Publ. RIMS, Kyoto Univ.
17 (1981), 137-151

Supplement to Holonomic Quantum Fields. IV

By

Michio JIMBO*, Tetsuji MIWA* and Mikio SATO*

This note is a supplementary paragraph to our preceding paper IV [1]. Our aim here is to relate the field operators in [1], constructed directly from the commutation relations, to the known models of Lagrangian field theory in two space-time dimensions; namely the Federbush model ([2]) and the massless Thirring model ([3]). In fact this connection has been known in the literature ([4] [5]), which we have come to know only lately. We wish to thank Professor N. Nakanishi for drawing our attention to the articles [5] [6]. As such, the content of this note is not essentially new, except for the exact computation of the n-point functions for the Federbush model (see § 3).

The plan of this note is as follows. In Section 1 we prepare several formulas needed in subsequent paragraphs. In Section 2 we give the operator solutions of the Federbush model ([4] [5] [6]) in terms of the operators introduced in Section 1. By identifying the current with the free one we check the validity of the microcausality and the equations of motion. In Section 3 we calculate their asymptotic fields, S-matrix ([5]) and the n-point functions by appealing to the results of IV [1]. In Section 4 we follow the analogue of Section 2 for the Thirring model, by using the operators introduced in II [7].

§ 1.

Let $\psi(x)={}^t(\psi_+(x),\psi_-(x))$, $\psi^*(x)={}^t(\psi^*_+(x),\psi^*_-(x))$ denote the free Dirac fields with positive mass m in two space-time dimensions. We choose the normalization(*)

Received November 13, 1979.

* Research Institute for Mathematical Sciences, Kyoto University, Kyoto 606, Japan.

(*) Our definition here differs from [1] by a factor $\sqrt{m/2}$.

(1. 1) $$\psi_\pm(x) = \sqrt{\frac{m}{2}} \int \underline{du} \sqrt{0+iu}^{\pm 1} e^{-im(x^- u + x^+ u^{-1})} \psi(u) = \psi_\pm(x; m),$$

$$\psi_\pm^*(x) = \sqrt{\frac{m}{2}} \int \underline{du} \sqrt{0+iu}^{\pm 1} e^{-im(x^- u + x^+ u^{-1})} \psi^*(u) = \psi_\pm^*(x; m),$$

so that the canonical equal time anti-commutation relations hold:

(1. 2) $$[\psi_\varepsilon(0, x^1), \psi_{\varepsilon'}^*(0, x^{1\prime})]_+ = \delta_{\varepsilon\varepsilon'} \delta(x^1 - x^{1\prime})$$

$$[\psi_\varepsilon(0, x^1), \psi_{\varepsilon'}(0, x^{1\prime})]_+ = 0, \ [\psi_\varepsilon^*(0, x^1), \psi_{\varepsilon'}^*(0, x^{1\prime})]_+ = 0 \qquad (\varepsilon, \varepsilon' = \pm).$$

In (1. 2) $\psi(u)$, $\psi^*(u)$ signify the creation $(u<0)$-annihilation $(u>0)$ operators in (4. 3. 48), (4. 3. 49) of [1], carrying the energy-momentum $p = (p^0, p^1) = \left(m\frac{u+u^{-1}}{2}, m\frac{u-u^{-1}}{2}\right)$ on the mass shell.

In the sequel we adopt the following convention:

(1. 3) $$\gamma^0 = \begin{pmatrix} & i \\ -i & \end{pmatrix}, \ \gamma^1 = \begin{pmatrix} & -i \\ -i & \end{pmatrix}, \ \gamma^5 = \gamma^0\gamma^1 = \begin{pmatrix} 1 & \\ & -1 \end{pmatrix}$$

$$\gamma^\pm = \frac{1}{2}(\gamma^0 \pm \gamma^1), \quad x^\pm = \frac{1}{2}(x^0 \pm x^1)$$

$$i\partial\!\!\!/ = i(\gamma^0\partial_0 + \gamma^1\partial_1) = \begin{pmatrix} 0 & -\partial_0 + \partial_1 \\ \partial_0 + \partial_1 & 0 \end{pmatrix}$$

$$\bar{\psi}(x) = {}^t\psi^*(x)\gamma^0 = (-i\psi_-^*(x), \ i\psi_+^*(x)).$$

The free current $j^\pm(x)$ and pseudo-current $\tilde{j}^\pm(x)$ are, by definition,

