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On $H^{p,q}(X, \mathbf{B})$ of Weakly 1-Complete Manifolds

By

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§ 1. Introduction

This note is a continuation of the author's previous work [4]. Let X be a complex manifold of dimension n and $\pi: \mathbf{B} \to X$ a holomorphic line bundle. We want to know the q-th cohomology group of X with coefficients in the sheaf of germs of B-valued holomorphic p-forms on X. We denote it by $H^{p,q}(X, \mathbf{B})$.

X is called weakly 1-complete if there exists an exhausting C^∞ plurisubharmonic function φ on X. φ is called an exhaustion function. In [4], we considered the case that $\mathbf{B}$ has a metric along the fibers whose curvature form is positive outside a compact subset K of X, and obtained the following theorems.

Theorem 1′. *Let X be a weakly 1-complete manifold of dimension n. Then under the above situation,*

$$\dim H^{n,q}(X, \mathbf{B}) < \infty \quad \text{for} \quad q \geqq 1 .$$

Theorem 2′. *Let X be a weakly 1-complete manifold of dimension n with an exhaustion function φ. Under the above situation, if $X_c := \{x \in X; \varphi(x) < c\}$ contains K, then the restriction map*

$$\rho_c : H^{n,q}(X, \mathbf{B}) \to H^{n,q}(X_c, \mathbf{B})$$

is bijective for $q \geqq 1$.

The purpose of this note is to prove the following theorems under the same conditions for X and $\mathbf{B}$ as above.

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Theorem 1. *Under the above situation, we have*

$$\dim H^{p,q}(X, \mathbf{B}) < \infty \quad \textit{for} \quad p+q>n .$$

Theorem 2. *Under the above situation, if X_c contains K, then the restriction map*

$$\rho_c \colon H^{p,q}(X, \mathbf{B}) \ \rightarrow \ H^{p,q}(X_c, \mathbf{B})$$

is bijective for $p+q>n$.

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§ 2. Preliminaries

Let X be a weakly 1-complete manifold of dimension n with an exhaustion function φ and $\mathbf{B}$ a holomorphic line bundle over X such that $\mathbf{B}$ has a metric $\boldsymbol{a}$ along the fibers whose curvature form $\boldsymbol{\Theta}(\boldsymbol{a})$ is positive outside a compact subset K of X. Let ds^2 be a hermitian metric on X whose fundamental form ω is $\sqrt{-1}\boldsymbol{\Theta}(\boldsymbol{a})$ outside a neighbourhood U of K. Let $\lambda(t)$ be a C^∞ convex nondecreasing function on $(-\infty, c)$, where $c \in \mathbf{R} \cup \{\infty\}$, and ds_λ^2 the hermitian metric on X_c associated to $\omega + \sqrt{-1}\partial\bar{\partial}\lambda(\varphi)$.

Proposition 1. If $\int_d^c \sqrt{\lambda''(t)}\,dt = \infty$ *for some $d<c$, then ds_λ^2 is a complete hermitian metric on X_c.*

Proof. Similar as Proposition 1 in [3].

We denote by $L_\lambda^{p,q}(X_c, \mathbf{B})$ the space of the square integrable $\mathbf{B}$-valued (p, q)-forms with respect to $\boldsymbol{a}e^{-\lambda(\varphi)}$ and ds_λ^2. We denote by $(f, g)_\lambda (\|f\|_\lambda)$ the inner product (the norm) in $L_\lambda^{p,q}(X_c, \mathbf{B})$. $\|f\|_\lambda^2$ is expressed in the form

$$\int_{X_c} |f|_\lambda^2 dv_\lambda .$$

Here dv_λ denotes the volume form for ds_λ^2, and $|f|_\lambda$ denotes the length of f with respect to $\boldsymbol{a}e^{-\lambda(\varphi)}$ and ds_λ^2.

We denote the adjoint of $\bar{\partial}: L_\lambda^{p,q-1}(X_c, \mathbf{B}) \to L_\lambda^{p,q}(X_c, \mathbf{B})$ by $\bar{\partial}_\lambda^*$. The domain, the range, and the kernel of $\bar{\partial}$ ($\bar{\partial}_\lambda^*$) are denoted by $D_\lambda^{p,q-1}$, $R_\lambda^{p,q}$, and $N_\lambda^{p,q-1}$ ($D_{\lambda*}^{p,q}$, $R_{\lambda*}^{p,q-1}$, and $N_{\lambda*}^{p,q}$), respectively.

We denote the quotient space $N_\lambda^{p,q}/R_\lambda^{p,q}$ by $H_\lambda^{p,q}(X_c, \mathbf{B})$.

Proposition 2 ([4], *Theorem* (2.6)). *If $X_c \supset \bar{U}$ and ds_λ^2 is a complete hermitian metric on X_c, then*

$$\dim H_\lambda^{p,q}(X_c, \mathbf{B}) < \infty \quad \text{for} \quad p+q>n .$$

We denote by $L_{\mathrm{loc}}^{p,q}(X_c, \mathbf{B})$ the space of locally square integrable $\mathbf{B}$-valued (p, q)-forms on X_c.

