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A New Class of Domains of Holomorphy (III) (Reinhardt domains of holomorphy on a 3-dimensional analytic set with a $(\mathbf{C}^*)^2$-action)

By

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Introduction

The present paper is the third part of the study on domains of holomorphy under the same title ([1] and [2]). In this paper we shall give a supplement of the second paper and complete the discussions there. We remark on the notations in this paper. If we say an analytic set $\underline{M}$, we mean that it is a 3-dimensional analytic set which is defined by polynomials in $\mathbf{C}^N$ and with an isolated singularity p_0. p_0 is assumed to be the origin of $\mathbf{C}^N$.

In the first paper, a new class of domains of holomorphy is introduced and they are called L-manifolds ([1]). In the second paper, we have treated domains of holomorphy on 3-dimensional Stein spaces and we have given examples of L-manifolds ([2]). There we have shown that certain domains of holomorphy are L-manifolds under certain conditions. The condition is stated as the condition A and domains are called simple domains (see Introduction in [2]). Unfortunately function-theoretic meanings of the condition A and simple domains have not been given there.

In this paper we shall remove these additional restrictions and generalize the examples to a certain general situation. For this purpose we consider a 3-dimensional analytic set with an isolated singularity which admits a $(\mathbf{C}^*)^2$-action (§1). Then we see that the condition A can be satisfied on such an analytic set. Moreover, if we define the concept

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of Reinhardt domains with respect to the $(\mathbf{C}^*)^2$-action on the analytic set, then the discussions for simple domains can be applied to Reinhardt domains (§ 2). Then we can state our results as follows:

Main Theorem. *Let $\underline{M}$ be a 3-dimensional analytic set with an isolated singularity which admits a $(\mathbf{C}^*)^2$-action and let $\underline{\Delta}$ be a relatively compact Reihardt domain on $\underline{M}$. Then $\underline{\Delta}$ is a domain of holomorphy if and only if it is an* L-*manifold.*

Detailed results will be stated at the end of Section 2. We prepare a proposition (see (3. 2)) and by using this proposition, we shall show that we can reduce essentially the proof of Main Theorem to the results in [2].

§ 1. An Analytic Set with a $(\mathbf{C}^*)^2$-Action

In this section we define an analytic set $\underline{M}$ with an isolated singularity p_0 which admits a $(\mathbf{C}^*)^2$-action and construct the canonical resolution of the singularity.

Definition (1. 1). *The following transformation group τ: $(\mathbf{C}^*)^2 \to \mathrm{Aut}(\mathbf{C}^N)$ is called a $(\mathbf{C}^*)^2$-action on $\mathbf{C}^N$ if there exist two euclidian subspaces $\mathbf{C}^n$, $\mathbf{C}^m$ with $\mathbf{C}^N = \mathbf{C}^n \times \mathbf{C}^m$ such that τ can be given as follows*:

$$\tau_g : \begin{cases} z^{i\prime} = \lambda z^i & (i = 1, 2, \cdots, n) \\ w^{j\prime} = \mu w^j & (j = 1, 2, \cdots, m), \end{cases}$$

where $g = (\lambda, \mu) \in (\mathbf{C}^*)^2$ and z^i, w^j denote the coordinates of $\mathbf{C}^n$, $\mathbf{C}^m$ respectively.

Definition (1. 2). *An analytic set $\underline{M}$ in $\mathbf{C}^N$ is said to have a $(\mathbf{C}^*)^2$-action if* (1) *$\underline{M} \not\subset \mathbf{C}^n$ and $\underline{M} \not\subset \mathbf{C}^m$ hold and* (2) *if $\underline{p} \in \underline{M}$, then $\tau_g(\underline{p}) \in \underline{M}$ for every $g \in (\mathbf{C}^*)^2$.*

Let $\underline{M}$ be an analytic set which is defined by the following polynomials:

$$\underline{M}:\ F_j=0,\quad j=1,2,\cdots,r.$$

Then, if $\underline{M}$ has a $(\mathbf{C}^*)^2$-action, F_j may be assumed to have the following form:

$$F_j=\sum_k \varphi_k^{(j)}(z^1,z^2,\cdots,z^n)\,\psi_k^{(j)}(w^1,w^2,\cdots,w^m), \tag{1.3}$$

where $\varphi_k^{(j)}$ and $\psi_k^{(j)}$ are homogenous polynomials with respect to $z^1,z^2,\cdots,z^n$ and $w^1,w^2,\cdots,w^m$ respectively whose degrees depend only on j and not on k.

