18) $$P_\Phi(\bar\partial u) + P_\Phi(\bar\partial_\Phi^* u) \geqq (\Theta\Lambda u, u)_\Phi \quad \text{for} \quad u \in C_0^{n,q}(X).$$

So it remains to show that

(19) $$C_f(\Theta\Lambda u, u)_\Phi \geqq |(f, u)_\Phi|^2 \quad \text{for} \quad f, u \in C_0^{n,q}(X) \quad q \geqq 1.$$

For this purpose it is enough to show that

(20) $$q|(f, u)_\Phi|^2 \leqq (\Lambda^{-1}\Theta^{-1}f, f)_\Phi (\Theta\Lambda u, u)_\Phi$$

and

(21) $$(\Lambda^{-1}\Theta^{-1}f, f)_\Phi \leqq qCQ(f)^2$$

for $f, u \in C_0^{n,q}(X)$, $q \geqq 1$, where $C := \sup_{x \in X} \exp(-\Phi(x))$ and we denote by Θ^{-1} and Λ^{-1} the inverses of $\Theta : C_0^{n-1,q-1}(X) \to C_0^{n,q}(X)$ and $\Lambda : C_0^{n,q}(X) \to C_0^{n-1,q-1}(X)$, respectively.

First we prove (21).   Letting

$$(22) \qquad f = \sum_{i_1 < \cdots < i_q} f_{i_1 \cdots i_q} \sigma_1 \wedge \cdots \wedge \sigma_n \wedge \bar{\sigma}_{i_1} \wedge \cdots \wedge \bar{\sigma}_{i_q}$$

we have

$$(23) \qquad \Lambda^{-1} \Theta^{-1} f = \sum_{i_1 < \cdots < i_q} \left(\sum_{k=1}^{q} \lambda_{i_k}^2 \right) f_{i_1 \cdots i_q} \sigma_1 \wedge \cdots \wedge \sigma_n \wedge \bar{\sigma}_{i_1} \wedge \cdots \wedge \bar{\sigma}_{i_q}$$

and

$$(24) \qquad \bar{*} f = K \sum_{i_1 < \cdots < i_q} \mathrm{sgn} \begin{pmatrix} 1 & 2 \cdots\cdots\cdots n \\ i_1 \cdots i_q j_1 \cdots j_{n-q} \end{pmatrix} \frac{\overline{f_{i_1 \cdots i_q}}}{\lambda_{i_1}^2 \cdots \lambda_{i_q}^2} \bar{\sigma}_{j_1} \wedge \cdots \wedge \bar{\sigma}_{j_{n-q}}.$$

Thus we have

$$(25) \quad \exp(-\Phi) \Lambda^{-1} \Theta^{-1} f \wedge \bar{*} f$$

$$= K \exp(-\Phi) \sum_{i_1 < \cdots < i_q} \frac{\sum\limits_{k=1}^{q} \lambda_{i_k}^2}{\lambda_{i_1}^2 \cdots \lambda_{i_q}^2} |f_{i_1 \cdots i_q}|^2 \sigma_1 \wedge \cdots \wedge \sigma_n \wedge \bar{\sigma}_1 \wedge \cdots \wedge \bar{\sigma}_n$$

$$\leq q K \exp(-\Phi) \sum_{i_1 < \cdots < i_q} |f_{i_1 \cdots i_q}|^2 \sigma_1 \wedge \cdots \wedge \sigma_n \wedge \bar{\sigma}_1 \wedge \cdots \wedge \bar{\sigma}_n$$

$$= q \exp(-\Phi) f \wedge \overline{\bar{\star}} f.$$

Therefore

$$(26) \qquad (\Lambda^{-1} \Theta^{-1} f, f)_{\Phi} \leq q C \int_X f \wedge \overline{\bar{\star}} f = q C Q(f)^2.$$