Lemma 1. *For any countably many elements $f_\mu \in L_{\mathrm{loc}}^{p,q}(X_c, \mathbf{B})$, $\mu = 1, 2, \cdots$, there exists a C^∞ strictly convex increasing function λ_0* on $(-\infty, c)$ *and a constant $A>0$ such that $ds_{\lambda_0}^2$ is complete, $\|f_\mu\|_{\lambda_0} < \infty$ for every μ, and $A\|f\|_0 \geqq \|f\|_{\lambda_0}$ for every $f \in L_0^{p,q}(X_c, \mathbf{B})$.*

Proof. Similar as Lemma 2.4 in [4].

We denote by $C_0^{p,q}(X_c, \mathbf{B})$ the space of C^∞ $\mathbf{B}$-valued (p, q)-forms on X_c with compact support. We denote by ϑ_λ the formal adjoint of $\bar{\partial}: C_0^{p,q}(X_c, \mathbf{B}) \to C_0^{p,q+1}(X_c, \mathbf{B})$ with respect to the metrics ds_λ^2 and $\boldsymbol{a}e^{-\lambda(\varphi)}$.

Proposition 3 (*cf.* [5], *Theorem* 1.1). *If ds_λ^2 is complete, then*

i) $C_0^{p,q}(X_c, \mathbf{B})$ *is dense in the space*

$$\{\psi; \psi \in L_\lambda^{p,q}(X_c, \mathbf{B}), \ \bar{\partial}\psi \in L_\lambda^{p,q+1}(X_c, \mathbf{B})\}$$

with respect to the graph norm of $\bar{\partial}$.

ii) $C_0^{p,q}(X_c, \mathbf{B})$ *is dense in the space*

$$\{\psi; \psi \in L_\lambda^{p,q}(X_c, \mathbf{B}), \ \vartheta_\lambda\psi \in L_\lambda^{p,q-1}(X_c, \mathbf{B})\}$$

with respect to the graph norm of ϑ_λ.

In what follows let λ be a C^∞ strictly convex increasing function on $(-\infty, c)$ satisfying $\int_d^c \sqrt{\lambda''(t)}\, dt = \infty$ for some $d<c$. Let $\{\lambda_\mu\}$ ($\mu = 1, 2, \cdots$) be a sequence of C^∞ convex increasing functions on $\mathbf{R}$ such that

$\int_0^{\infty}\sqrt{\lambda''_{\mu}}(t)\,dt=\infty$ and for every $c'<c$ and every nonnegative integer ν

$$\lim_{\mu\to\infty}\sup_{t\in(-\infty,c')}|\lambda_{\mu}^{(\nu)}(t)-\lambda^{(\nu)}(t)|=0\,.$$

Here $\lambda_{\mu}^{(\nu)}(t)$ $(\lambda^{(\nu)}(t))$ denotes the ν-th derivative of $\lambda_{\mu}(t)$ $(\lambda(t))$.

Lemma 2. *Let λ_{μ} be as above. If $p+q>n$ and $X_c\supset X_d\supset\bar{U}$, then there exists a constant C_0 such that for every μ,*

$$\int_{X-X_d}|f|^2_{\lambda_\mu}dv_{\lambda_\mu}\leqq C_0\left\{\|\bar\partial f\|^2_{\lambda_\mu}+\|\bar\partial^*_{\lambda_\mu}f\|^2_{\lambda_\mu}+\int_{X_d}|f|^2_{\lambda_\mu}dv_{\lambda_\mu}\right\}$$

for $f\in D^{p,q}_{\lambda_\mu}\cap D^{p,q}_{\lambda_\mu *}$.

Proof. Similar as Lemma 3.3 in [4].

Proposition 4. *Let λ and λ_{μ} be as above, $p+q>n$, and $X_c\supset\bar{U}$. Assume that there exists a constant $C_1>0$ such that for every μ and $f\in L^{p,q}_{\lambda_\mu}(X,\mathbf{B})$ we have $\|f|_{X_c}\|_{\lambda}\leqq C_1\|f\|_{\lambda_\mu}$, then there exists a constant $C>0$ and an integer μ_0 such that for every $\mu>\mu_0$ we have*

$$C^2\{\|\bar\partial f\|^2_{\lambda_\mu}+\|\bar\partial^*_{\lambda_\mu}f\|^2_{\lambda_\mu}\}\geqq\|f\|^2_{\lambda_\mu}$$

for any $f\in D^{p,q}_{\lambda_\mu}\cap D^{p,q}_{\lambda_\mu *}$ satisfying $(f|_{X_c},g)_{\lambda}=0$ for $g\in N^{p,q}_{\lambda}\cap N^{p,q}_{\lambda *}$.

Proof. If the proposition is false, then choosing a subsequence of $\{\lambda_\mu\}$ if necessary, we may assume that there exists a sequence $f_\mu\in L^{p,q}_{\lambda_\mu}(X,\mathbf{B})$ $(\mu=1,2,\cdots)$ such that

(1) $$f_\mu\in D^{p,q}_{\lambda_\mu}\cap D^{p,q}_{\lambda_\mu *}$$

(2) $$\|f_\mu\|_{\lambda_\mu}=1$$

(3) $$\|\bar\partial f_\mu\|_{\lambda_\mu}<1/\mu$$

(4) $$\|\bar\partial^*_{\lambda_\mu}f_\mu\|_{\lambda_\mu}<1/\mu$$

(5) $$(f_\mu|_{X_c},g)_{\lambda}=0 \text{ for any } g\in N^{p,q}_{\lambda}\cap N^{p,q}_{\lambda *}\,.$$