(1. 4) $$j^\pm(x) = :\bar{\psi}(x)\gamma^\pm\psi(x):, \ \tilde{j}^\pm(x) = :\bar{\psi}(x)\gamma^5\gamma^\pm\psi(x):$$

$$j^\pm(x) = \mp\tilde{j}^\pm(x) = :\psi_\pm^*(x)\psi_\pm(x):.$$

Let now $\varphi_F(a) = \varphi_F(a; l', l'')$ be a field operator satisfying the following commutation relations with the free fields:

(1. 5) $$\varphi_F(a)\psi_\pm(x) = \begin{cases} e^{2\pi i l'}\psi_\pm(x)\varphi_F(a) & \text{if } x^+<a^+ \text{ and } x^->a^- \\ e^{2\pi i l''}\psi_\pm(x)\varphi_F(a) & \text{if } x^+>a^+ \text{ and } x^-<a^- \end{cases}$$

$$\varphi_F(a)\psi_\pm^*(x) = \begin{cases} e^{-2\pi i l'}\psi_\pm^*(x)\varphi_F(a) & \text{if } x^+<a^+ \text{ and } x^->a^- \\ e^{-2\pi i l''}\psi_\pm^*(x)\varphi_F(a) & \text{if } x^+>a^+ \text{ and } x^-<a^-. \end{cases}$$

In the case $l''=0$ this type of operator has been constructed in [1] ((4. 3. 67), (4. 3. 68) with l shifted by a half odd integer). The general case is obtained by multiplying $e^{-2\pi i l'' \mathsf{N}}$ with $\varphi_F(a; l'-l''+1/2)$ in [1], where $\mathsf{N}=\int \underline{du} : \psi^*(-u)\psi(u):$. The result reads as follows.

$$(1.6)\qquad \varphi_F(a; l', l'') =: e^{\rho_F(a; l', l'')/2}:$$

$$\rho_F(a; l', l'')/2$$

$$= \iint \underline{du}\, \underline{du'} R_F(u, u'; l', l'') e^{-im(a^-(u+u')+a^+(u^{-1}+u'^{-1}))}\psi^*(u)\psi(u')$$

$$R_F(u, u'; l', l'')$$

$$= -2i \sin\pi(l'-l'')\cdot e^{\pi i l''(\varepsilon(u)-\varepsilon(u'))}(u-i0)^{l'-l''+1/2}(u'-i0)^{-l'+l''+1/2}$$

$$\times P\frac{1}{u+u'} - i(e^{\pi i l'\varepsilon(u)}\sin\pi l' + e^{\pi i l''\varepsilon(u)}\sin\pi l'')2\pi|u|\delta(u+u')$$

$$= \overline{R_F(-u', -u; -\bar{l}', -\bar{l}'')}.$$

Formulas needed to derive (1.6) are given in the Appendix. Note that the n-point functions for $\varphi_F(a; l', l'')$'s are essentially those in IV since they commute with N. For instance $\langle\varphi_F(a_1; l_1', l_1'')\varphi_F(a_2; l_2', l_2'')\rangle = \langle e^{-2\pi i l_1''\mathsf{N}}\varphi_F(a_1; l_1'-l_1''+1/2)e^{-2\pi i l_2''\mathsf{N}}\varphi_F(a_2; l_2'-l_2''+1/2)\rangle = \langle\varphi_F(a_1; l_1'-l_1''+1/2)\varphi_F(a_2; l_2'-l_2''+1/2)\rangle$, and so forth.

