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The Theory of Vector Valued Fourier Hyperfunctions of Mixed Type. II

By

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Abstract

The soft resolution $(\mathcal{G}'_{(0,p)}, \bar{\partial})$ of the sheaf $\underset{\sim}{\mathcal{O}}_{k,l}$ of rapidly decreasing holomorphic functions of (k,l) type is constructed. Using the above resolution, we prove $H^n_K(V, {}^E\widetilde{\mathcal{O}}_{k,l}) \cong L(\underset{\sim}{\mathcal{O}}_{k,l}(K), E)$.

§ 1. Introduction

In the first part of the present paper (S. Nagamachi [4]), which will be referred to as [I], we defined the mixed type Fourier hyperfunctions which take values in a Fréchet space E. The purpose of this second part is to prove that the space $H^n_K(V, {}^E\widetilde{\mathcal{O}}_{k,l})$ of E-valued Fourier hyperfunctions with support contained in a compact set K is isomorphic to the space $L(\underset{\sim}{\mathcal{O}}_{k,l}(K), E)$ of continuous linear mappings of $\underset{\sim}{\mathcal{O}}_{k,l}(K)$ into E. We proved this theorem in [I] only for $E=\mathbb{C}$ (Theorem 5. 13 of [I]).

In Section 2, we study the Fourier transformation for slowly increasing C^∞ functions and rapidly decreasing distributions. In Section 3, we prepare the theory of cohomology with bounds in an appropriate form.

In Section 4, we construct a soft resolution of the sheaf $\underset{\sim}{\mathcal{O}}_{k,l}$ of rapidly decreasing holomorphic functions (Theorem 4. 9),

$$0 \to \underset{\sim}{\mathcal{O}}_{k,l} \to \mathcal{G}'_{(0,0)} \xrightarrow{\bar{\partial}} \cdots \xrightarrow{\bar{\partial}} \mathcal{G}'_{(0,n)} \to 0 ,$$

where $\mathcal{G}'_{(0,p)}$ is the sheaf subordinate to the presheaf $\{\mathcal{G}'_{(0,p)}(\Omega)\}$ of $(0,p)$-forms whose coefficients are rapidly decreasing distributions in Ω

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(Definition 4. 1). To do this, we use the method similar to that developed in 7. 6 of L. Hörmander [1], that is, the duality arguments, using the property of the Fourier transformation (Propositions in § 2) and the estimate of the solutions of certain system of linear equations with polynomial coefficients (Proposition 3. 5, which is an extension of Theorem 7. 6. 11 of L. Hörmander [1]).

Using this method, we construct also the following resolution of $\tilde{\mathcal{O}}_{k,l}$, on $\boldsymbol{Q}^{k,l}$:

$$0 \to \tilde{\mathcal{O}}_{k,l} \to \mathcal{F}_{(0,0)} \xrightarrow{\bar{\partial}} \cdots \xrightarrow{\bar{\partial}} \mathcal{F}_{(0,n)} \to 0\,,$$

which is an extension of Theorem 4. 11 of [I], where the resolution has been obtained only on the open subset Ω of $\boldsymbol{Q}^{k,l}$ satisfying a certain condition.

In Section 5, we prove $H_K^n(V, {}^E\tilde{\mathcal{O}}_{k,l}) \cong L(\underset{\sim}{\mathcal{Q}}_{k,l}(K), E)$ (Theorem 5.7) using the Serre-Komatsu duality theorem and properties of tensor products of E with nuclear Fréchet spaces.

We continue to use the same notions and notations as those in [I].

§ 2. Function Spaces

In this section we study the Fourier transformation for slowly increasing C^∞ functions and rapidly decreasing distributions.

Definition 2. 1. Let K be the closure of $\prod_{i=1}^{j} (\Gamma_i \times B_i)$ in $\boldsymbol{Q}^{k,l}$, where Γ_i is the strictly convex closed cone in $\boldsymbol{R}^{2k_i+l_i}$ whose vertex is at the origin and B_i is the closed ball in $\boldsymbol{R}^{l_i}$ whose center is at the origin.

In this section we always denote by K the compact set defined in Definition 2. 1.

We identify $\boldsymbol{C}^n$ with $\boldsymbol{R}^{2n}$ and denote by $\langle x, \eta \rangle$ the inner product in $\boldsymbol{R}^{2n}$, i.e., $\langle x, \eta \rangle = \sum_{i=1}^{2n} x_i \eta_i$.

Definition 2. 2. Let $h_{K,\varepsilon}(\eta) = \sup_{x \in K \cap \boldsymbol{R}^{2n}} (-\langle x, \eta \rangle + \varepsilon |x|)$. Define $K_\varepsilon^\circ = \{\eta \in \boldsymbol{R}^{2n}; h_{K,\varepsilon}(\eta) < \infty\}$ and $K^\circ = \prod_{i=1}^{j} (\Gamma_i^\circ \times \boldsymbol{R}^{l_i})$, where $\Gamma_i^\circ =$

$\{\eta\in\mathbf{R}^{2k_i+l_i};\ \langle x,\eta\rangle>0$ for all $0\neq x\in\Gamma_i\}$.

Proposition 2. 3. $K^\circ=\cup_{\varepsilon>0}K^\circ_\varepsilon$.

Proof. Let $\eta_i=(\alpha_i,\beta_i)\in\Gamma^\circ_i\times\mathbf{R}^{l_i}$. Then

$$\begin{aligned}h_{K,\varepsilon}(\eta)&=\sup_{x\in K\cap\mathbf{R}^{2n}}(-\langle x,\eta\rangle+\varepsilon|x|)\\&=\sum_{i=1}^{j}\sup_{x_i\in\Gamma_i}(-\langle x_i,\alpha_i\rangle+\varepsilon|x_i|)+\sum_{i=1}^{j}\sup_{y_i\in B_i}(-\langle y_i,\beta_i\rangle+\varepsilon|y_i|)\\&=\sum_{i=1}^{j}h_{\Gamma_i,\varepsilon}(\alpha_i)+\sum_{i=1}^{j}h_{B_i,\varepsilon}(\beta_i).\end{aligned}$$

$h_{K,\varepsilon}(\eta)<\infty$ implies that $h_{\Gamma_i,\varepsilon}(\alpha_i)<\infty$ for all i and this shows that $\langle x_i,\alpha_i\rangle>0$ for $0\neq x_i\in\Gamma_i$ because if $\langle x_i,\alpha_i\rangle\leqq 0$ for some $0\neq x_i\in\Gamma_i$, then $-\langle tx_i,\alpha_i\rangle+\varepsilon|tx_i|$ tends to infinity as $t\to\infty$, this is a contradiction. Thus we have $K^\circ\supset K^\circ_\varepsilon$ and $K^\circ\supset\cup_{\varepsilon>0}K^\circ_\varepsilon$. Conversely if $\eta\in K^\circ$, then let $\inf_{x_i\in\Gamma_i,|x_i|=1}\langle x_i,\alpha_i\rangle=\delta_i>0$ and choose $\varepsilon>0$ satisfying $\varepsilon<\delta_i$ for all i. Then we have $-\langle tx_i,\alpha_i\rangle+\varepsilon|tx_i|\leqq 0$ for $x_i\in\Gamma_i$, $|x_i|=1$ and $t\geqq 0$, consequently $h_{\Gamma_i,\varepsilon}(\alpha_i)\leqq 0$ for all i. Since $h_{B_i,\varepsilon}(\beta_i)<\infty$ for all i, $h_{K,\varepsilon}(\eta)=\sum_{i=1}^{j}h_{\Gamma_i,\varepsilon}(\alpha_i)+\sum_{i=1}^{j}h_{B_i,\varepsilon}(\beta_i)<\infty$. Thus we have $K^\circ\subset\cup_{\varepsilon>0}K^\circ_\varepsilon$.

Proposition 2. 4. *If* $\eta=((\alpha_1,\beta_1),\cdots,(\alpha_j,\beta_j))\in K^\circ_\varepsilon$, *then* $((t_1\alpha_1,s_1\beta_1),\cdots,(t_j\alpha_j,s_j\beta_j))\in K^\circ_\varepsilon$ *for* $t_i\geqq 1$ *and arbitrary real* s_i, $1\leqq i\leqq j$.

Proof. $\eta\in K^\circ_\varepsilon$ is equivalent to $h_{\Gamma_i,\varepsilon}(\alpha_i)<\infty$ for $1\leqq i\leqq j$. Since $h_{\Gamma_i,\varepsilon}(\alpha_i)=\sup_{x_i\in\Gamma_i,|x_i|=1,s\geqq 0}(-\langle x_i,\alpha_i\rangle+\varepsilon)s$, $h_{\Gamma_i,\varepsilon}(\alpha_i)<\infty$ is equivalent to $\inf_{x_i\in\Gamma_i,|x_i|=1}\langle x_i,\alpha_i\rangle\geqq\varepsilon$. $\inf_{x_i\in\Gamma_i,|x_i|=1}\langle x_i,\alpha_i\rangle\geqq\varepsilon$ implies $\inf_{x_i\in\Gamma_i,|x_i|=1}\langle x_i,t_i\alpha_i\rangle\geqq\varepsilon$ for $t_i\geqq 1$. Thus we have $((t_1\alpha_1,s_1\beta_1),\cdots,(t_j\alpha_j,s_j\beta_j))\in K^\circ_\varepsilon$.

Corollary 2. 5. *Let* Int K°_ε *be the interior of* K°_ε. *If* $\eta=((\alpha_1,\beta_1),\cdots,(\alpha_j,\beta_j))\in\operatorname{Int}K^\circ_\varepsilon$, *then for* $t_i\geqq 1$ *and arbitrary real* s_i, $1\leqq i\leqq j$, $\eta(t,s)=((t_1\alpha_1,s_1\beta_1),\cdots,(t_j\alpha_j,s_j\beta_j))\in\operatorname{Int}K^\circ_\varepsilon$.

Proof. If $\eta\in\operatorname{Int}K^\circ_\varepsilon$, then there exists a neighbourhood V of zero such that $\eta+V\subset K^\circ_\varepsilon$. By Proposition 2. 4 we have $\eta(t,s)+V(t,1)\subset K^\circ_\varepsilon$, where $V(t,1)=\{\xi(t,1);\xi\in V\}$ is a neighbourhood of zero. Thus we have $\eta(t, s) \in \operatorname{Int} K_\varepsilon^\circ$.

Proposition 2.6. *Let $0<\delta<\varepsilon$. Then K_ε° is strictly contained in K_δ°, that is, the distance between K_ε° and the complement $(K_\delta^\circ)^c$ of K_δ° is positive. Therefore $K_\varepsilon^\circ \subset \operatorname{Int} K_\delta^\circ$.*

Proof. Let $\eta \in K_\varepsilon^\circ$ and $e \in \boldsymbol{C}^n$ with $|e|<\varepsilon-\delta$. Since $\eta \in K_\varepsilon^\circ$ is equivalent to $\inf_{x_i \in \Gamma_i, |x_i|=1} \langle x_i, \alpha_i\rangle \geqq \varepsilon$ for $i=1, \cdots, j$,

$$\inf_{x_i \in \Gamma_i, |x_i|=1} \langle x_i, \alpha_i+e_i\rangle \geqq \inf_{x_i \in \Gamma_i, |x_i|=1} \langle x_i, \alpha_i\rangle - \sup_{x_i \in \Gamma_i, |x_i|=1} \langle x_i, e_i\rangle$$
$$\geqq \varepsilon-(\varepsilon-\delta)=\delta.$$

Thus we have shown that $\eta+e \in K_\delta^\circ$ for all $\eta \in K_\varepsilon^\circ$ and $|e|<\varepsilon-\delta$. This shows that K_ε° is strictly contained in K_δ°.

Proposition 2.7. *Let f be a C^N function with support contained in $K \cap \boldsymbol{R}^{2n}$. Suppose there exist positive constants δ and C such that $|D^\alpha f(x)| \leqq C e^{\delta|x|}$ for all $|\alpha| \leqq N$. Define*

$$\tilde{f}(\zeta)=(2\pi)^{-n} \int_{\boldsymbol{R}^{2n}} e^{i\langle x, \zeta\rangle} f(x)\, dx .$$

Then $\tilde{f}(\zeta)$ is an analytic function defined in $\{\zeta \in \boldsymbol{C}^{2n}, \operatorname{Im} \zeta \in \operatorname{Int} K_\varepsilon^\circ\}$ for any $\varepsilon>\delta$, and satisfies

$$|\tilde{f}(\zeta)| \leqq C_\varepsilon' e^{h_{K,\varepsilon}(\operatorname{Im}\zeta)} /(1+|\zeta|)^N \tag{2.1}$$

for some constant $C_\varepsilon'>0$ and $\operatorname{Im} \zeta \in K_\varepsilon^\circ$.