By (2) and the assumption $\|f_\mu|_{X_c}\|_{\lambda}\leqq C_1\|f_\mu\|_{\lambda_\mu}$, we have $\|f_\mu|_{X_c}\|_{\lambda}\leqq C_1$. Thus there exists a subsequence of $\{f_\mu|_{X_c}\}$ which has a weak limit f in $L^{p,q}_{\lambda}(X_c,\mathbf{B})$. Since the coefficients of ϑ_{λ_μ} converge uniformly on every

X_d $(d<c)$ to the coefficients of ϑ_λ, by (4) we have $\vartheta_\lambda f=0$. Since ds_λ^2 is complete, we have $\bar{\partial}_\lambda^* = \vartheta_\lambda$ by Proposition 3. Hence we obtain $\bar{\partial}_\lambda^* f=0$. By (3) we have $\bar{\partial} f=0$. On the other hand, by (2), (3), (4), and Lemma 2, for every μ,

$$\int_{X_d} |f|_{\lambda_\mu}^2 dv_{\lambda_\mu} > 1/C_0 - 2/\mu^2 ,$$

hence

$$\int_{X_d} |f|_{\lambda_\mu}^2 dv_{\lambda_\mu} \geqq 1/C_0'$$

for some constant C_0' and $d<c$. Thus $f\neq 0$. But by (5) $(f, g)_\lambda = 0$ for any $g\in N_\lambda^{p,q} \cap N_{\lambda *}^{p,q}$. This is a contradiction. Q.E.D.

Let H_i $(i=1, 2)$ be Hilbert spaces. For a densely defined closed linear operator $T: H_1 \to H_2$, we denote by T^* the adjoint of T and by D_T, R_T and $N_T (D_{T^*}$, R_{T^*} and $N_{T^*})$ the domain, the range and the kernel of $T(T^*)$, respectively.

Proposition 5 (*cf.* [1], *Theorem* 1.1.4). *Let H_i $(i=1, 2, 3)$ be Hilbert spaces, $S(T)$ a densely defined closed linear operator from $H_2(H_1)$ to $H_3(H_2)$, satisfying $S\circ T=0$, and F a closed linear subspace of H_2 containing R_T. Assume that the following estimate holds for some constant $C>0$.*

$$C^2\{\|T^*f\|_1^2 + \|Sf\|_3^2\} \geqq \|f\|_2^2, \quad \textit{for every} \quad f\in D_{T^*} \cap D_S \cap F.$$

Here $\|\ \|_i$ $(i=1, 2, 3)$ denote the norms of H_i. Then

(I) *for every $u\in F$ satisfying $Su=0$, there exists $v\in D_T$ such that $Tv=u$,*

(II) *R_{T^*} is closed in H_1 and for every $u\in R_{T^*}$ there exists $v\in D_{T^*}$ such that $T^*v=u$ and $\|v\|_2 \leqq C\|u\|_1$.*

Proposition 6. *Let λ and λ_μ be as above, $p+q\geqq n$, and $X_c \supset \bar{U}$. Assume that there exists a constant $C_1>0$ such that for every μ, $i=0, 1$, and $u\in L_{\lambda_\mu}^{p,q+i}(X, B)$, we have $\|u|_{X_c}\|_\lambda \leqq C_1 \|u\|_{\lambda_\mu}$. Then, for every $f\in L_\lambda^{p,q}(X_c, \mathbf{B})$ satisfying $\bar{\partial} f=0$, and for every $\varepsilon>0$, there exist an integer μ_0 and $\tilde{f}\in L_{\lambda_{\mu_0}}^{p,q}(X, \mathbf{B})$ satisfying $\bar{\partial}\tilde{f}=0$ and $\|\tilde{f}|_{X_c} - f\|_\lambda < \varepsilon$.*

Proof. Let g be an element of $L_\lambda^{p,q}(X_c, \mathbf{B})$ such that $(g, \tilde{f}|_{X_c})_\lambda = 0$ for any $\tilde{f} \in L_{\lambda_\mu}^{p,q}(X, \mathbf{B})$ $(\mu = 1, 2, \cdots)$ satisfying $\bar{\partial}\tilde{f} = 0$. If we prove $(g, f)_\lambda = 0$ for any $f \in L_\lambda^{p,q}(X_c, \mathbf{B})$ satisfying $\bar{\partial}f = 0$, then the proposition follows from the Hahn-Banach's theorem.