Now we shall construct a resolution of the singularity of $\underline{M}$ with a $(\mathbf{C}^*)^2$-action: From (1.3), $\{F_j=0\}$ can be regarded as an analytic set in $\mathbf{P}^{n-1}\times\mathbf{P}^{m-1}$, which we denote by A, i.e.,

$$A:\ F_j=0\,,\quad j=1,2,\cdots,r\qquad \text{in } \mathbf{P}^{n-1}\times\mathbf{P}^{m-1}.$$

Let $[e_1]$ be the canonical positive divisor in $\mathbf{P}^{n-1}$ whose dual bundle is denoted by F. Then we have the following monoidal transform at the center $0\in\mathbf{C}^n$,

$$\rho':\ F\to\mathbf{C}^n\,.$$

We denote the zero-section of F by O'. Then we see that

$$\rho'^{-1}(0)=O'\quad\text{and}\quad \rho': F-O'\cong\mathbf{C}^n-\{0\}.$$

In the same manner, from the canonical positive divisor $[e_2]$ in $\mathbf{P}^{m-1}$, we can get the monoidal transform

$$\rho'':\ G\to\mathbf{C}^m,$$

where G is the dual bundle of $[e_2]$. Then we see that

$$\rho''^{-1}(0)=O''\quad\text{and}\quad \rho'': G-O''\cong\mathbf{C}^m-\{0\},$$

where O'' is the zero-section of G. Now we set

$$M=F\oplus G_{|A},\quad \rho=\rho'\oplus\rho''_{|M}\quad\text{and}\quad \pi: M\to A. \tag{1.4}$$

Then we see that

$$\rho: M\to\underline{M}\quad\text{and}\quad \rho: M-O\cong\underline{M}-\{p_0\},$$

where O is the zero-section of M. Since $\underline{M}$ admits only one isolated singularity, it is easily seen that A is a compact non-singular algebraic curve and M gives a resolution of the singularity of $\underline{M}$.

Definition (1. 5). *ρ: $M \to \underline{M}$ is called the canonical resolution of the singularity of $\underline{M}$.*

§ 2. A Reinhardt Domain

We define Reinhardt domains on an analytic set with a $(\mathbf{C}^*)^2$-action. We write

$$U = \{\theta = (\lambda, \mu) \in (\mathbf{C}^*)^2 \colon |\lambda| = 1 \quad \text{and} \quad |\mu| = 1\}.$$

Restricting the $(\mathbf{C}^*)^2$-action on U, we get $U \to \mathrm{Aut}(\mathbf{C}^N)$.

Definition (2. 1). *A relatively compact domain $\underline{\Delta}$ on $\underline{M}$ is called a Reinhardt domain if the following holds: If $\underline{p} \in \underline{\Delta}$, then $\tau_\theta(\underline{p}) \in \underline{\Delta}$ for any $\theta \in U$.*

We assume, for the sake of simplicity, that the following condition is satisfied when we talk about a domain $\underline{\Delta}$ with $p_0 \in \partial \underline{\Delta}$: For any neighborhood U of p_0, there exists uniquely a connected component $\underline{\Delta}'$ of $\underline{\Delta} \cap U$ such that $p_0 \in \partial \underline{\Delta}'$ holds.