For later reference we calculate the short distance behavior of the product $j^\pm(x)\varphi_F(a; l', l'')$. We have

$$(1.7)\qquad j^\pm(x)\varphi_F(a; l', l'') = \langle j^\pm(x)\varphi_F(a; l', l'')\rangle\varphi_F(a; l', l'')$$
$$+ :\psi_\pm^*(x, a; l', l'')\psi_\pm(x, a; l', l'')e^{\rho_F(a; l', l'')/2}:$$

where

$$(1.8)\qquad \langle j^\pm(x)\varphi_F(a; l', l'')\rangle = \frac{\mp im^2\sin\pi(l'-l'')}{\pi^2}(x-a)^\pm$$
$$\times(K_{-l'+l''}(mr)K_{l'-l''}(mr) - K_{l'-l''+1}(mr)K_{-l'+l''+1}(mr)),$$
$$(r=2\sqrt{-(x-a)^+(x-a)^-})$$

$$(1.9)\qquad \psi_\pm^*(x, a; l', l'')$$

$$= \sum_{j=0}^{\infty}\phi_{l+j}^*(a; l'')\cdot w_{l+j}(-(x-a)^-+i0, (x-a)^+-i0)_\pm$$

$$+ \sum_{j=1}^{\infty}\phi_{l-j}^*(a; l'')\cdot w_{-l+j}^*(-(x-a)^-+i0, (x-a)^+-i0)_\pm$$

$$\psi_{\pm}(x, a; l', l'')$$

$$= \sum_{j=1}^{\infty} \psi_{-l+j}(a; l'') \cdot w_{-l+j}(-(x-a)^{-}+i0, (x-a)^{+}-i0)_{\pm}$$

$$+ \sum_{j=0}^{\infty} \psi_{-l-j}(a; l'') \cdot w^{*}_{l+j}(-(x-a)^{-}+i0, (x-a)^{+}-i0)_{\pm}$$

with $l=l'-l''+1/2$ and

$$(1.10) \quad \psi^{*}_{k}(a;l'') = \sqrt{\frac{m}{2}} \int \underline{du} (0+iu)^{k} e^{+2\pi i l'' \theta(u)} \psi^{*}(u) e^{-im(a^{-}u+a^{+}u^{-1})}$$

$$\psi_{k}(a; l'') = \sqrt{\frac{m}{2}} \int \underline{du} (0+iu)^{k} e^{-2\pi i l'' \theta(u)} \psi(u) e^{-im(a^{-}u+a^{+}u^{-1})}.$$

The functions $w_l = {}^t(w_{l+}, w_{l-})$, $w^{*}_l = {}^t(w^{*}_{l+}, w^{*}_{l-})$ appearing in (1. 9) are those in IV [1].

In particular, assuming $0<|l'-l''|<1/2$ we have from (1. 7)~(1. 10)

$$(1.11) \quad \langle j^{\pm}(x)\varphi_F(a; l', l'')\rangle = \pm \frac{l'-l''}{4\pi i} \frac{1}{(x-a)^{\mp}} + 0(1) \quad (x \to a)$$

$$(1.12) \quad \lim_{x\to a} : \psi^{*}_{\pm}(x, a; l', l'')\psi_{\pm}(x, a; l', l'') e^{\rho_F(a; l', l'')/2}:$$

$$= \lim_{x\to a} : \Bigg(\psi^{*}_{l}(a;l'') \frac{(-m(x-a)^{-})^{l\mp1/2}}{\left(l\mp\frac{1}{2}\right)!}$$

$$+ \cdots + \psi^{*}_{l-1}(a;l'') \frac{(m(x-a)^{+})^{1-l\pm1/2}}{\left(1-l\pm\frac{1}{2}\right)!} + \cdots\Bigg)$$

$$\times \Bigg(\psi_{1-l}(a; l'') \cdot \frac{(-m(x-a)^{-})^{1-l\mp1/2}}{\left(1-l\mp\frac{1}{2}\right)!}$$

$$+ \cdots + \psi_{-l}(a; l'') \frac{(m(x-a)^{+})^{l\pm1/2}}{\left(l\pm\frac{1}{2}\right)!} + \cdots\Bigg) e^{\rho_F(a; l', l'')/2}:$$

$$= \frac{\sin \pi(l'-l'')}{\pi(l'-l'')} : \psi^{*}_{l'-l''\pm1/2}(a; l'')\psi_{-l'+l''\mp1/2}(a; l'') e^{\rho_F(a; l', l'')/2}:$$

$$= \pm \frac{1}{4\pi i(l'-l'')} \frac{\partial}{\partial a^{\mp}} \varphi_F(a; l', l'').$$