Proof. Let $\operatorname{Im} \zeta \in K_\varepsilon^\circ$. The inequalities

$$|\zeta^\alpha \tilde{f}(\zeta)| \leqq (2\pi)^{-n} \int_{\boldsymbol{R}^{2n}} e^{(-\langle x, \operatorname{Im}\zeta\rangle+\varepsilon|x|)} e^{-\varepsilon|x|} |D^\alpha f(x)|\, dx$$
$$\leqq C'' e^{h_{K,\varepsilon}(\operatorname{Im}\zeta)}$$

imply that $\tilde{f}(\zeta)$ is analytic in $\operatorname{Im} \zeta \in \operatorname{Int} K_\varepsilon^\circ$ and satisfies (2.1) for $\operatorname{Im} \zeta \in K_\varepsilon^\circ$.

Corollary 2.8. *Let $f \in \mathfrak{F}_c(K)$ (Definition 2.14 of [I]), then $\tilde{f}(\zeta)$ is an analytic function defined in $\{\zeta \in \boldsymbol{C}^{2n}; \operatorname{Im}\zeta \in K^\circ\}$ and satisfies*

$$|\tilde{f}(\zeta)| \leqq C_{N,\varepsilon} e^{h_{K,\varepsilon}(\operatorname{Im}\zeta)} / (1+|\zeta|)^N$$

in $\operatorname{Im}\zeta \in K_\varepsilon^\circ$ *for any* $\varepsilon > 0$ *and* $N > 0$, *where* $C_{N,\varepsilon}$ *is a positive number independent of* ζ.

Proof. The corollary follows from Propositions 2.3, 2.6 and 2.7.

Proposition 2.9. *Let* K *be the set defined in Definition* 2.1. *For any* $0 < \varepsilon \leqq 1$, *there exists an* $\eta_\varepsilon \in \operatorname{Int} K_\varepsilon^\circ$ satisfying $|\eta_\varepsilon| \leqq A\varepsilon$ *for some positive constant* A *not depending on* ε.

Proof. Let $\eta \in K_2^\circ$ and $A = |\eta|$. Define $\eta_\varepsilon = \varepsilon\eta$ for $0 < \varepsilon \leqq 1$, then $|\eta_\varepsilon| = A\varepsilon$ and

$$h_{K,2\varepsilon}(\eta_\varepsilon) = \sup_{x \in K \cap \mathbf{R}^{2n}} (-\langle x, \varepsilon\eta\rangle + 2\varepsilon|x|) = \varepsilon h_{K,2}(\eta) < \infty .$$

This shows that $\eta_\varepsilon \in K_{2\varepsilon}^\circ \subset \operatorname{Int} K_\varepsilon^\circ$.

Proposition 2.10. *Let* $N \geqq 3n$, *and let* $g(\zeta)$ *be an analytic function in* $\{\zeta \in \mathbf{C}^{2n}; \operatorname{Im}\zeta \in \operatorname{Int} K_\varepsilon^\circ\}$ *which satisfies*

$$|g(\zeta)| \leqq C \frac{1}{(1+|\zeta|)^N} e^{h_{K,\varepsilon}(\operatorname{Im}\zeta)}$$

for $\operatorname{Im}\zeta \in K_\varepsilon^\circ$. *If we define*

$$\hat{g}(x) = (2\pi)^{-n} \int_{\mathbf{R}^{2n}+i\eta} e^{i\langle x,\zeta\rangle} g(\zeta)\, d\zeta \quad \text{for } \eta \in \operatorname{Int} K_\varepsilon^\circ , \tag{2.2}$$

$\hat{g}(x)$ *is a* C^{N-3n} *function with support contained in* $K \cap \mathbf{R}^{2n}$, *satisfying* $|D^\alpha \hat{g}(x)| < M e^{\delta|x|}$ *for some constant* M *and* $\delta = A\varepsilon$, *where* A *is the constant appeared in Proposition* 2.9.

Proof. The inequalities

$$|\hat{g}(x)| = |(2\pi)^{-n} \int_{\mathbf{R}^{2n}} e^{-i\langle x,\xi\rangle} e^{\langle x,\eta\rangle} g(\xi + i\eta)\, d\xi|$$

$$\leqq C' e^{\langle x,\eta\rangle} \int |g(\xi + i\eta)|\, d\xi$$

$$\leqq C e^{\langle x,\eta\rangle} e^{h_{K,\varepsilon}(\eta)}$$

hold for $\eta\in\text{Int}\,K_\varepsilon^\circ$. $x\notin K$ implies $x_l\notin\Gamma_l$ or $y_k\notin B_k$ for some l, k. Hence there exists $\alpha_l\in\Gamma_l^\circ$ such that $\langle x_l, \alpha_l\rangle<0$ or y_k satisfies $\langle y_k, sy_k\rangle >h_{B_k,\varepsilon}(sy_k)$ for large $s>0$. Since (2.2) is independent of $\eta\in\text{Int}\,K_\varepsilon^\circ$ by the Cauchy-Poincaré theorem, we have, for large $s>0$,

$$|\hat{g}(x)|\leqq C\exp(\langle x_l, t\alpha_l\rangle+\langle y_k, -sy_k\rangle+h_{B_k,\varepsilon}(-sy_k)+\sum_{i\neq l}\langle x_i, \alpha_i\rangle+\sum_{i\neq k}\langle y_i, \beta_i\rangle+\sum_{i\neq k}h_{B_i,\varepsilon}(\beta_i)\ , \tag{2.3}$$

where we have used the facts that $h_{\Gamma_i,\varepsilon}(\alpha_i)\leqq 0$ and $\eta=((\alpha_1, \beta_1), \cdots, (t\alpha_l, \beta_l), \cdots, (\alpha_k, sy_k), \cdots, (\alpha_j, \beta_j))\in\text{Int}\,K_\varepsilon^\circ$ for large t, s (Proposition 2.4). The right hand side of (2.3) vanishes as t or s tends to infinity. Thus we have $g(x)=0$ if $x\notin K$.

Let $|\alpha|\leqq N-3n$. The inequalities

$$\begin{aligned}|D^\alpha\hat{g}(x)| &= (2\pi)^{-n}\left|\int_{\boldsymbol{R}^{2n}} e^{-i\langle x,\xi\rangle}e^{\langle x,\eta\rangle}(-i\xi+\eta)^\alpha g(\xi+i\eta)\,d\xi\right| \\ &\leqq Ce^{\langle x,\eta\rangle}e^{h_{K,\varepsilon}(\eta)} \\ &\leqq Ce^{|x|\cdot|\eta|}e^{h_{K,\varepsilon}(\eta)}\end{aligned} \tag{2.4}$$

hold for $\eta=\eta_\varepsilon\in\text{Int}\,K_\varepsilon^\circ$ such that $|\eta_\varepsilon|\leqq\delta=A\varepsilon$ by Proposition 2.9. Hence

$$|D^\alpha\hat{g}(x)|\leqq M\,e^{\delta|x|} \tag{2.5}$$

holds for some constant $M>0$.

Proposition 2.11. *Let f be a C^N function satisfying the conditions in Proposition 2.7, then $\hat{\hat{f}}=f$.*

Proof. Let $\eta\in\text{Int}\,K_\varepsilon^\circ$, then $e^{-\langle y,\eta\rangle}f(y)$ is rapidly decreasing. Therefore we have

$$\begin{aligned}\hat{\hat{f}}(x) &= (2\pi)^{-2n}\int_{\boldsymbol{R}^{2n}+i\eta} e^{-i\langle x,\zeta\rangle}\left(\int_{\boldsymbol{R}^{2n}} e^{i\langle y,\zeta\rangle}f(y)\,dy\right)d\zeta \\ &= (2\pi)^{-2n}\int_{\boldsymbol{R}^{2n}} e^{-i\langle x,\xi\rangle}e^{\langle x,\eta\rangle}\left(\int_{\boldsymbol{R}^{2n}} e^{i\langle y,\xi\rangle}e^{-\langle y,\eta\rangle}f(y)\,dy\right)d\xi \\ &= f(x).\end{aligned}$$

Proposition 2.12. *Let $g(\zeta)$ be an analytic function satisfying the condition in Proposition 2.10. Then $\tilde{\hat{g}}=g$.*

Proof. Let $\zeta=\xi+i\eta$ and $\eta\in\operatorname{Int}K_{\varepsilon}^{\circ}$, then $g(x+i\eta)$ is integrable with respect to x. Therefore we have

$$\tilde{\tilde{g}}(\zeta)=(2\pi)^{-2n}\int_{\boldsymbol{R}^{2n}}e^{i\langle u,\zeta\rangle}\left(\int_{\boldsymbol{R}^{2n}+i\eta}e^{-i\langle u,z\rangle}g(z)\,dz\right)du$$

$$=(2\pi)^{-2n}\int_{\boldsymbol{R}^{2n}}e^{-\langle u,\eta\rangle}e^{i\langle u,\xi\rangle}\left(\int_{\boldsymbol{R}^{2n}}e^{-i\langle u,x\rangle}e^{\langle u,\eta\rangle}g(x+i\eta)\,dx\right)du$$

$$=g(\xi+i\eta)=g(\zeta).$$

Proposition 2.13. *Let $f\in\mathcal{F}_c(K)$, we define*

$$|f|_{N,\varepsilon}^{2}=\int_{\boldsymbol{R}^{2n}-iK_{\varepsilon}^{\circ}}|\tilde{f}(\zeta)|^{2}e^{-2h_{K,\varepsilon}(\operatorname{Im}\zeta)}(1+|\zeta|^{2})^{N}d\lambda \tag{2.6}$$

then there exists a seminorm $\|f\|_{M,\delta}=\sup_{x\in\boldsymbol{R}^{2n},|\alpha|\leqq M}|e^{-\delta|x|}D^{\alpha}f(x)|$ of $\mathcal{F}_c(K)$ such that $|f|_{N,\varepsilon}\leqq C\|f\|_{M,\delta}$.

Proof. The inequalities

$$|e^{-h_{K,\varepsilon}(\operatorname{Im}\zeta)}\zeta^{\alpha}\tilde{f}(\zeta)|$$

$$=\frac{1}{(2\pi)^{n}}\left|\int_{\boldsymbol{R}^{2n}}e^{-h_{K,\varepsilon}(\operatorname{Im}\zeta)}e^{i\langle x,\zeta\rangle}D^{\alpha}f(x)\,dx\right.$$

$$\leqq\frac{1}{(2\pi)^{n}}\left|\int_{\boldsymbol{R}^{2n}}e^{\langle x,\operatorname{Im}\zeta\rangle-\varepsilon|x|}e^{i\langle x,\zeta\rangle}D^{\alpha}f(x)\,dx\right|$$

$$\leqq\frac{1}{(2\pi)^{n}}\int_{\boldsymbol{R}^{2n}}|e^{-\varepsilon|x|}D^{\alpha}f(x)|\,dx$$

$$\leqq C_{\delta}\|f\|_{N,\delta},$$

for $0<\delta<\varepsilon$ and $\operatorname{Im}\zeta\in K_{\varepsilon}^{\circ}$, show that

$$e^{-2h_{K,\varepsilon}(\operatorname{Im}\zeta)}(1+|\zeta|^{2})^{N}|\tilde{f}(\zeta)|^{2}\leqq C'\|f\|_{N,\delta}^{2}.$$

Then we have

$$|f|_{N-3n,\varepsilon}^{2}=\int_{\boldsymbol{R}^{2n}+iK_{\varepsilon}^{\circ}}e^{-2h_{K,\varepsilon}(\operatorname{Im}\zeta)}(1+|\zeta|^{2})^{N-3n}|\tilde{f}(\zeta)|^{2}d\lambda$$

$$\leqq C\|f\|_{N,\delta}^{2}.$$

Thus we have, for $M=N+3n$ and $\delta=\varepsilon/2$

$$|f|_{N,\varepsilon}\leqq C\|f\|_{M,\delta}.$$

Proposition 2.14. Int $K_\varepsilon^\circ = \cup_{\delta>\varepsilon} K_\delta^\circ$.