Reinhardt domains can be classified as follows:

Definition (2. 2). *Let $\underline{\Delta}$ be a Reinhardt domain.*

(I) *$\underline{\Delta}$ is called of Type I, if $p_0 \notin \partial \underline{\Delta}$.*

Furthermore, a domain with $p_0 \in \partial \underline{\Delta}$ can be classified as follows:

(II) *$\underline{\Delta}$ is called of Type II, if there exists a point $\underline{p} \in M$ ($\underline{p} \neq p_0$) such that $\tau_g(\underline{p}) \cap \underline{\Delta} = \emptyset$ for any $g \in (\mathbf{C}^*)^2$.*

(III) *Otherwise, $\underline{\Delta}$ is called of Type III.*

Let ρ: $M \to \underline{M}$ be the canonical resolution of $\underline{M}$. Then M can be written as in (1. 3). With respect to a certain covering $\{U_\lambda\}$ of A, we denote the fibre coordinates of F and G on U_λ by ζ_λ and η_λ respectively. We define analytic sets S and H on M by

(2. 3) $S=\{\zeta_\lambda=0\}$ and $H=\{\eta_\lambda=0\}$.

We set

$$\varDelta=\overline{(\rho^{-1}(\underline{\varDelta}))}^{\circ},$$

where $\bar{E}$ means the closure of E and E° means the open kernel of E. In the following, if we say that $\varDelta$ in M is a Reinhardt domain, $\varDelta$ satisfies a similar condition to (2. 1) with respect to the $(\mathbf{C}^*)^2$-action along the fibres of M. Then if $\underline{\varDelta}$ is a Reinhart domain in $\underline{M}$, then $\varDelta$ is a Reinhardt domain. Corresponding to the classification of Reinhardt domains $\underline{\varDelta}$ in $\underline{M}$, we obtain

(2. 4) (1) *$\underline{\varDelta}$ is of Type I* $\Leftrightarrow \partial\varDelta\cap A=\emptyset$,

(2) *$\underline{\varDelta}$ is of Type II* $\Leftrightarrow \partial\varDelta\cap A\neq\emptyset$ *and there exists a point p in A such that* $\pi^{-1}(p)\cap\varDelta=\emptyset$ *holds,*

(3) *$\underline{\varDelta}$ is of Type III* $\Leftrightarrow \partial\varDelta\cap A\neq\emptyset$ *and* $\pi^{-1}(p)\cap\varDelta\neq\emptyset$ *holds for every point* $p\in A$.

In the following we classify domains of Type III into two classes:

(4) *$\underline{\varDelta}$ is called of Type $III_{(1)}$* $\Leftrightarrow \varDelta\cap S\neq\emptyset$ *or* $\varDelta\cap H\neq\emptyset$,

(5) *$\underline{\varDelta}$ is called of type $III_{(2)}$* $\Leftrightarrow \varDelta\cap S=\emptyset$ *and* $\varDelta\cap H=\emptyset$ *hold.*

Then we can state our results as follows:

Theorem (2. 5). *Let $\underline{\varDelta}$ be a Reinhardt domain on $\underline{M}$ which is a domain of holomorphy. Then $\underline{\varDelta}$ is an* L*-manifold. Moreover, we can obtain the following results*:

(I) *If $\underline{\varDelta}$ is of Type I, then $\underline{\varDelta}$ is Stein.*

(II) *If $\underline{\varDelta}$ is of Type II, then $\varDelta$ is Stein.*

(III) (α) *If $\underline{\varDelta}$ is of Type $III_{(1)}$, then* E_M *is of infinite order and the algebra of holomorphic functions on $\underline{\varDelta}$ is not a Stein algebra.*

(β) *If $\underline{\varDelta}$ is of Type $III_{(2)}$, then* E_M *is of infinite order and $\underline{\varDelta}$ is not Stein.*

As for the definition of E_M, see [2], pages 527 and 528. Then we can summarize our results in the following table:

Type \ E_M	of finite order	of infinite order
I	Stein	Stein
II	Stein	Stein
$\mathrm{III}_{(1)}$	——	non Stein with a non-Stein algebra
$\mathrm{III}_{(2)}$	——	non Stein