Let $u \in L_{\lambda_\mu}^{p,q}(X, \mathbf{B})$. From the assumption we have

$$(g, u|_{X_c})_\lambda \leqq C_1 \|g\|_\lambda \|u\|_{\lambda_\mu}.$$

Hence $(g, \cdot|_{X_c})_\lambda$ is a continuous linear functional on $L_{\lambda_\mu}^{p,q}(X, \mathbf{B})$ and its norm is not greater than $C_1\|g\|_\lambda$. From the Riesz representation theorem there exists $g_\mu \in L_{\lambda_\mu}^{p,q}(X, \mathbf{B})$ such that $\|g_\mu\|_{\lambda_\mu} \leqq C_1\|g\|_\lambda$ and $(g_\mu, u)_{\lambda_\mu} = (g, u|_{X_c})_\lambda$ for every $u \in L_{\lambda_\mu}^{p,q}(X, \mathbf{B})$. Clearly $g_\mu = 0$ on $X - X_c$ and $\|g_\mu|_{X_c}\|_\lambda \leqq C_1^2\|g\|_\lambda$. On the other hand by the assumption we have $(g_\mu|_{X_c}, v)_\lambda \to (g, v)_\lambda\ (\mu \to \infty)$ for every $v \in C_0^{p,q}(X_c, \mathbf{B})$. Therefore $g_\mu|_{X_c}$ converges weakly to g in $L_\lambda^{p,q}(X_c, \mathbf{B})$. Since g_μ is orthogonal to $N_{\lambda_\mu}^{p,q}$, g_μ is contained in the closure of $R_{\lambda_\mu}^{p,q}$. Hence by Proposition 4, and Proposition 5, (II) there exist $C > 0$ and $w_\mu \in L_{\lambda_\mu}^{p,q+1}(X, \mathbf{B})$ for any μ such that $\|w_\mu\|_{\lambda_\mu} \leqq C\|g_\mu\|_{\lambda_\mu}$ and $\bar{\partial}_{\lambda_\mu}^* w_\mu = g_\mu$. We have $\|w_\mu|_{X_c}\|_\lambda \leqq C_1\|w_\mu\|_{\lambda_\mu} \leqq C \cdot C_1^2\|g\|_\lambda$. Hence a subsequence of $\{w_\mu|_{X_c}\}$ converges weakly in $L_\lambda^{p,q+1}(X_c, \mathbf{B})$. Let the weak limit be w. By Proposition 3, we obtain $\bar{\partial}_\lambda^* w = g$.

Therefore for every $h \in N_\lambda^{p,q}$ we have $(g, h)_\lambda = (\bar{\partial}_\lambda^* w, h)_\lambda = (w, \bar{\partial}h)_\lambda = 0$. Q.E.D.

Lemma 3. *Let $c \in \mathbf{R}$ and λ_μ $(\mu = 1, 2, \cdots)$, λ be as above. Assume that $d\varphi$ is nowhere zero on the boundary of X_c and there exists a constant $C_2 \geqq 1$ such that for every $\varepsilon > 0$, there exists an integer μ_0 so that the following inequalities are satisfied for $c - \varepsilon < t < c$ and $\mu > \mu_0$.*

(6) $$\frac{1}{C_2\varepsilon}\lambda_\mu'(t) \leqq \lambda_\mu''(t),$$

(7) $$\frac{1}{C_2\varepsilon}\lambda'(t) \leqq \lambda''(t),$$

(8) $$\lambda_\mu(t) \leqq \lambda(t),$$

(9) $$\lambda_\mu'(t) \leqq C_2\lambda'(t),$$

(10) $$\lambda_\mu''(t) \leqq C_2\lambda''(t),$$

$$(11)\qquad e^{-\lambda(t)/n}\lambda'(t) \leqq C_2 e^{-\lambda_\mu(t)/n}\lambda'_\mu(t),$$

$$(12)\qquad e^{-\lambda(t)/n}\lambda''(t) \leqq C_2 e^{-\lambda_\mu(t)/n}\lambda''_\mu(t).$$

Then there exists a constant C_3 such that, for every μ and $f \in L^{p,q}_{\lambda_\mu}(X, \mathbf{B})$,

$$(13)\qquad \|f|X_c\|_\lambda \leqq C_3 \|f\|_{\lambda_\mu}.$$

Proof. We choose c' satisfying $c'<c$ so that $d\varphi$ is nowhere zero on $X_c - X_{c'}$. Let $x \in X_c - X_{c'}$ be any point. We choose a basis $(\sigma_1, \cdots, \sigma_n)$ of the holomorphic cotangent space of X at x so that $(\sigma_1, \cdots, \sigma_n)$ is orthonormal with respect to the original metric ds^2 and satisfies the following conditions (14) and (15).

$$(14)\qquad \partial\varphi|_x = \eta\sigma_1$$

$$(15)\qquad \partial\bar\partial\varphi|_x = \sum_{i=1}^{l} \eta_i \sigma_i \wedge \bar\sigma_i + \sum_{i=1}^{l} \eta^1_i \sigma_1 \wedge \bar\sigma_i + \sum_{i=1}^{l} \bar\eta^1_i \sigma_i \wedge \bar\sigma_1 .$$

Here $\eta>0$, $\eta_1 \geqq 0$, $\eta_i>0$ for $2 \leqq i \leqq l$, and $\eta^1_i \in \mathbf{C}$. We set $\eta_i = 0$ for $i \geqq l+1$. Note that, by hypothesis, there exists $\eta_0>0$, which is independent of x, satisfying $\eta>\eta_0$.