Proof. Int $K_\varepsilon^\circ \supset \cup_{\delta>\varepsilon} K_\delta^\circ$ is clear from Proposition 2.6. Let $\eta \in \text{Int}\, K_\varepsilon^\circ$, then there exists a positive number γ such that for every $e \in C^n$ with $|e| \leqq \gamma$, $\eta + e \in K_\varepsilon^\circ$. Thus we have

$$0 \geqq \sup_{x_i \in \Gamma_i, |x_i|=1, |e_i| \leqq \gamma} (-\langle x_i, \alpha_i + e_i \rangle + \varepsilon)$$

$$= \sup_{x_i \in \Gamma_i, |x_i|=1} (-\langle x_i, \alpha_i \rangle + \varepsilon + \gamma).$$

This shows that $\eta \in K_{\varepsilon+\gamma}^\circ$ and Int $K_\varepsilon^\circ \subset \cup_{\delta>\varepsilon} K_\delta^\circ$.

Proposition 2.15. Int K_ε° *is a convex set and* $h_{K,\varepsilon}(\eta)$ *is a convex function in* Int K_ε°.

Proof. Let $\xi, \eta \in \text{Int}\, K_\varepsilon^\circ$, then there exist $\delta > \varepsilon$ such that $\xi, \eta \in K_\delta^\circ$. For $\lambda, \mu \geqq 0, \lambda + \mu = 1$, we have

$$\text{(2.7)} \qquad h_{K,\delta}(\lambda\xi + \mu\eta) = \sup_{x \in K \cap \boldsymbol{R}^{2n}} (-\langle x, \lambda\xi + \mu\eta \rangle + \delta|x|)$$

$$= \sup_{x \in K \cap \boldsymbol{R}^{2n}} (-\lambda\langle x, \xi \rangle - \mu\langle x, \eta \rangle + \delta(\lambda + \mu)|x|)$$

$$\leqq \lambda h_{K,\delta}(\xi) + \mu h_{K,\delta}(\eta) < \infty .$$

This shows that $\lambda\xi + \mu\eta \in K_\delta^\circ \subset \text{Int}\, K_\varepsilon^\circ$. Hence Int K_ε° is convex. The equation (2.7) shows that $h_{K,\delta}(\eta)$ is a convex function defined in K_δ°, hence $h_{K,\varepsilon}(\eta)$ is convex in Int K_ε°.

Proposition 2.16. $h_{K,\varepsilon}(\eta)$ *is Lipschitz continuous in* K_ε°, *that is,*

$$|h_{K,\varepsilon}(\eta) - h_{K,\varepsilon}(\eta')| \leqq C|\eta - \eta'|$$

for some constant C.

Proof. Let $h_{\Gamma_i,\varepsilon}(\alpha_i) = \sup_{x_i \in \Gamma_i} (-\langle x_i, \alpha_i \rangle + \varepsilon|x_i|)$ and $h_{B_i,\varepsilon}(\beta_i) = \sup_{y_i \in B_i} (-\langle y_i, \beta_i \rangle + \varepsilon|y_i|)$. Then

$$h_{K,\varepsilon}(\eta) = \sum_{i=1}^{j} h_{\Gamma_i,\varepsilon}(\alpha_i) + \sum_{i=1}^{j} h_{B_i,\varepsilon}(\beta_i)$$

where $\eta = ((\alpha_1, \beta_1), \cdots, (\alpha_j, \beta_j))$ and $x = ((x_1, y_1), \cdots, (x_j, y_j))$. Let $|B_i|$

be the diameter of the ball B_i. We have

$$|h_{B_i,\varepsilon}(\beta_i)-h_{B_i,\varepsilon}(\beta_i')|\leqq \sup_{x_i\in B_i}|\langle x_i,\beta_i-\beta_i'\rangle|\leqq|B_i||\beta_i-\beta_i'|.$$

Since $\eta\in K_\varepsilon^\circ$ implies that $h_{\Gamma_i,\varepsilon}(\alpha_i)=0$ for all i, we have

$$|h_{K,\varepsilon}(\eta)-h_{K,\varepsilon}(\eta')|\leqq C|\eta-\eta'|$$

where $C=\max_i(|B_i|)$.

§ 3. Cohomology with Bounds

For the later use, we develope the theory of cohomology with bounds on the pseudoconvex domain Ω in $\boldsymbol{C}^n$, which is an extension of what is developed in 7. 6 of L. Hörmander [1], where the case $\Omega=\boldsymbol{C}^n$ is treaded.

Here we use the same notation that is used in 7. 6 of L. Hörmander [1]. We denote by $\mathcal{U}^{(\nu)}$ the covering of $\boldsymbol{C}^n$ which consists of the cubes $U_g^{(\nu)}$ with side equal to $2\cdot 3^{-\nu}$ and center at $g\cdot 3^{-\nu}$, where g runs through the set I of points in $\boldsymbol{C}^n$ with integral coordinates. For every ν and g we can find precisely one g' such that $U_{g'}^{(\nu)}$ contains the cube with the same center as $U_g^{(\nu+1)}$ but twice the side; we set $\rho_{\nu,\nu+1}g=g'$. More generally if $\nu<\mu$, we define

$$\rho_{\nu,\mu}g=\rho_{\nu,\nu+1}\rho_{\nu+1,\nu+2}\cdots\rho_{\mu-1,\mu}g\,.$$

Let Ω be an open subset of $\boldsymbol{C}^n$, then $\mathcal{U}^{(\nu)}\cap\Omega=\{U_g^{(\nu)}\cap\Omega;\,g\in I\}$ is an open covering of Ω. We also define

$$\Omega^{\nu,\mu}=\cup\{U_g^{(\mu)};\,U_{g'}^{(\nu)}\subset\Omega\quad\text{for }g'=\rho_{\nu,\mu}g\}$$

and

$$\Omega_\varepsilon=\{z\in\Omega;\,\mathrm{dist}\,(z,\Omega^c)>\varepsilon\}$$

where dist (z,Ω^c) is the distance between the point z and the complement Ω^c of Ω. We use the abbreviation $\Omega_\varepsilon^{\nu,\mu}$ for $(\Omega_\varepsilon)^{\nu,\mu}$.

Let $P=(P_{j,k}),\,j=1,\cdots,p,\,k=1,\cdots,q$ be the matrix with polynomial entries, and consider the sheaf homomorphism

$$P:\mathcal{O}^q\to\mathcal{O}^p \tag{3. 1}$$

defined by the mapping $(f_1,\cdots,f_q)\in\mathcal{O}^q$ to $\{\sum P_{j,k}f_k\}_{j=1}^p$. Let $\mathfrak{R}_P$ be the kernel of the sheaf homomorphism (3.1). It is known that $\mathfrak{R}_P$ is a coherent analytic sheaf and finitely generated by the germs of q-tuples $Q=(Q_1, \cdots, Q_q)$ with polynomial components such that

$$\sum_{k=1}^{q} P_{j,k}Q_k=0, \quad j=1, \cdots, p.$$

(See Lemma 7.6.3 of L. Hörmander [1].)

If ϕ is a continuous function, we define $C^\sigma(\mathfrak{U}^{(\nu)}\cap\Omega, \mathfrak{R}_P, \phi)$ as the set of alternating cochains $c=\{c_s\}$, $s\in I^{\sigma+1}$ where $c_s\in\Gamma(U^{(\nu)}\cap\Omega, \mathfrak{R}_P)$, and

$$\|c\|_\phi=\sum_{|s|=\sigma+1}\int_{U_s^{(\nu)}\cap\Omega}|c_s|^2e^{-\phi}d\lambda<\infty.$$

We define $\rho_{\nu,\mu}^*: C^\sigma(\mathfrak{U}^{(\nu)}\cap\Omega, \mathfrak{R}_P, \phi)\to C^\sigma(\mathfrak{U}^{(\mu)}\cap\Omega, \mathfrak{R}_P, \phi)$ by setting $(\rho_{\nu,\mu}^*c)_s$ equal to the restriction of $c_{\rho_{\nu,\mu}(s_0)\cdots\rho_{\nu,\mu}(s_\sigma)}$ to $U_s^{(\mu)}$.

Proposition 3.1. *Let ϕ be a plurisubharmonic function in an open set V in C^n, and Ω be a pseudoconvex domain contained in V. For every cochain $c\in C^\sigma(\mathfrak{U}^{(\nu)}\cap V, \mathcal{O}, \phi)$ with $\delta c=0$, one can find a cochain $c'\in C^{\sigma-1}(\mathfrak{U}^{(\nu+\sigma-1)}\cap\Omega^{\nu,\nu-\sigma-1}, \mathcal{O}, \psi)$ so that $\delta c'=\rho_{\nu,\nu+\sigma-1}^*c$ and*

$$\|c'\|_\psi\leqq K\|c\|_\phi. \tag{3.2}$$

Here K is a constant independent of ϕ and c, and ψ is defined by $\psi(z)=\phi(z)+2\log(1+|z|^2)$.

We prove this in a way similar to Proposition 7.6.1 of L. Hörmander [1], so that we need the following lemma.

Lemma 3.2. *Let Ω be a pseudoconvex domain and let Ω' be a relatively compact subset of Ω. For every plurisubharmonic function ϕ in Ω and every $f\in L^2_{(0,q+1)}(\Omega, \phi)$ with $\bar\partial f=0$, one can find $u\in L^2_{(0,q)}(\Omega,$ loc) with $\bar\partial u=f$ and*

$$\int_{\Omega'}|u|^2e^{-\phi}d\lambda\leqq K\int_\Omega|f|^2e^{-\phi}d\lambda$$

where K is independent of u and ϕ.

Proof. See Lemma 7.6.2 of L. Hörmander [1].

Proof of Proposition 3. 1. We introduce the space $C^p(\mathcal{U}^{(\nu)}\cap V, \mathcal{Z}_q, \phi)$ of all alternating cochains $c=\{c_s\}$, $s\in I^{p+1}$, where $c_s\in L^2_{(0,q)}(U_s^{(\nu)}\cap V, \phi)$, $\bar\partial c_s=0$ and

$$\|c\|_\phi^2=\sum_{|s|=p+1}\int_{U_s^{(\nu)}\cap V}|c_s|^2e^{-\phi}d\lambda<\infty .$$

We wish to prove that if $\delta c=0\,(p>0)$, then one can find $c'\in C^{p-1}(\mathcal{U}^{(\nu+p-1)}\cap \Omega^{\nu,\nu+p-1}, \mathcal{Z}_q, \psi)$ so that $\delta c'=\rho^*_{\nu,\nu+p-1}c$ and (3. 2) hold. For $q=0$, this assertion is precisely Proposition 3. 1. We shall prove it assuming, if $p>1$, that it has already been proved for smaller values of p and all q.

Choose a non-negative function $\chi\in C_0^\infty(U_0^{(\nu)})$ such that $\sum_g\chi(z-g)=1$. Now set $b_s=\sum\chi(z-g)\,c_{g,s}$, $s\in I^p$, then we have $\delta b=c$ and

$$|b_s|^2\leqq\sum\chi(z-g)\,|c_{g,s}|^2 ,$$

hence

$$\|b\|_\phi^2\leqq\|c\|_\phi^2 .$$

Let $\bar\partial b$ be the cochain belonging to $C^{p-1}(\mathcal{U}^{(\nu)}\cap V, \mathcal{Z}_{q+1}, \phi)$ defined by $(\bar\partial b)_s=\bar\partial b_s=\sum\bar\partial\chi(z-g)\wedge c_{g,s}$. Then we obtain with a constant K

$$\|\bar\partial b\|_\phi\leqq K\|c\|_\phi .$$

Now $\delta\bar\partial b=\bar\partial\delta b=\bar\partial c=0$. If $p>1$, we can by the inductive hypothesis find a cochain $b'\in C^{p-2}(\mathcal{U}^{(\nu+p-2)}\cap\Omega^{\nu,\nu-p-2}, \mathcal{Z}_{q+1}, \psi)$ such that $\delta b'=\rho^*_{\nu,\nu+p-2}\bar\partial b$ and for some constant K_1

$$\|b'\|_\psi\leqq K_1\|\bar\partial b\|_\phi\leqq KK_1\|c\|_\phi .$$

Since $\bar\partial b'_s=0$ and ψ is plurisubharmonic, by Lemma 3. 2 we can choose $b''_s\in L^2_{(0,q)}(U_s^{(\nu+p-1)}, \psi)$ for every $s\in I^{p-1}$ satisfying $U_{s'}^{(\nu+p-2)}\subset\Omega^{\nu,\nu+p-2}$, $s'=\rho_{\nu+p-2,\nu+p-1}s$ so that $\bar\partial b''_s=b'_{s'}$ in $U_s^{(\nu+p-1)}$ and with a constant K_2,

$$\int_{U_s^{(\nu+p-1)}}|b''_s|^2e^{-\psi}d\lambda\leqq K_2\int_{U_s^{(\nu+p-2)}}|b'_{s'}|^2e^{-\psi}d\lambda .$$

Now set

$$c'=\rho^*_{\nu,\nu+p-1}b-\delta b'' .$$

Then $\delta c'=\rho^*_{\nu,\nu+p-1}\delta b=\rho^*_{\nu,\nu+p-1}c$, and

$$\bar\partial c'=\rho^*_{\nu,\nu+p-1}\bar\partial b-\delta\bar\partial b''=\rho^*_{\nu,\nu+p-1}\bar\partial b-\delta\rho^*_{\nu+p-2,\nu+p-1}b'$$

$$=\rho^*_{\nu,\nu+p-1}\bar{\partial}b-\rho^*_{\nu+p-2,\nu+p-1}\rho^*_{\nu,\nu+p-2}\bar{\partial}b=0 .$$

Summing up the estimates for b, b' and b'' given above, we obtain $c'\in C^{p-1}(\mathcal{U}^{(\nu+p-1)}\cap\Omega^{\nu,\nu+p-1}, \mathcal{Z}_q, \psi)$ and the estimate (3. 2).