§ 3. Proof of Theorem (2. 5)

In this section we give a proof of Theorem (2. 5). At first we prove the following

Proposition (3. 1). *In the case of Type I or II, a domain of holomorphy is Stein.*

Proof. We have given a proof for a domain of Type I in (3. 2) in [2]. Hence we consider a domain of Type II. From (2) in (2. 4), there exists a point $p \in A$ such that $\varDelta$ is contained in $M-\pi^{-1}(p)$. Since $A-p$ is Stein, $M-\pi^{-1}(p)$ is Stein. Hence $\varDelta$ is Stein. If $\varDelta \cap A=\emptyset$, then $\underline{\varDelta}$ is also Stein. If $\varDelta \cap A \neq \emptyset$, then $\underline{\varDelta}$ is not a domain of holomorphy, because codimension of A is two. In this case the K-hull of $\underline{\varDelta}$ is $\varDelta$ (see (2. 5) in [1]). Hence we prove the assertion.

Here we prepare a proposition. We choose an integer $l\,(l>0)$ and take a metric of $\{a_\lambda\}$ of $F \otimes G^{-l}$ with respect to some covering $\{U_\lambda\}$ of A (see (1. 4)). We denote the fiber coordinates of F and G as in (2. 3). We put

$$\phi_{1,l}=a_\lambda|\zeta_\lambda|^2/|\eta_\lambda|^{2l}.$$

Then we can show the following

Proposition (3. 2). *Let $\varDelta$ be a Reinhardt domain on M such that $\pi^{-1}(p) \cap \varDelta \neq O$ for every $p \in A$. Moreover, we assume that $\varDelta \cap S \neq \emptyset$ holds. Then we can find an integer $l\,(l>0)$ and a neighborhood U of A such that every holomorphic function on $\varDelta$ can be extended to a domain satisfying $\{\phi_{1,l}<\varepsilon\} \cap U$, where U is a neighborhood of A*

and ε *is a positive number.*

Proof. We make a resolution of singularities of indeterminancy of $\phi_{1,l}$ as in Section 4 in [2] (see (II) in p. 551), which we denoted by $Q^{(l)}: M_{(l)} \to M$. We write

$$Q^{(l)^{-1}}(A) = \Sigma_{(l)}, \quad \overline{Q^{(l)^{-1}}(S) - \Sigma_{(l)}} = S_1^* \quad \text{and} \quad \overline{(Q^{(l)^{-1}}(\varDelta))}^{\circ} = \varDelta_{(l)}.$$

As we have defined in [2] (see p. 552), we choose local coordinates z_λ, $\xi^{(2)}_{\lambda|l}$ and $u^{(2)}_{\lambda|l}$ such that

$$\{u^{(2)}_{\lambda|l} = 0\} = S_1^* \quad \text{and} \quad \{\xi^{(2)}_{\lambda|l} = 0\} = L_l.$$

We set

$$\Omega^\lambda_l = \{(z_\lambda, \xi^{(2)}_{\lambda|l}, u^{(2)}_{\lambda|l}) : |\xi^{(2)}_{\lambda|l}| < +\infty \quad \text{and} \quad |u^{(2)}_{\lambda|l}| < +\infty\}$$

and $\Omega_l = \bigcup_\lambda \Omega^\lambda_l$. We take a holomorphic function f on $\varDelta$. Since $\varDelta$ is a Reinhardt domain, we see that

$$Q^{(l)*}f = \sum a_\lambda^{(j,k)}(z_\lambda)\, \xi^{(2)^j}_{\lambda|l}\, u^{(2)^k}_{\lambda|l} \qquad \text{on } \varDelta_{(l)} \cap \Omega^l_\lambda,$$

where

$$\{a_\lambda^{(j,k)}(z_\lambda)\} \in H^\circ(A, \mathcal{O}([L_l]^{-j} \otimes [S_1^*]^{-k}_{|A})).$$