We have

$$(16)\qquad ds^2_{\lambda_\mu}|_x = (1+\lambda''_\mu(\varphi(x))\eta^2 + \lambda'_\mu(\varphi(x))\eta_1)\sigma_1\cdot\bar\sigma_1 + \sum_{i=2}^{n}(1+\lambda'_\mu(\varphi(x))\eta_i)\sigma_i\cdot\bar\sigma_i + \lambda'_\mu(\varphi(x))\Big(\sum_{i=2}^{n}\eta^1_i\sigma_1\cdot\bar\sigma_i + \sum_{i=2}^{n}\bar\eta^1_i\sigma_i\cdot\bar\sigma_1\Big).$$

Since φ is plurisubharmonic, we have

$$|2\mathrm{Re}(\eta^1_i u_1 \bar u_i)| \leqq 2\eta_1|u_1|^2 + \frac{1}{2}\eta_i|u_i|^2, \quad \text{for any } u_i \in \mathbf{C} \ \ (1 \leqq i \leqq n).$$

Hence we have

$$(17)\qquad |2\mathrm{Re}\sum_{i=1}^{n}\eta^1_i u_1 \bar u_i| \leqq 2(n-1)\eta_1|u_1|^2 + \frac{1}{2}\sum_{i=2}^{n}\eta_i|u_i|^2.$$

Therefore we obtain

$$(18)\qquad ((1+\lambda''_\mu(\varphi(x))\eta^2 - (2n-3)\lambda'_\mu(\varphi(x))\eta_1)\sigma_1\cdot\bar\sigma_1$$

$$+\sum_{i=2}^{n}\left(1+\frac{1}{2}\lambda'_\mu(\varphi(x))\eta_i\right)\sigma_i\cdot\bar{\sigma}_i$$

$$\leqq ds^2_{\lambda_\mu}|_x$$

$$\leqq((1+\lambda''_\mu(\varphi(x)))\eta^2+(2n-3)\lambda'_\mu(\varphi(x))\eta_1)\sigma_1\cdot\bar{\sigma}_1$$

$$+\sum_{i=2}^{n}\left(1+\frac{3}{2}\lambda'_\mu(\varphi(x))\eta_i\right)\sigma_i\cdot\bar{\sigma}_i\,.$$

We set

$$S_\mu=\sum_{i=1}^{n}\xi_{\mu i}\sigma_i\cdot\bar{\sigma}_i \tag{19}$$

and

$$S=\sum_{i=1}^{n}\xi_i\sigma_i\cdot\bar{\sigma}_i\,, \tag{20}$$

where

$$\begin{cases}\xi_{\mu 1}=1+\lambda''_\mu(\varphi(x))\eta^2+\lambda'_\mu(\varphi(x))\eta_1\\ \xi_{\mu i}=1+\lambda'_\mu(\varphi(x))\eta_i\,,\ \text{for}\quad 2\leqq i\leqq n\,,\\ \xi_1=1+\lambda''(\varphi(x))\eta^2+\lambda'(\varphi(x))\eta_1\\ \xi_i=1+\lambda'(\varphi(x))\eta_i\quad\text{for}\quad 2\leqq i\leqq n\,.\end{cases} \tag{21}$$

First we compare $ds^2_{\lambda_\mu}|_x$ ($ds^2_\lambda|_x$) with S_μ (with S). We choose c' in advance sufficiently close to c so that

$$3(2n-3)C_2(c-c')\eta_1<\eta^2\,. \tag{22}$$

By the assumption, there exists an integer μ_0 such that the inequalities (6) to (12) are satisfied with $\varepsilon=c-c'$, $c'<t<c$, and $\mu>\mu_0$. Hence, by (6), (18), and (22), we have

$$\frac{1}{3}S_\mu<ds^2_{\lambda_\mu}|_x<3S_\mu\,,\quad\text{for}\quad \mu>\mu_0\,. \tag{23}$$

Similarly as above, we have

$$\frac{1}{3}S<ds^2_\lambda|_x<3S\,. \tag{24}$$

Let $(\tau_{\mu 1},\cdots,\tau_{\mu n})$ be a basis of the holomorphic cotangent space of X at x such that

$$S_\mu = \sum_{i=1}^n \tau_{\mu i} \cdot \bar{\tau}_{\mu i} \tag{25}$$

and

$$ds^2_{\lambda_\mu}|_x = \sum_{i=1}^n \gamma_{\mu i} \tau_{\mu i} \cdot \bar{\tau}_{\mu i}\,, \quad \gamma_{\mu i} > 0\,. \tag{26}$$

By (24), we have

$$\frac{1}{3} < \gamma_{\mu i} < 3, \quad \text{for} \quad \mu > \mu_0\,. \tag{27}$$

We put

$$V_\mu = \left(\frac{\sqrt{-1}}{2}\right)^n \tau_{\mu 1} \wedge \bar{\tau}_{\mu 1} \wedge \cdots \wedge \tau_{\mu n} \wedge \bar{\tau}_{\mu n}\,. \tag{28}$$

Then we have

$$dv_{\lambda_\mu}|_x = \Big(\prod_{i=1}^n \gamma_{\mu i}\Big) V_\mu = \Big(\prod_{i=1}^n \gamma_{\mu i}\Big)\Big(\prod_{i=1}^n \xi_{\mu i}\Big) dv|_x\,. \tag{29}$$

Let $f \in C_0^{p,q}(X, \mathbf{B})$. We put

$$f|_x = \sum_{\substack{i_1 < \cdots < i_p \\ j_1 < \cdots < j_q}} f_{i_1 \cdots i_p \bar{j}_1 \cdots \bar{j}_q} \sigma_{i_1} \wedge \cdots \wedge \sigma_{i_p} \wedge \bar{\sigma}_{j_1} \wedge \cdots \wedge \bar{\sigma}_{j_q}\,. \tag{30}$$