Hence

(1.13) $$\lim_{\varepsilon\to 0}\frac{1}{2}(j^{\pm}(a+\varepsilon)\varphi_F(a;l',l'')+j^{\pm}(a-\varepsilon)\varphi_F(a;l',l''))$$
$$=\pm\frac{1}{4\pi i(l'-l'')}\frac{\partial}{\partial a^{\mp}}\varphi_F(a;l',l'')\qquad \left(0<|l'-l''|<\frac{1}{2}\right).$$

§ 2.

The Federbush model ([2]) is a 2-dimensional model theory in which two species $\Psi_\alpha(x)={}^t(\Psi_{\alpha+}(x),\Psi_{\alpha-}(x))$, $\bar{\Psi}_\alpha(x)=(\Psi^*_{\alpha+}(x),\Psi^*_{\alpha-}(x))\gamma^0$ ($\alpha=I, II$) of massive Dirac fields are interacting via current-pseudocurrent interaction. Its Lagrangian is written as

(2.1) $$\mathcal{L}_{FED}=\mathcal{L}_0+\mathcal{L}_{int},$$
$$\mathcal{L}_0=\sum_{\alpha=I,II}\bar{\Psi}_\alpha(x)\,(i\partial\!\!\!/-m_\alpha)\Psi_\alpha(x)$$
$$\mathcal{L}_{int}=-2g(J_I^-(x)\tilde{J}_{II}^+(x)+J_I^+(x)\tilde{J}_{II}^-(x))$$
$$=-g\sum_{\mu,\nu=0,1}\varepsilon_{\mu\nu}J_I^\mu(x)J_{II}^\nu(x)$$

where $J_\alpha^\pm(x)=\bar{\Psi}_\alpha(x)\gamma^\pm\Psi_\alpha(x)=\mp\tilde{J}_\alpha^\pm(x)=1/2(J_\alpha^0(x)\pm J_\alpha^1(x))$ ($\alpha=I, II$), $\varepsilon_{\mu\nu}=-\varepsilon_{\nu\mu}$, $\varepsilon_{10}=1$. Accordingly the equations of motion are

(2.2) $$(i\partial\!\!\!/-m_I)\Psi_I(x)-2g(\tilde{J}_{II}^+(x)\gamma^-+\tilde{J}_{II}^-(x)\gamma^+)\Psi_I(x)=0$$
$$(i\partial\!\!\!/-m_{II})\Psi_{II}(x)+2g(\tilde{J}_I^+(x)\gamma^-+\tilde{J}_I^-(x)\gamma^+)\Psi_{II}(x)=0\,.$$

As has been pointed out previously in the literature ([3] [4] [5]) this model has an explicit operator solution. Namely let $\psi_\alpha(x;m_\alpha)$ ($\alpha=I, II$) denote independent(*) free fermion fields of mass m_α, and let $\varphi_{F\alpha}(x;l',l'';m_\alpha)$ be the corresponding operators introduced in Section 1. Then the products

(2.3) $$\Psi_I(x)=\psi_I(x;m_I)\cdot\varphi_{FII}(x;l',l'';m_{II})$$
$$\Psi_{II}(x)=\psi_{II}(x;m_{II})\cdot\varphi_{FI}(x;l'',l';m_I)$$
$$\bar{\Psi}_I(x)=\bar{\psi}_I(x;m_I)\cdot\varphi_{FII}(x;-l',-l'';m_{II})$$
$$\bar{\Psi}_{II}(x)=\bar{\psi}_{II}(x;m_{II})\cdot\varphi_{FI}(x;-l'',-l';m_I)$$