Now we prove (20).   In the proof of (21) we showed that

$$(27) \qquad \Lambda^{-1} \Theta^{-1} f = \sum_{i_1 < \cdots < i_q} \left(\sum_{k=1}^{q} \lambda_{i_k}^2 \right) f_{i_1 \cdots i_q} \sigma_1 \wedge \cdots \wedge \sigma_n \wedge \bar{\sigma}_{i_1} \wedge \cdots \wedge \bar{\sigma}_{i_q}$$

$$= \sum_{i_1 < \cdots < i_q} \left(\sum_{k=1}^{q} \frac{\lambda_{i_k}^2}{\lambda_1 \cdots \lambda_n \lambda_{i_1} \cdots \lambda_{i_q}} \right) f_{i_1 \cdots i_k} s_1 \wedge \cdots \wedge s_n \wedge \bar{s}_{i_1} \wedge \cdots \wedge \bar{s}_{i_q}$$

and

$$(28) \quad \bar{*} f = K \sum_{j_1 < \cdots < j_{n-q}} \mathrm{sgn} \begin{pmatrix} 1 & 2 \cdots\cdots\cdots n \\ i_1 \cdots i_q j_1 \cdots j_{n-q} \end{pmatrix} \frac{\overline{f_{i_1 \cdots i_q}}}{\lambda_1 \cdots \lambda_n \lambda_{i_1} \cdots \lambda_{i_q}} \bar{s}_{j_1} \wedge \cdots \wedge \bar{s}_{j_{n-q}}.$$

Hence if we let

$$(29) \qquad \tilde{f} = \sum_{i_1 < \cdots < i_q} \tilde{f}_{i_1 \cdots i_q} s_1 \wedge \cdots \wedge s_n \wedge \bar{s}_{i_1} \wedge \cdots \wedge \bar{s}_{i_q}$$

$$:= \sum_{i_1 < \cdots < i_q} \sqrt{\sum_{k=1}^{q} \lambda_{i_k}^2} \frac{f_{i_1 \cdots i_q}}{\lambda_1 \cdots \lambda_n \lambda_{i_1} \cdots \lambda_{i_q}} s_i \wedge \cdots \wedge s_n \wedge \bar{s}_{i_1} \wedge \cdots \wedge \bar{s}_{i_q}$$

we have

$$(30) \qquad (\Lambda^{-1} \Theta^{-1} f, f)_{\Phi} = (\tilde{f}, \tilde{f})_{\Phi}.$$

As for $u = \Sigma u_{i_1 \cdots i_q} s_1 \wedge \cdots \wedge s_n \wedge \bar{s}_{i_1} \wedge \cdots \wedge \bar{s}_{i_q}$ we have

$$(31) \qquad \Theta \Lambda u = \sum_{i_1 < \cdots < i_q} \sum_{k=1}^{q} \frac{u_{i_1 \cdots i_q}}{\lambda_{i_k}^2} s_1 \wedge \cdots \wedge s_n \wedge \bar{s}_{i_1} \wedge \cdots \wedge \bar{s}_{i_q},$$

Hence if we let

$$(32) \qquad \tilde{u} = \sum_{i_1 < \cdots < i_q} \tilde{u}_{i_1 \cdots i_q} s_1 \wedge \cdots \wedge s_n \wedge \bar{s}_{i_1} \wedge \cdots \wedge \bar{s}_{i_q}$$

$$:= \sum_{i_1 < \cdots < i_q} \sqrt{\sum_{k=1}^{q} \frac{1}{\lambda_{i_k}^2}}\, u_{i_1 \cdots i_q} s_1 \wedge \cdots \wedge s_n \wedge \bar{s}_{i_1} \wedge \cdots \wedge \bar{s}_{i_q}.$$

we have

$$(33) \qquad (\Theta \Lambda u, u)_\Phi = (\tilde{u}, \tilde{u})_\Phi.$$

On the other hand we have

$$(34) \qquad \exp(-\Phi) u \wedge \bar{*} f$$

$$= K \exp(\Phi) \sum_{i_1 < \cdots < i_q} u_{i_1 \cdots i_q} \overline{f'_{i_1 \cdots i_q}} s_1 \wedge \cdots \wedge s_n \wedge \bar{s}_1 \wedge \cdots \wedge s_n,$$

where

$$(35) \qquad f'_{i_1 \cdots i_q} = \frac{1}{\lambda_1 \cdots \lambda_n \lambda_{i_1} \cdots \lambda_{i_q}} f_{i_1 \cdots i_q}.$$