Here $f_{i_1 \cdots i_p \bar{j}_1 \cdots \bar{j}_q} \in \mathbf{B}|_x$. Let $|f|_x|'_{\lambda_\mu}$ denote the length of $f|_x$ with respect to $\boldsymbol{a}e^{-\lambda_\mu(\varphi)}$ and S_μ. By (19), we have

$$|f|_x|'^2_{\lambda_\mu} = \sum_{\substack{i_1 < \cdots < i_p \\ j_1 < \cdots < j_q}} \frac{|f_{i_1 \cdots i_p \bar{j}_1 \cdots \bar{j}_q}|^2_{\lambda_\mu}}{\Big(\prod_{\alpha=1}^p \xi_{\mu i_\alpha}\Big)\Big(\prod_{\beta=1}^q \xi_{\mu j_\beta}\Big)}\,. \tag{31}$$

On the other hand, if we set

$$f|_x = \sum_{\substack{i_1 < \cdots < i_p \\ j_1 < \cdots < j_q}} f_{i_1 \cdots i_p \bar{j}_1 \cdots \bar{j}_q, \mu} \tau_{\mu i_1} \wedge \cdots \wedge \tau_{\mu i_p} \wedge \bar{\tau}_{\mu j_1} \wedge \cdots \wedge \bar{\tau}_{\mu j_q}\,, \tag{32}$$

we have

$$|f|_x|^2_{\lambda_\mu} = \sum_{\substack{i_1 < \cdots < i_p \\ j_1 < \cdots < j_q}} \frac{|f_{i_1 \cdots i_p \bar{j}_1 \cdots \bar{j}_q, \mu}|^2_{\lambda_\mu}}{\Big(\prod_{\alpha=1}^p \gamma_{\mu i_\alpha}\Big)\Big(\prod_{\beta=1}^q \gamma_{\mu j_\beta}\Big)}\,. \tag{33}$$

Therefore, in virtue of (27), we have

$$|f|_x|^2_{\lambda_\mu} \geqq 3^{-p-q} \sum_{\substack{i_1 < \cdots < i_p \\ j_1 < \cdots < j_q}} |f_{i_1 \cdots i_p \bar{j}_1 \cdots \bar{j}_q, \mu}|^2_{\lambda_\mu} \tag{34}$$

$$=3^{-p-q}|f|_x|'^2_{\lambda_\mu}$$

$$=3^{-p-q}\sum_{\substack{i_1<\cdots<i_p\\ j_1<\cdots<j_q}}\frac{|f_{i_1\cdots i_p\bar{j}_1\cdots\bar{j}_q}|^2_{\lambda_\mu}}{(\prod_{\alpha=1}^{p}\xi_{\mu i_\alpha})(\prod_{\beta=1}^{q}\xi_{\mu j_\beta})}\,,\quad \text{for}\quad \mu>\mu_0\,.$$

Combining (27), (29) and (34), we have

(35) $$|f|^2_{\lambda_\mu}dv_{\lambda_\mu}|_x$$

$$\geqq 3^{-p-q-n}\sum_{\substack{i_1<\cdots<i_p\\ j_1<\cdots<j_q}}\frac{|f_{i_1\cdots i_p\bar{j}_1\cdots\bar{j}_q}|^2_{\lambda_\mu}\prod_{i=1}^{n}\xi_{\mu i}}{(\prod_{\alpha=1}^{p}\xi_{\mu i_\alpha})(\prod_{\beta=1}^{q}\xi_{\mu j_\beta})}dv|_x\,,\quad \text{for}\quad \mu>\mu_0\,.$$

Similarly as above, we have

(36) $$|f|^2_\lambda dv_\lambda|_x$$

$$\leqq 3^{p+q+n}\sum_{\substack{i_1<\cdots<i_p\\ j_1<\cdots<j_q}}|f_{i_1\cdots i_p\bar{j}_1\cdots\bar{j}_q}|^2_\lambda\frac{\prod_{i=1}^{n}\xi_i}{(\prod_{\alpha=1}^{p}\xi_{i_\alpha})(\prod_{\beta=1}^{q}\xi_{j_\beta})}dv|_x\,.$$

By (9) and (10), we have

(37) $$\xi_{\mu i}\leqq C_2\xi_i,\quad \text{for}\quad 1\leqq i\leqq n \text{ and } \mu>\mu_0\,.$$

Therefore we obtain

(38) $$\frac{C_2^{p+q}}{(\prod_{\alpha=1}^{p}\xi_{\mu i_\alpha})(\prod_{\beta=1}^{q}\xi_{\mu j_\beta})}$$

$$\geqq\frac{1}{(\prod_{\alpha=1}^{p}\xi_{i_\alpha})(\prod_{\beta=1}^{q}\xi_{j_\beta})}\,,\quad \text{for}\quad \mu>\mu_0\,.$$

On the other hand, by (8), (11) and (12),

(39) $$C_2e^{-\lambda_\mu(\varphi(x))/n}\xi_{\mu i}\geqq e^{-\lambda(\varphi(x))/n}\xi_i\,,\quad \text{for}\quad 1\leqq i\leqq n\,,$$

whence

(40) $$C_2^n\sum_{\substack{i_1<\cdots<i_p\\ j_1<\cdots<j_q}}|f_{i_1\cdots i_p\bar{j}_1\cdots\bar{j}_q}|^2_{\lambda_\mu}(\prod_{i=1}^{n}\xi_{\mu i})$$

$$\geqq\sum_{\substack{i_1<\cdots<i_p\\ j_1<\cdots<j_q}}|f_{i_1\cdots i_p\bar{j}_1\cdots\bar{j}_q}|^2_{\lambda_\mu}e^{\lambda(\varphi(x))-\lambda_\mu(\varphi(x))}\prod_{i=1}^{n}\xi_i$$

$$=\sum_{\substack{i_1<\cdots<i_p\\ j_1<\cdots<j_q}}|f_{i_1\cdots i_p\bar{j}_1\cdots\bar{j}_q}|^2_\lambda\prod_{i=1}^{n}\xi_i\,,\quad \text{for}\quad \mu>\mu_0\,.$$