It remains to consider the case $p=1$. The fact that $\delta\bar{\partial}b=0$ then means that $\bar{\partial}b$ defines uniquely a form f of type $(0, q+1)$ in V with $\bar{\partial}f=0$ and

$$\int|f|^2e^{-\phi}d\lambda\leqq\|\bar{\partial}b\|^2_\phi\leqq K^2\|c\|^2_\phi .$$

By Theorem 4. 4. 2 of L. Hörmander [1], we can find a form $u\in L^2_{(0,q)}(\Omega, \psi)$ so that $\bar{\partial}u=f$ and

$$\int_\Omega|u|^2e^{-\psi}d\lambda\leqq\int_\Omega|f|^2e^{-\phi}d\lambda .$$

Setting $c'_s=b_s-u$, we obtain $c'\in C^0(\mathcal{U}^{(\nu)}\cap\Omega, \mathcal{Z}_q, \psi)$ and the estimate (3. 2).

Proposition 3. 3. *Let P be a matrix with polynomial entries and Ω be a neighbourhood of 0. Then there exists a neighbourhood Ω' of 0 such that for every $u\in\mathcal{O}(\Omega+z)^q$ one can find $v\in\mathcal{O}(\Omega'+z)^q$ satisfying $Pv=Pu$, and*

$$\text{(3. 3)}\qquad \sup_{\Omega'+z}|v|\leqq C(1+|z|)^N\sup_{\Omega+z}|Pu| ,$$

where the constants C and N are independent of u and $z\in\boldsymbol{C}^n$.

Proof. See Proposition 7. 6. 5 of L. Hörmander [1].

Proposition 3. 4. *Let a matrix P and an integer ν be given. Then there exist integers μ and N such that, if ϕ is plurisubharmonic in a pseudoconvex domain Ω and for some constant $C>0$*

$$\text{(3. 4)}\qquad |\phi(z)-\phi(z')|<C, \quad |z-z'|<1 ,$$

then for every $c\in C^\sigma(\mathcal{U}^{(\nu)}\cap\Omega^{\lambda,\nu}, \mathcal{R}_P, \phi)$ with $\delta c=0, \sigma>0, \lambda\leqq\nu$, one can find $c'\in C^{\sigma-1}(\mathcal{U}^{(\mu)}\cap\Omega^{\lambda,\mu}_{(\tau-\sigma+1)\varepsilon}, \mathcal{R}_P, \phi_N)$ so that $\delta c'=\rho^*_{\nu,\mu}c$ and for some constant K

$$\text{(3. 5)}\qquad \|c'\|_{\phi_N}\leqq K\|c\|_\phi .$$

Here $\phi_N(z)=\phi(z)+N\log(1+|z|^2)$, $\tau=2^{2n}$ *and* $\varepsilon\geqq\sqrt{2n}\,3^{1-\lambda}$.

Proof. We can also prove the proposition in a way similar to the proof of Theorem 7. 6. 10 of L. Hörmander [1]. We shall prove it by induction for decreasing σ, noting that it is valid when $\sigma>2^{2n}$, since there are no non-zero $c\in C^\sigma(\mathcal{U}^{(\nu)}\cap\Omega^{\lambda,\nu},\mathcal{R}_P,\phi)$. Thus assume that the theorem has been proved for all P when σ is replaced by $\sigma+1$. By Lemma 7. 6. 4 of L. Hörmander [1], we have $c_s=Qd_s$ for $d\in C(\mathcal{U}^{(\nu)}\cap\Omega^{\lambda,\nu},\mathcal{O}^r)$. By Proposition 3. 3 and the condition (3. 4), if μ is large we can choose $d'_s\in\mathcal{O}(U_s^{(\mu)})^r$ so that $Qd'_s=Qd_{s'}=c_{s'}$ in $U_s^{(\mu)}$ and

$$\int_{U_s^{(\mu)}}|d'_s|^2(1+|z|^2)^{-N}e^{-\phi(z)}d\lambda\leqq C\int_{U_{s'}^{(\nu)}}|c_{s'}|^2e^{-\phi(z)}d\lambda$$

for $s'=\rho_{\nu,\mu}s$ and $U_{s'}^{(\nu)}\subset\Omega^{\lambda,\nu}$. Thus we have $d'\in C^\sigma(\mathcal{U}^{(\mu)}\cap\Omega^{\lambda,\mu},\mathcal{O}^r,\phi_N)$, $\rho^*_{\nu,\mu}c=Qd'$ and

$$\|d'\|_{\phi_N}\leqq C_1\|c\|_\phi\,.$$

Since $\delta c=0$, it follows that $\delta Qd'=Q\delta d'=0$. Thus $\delta d'=d''\in C^{\sigma+1}(\mathcal{U}^{(\mu)}\cap\Omega^{\lambda,\mu},\mathcal{R}_Q,\phi_N)$, and since $\delta d''=0$ and ϕ_N is plurisubharmonic, it follows by the inductive hypothesis that for suitable N' and $\mu'>\mu$ we can find $d'''\in C^\sigma(\mathcal{U}^{(\mu')}\cap\Omega^{\lambda,\mu'}_{(\tau-\sigma)\varepsilon},\mathcal{R}_Q,\phi_{N'})$ so that $\delta d'''=\rho^*_{\mu,\mu'}d''$ and

$$\|d'''\|_{\phi_{N'}}\leqq C_2\|d''\|_{\phi_N}\,.$$

Setting $\gamma=\rho^*_{\mu,\mu'}d'-d'''\in C^\sigma(\mathcal{U}^{(\mu')}\cap\Omega^{\lambda,\mu'}_{(\tau-\sigma)\varepsilon},\mathcal{O}^r,\phi_{N'})$, we have $\delta\gamma=\rho^*_{\mu,\mu'}d''-\delta d'''=0$ and

$$\|\gamma\|_{\phi_{N'}}\leqq C_3\|c\|_\phi\,.$$

Hence Proposition 3. 1 shows that for some $\mu''>\mu'$ and $N''>N'$ one can fined $\gamma'\in C^{\sigma-1}(\mathcal{U}^{(\mu'')}\cap\Omega^{\lambda,\mu''}_{(\tau-\sigma+1)\varepsilon},\mathcal{O}^r,\phi_{N''})$ so that $\rho^*_{\mu',\mu''}\gamma=\delta\gamma'$ and

$$\|\gamma'\|_{\phi_{N''}}\leqq C_4\|\gamma\|_{\phi_{N'}}\leqq C_5\|c\|_\phi\,. \tag{3.6}$$

Here we used the fact that $\Omega_{(\tau-\sigma+1)\varepsilon}$ is a pseudoconvex domain contained in $\Omega^{\lambda,\mu'}_{(\tau-\sigma)\varepsilon}$ as $\varepsilon\geqq\sqrt{2n}\,3^{1-\lambda}$. If we set $c'=Q\gamma'$, it follows that

$$\begin{aligned}\delta c'=Q\delta\gamma'=Q\rho^*_{\mu',\mu''}\gamma&=Q\rho^*_{\mu',\mu''}\rho^*_{\mu,\mu'}d'-\rho^*_{\mu',\mu''}Qd'''\\&=\rho^*_{\mu,\mu''}Qd'=\rho^*_{\mu,\mu''}\rho^*_{\nu,\mu}c=\rho^*_{\nu,\mu''}c\,.\end{aligned}$$

Since (3. 6) implies (3. 5) for suitable μ and N, the proposition is proved.

Proposition 3.5. *Let Ω' be an open set which is strictly contained in a pseudoconvex domain Ω of C^n* $(\mathrm{dist}(\Omega', \Omega^c) \geqq \delta > 0)$. *Given the system P there is a constant N such that, if ϕ is a plurisubharmonic function satisfying* (3.4), *then for all $u \in \mathcal{O}(\Omega)^q$ one can find $v \in \mathcal{O}(\Omega')^q$ with $Pv = Pu$ and*

$$(3.7) \qquad \int_{\Omega'} |v|^2 e^{-\phi}(1+|z|^2)^{-N} d\lambda \leqq C \int_{\Omega} |Pu|^2 e^{-\phi} d\lambda$$

where C is a constant independent of u.

Proof. First, choose ν so that $\delta > \tau\varepsilon = 2^{2n}\sqrt{2n}\, 3^{1-\nu}$. By Proposition 3.3 we can shoose $\nu < \mu$ so that there exists an element $u_g \in \mathcal{O}(U_g^{(\mu)})^q$ such that $Pu_g = Pu$ in $U_g^{(\mu)} \subset U_{g'}^{(\nu)} \subset \Omega$, and for some constants C and N independent of u and $g \in I$

$$(3.8) \qquad \int_{U_g^{(\mu)}} |u_g|^2 e^{-\phi}(1+|z|^2)^{-N} d\lambda \leqq C \int_{U_{g'}^{(\nu)}} |Pu|^2 e^{-\phi} d\lambda$$

where $g' = \rho_{\nu,\mu} g$. Let $c_{g_1 g_2} = u_{g_1} - u_{g_2}$. This defines a cocycle $c \in C^1(\mathcal{U}^{(\mu)} \cap \Omega^{\nu,\mu}, \mathcal{R}_P, \phi_N)$ and by (3.8) we obtain

$$(3.9) \qquad \|c\|^2_{\phi_N} \leqq C' \int_{\Omega} |Pu|^2 e^{-\phi} d\lambda\,.$$

Proposition 3.4 asserts that for some $\lambda > \mu$ and $N' > N$ there exists a cochain $c' \in C^0(\mathcal{U}^{(\lambda)} \cap \Omega', \mathcal{R}_P, \phi_{N'})$ such that $\delta c' = \rho^*_{\mu,\lambda} c | \Omega'$ and

$$(3.10) \qquad \|c'\|_{\phi_{N'}} \leqq C'' \|c\|_{\phi_N}\,.$$

Here we used the fact that Ω' is contained in $\Omega^{\nu,\lambda}_{\tau\varepsilon}$ as $\delta > \tau\varepsilon$. This means that if we set $v = u_{\rho_{\mu,\nu} g} + c'_g$ in $U_g^{(\lambda)} \cap \Omega'$, we define uniquely an element $v \in \mathcal{O}(\Omega')^q$. Since $Pc'_g = 0$, it follows that $Pv = Pu$, and from the estimates (3.8), (3.9) and (3.10) we obtain (3.7) with N replaced by N'.

§4. Soft Resolution of $\underset{\sim}{\mathcal{O}}_{k,l}$

In this section, we define the space $\mathcal{G}'(\Omega)$ of rapidly decreasing distributions, and using this space we make a resolution of $\underset{\sim}{\mathcal{O}}_{k,l}$, that is,

$$0 \to \underset{\sim}{\mathcal{O}}_{k,l} \to \mathcal{G}'_{(0,0)} \xrightarrow{\bar\partial} \mathcal{G}'_{(0,1)} \xrightarrow{\bar\partial} \cdots \xrightarrow{\bar\partial} \mathcal{G}'_{(0,n)} \to 0 .$$

Definition 4. 1. Let Ω be an open set in $\boldsymbol{Q}^{k,l}$. We denote by $\mathcal{G}(\Omega)$ the inductive limit $\varinjlim_{K\subset\Omega}\mathcal{F}_c(K)$ of $\mathcal{F}_c(K)$, where K is a compact set in Ω. We denote by $\mathcal{G}'(\Omega)$ the dual space of $\mathcal{G}(\Omega)$.