By (29), (32) and (35) we have

$$(36) \qquad |u_{i_1 \cdots i_q} \overline{f'_{i_1 \cdots i_q}}|$$

$$= \frac{1}{\sqrt{\sum_{k=1}^{q} \lambda_{i_k}^2} \sqrt{\sum_{k=1}^{q} \lambda_{i_k}^{-2}}} |\tilde{u}_{i_1 \cdots i_q} \tilde{f}_{i_1 \cdots i_q}|$$

$$\leqq \frac{1}{q} |\tilde{u}_{i_1 \cdots i_q} \tilde{f}_{i_1 \cdots i_q}|.$$

Therefore by Cauchy-Schwarz inequality we have

$$(37) \qquad \left| \int_U \exp(-\Phi) u \wedge \bar{*} f \right|^2$$

$$\leqq \frac{1}{q} \int_U \exp(-\Phi) \tilde{u} \wedge \bar{*} \tilde{u} \int_U \exp(-\Phi) \tilde{f} \wedge \bar{*} \tilde{f},$$

where U is a sufficiently small neighbourhood of x_0. Since x_0 was arbitrary we obtain

$$(38) \qquad q |(u, f)_\Phi|^2 \leqq (\tilde{u}, \tilde{u})_\Phi (\tilde{f}, \tilde{f})_\Phi = (\Theta \Lambda u, u)_\Phi (\Lambda^{-1} \Theta^{-1} f, f)_\Phi.$$

Thus we obtain (20). $\hfill$ Q.E.D.

Combining Propositon 1.4 with a well known theorem (cf. e.g. [5, Theorem 1.1.4]) we obtain the following theorem (see also [6]).

Theorem 1.5. *Let X, ds^2 and Φ be as above. Let $f \in L_2^{n,q}(X)$, $q \geqq 1$. Then there exists a $g \in L_1^{n,q-1}(X)$ satisfying $\bar{\partial}g = f$ if and only if $\bar{\partial}f = 0$.*

Since $L_1^{n,0}(X) = L_2^{n,0}(X)$, we have

Corollary 1.6. *Let X, ds^2, and Φ be as above and let $f \in L_2^{n,1}(X)$, then there exists a $g \in L_2^{n,0}(X)$ such that $\bar{\partial}g = f$ if and only if $\bar{\partial}f = 0$.*

Corollary 1.7. *Let X be a complex manifold of dimension n with a C^2 complete Kähler metric, and Φ a bounded strictly plurisubharmonic function on X. Assume that Φ is of class C^4. Then, for any $f \in L_2^{n,1}(X)$ with $\bar{\partial}f = 0$, there exists a $g \in L_2^{n,0}(X)$ such that $f = \bar{\partial}g$.*

Proof. Let ds^2 be the given complete Kähler metric on X. We set

$$d\tilde{s}^2 := ds^2 + d\sigma^2.$$

Then the eigenvalues of $d\tilde{s}^2$ with respect to $d\sigma^2$ is larger than 1. Thus the hypothesis of Theorem 1.5 is satisfied.

§2. Complete Kähler Domains with C^1-Boundary

Let M be a complex manifold of dimension n. Let X be a domain, i.e., a connected open set in M. X is called a domain with C^1-boundary if there exists a real valued continuously differentiable function ρ defined on a neighbourhood $\tilde{V}$ of the boundary ∂X of X such that $\tilde{V} \cap X = \{x \in \tilde{V}; \rho(x) < 0\}$ and grad ρ is nonzero everywhere on ∂X. We call ρ a defining function of ∂X. X is called a complete Kähler domain if there exists a complete Kähler metric on X.

Example. Let $\mathbf{C}^n$ be the complex number space of dimension n. We let

$$(40) \qquad B(r) = \{(x_1, \ldots, x_n) \in \mathbf{C}^n; \sum_{i=1}^{n} |x_i|^2 < r\}, \qquad r > 0.$$

Then $B(r)$ is a complete Kähler domain with C^1-boundary (in $\mathbf{C}^n$). As a complete Kähler metric on $B(r)$ we may choose

$$(41) \qquad ds_r^2 = \sum_{\alpha,\beta} \frac{\partial^2 \eta(z)}{\partial z^\alpha \partial \bar{z}^\beta} dz^\alpha \cdot d\bar{z}^\beta,$$

where $z = (z^1, \ldots, z^n)$ and

$$(42) \qquad \eta(z) = \frac{1}{r - \sum_{\alpha=1}^{n} |z^\alpha|^2}.$$

More generally holomorphically convex domains in Stein manifolds are complete Kähler domains. The following lemma is obvious.

Lemma 2.1. *Let X_1 and X_2 be complete Kähler domains with C^1-boundary in M. Then $X_1 \cap X_2$ is a complete Kähler domain with C^1-boundary in X_1.*

Definition 2.2. A domain X in a complex manifold M is called *pseudoconvex* if for every $x_0 \in \partial X$ there exists a neighbourhood U of x_0 in M such that $U \cap X$ is a Stein manifold.

The following theorem is the main theorem of this section.