Combining (35), (36), (38) and (40), we obtain

$$(41)\qquad 3^{2(p+q+n)}C_2^{n+p+q}|f|^2_{\lambda_\mu}dv_{\lambda_\mu}|_x \geqq |f|^2_\lambda dv_\lambda|_x, \quad \text{for} \quad \mu>\mu_0.$$

Since $x\in X_c - X_{c'}$ was arbitrary, we have

$$(42)\qquad 3^{2(p+q+n)}C_2^{p+q+n}\int_{X_c-X_{c'}}|f|^2_{\lambda_\mu}dv_{\lambda_\mu} \geqq \int_{X_c-X_{c'}}|f|^2_\lambda dv_\lambda,$$

for $f\in L^{p,q}_{\lambda_\mu}(X, \mathbf{B})$, if $\mu>\mu_0$.

On the other hand, since $X_{c'}$ is relatively compact in X_c, $ds^2_{\lambda_\mu}$ and $ae^{-\lambda_\mu(\varphi)}$ converge to ds^2_λ and $ae^{-\lambda(\varphi)}$, respectively, uniformly on $X_{c'}$. Therefore, there is a constant C' such that for any $\mu\geqq 1$ and $f\in L^{p,q}_{\lambda_\mu}(X,\mathbf{B})$,

$$(43)\qquad C'\int_{X_{c'}}|f|^2_{\lambda_\mu}dv_{\lambda_\mu}\geqq \int_{X_{c'}}|f|^2_\lambda dv_\lambda.$$

Combining (42) with (43), we have

$$(44)\qquad \int_{X_c}|f|^2_\lambda dv_\lambda \leqq C_3'\int_{X_c}|f|^2_{\lambda_\mu}dv_{\lambda_\mu},$$

for $\mu\geqq\mu_0$, where $C_3'=\max\{3^{2(p+q+n)}C_2^{p+q+n}, C'\}$. Since X_c is relatively compact in X, there is a constant C_3 such that $C_3>C_3'$ and

$$(45)\qquad C_3'\int_{X_c}|f|^2_{\lambda_{\mu_0+1}}dv_{\lambda_{\mu_0+1}} \leqq C_3\int_{X_c}|f|^2_{\lambda_\mu}dv_{\lambda_\mu},$$

for $1\leqq\mu\leqq\mu_0$ and $f\in L^{p,q}_{\lambda_\mu}(X, \mathbf{B})$. Thus we obtain

$$(46)\qquad \|f|_{X_c}\|^2_\lambda \leqq C_3\|f\|^2_{\lambda_\mu},$$

for any μ and $f\in L^{p,q}_{\lambda_\mu}(X, \mathbf{B})$. Q.E.D.

§ 3. Proof of Theorem 1

Let $N^{p,q}_{loc}(X_c, \mathbf{B})$ $(R^{p,q}_{loc}(X_c, \mathbf{B}))$ be the subspace of $L^{p,q}_{loc}(X_c, \mathbf{B})$ consisting of the $\bar\partial$-closed ($\bar\partial$-exact) forms. When U_c is a domain of holomorphy in $\mathbf{C}^n$, we have $N^{p,q}_{\text{loc}}(U_c, \mathbf{B}) = R^{p,q}_{\text{loc}}(U_c, \mathbf{B})$ (cf. Theorem 4.2.5 in [2]). Therefore by a standard argument we have $H^{p,q}(X_c, \mathbf{B}) \cong N^{p,q}_{\text{loc}}(X_c, \mathbf{B})/R^{p,q}_{\text{loc}}(X_c, \mathbf{B})$ (cf. Theorem 7.4.1 in [2]).

Theorem 1 immediately follows from Proposition 2 and Lemma 1, combined with the above expression of $H^{p,q}(X_c, \mathbf{B})$. In fact, let $p+q>n$ and $f_\mu \in N^{p,q}_{\text{loc}}(X, \mathbf{B})$ $(\mu=1, 2, \cdots)$. By Lemma 1, there exists a C^∞ strictly convex increasing function λ_0 on $\mathbf{R}$ such that $f_\mu \in L^{p,q}_{\lambda_0}(X, \mathbf{B})$ for any μ. Therefore by Proposition 2, there exists $g \in L^{p,q-1}_{\lambda_0}(X, \mathbf{B})$ such that $\bar{\partial} g$ is a nontrivial linear combination of f_μ. Hence $H^{p,q}(X, \mathbf{B})$ cannot be infinite dimensional. Q.E.D.

§ 4. Proof of Theorem 2

Let $p+q>n$ and $X_d \supset \bar{U}$ $(d \in \mathbf{R})$. We assume that $d\varphi$ is nowhere zero on the boundary of X_d.