Since the injection of $\mathcal{G}(\Omega)$ into $\mathcal{F}(\Omega)$ (Definition 2. 13 of [I]) is continuous and of dense range, $\mathcal{F}'(\Omega)$ is a linear subspace of $\mathcal{G}'(\Omega)$. Moreover, we have the following proposition.

Proposition 4. 2. *An element of $\mathcal{G}'(\Omega)$ belongs to $\mathcal{F}'(\Omega)$ if and only if it has a compact support.*

Proof. Let $T\in\mathcal{F}'(\Omega)$. By the definition of the topology of $\mathcal{F}(\Omega)$ (see Definition 2. 13 of [I]), there are a compact set K in Ω, an integer $m\geqq 0$, and a constant $C>0$ such that for all $\phi\in\mathcal{F}(\Omega)$,

$$|\langle T, \phi\rangle|\leqq C \sup_{|\alpha|\leqq m,\, x\in K\cap \boldsymbol{C}^n} |D^\alpha\phi(x)|e^{-|x|/(m+1)}.$$

This implies immediately that $\langle T, \phi\rangle=0$ whenever the support of ϕ is contained in the complement of K, which means that supp $T\subset K$.

Conversely if T is an element of $\mathcal{G}'(\Omega)$ with the compact support K. Let $\alpha(x)\in\mathcal{F}_c(\Omega)$ be equal to one in some neighbourhood of K. Then $\langle T, \phi\rangle=\langle T, \alpha\phi\rangle$ and if ϕ_ν converges to zero in $\mathcal{F}(\Omega)$, $\alpha\phi_\nu$ converges to zero in $\mathcal{G}(\Omega)$. Therefore $\mathcal{F}(\Omega)\ni\phi\to\langle T, \phi\rangle$ is continuous, hence $T\in\mathcal{F}'(\Omega)$.

Proposition 4. 3. *If Ω is a bounded open set in $\boldsymbol{C}^n$ then $\mathcal{G}'(\Omega) = \mathcal{D}'(\Omega)$.*

Proof. It is obvious, since $\mathcal{G}(\Omega)=\mathcal{D}(\Omega)$.

Proposition 4. 4. *Let K be a compact subset of $\boldsymbol{Q}^{k,l}$ defined in Definition 2. 1, and Ω be a neighbourhood of K. For $f\in\mathcal{F}'(\Omega)$, define*

(4. 1) $$\hat f(\zeta)=\langle f, e^{-i\langle x,\zeta\rangle}\rangle/(2\pi)^n$$

then $\hat{f}(\zeta)$ *is analytic in* $\{\zeta\in \boldsymbol{C}^{2n}; |\mathrm{Im}\,\zeta|<\varepsilon\}$ *for some* $\varepsilon>0$ *and there exists an* N *satisfying* $|\hat{f}(\zeta)|\leqq C(1+|\zeta|)^N$ *for* $|\mathrm{Im}\,\zeta|<\varepsilon$. *The equality*

$$(4.2)\qquad \langle f, v\rangle=\int_{\boldsymbol{R}^{2n}}\hat{f}(\xi+i\eta)\tilde{v}(\xi+i\eta)d\xi$$

holds for $v\in\mathcal{F}_c(K)$ *and* $\eta\in K^\circ$ *with* $|\eta|<\varepsilon$.

Proof. By the definition of the topology of $\mathcal{F}(\Omega)$, there exists a seminorm $\|\cdot\|_{L,N,\varepsilon}$ satisfying $|\langle f, v\rangle|\leqq C\|v\|_{L,N,\varepsilon}$ for some constant C, where $\|v\|_{L,N,\varepsilon}=\sup_{x\in L\cap\boldsymbol{R}^{2n},|\alpha|\leqq N}|D^\alpha f(x)|e^{-\varepsilon|x|}$ for the compact set L in Ω and $\varepsilon>0$, $N>0$. If $|\mathrm{Im}\,\zeta|<\varepsilon$, then

$$\|e^{-i\langle x,\zeta\rangle}\|_{L,N,\varepsilon}=\sup_{x\in L\cap\boldsymbol{R}^{2n},|\alpha|\leqq N}|\zeta^\alpha e^{-i\langle x,\zeta\rangle}|e^{-\varepsilon|x|}$$
$$\leqq\sup_{|\alpha|\leqq N}\{|\zeta^\alpha|\}\leqq(1+|\zeta|)^N<\infty\,.$$

Hence $\hat{f}(\zeta)=\langle f, e^{-i\langle x,\zeta\rangle}\rangle/(2\pi)^n$ is analytic in $|\mathrm{Im}\,\zeta|<\varepsilon$ and satisfies $|\hat{f}(\zeta)|\leqq C(1+|\zeta|)^N$. Since

$$v(x)=\frac{1}{(2\pi)^n}\int_{\boldsymbol{R}^{2n}}e^{-i\langle x,\xi+i\eta\rangle}\tilde{v}(\xi+i\eta)d\xi$$

by Proposition 2.11, and the Riemann sum converges with respect to the seminorm $\|\cdot\|_{L,N,\varepsilon}$, then

$$\langle f, v\rangle=\langle f,\frac{1}{(2\pi)^n}\int_{\boldsymbol{R}^{2n}}e^{-i\langle x,\xi+i\eta\rangle}\tilde{v}(\xi+i\eta)d\xi\rangle$$
$$=\int_{\boldsymbol{R}^{2n}}\hat{f}(\xi+i\eta)\tilde{v}(\xi+i\eta)d\xi\,.$$

Remark 4.5. The equality (4.2) holds when v satisfies $|D^\alpha v(x)|\leqq Ce^{\delta|x|}$ for $|\alpha|\leqq N+3n$ and $\delta>0$ such that K_δ° has an element η satisfying $|\eta|<\varepsilon$.

Let $\bar{\partial}$ be the Cauchy-Riemann operator defined by

$$\bar{\partial}_p: u=\sum_{i_1<,\cdots,<i_p}u_{i_1,\cdots,i_p}d\bar{z}_{i_1}\wedge\cdots\wedge d\bar{z}_{i_p}\to$$
$$w=\sum_{i_1<,\cdots,<i_p,j}(\partial u_{i_1,\cdots,i_p}/\partial\bar{z}_j)d\bar{z}_j\wedge d\bar{z}_{i_1}\wedge\cdots\wedge d\bar{z}_{i_p}\,.$$

If we identify forms u and w with vector functions $\check{u}$ and $\check{w}$ having $\binom{n}{p}$ and $\binom{n}{p+1}$ components respectively, $\bar{\partial}_p$ can be represented by $P_p(D)$ where $P_p(\zeta)$ is a $\binom{n}{p}-\binom{n}{p+1}$ matrix with polynomial entries, and $D=i\partial/\partial x$. It is known as the Koszul resolution that the following sequence is exact:

$$0\to A\xrightarrow{{}^tP_{n-1}(\zeta)} A^n\to\cdots\to A^{\binom{n}{p+1}}\xrightarrow{{}^tP_p(\zeta)}\cdots$$

$$\cdots\xrightarrow{{}^tP_0(\zeta)} A\to\mathrm{Coker}\,{}^tP_0(\zeta)\to 0\,,$$

where A is the polynomial ring of the variable $\zeta=(\zeta_1,\cdots,\zeta_{2n})$ and ${}^tP_p(\zeta)$ is the transpose of $P_p(-\zeta)$ (see Example 4 in § 7 of Chapter VII of V. P. Palamodov [5]). It is known that $\mathcal{R}_{{}^tP_p}$ is generated by the germs of the lows of the matrix $P_{p+1}(\zeta)$ (see Lemma 7. 6. 3 of L. Hörmander [1]). Since $\mathcal{R}_{{}^tP_p}$ is a coherent analytic sheaf, we have the following proposition.

Proposition 4. 6. *Let Ω be a pseudoconvex domain. If $f\in\mathcal{O}^r(\Omega)$ satisfies the equation ${}^tP_p(\zeta)f(\zeta)=0$, then there exists a $g\in\mathcal{O}^s(\Omega)$ satisfying $f(\zeta)={}^tP_{p+1}(\zeta)g(\zeta)$, where $r=\binom{n}{p+1}$ and $s=\binom{n}{p+2}$.*

Proof. See Theorem 7. 2. 9 of L. Hörmander [1].

Definition 4. 7. (The sheaf of rapidly decreasing distributions.) We denote by $\mathcal{G}'$ the sheaf determined by a presheaf $\{\mathcal{G}'(\Omega)\}$, where Ω is an open set in $\mathbf{Q}^{k,l}$.

For any locally finite covering $\{U_\alpha\}$ of Ω, there exists a partition of unity $\{\phi_\alpha\}$ subordinate to the covering $\{U_\alpha\cap\mathbf{C}^n\}$ such that all derivatives of ϕ_α are bounded. Then $\mathcal{G}'(\Omega)$ is the section module of the sheaf $\mathcal{G}'$ and $\mathcal{G}'$ is a soft sheaf.

Theorem 4. 8. *Let Ω be a neighbourhood of a point z_∞ at infinity in $\mathbf{Q}^{k,l}$. If $f\in\mathcal{G}'_{(0,p)}(\Omega)$ satisfies $\bar{\partial}f=0$, then there exists a neighbourhood ω of z_∞ with $\omega\subset\Omega$ and $u\in\mathcal{G}'_{(0,p-1)}(\omega)$ such that $\bar{\partial}u=f$ in ω.*

Proof. First we choose a neighbourhood ω of z_∞ having the form $\omega=a+\operatorname{Int} K$, where K is the compact set in $\boldsymbol{Q}^{k,l}$ defined in Definition 2.1 and $a\in\boldsymbol{R}^{2n}$.

Let L be a compact set in Ω containing ω. Then $f\in\mathcal{F}_c'(L)^J$ and satisfies, for some $m>0$, $\varepsilon>0$, $|\langle f,\psi\rangle|\leqq C\|\psi\|_{m,\varepsilon}$ for all $\psi\in\mathcal{F}_c(L)^J$, where $J=\binom{n}{p}$. Hence

$$(4.3)\qquad |\langle f,\phi\rangle|\leqq C\|\phi\|_{m,\varepsilon}\qquad\text{for}\quad \phi\in\mathcal{G}(\omega)^J,$$

where $\|\phi\|_{m,\varepsilon}=\sum_{j=1}^{J}\sup_{x\in\boldsymbol{R}^{2n},|\alpha|\leqq m}|D^\alpha\phi_j(x)|e^{-\varepsilon|x|}$. If we can show that there exist $M>0$ and $\delta>0$ satisfying

$$(4.4)\qquad |\langle f,v\rangle|\leqq C\|\vartheta v\|_{M,\delta}\quad\text{for all}\quad v\in\mathcal{G}_{(0,n-p)}(\omega),$$

by the Hahn-Banach theorem there exists a $u\in\mathcal{G}'_{(0,p-1)}(\omega)$ satisfying $\langle f,v\rangle=\langle u,\vartheta v\rangle$, that is, $\bar\partial u=f$ in ω, where ϑ is the dual operator of $\bar\partial$. Let $v\in\mathcal{G}(\omega)^J$, then $\operatorname{supp} v\subset a+K$. By the coordinate transformation (translation) we may assume $\operatorname{supp} v\subset K$. Then, by Corollary 2.8, $\tilde v(\zeta)$ is analytic for $\operatorname{Im}\zeta\in K^\circ$ and satisfies, for any $\varepsilon>0$ and $\nu>0$,

$$|\tilde v(\zeta)|\leqq C_{\varepsilon,\nu}\frac{1}{(1+|\zeta|)^\nu}e^{h_{K,\varepsilon}(\operatorname{Im}\zeta)}\quad\text{for}\quad \operatorname{Im}\zeta\in K_\varepsilon^\circ.$$