Theorem 2.3. *Let X be a complete Kähler domain with C^1-boundary in a complex manifold M. Then X is pseudoconvex.*

Proof. Let X be a complete Kähler domain with C^1-boundary in M and $x_0 \in \partial X$. Let W be a coordinate neighbourhood of x_0 with coordinate $(z^1, \ldots, z^n)$ such that z^i are holomorphic on a neighbourhood of the closure of W and

$$(43) \qquad W = \{ x \in M ; \sum_{i=1}^{n} |z^i(x)|^2 < 1 \}.$$

After a linear change of coordinates we may assume that if we let $W_\gamma := \{ x \in W ; z^n(x) = \gamma \}$ ($|\gamma| < 1/2$), $X \cap W_\gamma$ is a domain with C^1-boundary in W_γ. By Lemma 2.1, $X \cap W$ and $X \cap W_\gamma$ are complete Kähler domains with C^1-boundary in W and W_γ, respectively. We let

$$(44) \qquad B(x_0, r) = \{ x \in W ; \sum_{i=1}^{n} |z^i(x)|^2 < r \} \qquad \text{for} \quad r < 1.$$

Let ρ be a defining function of ∂X. We choose $(\xi^1, \ldots, \xi^n, \eta^1, \ldots, \eta^n)$, $\xi^i, \eta^i \in \mathbb{R}$, $i = 1, 2, \ldots, n$ satisfying

$$(45) \qquad \sum_{i=1}^{n} |\xi^i|^2 + \sum_{i=1}^{n} |\eta^i|^2 = 1$$

and

$$(46) \qquad \sum_{i=1}^{n} \frac{\partial \rho}{\partial u^i} \xi^i + \sum_{i=1}^{n} \frac{\partial \rho}{\partial v^i} \eta^i > 0 \qquad \text{at} \quad x_0,$$

where we set $z^i = u^i + \sqrt{-1}\, v^i$. Choosing W smaller if necessary we may assume that ρ is defined on W. For $r < 1$ and $\varepsilon < 1 - r$ we define

$$(47) \qquad X_r^\varepsilon = \{ x \in B(x_0, r) ; \rho(u(x) - \varepsilon \xi, v(x) - \varepsilon \eta) < 0 \},$$

where $u(x) = u = (u^1, \ldots, u^n)$, $v(x) = v = (v^1, \ldots, v^n)$ and so on. X_r^ε is the domain

in $B(x_0, r)$ obtained by shifting ∂X by ε in the direction of (ξ, η). It is clear that for sufficiently small r there exists $\varepsilon_0 > 0$ such that for any $\varepsilon < \varepsilon_0$ and $|\gamma| < r$, X_r^ε and $X_r^\varepsilon \cap W_\gamma$ are complete Kähler domains with C^1-boundary in $B(x_0, r)$ and $B(x_0, r) \cap W_\gamma$, respectively. Moreover by (46) we may assume that for any $\varepsilon < \varepsilon_0$ the closures of $X \cap B(x_0, r)$ and $X \cap B(x_0, r) \cap W_\gamma$ in $B(x_0, r)$ are contained in X_r^ε and $X_r^\varepsilon \cap W_\gamma$, respectively. We fix such r and ε_0.

Now we prove the theorem by induction on the dimension n of X. The case $n = 1$ being well known we assume that $n \geq 2$ and the theorem is valid for $k < n$. We are going to prove that $X \cap B(x_0, r/2)$ is a Stein manifold. For this purpose it suffices to prove the following assertion (see [6] Theorem 2.6.5 and Theorem 5.4.6).

Assertion: There exists no $x_1 \in \partial(X \cap B(x_0, r/2))$ satisfying the property that there is a neighbourhood V of x_1 such that for any holomorphic function f on $X \cap B(x_0, r/2)$ there is a holomorphic function $\tilde{f}$ on V which coincides with f on $V \cap X \cap B(x_0, r/2)$.