(*) That is, they are regarded as elements of a Clifford algebra $A(W_I\oplus W_{II})\cong A(W_I)\otimes A(W_{II})$ of a direct sum of orthogonal spaces W_I, W_{II}. In particular ψ_I or ψ_I^* and ψ_{II} or ψ_{II}^* totally anticommute with each other.

provide an exact solution of the Federbush model (2. 1). The parameters l', l'' are related to the coupling constant g through

$$g=2\pi(l'-l''). \tag{2.4}$$

In the sequel we shall check the validity of the equations of motion (2. 2), examining the precise meaning of the current, product at an equal space-time point, etc.

(1) **Microcausality.** Making use of the commutation relation (1. 5) and the local commutativity of φ_F-fields (cf. pp. 102~107 [1]), we may verify the microcausality for $\Psi_I(x), \Psi_{II}(x)$. For example if $(x-x')^+<0$ and $(x-x')^->0$ we have

$$\begin{aligned}
&\Psi_I(x)\Psi_{II}(x')\\
&=\psi_I(x;m_I)\varphi_{FII}(x;l',l'';m_{II})\cdot\psi_{II}(x';m_{II})\varphi_{FI}(x';l'',l';m_I)\\
&=e^{2\pi il''}\psi_I(x;m_I)\psi_{II}(x';m_{II})\varphi_{FII}(x;l',l'',m_{II})\varphi_{FI}(x';l'',l';m_I)\\
&=-e^{2\pi il''}\psi_{II}(x';m_{II})\psi_I(x;m_I)\varphi_{FI}(x';l'',l',m_I)\varphi_{FII}(x;l',l'';m_{II})\\
&=-\psi_{II}(x';m_{II})\varphi_{FI}(x';l'',l';m_I)\psi_I(x;m_I)\varphi_{FII}(x;l',l'',m_{II})\\
&=-\Psi_{II}(x')\Psi_I(x).
\end{aligned}$$

Similar argument shows that any two of $\Psi_I(x)$, $\Psi_{II}(x)$, $\overline{\Psi}_I(x)$ and $\overline{\Psi}_{II}(x)$ anticommute for spacelike seperation of variables.

(2) **Current.** Formally the current for $\Psi_I(x)$ is given by $J_I^{\pm}(x)=\overline{\Psi}_I(x)\gamma^{\pm}\Psi_I(x)=\psi_{I\pm}^*(x)\psi_{I\pm}(x)\cdot\varphi_{FII}(x;-l',-l'';m_{II})\varphi_{FII}(x;l',l'';m_{II})$. The second factor, if it has a meaning, should again be an element of the Clifford group whose induced rotation is 1. It is natural therefore to regard this factor as a constant. Thus *we identify the interacting currents with the free ones*: $J_\alpha^{\pm}(x)\doteqdot j_\alpha^{\pm}(x)$ $(\alpha=I, II)$.

(3) **Equation of motion.** Applying the Dirac operator $i\partial-m_I$ to (2. 3) we obtain

$$\begin{aligned}(i\partial-m_I)\Psi_I(x)&=(i\partial-m_I)\psi_I(x;m_I)\cdot\varphi_{FII}(x;l',l'';m_{II})\\&\quad+\psi_I(x,m_I)\cdot i\partial\varphi_{FII}(x;l',l'';m_{II})\end{aligned} \tag{2.5}$$

$$=\psi_I(x;m_I)\cdot i\Big(\gamma^+\frac{\partial}{\partial x^+}\varphi_{FII}(x)+\gamma^-\frac{\partial}{\partial x^-}\varphi_{FII}(x)\Big).$$

On the other hand we have from (1. 13)

$$\frac{\partial}{\partial x^\pm}\varphi_F(x;l',l'';m_{II})=-2\,ig\tilde{j}_{II}^{\mp}(x)\cdot\varphi_F(x;l',l'';m_{II}) \tag{2.6}$$

where the product of the right hand side is defined through the point separating limit (1. 13). This shows that

$$(i\partial\!\!\!/-m_I)\Psi_I(x)=2g(\gamma^+\tilde{j}_{II}^-(x)+\gamma^-\tilde{j}_{II}^+(x))\cdot\Psi_I(x)$$
$$=2g(\gamma^+\tilde{J}_{II}^-(x)+\gamma^-\tilde{J}_{II}^+(x))\cdot\Psi_I(x).$$

The second equation of (2. 2) is checked similarly.