Let $\bar\partial_p$ be represented by $P_p(D)$, then by Proposition 3.5 there exists an N such that for any ν there exists a function $V(\zeta)$ analytic for $\operatorname{Im}\zeta\in\operatorname{Int}K_{2\varepsilon}^\circ$ and satisfying

$${}^tP_{p-1}(\zeta)V(\zeta)={}^tP_{p-1}(\zeta)\tilde v(\zeta)$$

and

$$\int_{\boldsymbol{R}^{2n}+i\operatorname{Int}K_{2\varepsilon}^\circ}|V(\zeta)|^2e^{-2h_{K,\varepsilon}(\operatorname{Im}\zeta)}(1+|\zeta|^2)^{\nu-N}d\lambda$$
$$\leqq\int_{\boldsymbol{R}^{2n}+iK_\varepsilon^\circ}|{}^tP_{p-1}(\zeta)\tilde v(\zeta)|^2e^{-2h_{K,\varepsilon}(\operatorname{Im}\zeta)}(1+|\zeta|^2)^\nu d\lambda<\infty,$$

where we have used the fact that $h_{K,\varepsilon}(\operatorname{Im}\zeta)$ is a convex (hence plurisubharmonic) function satisfying the condition (3.4) and $\boldsymbol{R}^{2n}+i\operatorname{Int}K_{2\varepsilon}^\circ$ is a pseudoconvex domain strictly contained in $\boldsymbol{R}^{2n}+i\operatorname{Int}K_\varepsilon^\circ$ (see Propositions 2.15 and 2.16). From the above inequality, we have

$$|V(\zeta)|\leqq C\frac{1}{(1+|\zeta|)^{\nu-N}}e^{h_{K,\varepsilon}(\operatorname{Im}\zeta)}\quad\text{for}\quad\operatorname{Im}\zeta\in K_{3\varepsilon}^\circ.$$

Propositions 2.10, 2.12 and the above inequality imply that $V(\zeta)=\tilde{v}_1(\zeta)$ for a $C^{\nu-N-3n}$ function v_1 with support contained in K satisfying $\|v_1\|_{\nu-N-3n,3A\varepsilon}<\infty$. From Propositions 3.5 and 4.6, there exists a function $\Phi(\zeta)$ analytic in $\{\zeta\in C^{2n}; \mathrm{Im}\zeta\in\mathrm{Int}\,K^{\circ}_{3\varepsilon}\}$ and satisfying $V(\zeta)-\tilde{v}(\zeta)={}^tP_p(\zeta)\Phi(\zeta)$ and

$$\int_{R^{2n}+iK^{\circ}_{3\varepsilon}}|\Phi(\zeta)|^2e^{-2h_{K,\varepsilon}(\mathrm{Im}\zeta)}(1+|\zeta|^2)^{\nu-N'}d\lambda<\infty,$$

for some constant N' depending only on $P_p(\zeta)$ and $P_{p-1}(\zeta)$. This implies that there exists a $C^{\nu-N'-3n}$ function ϕ with support contained in K, satisfying $\Phi(\zeta)=\tilde{\phi}(\zeta)$ and $\|\phi\|_{\nu-N'-3n,4A\varepsilon}<\infty$.

Considering the inequality (4.3), if we take sufficiently large $\nu>0$ and small $\varepsilon>0$, we have

$$\langle f, v_1\rangle-\langle f, v\rangle=\langle f, {}^tP_p(D)\phi\rangle=\langle P_p(D)f, \phi\rangle=\langle\bar{\partial}f, \phi\rangle=0.$$

Let $\alpha\in\mathcal{F}_c(L)$ with $\alpha(x)=1$ on a neighbourhood of $\omega\cap R^{2n}$. Define $f_0=\alpha f$, then $f_0\in\mathcal{F}'(\Omega)$ by Proposition 4.2, and $\langle f, v\rangle=\langle f_0, v\rangle$ for any C^N function v with support contained in $\bar{\omega}$ and satisfying $\|v\|_{m,\varepsilon}<\infty$.

By Remark 4.5 if we take sufficiently large $\nu>0$ and small $\varepsilon>0$, we have

$$\begin{aligned}|\langle f, v\rangle|^2&=|\langle f, v_1\rangle|^2=|\langle f_0, v_1\rangle|^2\\&\leqq\left(\int_{R^{2n}}|\hat{f}_0(\xi+i\eta)\tilde{v}_1(\xi+i\eta)|d\xi\right)^2\\&\leqq\int_{R^{2n}}|\hat{f}_0(\xi+i\eta)|^2(1+|\xi|^2)^{N-\nu}d\xi\\&\qquad\times\int_{R^{2n}}|V(\xi+i\eta)|^2(1+|\xi|^2)^{\nu-N}d\xi\\&\leqq C_1\int_{R^{2n}+iK^{\circ}_{2\varepsilon}}|V(\zeta)|^2e^{-2h_{K,\varepsilon}(\mathrm{Im}\zeta)}(1+|\zeta|^2)^{\nu-N}d\lambda\\&\leqq C\int_{R^{2n}+iK^{\circ}_{\varepsilon}}|{}^tP_{p-1}(\zeta)\tilde{v}(\zeta)|^2e^{-2h_{K,\varepsilon}(\mathrm{Im}\zeta)}(1+|\zeta|^2)^{\nu}d\lambda\\&\leqq C\|{}^tP_{p-1}(D)v\|_{M,\delta}=C\|\vartheta v\|_{M,\delta}.\end{aligned}$$

The last inequality follows from Proposition 2.13. Thus we have shown (4.4), and completed the proof.

Theorem 4. 9. *We have the following soft resolution of the sheaf* $\underset{\sim}{\mathcal{O}}_{k,l}$:

$$(4.5)\qquad 0\to\underset{\sim}{\mathcal{O}}_{k,l}\to\mathcal{G}'_{(0,0)}\xrightarrow{\bar\partial}\mathcal{G}'_{(0,1)}\xrightarrow{\bar\partial}\cdots\xrightarrow{\bar\partial}\mathcal{G}'_{(0,n)}\to 0\,.$$

Proof. Since the restriction of $\underset{\sim}{\mathcal{O}}_{k,l}$ or $\mathcal{G}'$ to $\boldsymbol{C}^n$ is $\mathcal{O}$ or $\mathcal{D}'$, respectively, and it is well known that the following sequence is exact:

$$0\to\mathcal{O}\to\mathcal{D}'_{(0,0)}\xrightarrow{\bar\partial}\cdots\xrightarrow{\bar\partial}\mathcal{D}'_{(0,n)}\to 0\,.$$

In order to obtain the resolution (4. 5), we have only to make it at points at infinity. It is done in Theorem 4. 8.

Definition 4. 10. Let K be the compact set in $\boldsymbol{Q}^{k,l}$ defined in Definition 2. 1. Define $I_{K,\varepsilon}(\eta)=\sup_{x\in K\cap\boldsymbol{R}^{2n}}(\langle x,\eta\rangle-\varepsilon|x|)$ and $K^\circ_{(\varepsilon)}=\{\eta\in\boldsymbol{R}^{2n};\ I_{K,\varepsilon}(\eta)<\infty\}$.

Proposition 4. 11. *Let Ω be an open set in $\boldsymbol{Q}^{k,l}$ containing K. If $f\in\mathcal{F}'(\Omega)$ satisfies the inequality $|\langle f,v\rangle|\leqq C\|v\|_{K,N,\varepsilon}$ for all $v\in\mathcal{F}(\Omega)$, where $\|v\|_{K,N,\varepsilon}=\sup_{x\in K\cap\boldsymbol{R}^{2n},|\alpha|\leqq N}|D^\alpha f(x)|e^{-\varepsilon|x|}$, then $\hat f(\zeta)=\langle f,e^{-i\langle x,\zeta\rangle}\rangle/(2\pi)^n$ is analytic in $\{\zeta\in\boldsymbol{C}^{2n};\mathrm{Im}\zeta\in\mathrm{Int}\,K^\circ_{(\varepsilon)}\}$ and satisfies, for some constant $C>0$,*

$$(4.6)\qquad |\hat f(\zeta)|\leqq C(1+|\zeta|)^N e^{I_{K,\varepsilon}(\mathrm{Im}\zeta)}\qquad\text{for}\quad \mathrm{Im}\,\zeta\in K^\circ_{(\varepsilon)}\,.$$

Proof. Let $\zeta=\xi+i\eta$ and $\eta\in K^\circ_{(\varepsilon)}$. Then we have

$$\|e^{-i\langle x,\zeta\rangle}\|_{K,N,\varepsilon}=\sup_{x\in K\cap\boldsymbol{R}^{2n},|\alpha|\leqq N}|\zeta^\alpha e^{\langle x,\eta\rangle}|e^{-\varepsilon|x|}$$
$$\leqq(1+|\zeta|)^N e^{I_{K,\varepsilon}(\eta)}.$$

Since $(e^{-i\langle x,\zeta+h\rangle}-e^{-i\langle x,\zeta\rangle})/h$ converges to $-ixe^{-i\langle x,\zeta\rangle}$ as $h\to 0$ with respect to $\|\cdot\|_{K,N,\varepsilon}$ for $\mathrm{Im}\,\zeta\in\mathrm{Int}\,K^\circ_{(\varepsilon)}$, $\hat f(\zeta)$ is analytic.

Proposition 4. 12. *Let $F(\zeta)$ be an analytic function in $\{\zeta\in\boldsymbol{C}^{2n};\ \mathrm{Im}\zeta\in\mathrm{Int}\,K^\circ_{(\varepsilon)}\}$ satisfying the inequality* (4. 6). *Then $F(\zeta)$ defines an element $f\in\mathcal{F}'(\boldsymbol{Q}^{k,l})$ with support contained in K satisfying*

$$\langle f,\phi\rangle=\int_{\boldsymbol{R}^{2n}+i\eta}F(\zeta)\tilde\phi(\zeta)d\zeta\qquad\text{for}\quad\phi\in C_0^\infty(\boldsymbol{R}^{2n}).$$

Proof. If $\phi \in C_0^\infty(\boldsymbol{R}^{2n})$, then $\tilde{\phi}(\zeta)$ is an entire function satisfying for any $\nu > 0$

$$|\tilde{\phi}(\zeta)| < C\, e^{h_B(\mathrm{Im}\,\zeta)}/(1+|\zeta|)^\nu,$$

where B is the support of ϕ and $h_B(\eta) = \sup_{x\in B}(-\langle x, \eta\rangle)$. Hence the linear form

$$\int_{\boldsymbol{R}^{2n}+i\eta} F(\zeta)\tilde{\phi}(\zeta)\,d\zeta = \langle f, \phi\rangle$$

defines a distribution f. Let B be convex and $B\cap K=\phi$, then there exists a vector $\eta \in (-K^\circ) \subset K^\circ_{(\varepsilon)}$ such that for some $\delta > 0$

$$\sup_{x\in K\cap \boldsymbol{R}^{2n}} \langle x, \eta\rangle \leqq \langle y, \eta\rangle - \delta|\eta| \qquad \text{for all} \quad y \in B,$$

hence $I_{K,\varepsilon}(\eta) + h_B(\eta) \leqq -\delta|\eta|$. Thus we have

$$\lim_{t\to\infty} \int_{\boldsymbol{R}^{2n}+it\eta} F(\zeta)\tilde{\phi}(\zeta)\,d\zeta$$
$$\leqq \lim_{t\to\infty} C\, e^{I_{K,\varepsilon}(t\eta)+h_B(t\eta)} \leqq \lim_{t\to\infty} C\, e^{-t\delta|\eta|} = 0\,.$$

Hence the support of f is contained in K. Let L be a neighbourhood of K having the form of Definition 2.1. If $\psi \in \mathcal{F}_c(L)$ then $\tilde{\psi}(\zeta)$ is analytic in $\{\zeta \in \boldsymbol{C}^{2n}; \mathrm{Im}\,\zeta \in L^\circ\}$ and satisfies for any $\nu > 0$ and $\varepsilon > 0$

$$|\tilde{\psi}(\zeta)| \leqq C\, e^{h_{L,\varepsilon}(\mathrm{Im}\,\zeta)}/(1+|\zeta|)^\nu \qquad \text{for} \quad \mathrm{Im}\,\zeta \in L^\circ_\varepsilon\,.$$

Hence it follows from the formula

$$\int_{\boldsymbol{R}^{2n}+i\eta} F(\zeta)\tilde{\psi}(\zeta)\,d\zeta$$

that the distribution f can be extended to $\mathcal{F}_c(L)$. Let $\alpha \in \mathcal{F}_c(L)$ such that $\alpha(x)=1$ in a neighbourhood of K, then $\alpha v \in \mathcal{F}_c(L)$ for $v \in \mathcal{F}(\boldsymbol{Q}^{k,l})$. Since the support of f is contained in K, we have $\langle f, v\rangle = \langle f, \alpha v\rangle$. This shows that $f \in \mathcal{F}'(\boldsymbol{Q}^{k,l})$.

Let Ω be an open set in $\boldsymbol{Q}^{k,l}$ which has the form $a + \mathrm{Int}\,K$, where K is the convex set defined in Definition 2.1 and $a \in \boldsymbol{C}^n$.