We prove this assertion by contradiction. Suppose that there exists such $x_1 \in \partial(X \cap B(x_0, r/2))$ and V. Since $B(x_0, r/2)$ is a Stein manifold $x_1 \in \partial X \cap B(x_0, r/2)$. We fix γ in such a way that $W_\gamma \cap V - X \neq \emptyset$. There exists $\varepsilon_1 > 0$ such that $\varepsilon_0 > \varepsilon_1$ and $\partial(X_r^{\varepsilon_1} \cap W_\gamma) \cap V \neq \emptyset$. Since by the induction hypothesis $X_r^{\varepsilon_1} \cap W_\gamma$ is pseudoconvex in $B(x_0, r) \cap W_\gamma$, it follows that $X_r^{\varepsilon_1} \cap W_\gamma$ is Stein (see [6] Theorem 5.4.6). Hence there exists a holomorphic function f on $X_r^{\varepsilon_1} \cap W_\gamma$ such that

$$(48) \qquad \sup_{x \in V \cap X_r^{\varepsilon_1} \cap W_\gamma} |f(x)| = \infty .$$

We define a holomorphic map $\pi : B(x_0, r/2) \to W_\gamma$ by

$$(49) \qquad \pi(z^1, \ldots, z^n) = (z^1, \ldots, z^{n-1}, \gamma) .$$

Since the closure of $X \cap B(x_0, r)$ in $B(x_0, r)$ is contained in $X_r^{\varepsilon_1}$, we can define a positive number δ_0 as follows. If

$$(50) \qquad X \cap B\left(x_0, \frac{r}{2}\right) \subset \pi^{-1}(X_r^{\varepsilon_1} \cap W_\gamma)$$

we let $\delta_0 := 1$. Otherwise we let

$$(51) \qquad \delta_0 := \inf_{x \in X \cap B(x_0, r/2) - \pi^{-1}(X_r^\varepsilon \cap W_\delta)} |z^n(x) - \gamma| > 0 .$$

We fix $\delta > 0$ so that

(52) $\delta < \delta_0$.

Let χ be an infinitely differentiable function from $\mathbb{C}$ to $\mathbb{R}$ such that $\chi(w) = 0$ if $|w| \geq \delta$, $\chi(w) = 1$ if $|w| \leq \delta/2$ and $0 \leq \chi(w) \leq 1$. We define a function h on $X \cap B(x_0, r/2)$ by

(53) $h(x) = \chi(z^n(x) - \gamma) \cdot f(\pi(z^1(x), \ldots, z^n(x)))$ if $|z^n(x) - \gamma| < \delta_0$

and

(54) $h(x) = 0$ otherwise .

Then

(55) $\bar{\partial} h \wedge \dfrac{dz^1 \wedge \cdots \wedge dz^n}{z^n - \gamma}$

is a $\bar{\partial}$ closed $(n, 1)$-form on $X \cap B(x_0, r/2)$ which is square integrable with respect to the metric

(56) $d\sigma^2 = \sum_{\alpha, \beta} \dfrac{\partial^2(\sum_{i=1}^n |z^i|^2)}{\partial z^\alpha \partial \bar{z}^\beta} dz^\alpha \cdot d\bar{z}^\beta$.

Since $X \cap B(x_0, r/2)$ is a relatively compact complete Kähler domain in W, by Corollary 1.7 there exists a square integrable $(n, 0)$ form g on $X \cap B(x_0, r/2)$ such that

(57) $\bar{\partial} g = \bar{\partial} h \wedge \dfrac{dz^1 \wedge \cdots \wedge dz^n}{z^n - \gamma}$.

Hence letting $g = g^1 dz^1 \wedge \cdots \wedge dz^n$ we get

(58) $\bar{\partial}(h - (z^n - \gamma)g^1) = 0$.

Therefore $h - (z^n - \gamma)g^1$ is a holomorphic function on $X \cap B(x_0, r/2)$ and it coincides with f on $W_\gamma \cap X \cap B(x_0, r/2)$. Note that by the hypothesis x_1 is a point through which every holomorphic function on $X \cap B(x_0, r/2)$ can be continued analytically to a holomorphic function on V. But this contradicts the fact that $h - (z^n - \gamma)g^1$ is holomorphic on $X \cap B(x_0, r/2)$ and, by (48) combined with the theorem of identity, cannot be continued analytically across $\partial(X_r^{\varrho_1} \cap W_\gamma) \cap V$.

<div align="right">Q. E. D.</div>

By the theorem of Docquier-Grauert [3] we have

Corollary 2.4. *Let M be a Stein manifold and X be a domain with C^1-boundary in M. Then X is a Stein manifold if and only if X has a complete Kähler metric.*

The corollary generalizes Satz C of Grauert [4].

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Added in proof: The author has obtained in a forthcoming paper 'Analyticity of complements of complete Kähler domains' that if X is a domain in a complex manifold M and M-X is a regular C^1-submanifold of real codimension 2, then X is pseudoconvex (equivalently, M-X is a complex submanifold of M).