Since $S_1^* \cap \varDelta \neq \emptyset$, we see that $k \geqq 0$. If we choose a positive integer l very large, then from (i), (1) and (ii), (1) in (4. 14) in [2] (also see (4. 12) in [2]), we get

$$[S_1^*]_{|A} > 0 \quad \text{and} \quad [L_l]_{|A} < 0.$$

Hence, if $j < 0$, we see that $[S_1^*]^{-k} \otimes [L_l]^{-j}_{|A} < 0$. Therefore we see that $\{a_\lambda^{(j,k)}\} = 0$ in this case, which proves the assertion in (3. 2).

Now we return to the proof of our Theorem in the case of domains of Type III. At first we consider domains of Type $\mathrm{III}_{(1)}$. From the construction of the canonical resolution $\rho: M \to \underline{M}$, we see that M satisfies the conditions $A_{(0)}$ and $B_{(0)}$ in page 547 in [2]. From (3. 2), we see that $\varDelta$ satisfies the condition $C_{(0)}$ in page 554 in [2]. Hence following the discussions in Sections 5 and 6 in [2], we get our conclusions in this case. Next we treat domains of Type $\mathrm{III}_{(2)}$. We take a holomorphic function f on $\varDelta$. Then f can be written as

$$f=\sum a_\lambda^{(j,k)}(z_\lambda)\xi_\lambda^j\eta_\lambda^k ,$$

where

$$\{a_\lambda^{(j,k)}\}\in H^\circ(A, \mathcal{O}([F]^{-j}\otimes[G]^{-k})).$$

From the assumption on a domain of Type $\mathrm{III}_{(2)}$, we have $\Delta\cap S=\emptyset$ and $\Delta\cap H=\emptyset$. Since F and G are negative, we see that both j and k can never be negative at the same time. Hence we have the following decompositions of f:

$$f=f_1+f_2+f_3$$

where

f_1: the singular part of f which is singular at H,

f_2: the singular part of f which is singular at S,

f_3: the holomorphic part of f.

Then we see that f_1 (resp. f_2) is a holomorphic function at S (resp. H). Therefore by using (3. 2) and following the proof of Theorem (5. 1) in [2], we see that there exist small positive constants ε, ε' and a neighborhood U of A such that f_1 (resp. f_2) is holomorphic on $\{\phi^*<\varepsilon\}\cap U$ (resp. on $\{\phi^{*^{-1}}<\varepsilon'\}\cap U$), where ϕ^* is the characteristic function of M (see (1. 6) in [2]). We construct the resolution of singularities of indeterminancy of ϕ^*, μ: $M_*\to M$. We follow the notations in (5. 13) in [2]. By using (6. 1) in [2], we can find positive constants c, c' such that

$$E_*\cap\Delta_*=\{c'<h_{|E_*}<c\} , \tag{3. 3}$$

where E_* is the exceptional set which is inserted at the final step of the construction of μ and $\Delta_*=\overline{(\mu^{-1}(\Delta))}^\circ$. Since Δ_* is a Reinhardt domain, we see that

$$\Delta_*\subset\{c'<h<c\} . \tag{3. 4}$$

From (3. 3), (3. 4) and by using (3. 5) in [2], we see that Δ_* is a weakly 1-complete manifold. Moreover, $[E_*]$ is a negative line bundle on Δ_*. From these facts, we see that Δ_* is a B-resolution of $\underline{\Delta}$ if E_M is of infinite order (see (5. 13) in [2]). Hence we see that $\underline{\Delta}$ is an L-manifold if E_M is of infinite order. Detailed discussions can be done in a similar as in Section 6 in [2]. Therefore we need not repeat them.

Hereby we complete the proof of Theorem (2. 5).

References

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[2] ———, A new class of domains of holomorphy (II) (Domains of holomorphy on a three dimensional Stein space with an isolated singularity), *ibid.*, **13** (1977), 523-571.