Theorem 4.13. *If $\bar{\partial}_p v = 0$ for $v \in \mathcal{F}_{(0,p)}(\Omega)$, then there exists $u \in \mathcal{F}_{(0,p-1)}(\Omega)$ satisfying $\bar{\partial}_{p-1} u = v$.*

Proof. We represent $\bar{\partial}_p$ by $P_p(D)$. Since all the spaces of the sequence

$$\mathcal{F}(\Omega)^q \xrightarrow{P_{p-1}(D)} \mathcal{F}(\Omega)^r \xrightarrow{P_p(D)} \mathcal{F}(\Omega)^s$$

are FS spaces (see Remark 2. 27 in [I]), we have only to show that the dual sequence

$$\mathcal{F}'(\Omega)^q \xleftarrow{{}^tP_{p-1}(D)} \mathcal{F}'(\Omega)^r \xleftarrow{{}^tP_p(D)} \mathcal{F}'(\Omega)^s$$

is exact and the range of ${}^tP_{p-1}(D)$ is closed.

Let $g \in \mathcal{F}'(\Omega)^r$, then there exist a convex set of the form $b+L$ contained in Ω and constants $N>0$, $\varepsilon>0$ such that the estimate

$$|\langle g, v\rangle| \leqq C\|v\|_{b+L,N,\varepsilon}$$

holds for all $v \in \mathcal{F}(\Omega)^r$. We may assume that L is also a convex set of the type in Definition 2. 1. By coordinate transformation (translation) we may also assume $b=0$. Then, by Proposition 4. 11, $\hat{g}(\zeta)$ is analytic in $\{\zeta \in \boldsymbol{C}^{2n}; \operatorname{Im}\zeta \in \operatorname{Int} L^{\circ}_{(\varepsilon)}\}$ and satisfies

$$|\hat{g}(\zeta)| \leqq C(1+|\zeta|)^N e^{I_{L,\varepsilon}(\operatorname{Im}\zeta)} \qquad \text{for} \quad \operatorname{Im}\zeta \in L^{\circ}_{(\varepsilon)}.$$

The equation ${}^tP_{p-1}(D)g=0$ implies ${}^tP_{p-1}(-\zeta)\hat{g}(\zeta)=0$ in $\{\zeta \in \boldsymbol{C}^{2n}; \operatorname{Im}\zeta \in \operatorname{Int} L^{\circ}_{(\varepsilon)}\}$. Then by Propositions 3. 5 and 4. 6, there exists an analytic function $F(\zeta)$ such that ${}^tP_p(-\zeta)F(\zeta)=\hat{g}(\zeta)$ for $\operatorname{Im}\zeta \in \operatorname{Int} L^{\circ}_{(\varepsilon/2)}$ and satisfying for some $\nu>0$

$$|F(\zeta)| \leqq C(1+|\zeta|)^{\nu} e^{I_{L,\varepsilon/2}(\operatorname{Im}\zeta)} \qquad \text{for} \quad \operatorname{Im}\zeta \in L^{\circ}_{(\varepsilon/2)}.$$

Here we used the fact that $I_{L,\varepsilon}(\eta)$ is convex and Lipschitz continuous, and $L^{\circ}_{(\varepsilon/2)}$ is a convex set contained strictly in $L^{\circ}_{(\varepsilon)}$. This shows that there exists $f \in \mathcal{F}'(\Omega)^s$ such that

$$\begin{aligned}\langle f, P_k(D)v\rangle &= \int_{\boldsymbol{R}^{2n}+i\eta} F(\zeta)P_k(\zeta)\tilde{v}(\zeta)\,d\zeta \\ &= \int_{\boldsymbol{R}^{2n}+i\eta} {}^tP_k(-\zeta)F(\zeta)\tilde{v}(\zeta)\,d\zeta \\ &= \int_{\boldsymbol{R}^{2n}+i\eta} \hat{g}(\zeta)\tilde{v}(\zeta)\,d\zeta = \langle g, v\rangle\end{aligned}$$

for all $v \in \mathcal{F}_c(K)$, that is, ${}^tP_k(D)f=g$.

Next we prove the closedness of the range of ${}^tP_0(D)$. Assume

$F_j \to F$ in $\mathcal{F}'(\Omega)$ with $F_j = {}^tP_0(D)G_j$ for $G_j \in \mathcal{F}'(\Omega)^n$. Since the sequence $\{F_j\}$ is a bounded set in the DFS space $\mathcal{F}'(\Omega)$, there exist a compact set L in Ω (we may assume that L is a convex set of the type in Definition 2.1) and constants $C>0$, $\varepsilon>0$ satisfying

$$|\hat{F}_j(\zeta)| \leqq C(1+|\zeta|)^N e^{I_{L,\varepsilon}(\operatorname{Im}\zeta)} \qquad \text{for} \quad \operatorname{Im}\zeta \in L^\circ_{(\varepsilon)} .$$

By Proposition 3.5 we can choose $\Psi_j(\zeta)$ satisfying

$$(4.8) \qquad |\Psi_j(\zeta)| \leqq C'(1+|\zeta|)^{N'} e^{I_{L,\varepsilon}(\operatorname{Im}\zeta)} \qquad \text{for} \quad \operatorname{Im}\zeta \in L^\circ_{(\varepsilon/2)} .$$

Since $\{\Psi_j(\zeta)\}$ forms a normal family, there exists a subsequence which converges to $\Psi(\zeta)$ which also satisfies (4.8). Thus there exists $G \in \mathcal{F}'(\Omega)^n$ satisfying

$$\begin{aligned}\langle G, P_0(D)v\rangle &= \int_{\mathbf{R}^{2n}+i\eta} \Psi(\zeta) P_0(\zeta)\tilde{v}(\zeta)\, d\zeta \\ &= \lim_{k\to\infty} \int_{\mathbf{R}^{2n}+i\eta} {}^tP_0(-\zeta)\Psi_{j_k}(\zeta)\tilde{v}(\zeta)\, d\zeta \\ &= \lim_{k\to\infty} \langle F_{j_k}, v\rangle = \langle F, v\rangle .\end{aligned}$$

This shows that $F = {}^tP_0(D)G$, that is, the range of ${}^tP_0(D)$ is closed.

At the end of this section, we give an extension of Theorem 4.11 of [I].

Theorem 4.14. *We have the following soft resolution of the sheaf* $\tilde{\mathcal{O}}_{k,l}$ *on* $\mathbf{Q}^{k,l}$:

$$(4.7) \qquad 0 \to \tilde{\mathcal{O}}_{k,l} \to \mathcal{F}_{(0,0)} \xrightarrow{\bar\partial} \cdots \xrightarrow{\bar\partial} \mathcal{F}_{(0,n)} \to 0 .$$

Proof. Since the restriction of $\tilde{\mathcal{O}}_{k,l}$ or $\mathcal{F}$ to $\mathbf{C}^n$ is $\mathcal{O}$ or $\mathcal{E}$ respectively, and it is well known that the following sequence is exact:

$$0 \to \mathcal{O} \to \mathcal{E}_{(0,0)} \xrightarrow{\bar\partial} \cdots \xrightarrow{\bar\partial} \mathcal{E}_{(0,n)} \to 0 .$$

In order to obtain the resolution (4.7) of $\tilde{\mathcal{O}}_{k,l}$, we have only to make the resolution at points at infinity. Since the point z_∞ at infinity has a fundamental system of neighbourhoods whose member has the form $a + \operatorname{Int} K$, Theorem 4.13 gives the resolution at points at infinity.

Remark 4. 15. In the above theorem the resolution is obtained on the whole $\boldsymbol{Q}^{k,l}$, while in Theorem 4. 11 of [I], it is obtained on the open subset Ω which satisfies the condition (i) of Definition 4.5 of [I].

§ 5. Fourier Hyperfunctions with Compact Supports

In this section, we show that the space $H_K^n(V, {}^E\tilde{\mathcal{O}}_{k,l})$ of E-valued Fourier hyperfunctions is isomorphic to the space $L(\underset{\sim}{\mathcal{O}}_{k,l}(K), E)$ of continuous linear mappings from $\underset{\sim}{\mathcal{O}}_{k,l}(K)$ to a Fréchet space E.

Let K be a compact set in $\prod_{i=1}^{j} \boldsymbol{D}^{n_i}$ and V be an $\tilde{\mathcal{O}}_{k,l}$-pseudoconvex neighbourhood of K in $\boldsymbol{Q}^{k,l}$. From Theorem 5. 8 and Corollary 5. 10 of [I], we have $H_c^p(V, \underset{\sim}{\mathcal{O}}_{k,l})=0$ for $0\leqq p\leqq n-1$ and $H^p(K, \underset{\sim}{\mathcal{O}}_{k,l})=0$ for $p\geqq 1$. Therefore from the long exact sequence of cohomology groups with compact supports,

$$\begin{aligned}0\to H_c^0(V-K, \underset{\sim}{\mathcal{O}}_{k,l})&\to H_c^0(V, \underset{\sim}{\mathcal{O}}_{k,l})\to H^0(K, \underset{\sim}{\mathcal{O}}_{k,l})\\ &\xrightarrow{\delta} H_c^1(V-K, \underset{\sim}{\mathcal{O}}_{k,l})\xrightarrow{\rho} H_c^1(V, \underset{\sim}{\mathcal{O}}_{k,l})\to H^1(K, \underset{\sim}{\mathcal{O}}_{k,l})\\ &\to H_c^2(V-K, \underset{\sim}{\mathcal{O}}_{k,l})\to H_c^2(V, \underset{\sim}{\mathcal{O}}_{k,l})\to\cdots,\end{aligned}$$

follows that $\delta: H^0(K, \underset{\sim}{\mathcal{O}}_{k,l})\cong H_c^1(V-K, \underset{\sim}{\mathcal{O}}_{k,l})$ and $H_c^2(V-K, \underset{\sim}{\mathcal{O}}_{k,l})=0$, for $n\geqq 2$.

Since by Theorem 4. 9 we have the soft resolution

$$0\to\underset{\sim}{\mathcal{O}}_{k,l}\to\mathcal{G}'_{(0,0)}\xrightarrow{\bar\partial}\mathcal{G}'_{(0,1)}\xrightarrow{\bar\partial}\cdots\xrightarrow{\bar\partial}\mathcal{G}'_{(0,n)}\to 0,$$

$H_c^1(V-K, \underset{\sim}{\mathcal{O}}_{k,l})$ can be represented by the first cohomology group of the complex $(\mathcal{F}'_{(0,\cdot)}(V-K), \bar\partial)$. Then δ can be represented as the following continuous mapping. Let U be an open neighbourhood of K and $\alpha\in\mathcal{F}_c(U)$ such that $\alpha=1$ in $W\cap\boldsymbol{R}^{2n}$, where W is some neighbourhood of K in U. The map

$$\delta_{U,\alpha}: H^0(U, \underset{\sim}{\mathcal{O}}_{k,l})\to\{u\in\mathcal{F}'_{(0,1)}(V-K); \bar\partial u=0\}$$

defined by $\delta_{U,\alpha}(f)=\bar\partial(\alpha f)$ is continuous and induces a continuous map of $H^0(U, \underset{\sim}{\mathcal{O}}_{k,l})$ into $H_c^1(V-K, \underset{\sim}{\mathcal{O}}_{k,l})$. These maps define the map δ on the inductive limit $H^0(K, \underset{\sim}{\mathcal{O}}_{k,l})=\varinjlim_{U\supset K} H^0(U, \underset{\sim}{\mathcal{O}}_{k,l})$ of $H^0(U, \underset{\sim}{\mathcal{O}}_{k,l})$ and therefore δ is continuous. Moreover we can show that δ is an open mapping.

Proposition 5. 1. *Let $n\geqq 2$. Consider the dual complex,*

$$(5.1)\qquad \begin{array}{ccccc} \to \mathcal{F}_{(0,n-2)}(V-K) & \xrightarrow{\bar{\partial}_{n-2}} & \mathcal{F}_{(0,n-1)}(V-K) & \xrightarrow{\bar{\partial}_{n-1}} & \mathcal{F}_{(0,n)}(V-K)\to 0 \\ \updownarrow & & \updownarrow & & \updownarrow \\ \leftarrow \mathcal{F}'_{(0,2)}(V-K) & \xleftarrow{-\bar{\partial}_{1}} & \mathcal{F}'_{(0,1)}(V-K) & \xleftarrow{-\bar{\partial}_{0}} & \mathcal{F}'_{(0,0)}(V-K)\leftarrow 0\,. \end{array}$$

Then the ranges of the operators are all closed.