We set

$$\lambda(t) = -2n \log (d-t)$$

and

$$\begin{aligned} \hat{\lambda}_\mu(t) &= -2n \log (d+1/\mu - t), \quad \text{for} \quad t \leqq d+1/2\mu \\ &= 4n\mu(t-d-1/2\mu) - 4n \log 2\mu, \quad \text{for} \quad t > d+1/2\mu \end{aligned}$$

for $\mu = 1, 2, \cdots$.

Let $\tau(t)$ be a C^∞ convex increasing function such that $\int_0^\infty \sqrt{\tau''(t)}\, dt = \infty$ and the inclusion map

$$\iota_\tau : H^{p,q}_\tau(X, \mathbf{B}) \to H^{p,q}(X, \mathbf{B})$$

is surjective (cf. § 3). We may assume that $\tau(t) = 0$ on $(-\infty, d)$.

We set

$$\lambda_{\mu,k}(t) = k \int_{-\infty}^{\infty} \hat{\lambda}_\mu(s) \chi(k(s-t))\, ds + \tau(t).$$

Here χ is a nonnegative C^∞ function with compact support satisfying $\int_{-\infty}^{\infty} \chi(t)\, dt = 1$.

For every μ and k there exists $r_{\mu,k} \in \mathbf{R}$ such that

$$\lambda''_{\mu,k}(t) = \tau''(t) \quad \text{for} \quad t \geqq r_{\mu,k}.$$

Therefore $\int_0^\infty \sqrt{\lambda''_{\mu,k}(t)}\,dt=\infty$ and the inclusion maps

$$\iota_{\mu,k}: H^{p,q}_{\lambda_{\mu,k}}(X,\mathbf{B}) \to H^{p,q}(X,\mathbf{B})$$

are surjective.

We have

$$\int_{d-1}^{d} \sqrt{\lambda''(t)}\,dt=2n\int_{d-1}^{d}\frac{dt}{d-t}=\infty .$$

Since $\lambda^{(\nu)}_{\mu,k}(t)\to\lambda^{(\nu)}_{\mu}(t)$ $(k\to\infty)$ for every μ and ν, uniformly on $(-\infty,d)$, there exists a sequence $\{k_\mu\}$ of positive integers such that $\lambda_{\mu,k_\mu}(\mu=1,2,\cdots)$ and λ satisfy the requirements (6) to (10) in Lemma 3. Let $t<d$. Then

$$e^{-\hat{\lambda}_\mu(t)/n}\hat{\lambda}'_\mu(t)=d+1/\mu-t$$
$$>d-t=e^{-\lambda(t)/n}\lambda'(t)$$

and

$$e^{-\hat{\lambda}_\mu(t)/n}\hat{\lambda}''_\mu(t)=1$$
$$=e^{-\lambda(t)/n}\lambda''(t).$$

Therefore we may assume that λ_{μ,k_μ} and λ satisfy also (11) and (12) in Lemma 3. We set $\lambda_\mu=\lambda_{\mu,k_\mu}$.

By Lemma 3, λ and λ_μ satisfy the requirement in Proposition 4 and Proposition 6. Since $H^{p,q}_\lambda(X_d,\mathbf{B})$ is finite dimensional (cf. Proposition 2), from Proposition 6 we obtain the surjectivity of the restriction map

$$\rho_{d,\lambda}: H^{p,q}(X,\mathbf{B}) \to H^{p,q}_\lambda(X_d,\mathbf{B}).$$

On the other hand from Proposition 4, it follows that there exists μ_0 such that the restriction map

$$\rho^\mu_d: H^{p,q}_{\lambda_\mu}(X,\mathbf{B}) \to H^{p,q}_\lambda(X_d,\mathbf{B})$$

is injective if $\mu>\mu_0$.

Observing the following diagram

$$\begin{array}{ccc} H^{p,q}(X,\mathbf{B}) & \xrightarrow{\rho_{d,\lambda}} & H^{p,q}_{\lambda}(X_d,\mathbf{B}) \\ & \nwarrow \iota_{\mu} \quad \rho^{\mu}_{d} \nearrow & \\ & H^{p,q}_{\lambda_{\mu}}(X,\mathbf{B}) & \end{array}$$

we have the surjectivity of ρ^{μ}_{d}, since ι_{μ} and $\rho_{d,\lambda}$ are surjective. Since ρ^{μ}_{d} is injective for $\mu>\mu_0$, it follows that ρ^{μ}_{d} is bijective for $\mu>\mu_0$, whence ι_{μ} is bijective for $\mu>\mu_0$.

Hence $\rho_{d,\lambda}$ is bijective. Therefore, for every $c>d$, the restriction map $\rho^{c}_{d,\lambda}: H^{p,q}(X_c,\mathbf{B})\to H^{p,q}_{\lambda}(X_d,\mathbf{B})$ is bijective. Thus, the restriction map

$$\rho_c: H^{p,q}(X,\mathbf{B}) \to H^{p,q}(X_c,\mathbf{B})$$

is bijective, since $\rho^{c}_{d,\lambda}\circ\rho_c=\rho_{d,\lambda}$.

By Sard's theorem, for any X_c such that $X_c\supset\bar{U}$, there exists X_d such that $d\varphi$ is nowhere zero on the boundary of X_d and $X_c\supset X_d\supset\bar{U}$. Since U can be chosen arbitrarily close to K, ρ_c is bijective for every $c\in\mathbf{R}$ satisfying $X_c\supset K$. Q.E.D.

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