Proof. $H^2_c(V-K, \mathcal{O}_{k,l})=0$ shows that the range of $-\bar{\partial}_1$ is closed, and from Theorem 5. 11 of [I], it follows that the range of $\bar{\partial}_{n-1}$ is closed. The closedness of ranges of other operators is a consequence of the so-called Serre-Komatsu duality theorem (see Theorem 4. 7 of [I]).

Proposition 5. 2. *Let $n\geqq 2$, then $H^0(K, \mathcal{O}_{k,l})$ and $H^1_c(V-K, \mathcal{O}_{k,l})$ are DFS spaces.*

Proof. Proposition 2. 7 of [I] shows that $H^0(K, \mathcal{O}_{k,l})=\mathcal{O}_{k,l}(K)$ is a *DFS* space. $\mathcal{F}'_{(0,1)}(V-K)$ is a *DFS* space as the dual space of an *FS* space $\mathcal{F}_{(0,1)}(V-K)$ (see Remark 2. 27 of [I]). Since a closed subspace and a quotient space (by its closed subspace) of a *DFS* space are also *DFS* spaces, it follows from the fact that the range of $-\bar{\partial}_0$ is closed, that $H^1_c(V-K, \mathcal{O}_{k,l})$ is a *DFS* space.

Theorem 5. 3. *Let E be a fully complete space, and let F be a barrelled space. Let f be a linear mapping of a subspace $E_0\subset E$ onto F. Suppose that the graph of f is closed in $E\times F$. Then f is open.*

Proof. See Theorem 4. 10 of V. Pták [6].

Propositon 5. 4. *Let $n\geqq 2$, then $\delta: H^0(K, \mathcal{O}_{k,l})\to H^1_c(V-K, \mathcal{O}_{k,l})$ is a homeomorphism.*

Proof. It is known that *DFS* spaces are fully complete and barrelled spaces (see Theorems 4. 3. 28 and 4. 3. 40 of H. Komatsu [3]). Since δ is a one-to-one onto continuous mapping, it follows from Theorem

5. 3 that δ is a homeomorphism.

Proposition 5. 5. *Let* $n\geqq 2$, *then* $H^{n-1}(V-K, \tilde{\mathcal{O}}_{k,l})\cong[\underset{\sim}{\mathcal{O}}_{k,l}(K)]'$.

Proof. $H^{n-1}(V-K, \tilde{\mathcal{O}}_{k,l})$ is represented by the $(n-1)$-th cohomology group of the complex $(\mathcal{F}_{(0,\cdot)}(V-K), \bar{\partial})$. It follows from Proposition 5. 1 and the so-called Serre-Komatsu duality theorem (Theorem 4. 7 of [I]) that

$$H^{n-1}(V-K, \tilde{\mathcal{O}}_{k,l})\cong[H_c^1(V-K, \underset{\sim}{\mathcal{O}}_{k,l})]'\cong[\underset{\sim}{\mathcal{O}}_{k,l}(K)]'.$$

Let E be a Fréchet space. From the exact sequence,

$$\text{(5. 2)}\qquad \cdots\to H_K^p(V, {}^E\tilde{\mathcal{O}}_{k,l})\to H^p(V, {}^E\tilde{\mathcal{O}}_{k,l})$$
$$\to H^p(V-K, {}^E\tilde{\mathcal{O}}_{k,l})\to H_K^{p+1}(V, {}^E\tilde{\mathcal{O}}_{k,l})\to\cdots$$

and the fact that if V is $\tilde{\mathcal{O}}_{k,l}$-pseudoconvex, $H^p(V, {}^E\tilde{\mathcal{O}}_{k,l})=0$ for $p>0$ (see Theorem 6. 6 of [I]), it follows that $H_K^n(V, {}^E\tilde{\mathcal{O}}_{k,l})\cong H^{n-1}(V-K, {}^E\tilde{\mathcal{O}}_{k,l})$, for $n\geqq 2$.

Proposition 5. 6. *Let* $n\geqq 2$, then $H^{n-1}(V-K, {}^E\tilde{\mathcal{O}}_{k,l})\cong H^{n-1}(V-K, \tilde{\mathcal{O}}_{k,l})\hat{\otimes}E$ *for a Fréchet space E.*

Proof. We represent $H^{n-1}(V-K, \tilde{\mathcal{O}}_{k,l})$ by the $(n-1)$-th cohomology group of the complex,

$$\cdots\to\mathcal{F}_{(0,n-2)}(V-K)\xrightarrow{\bar{\partial}_{n-2}}\mathcal{F}_{(0,n-1)}(V-K)\xrightarrow{\bar{\partial}_{n-1}}\mathcal{F}_{(0,n)}(V-K)\to 0\,.$$

Since the range of $\bar{\partial}_{n-2}$ is closed by Proposition 5. 1 and $\mathcal{F}_{(0,n-1)}(V-K)$ is a Frécet nuclear space, we have the exact sequence

$$\text{(5. 3)}\qquad 0\to\text{im}\,\bar{\partial}_{n-2}\to\ker\bar{\partial}_{n-1}\to\ker\bar{\partial}_{n-1}/\text{im}\,\bar{\partial}_{n-2}\to 0$$

where all the spaces are Fréchet nuclear spaces. Since the tensoring by $\hat{\otimes}E$ is a exact functor (see Theorem 6. 5 of [I]), we have the following exact sequence:

$$\text{(5. 4)}\qquad 0\to(\text{im}\,\bar{\partial}_{n-2})\hat{\otimes}E\to(\ker\bar{\partial}_{n-1})\hat{\otimes}E\to H^{n-1}(V-K, \tilde{\mathcal{O}}_{k,l})\hat{\otimes}E\to 0\,.$$

If we denote the closed linear hull by [ ], we have

$$\ker(\bar{\partial}_{n-1}\hat{\otimes}1_E)=[f\otimes e\in\mathcal{F}_{(0,n-1)}(V-K)\hat{\otimes}E\,;\bar{\partial}_{n-1}f=0]$$
$$=(\ker\bar{\partial}_{n-1})\hat{\otimes}E\,.$$

By Proposition 43.9 of F. Treves [7], we also have $\operatorname{im}(\bar{\partial}_{n-2}\hat{\otimes}1_E)=(\operatorname{im}\bar{\partial}_{n-2})\hat{\otimes}E$. Since $H^{n-1}(V-K,{}^E\tilde{\mathcal{O}}_{k,l})$ can be represented by the $(n-1)$-th cohomology group of the complex $(\mathcal{F}_{(0,\cdot)}(V-K,E),{}^E\bar{\partial})$ and $\mathcal{F}_{(0,\cdot)}(V-K,E)\cong\mathcal{F}_{(0,\cdot)}(V-K)\hat{\otimes}E$ and ${}^E\bar{\partial}=\bar{\partial}\hat{\otimes}1_E$, we have $H^{n-1}(V-K,{}^E\tilde{\mathcal{O}}_{k,l})\cong H^{n-1}(V-K,\tilde{\mathcal{O}}_{k,l})\hat{\otimes}E$.

Theorem 5.7. *Let E be a Fréchet space and K be a compact set in $\prod_{i=1}^{j}\boldsymbol{D}^{n_i}$. Then $H^n_K(V,{}^E\tilde{\mathcal{O}}_{k,l})\cong L(\underset{\sim}{\mathcal{O}}_{k,l}(K),E)$.*

Proof. By Proposition 50.5 of F. Treves [7], we have $L(\underset{\sim}{\mathcal{O}}_{k,l}(K),E)\cong[\underset{\sim}{\mathcal{O}}_{k,l}(K)]'\hat{\otimes}E$. Propositions 5.5 and 5.6 show that $[\underset{\sim}{\mathcal{O}}_{k,l}(K)]'\hat{\otimes}E\cong H^{n-1}(V-K,{}^E\tilde{\mathcal{O}}_{k,l})$, for $n\geqq 2$. Thus we have $H^n_K(V,{}^E\tilde{\mathcal{O}}_{k,l})\cong L(\underset{\sim}{\mathcal{O}}_{k,l}(K),E)$, for $n\geqq 2$.

If $n=1$, $H^1(W,\tilde{\mathcal{O}}_{k,l})=0$ for any open set W in $\boldsymbol{Q}^{k,l}$ satisfying the condition (i) of Definition 5.1 of [I] (Theorem 5.11 of [I]). Consider the dual complex,

$$\begin{array}{ccc}0\to\mathcal{F}_{(0,0)}(W)&\xrightarrow{\bar{\partial}}&\mathcal{F}_{(0,1)}(W)\to 0\\ \updownarrow&&\updownarrow\\ 0\leftarrow\mathcal{F}'_{(0,1)}(W)&\xleftarrow{-\bar{\partial}}&\mathcal{F}'_{(0,0)}(W)\leftarrow 0\,.\end{array}$$

Then the range of $\bar{\partial}\,(=\mathcal{F}_{(0,1)}(W))$ is closed, therefore the range of $(-\bar{\partial})$ is closed and

$$\tilde{\mathcal{O}}_{k,l}(W)\cong[H^1_c(W,\underset{\sim}{\mathcal{O}}_{k,l})]'.$$

The mapping ρ of the exact sequence

$$0\to H^0(K,\underset{\sim}{\mathcal{O}}_{k,l})\xrightarrow{\delta}H^1_c(V-K,\underset{\sim}{\mathcal{O}}_{k,l})\xrightarrow{\rho}H^1_c(V,\underset{\sim}{\mathcal{O}}_{k,l})\to 0$$

is continuous since it is induced by the continuous injection of $\mathcal{F}'(V-K)$ into $\mathcal{F}'(V)$. Therefore the dual sequence

$$0\to\tilde{\mathcal{O}}_{k,l}(V)\xrightarrow{\rho^*}\tilde{\mathcal{O}}_{k,l}(V-K)\xrightarrow{\delta^*}[\underset{\sim}{\mathcal{O}}_{k,l}(K)]'\to 0$$

is exact. Since all the spaces of the above sequence are Fréchet nuclear, we have the exact sequence

$$0\to\tilde{\mathcal{O}}_{k,l}(V,E)\to\tilde{\mathcal{O}}_{k,l}(V-K,E)\to[\underset{\sim}{\mathcal{O}}_{k,l}(K)]'\hat{\otimes}E\to 0,$$

where we used the fact that $\tilde{\mathcal{O}}_{k,l}(W,E)\cong\tilde{\mathcal{O}}_{k,l}(W)\hat{\otimes}E$ for an open set W in $\boldsymbol{Q}^{k,l}$ ((6. 6) of [I]) and the tensoring $\hat{\otimes}E$ is an exact functor (Theorem 6. 5 of [I]). Thus we have

$$H_K^n(V,{}^E\tilde{\mathcal{O}}_{k,l})\cong\tilde{\mathcal{O}}_{k,l}(V-K,E)/\tilde{\mathcal{O}}_{k,l}(V,E)\cong[\underset{\sim}{\mathcal{O}}_{k,l}(K)]'\hat{\otimes}E$$
$$\cong L(\underset{\sim}{\mathcal{O}}_{k,l}(K),E),$$

for $n=1$.

Corollary 5. 8. *Let Ω be an open set in $\prod_{i=1}^{j}\boldsymbol{D}^{n_i}$. Then ${}^E\mathfrak{R}_{k,l}(\Omega)\cong L(\underset{\sim}{\mathcal{O}}_{k,l}(\Omega),E)/L(\underset{\sim}{\mathcal{O}}_{k\ l}(\partial\Omega),E)$.*

Proof. The corollary follows from Proposition 6. 10 of [I] and Theorem 5. 7.

Without changing the proof of Theorem 5. 7, we can prove the following theorem, which corresponds to Theorem 5. 12 of [I] in the scalar valued case.

Theorem 5. 9. *Let K be a compact set in $\boldsymbol{Q}^{k,l}$, and V be an $\tilde{\mathcal{O}}_{k,l}$-pseudoconvex domain containing K. Suppose $H^p(K,\underset{\sim}{\mathcal{O}}_{k,l})=0$ for $p\geqq 1$. Then we have*

$$H_K^n(V,{}^E\tilde{\mathcal{O}}_{k,l})\cong L(\underset{\sim}{\mathcal{O}}_{k,l}(K),E).$$

Remark 5. 10. We can also prove $H_K^p(V,{}^E\tilde{\mathcal{O}}_{k,l})=0$ for $p\neq n$, for a compact set K satisfying the condition of the above theorem, in the same way as Theorem 6. 8 of [I].

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