All these considerations have a straightfoward extension to include many species of fermions. Namely a model with Lagrangian

$$\mathcal{L}=\sum_{\alpha=1}^{A}\overline{\Psi}_\alpha(x)(i\partial\!\!\!/-m_\alpha)\Psi_\alpha(x)-\frac{1}{2}\sum_{\alpha,\beta=1}^{A}\lambda_{\alpha\beta}(J_\alpha^-(x)\tilde{J}_\beta^+(x)+J_\alpha^+(x)\tilde{J}_\beta^-(x)) \tag{2.7}$$

has an operator solution

$$\Psi_\alpha(x)=\psi_\alpha(x)\prod_{\beta(\neq\alpha)}\varphi_{F\beta}(x;l_{\alpha\beta},l_{\beta\alpha})$$
$$\overline{\Psi}_\alpha(x)=\overline{\psi}_\alpha(x)\prod_{\beta(\neq\alpha)}\varphi_{F\beta}(x;-l_{\alpha\beta},-l_{\beta\alpha}) \tag{2.8}$$

where

$$\lambda_{\alpha\beta}=2\pi(l_{\alpha\beta}-l_{\beta\alpha})=-\lambda_{\beta\alpha}\,. \tag{2.9}$$

§ 3.

The asymptotic fields for $\Psi_\alpha(x)$ are calculated in the same way as in Proposition 4. 6. 2 of IV [1]. We merely state the results here.

Define $\Psi_{\alpha\,{in \atop out}}(u)$ to be

$$\sqrt{\frac{m}{2}}\sqrt{0+iu}^{\,\varepsilon}\,\Psi_{\alpha\,{in \atop out}}(u)=\lim_{t\to\mp\infty}\frac{i\varepsilon(u)}{2}\int_{x^0=t}dx^1\Big(e^{im_\alpha(x^-u+x^+u^{-1})}\times\frac{\partial}{\partial x^0}\Psi_{\alpha\varepsilon}(x)-\Psi_{\alpha\varepsilon}(x)\frac{\partial}{\partial x^0}e^{im_\alpha(x^-u+x^+u^{-1})}\Big) \tag{3.1}$$

where $\alpha = I, II$ and $\varepsilon = \pm$ refers to the spinor component. We also define $\Psi^*_{\alpha \underset{out}{in}}(u)$ by replacing $\Psi_{\alpha\varepsilon}(x)$ by $\Psi^*_{\alpha\varepsilon}(x)$ in the right hand side of (3. 1). We set further

$$N^+_{\alpha u} = \int_0^{|u|} \underline{du'} (\psi^{*\dagger}_\alpha(u')\psi^*_\alpha(u') - \psi^\dagger_\alpha(u')\psi_\alpha(u')) \tag{3. 2}$$

$$N^-_{\alpha u} = \int_{|u|}^{\infty} \underline{du'} (\psi^{*\dagger}_\alpha(u')\psi^*_\alpha(u') - \psi^\dagger_\alpha(u')\psi_\alpha(u'))$$

$$N_\alpha(u; l', l'') = l' N^+_{\alpha u} + l'' N^-_{\alpha u}$$

$$(\alpha = I, II, \psi^{*\dagger}_\alpha(u) = \psi_\alpha(-u), \psi^\dagger_\alpha(u) = \psi^*_\alpha(-u)).$$

We have then the following (cf. (13), (14) of VIII [8]).

$$\Psi_{I\,in}(u) = \psi_I(u) e^{2\pi i N_{II}(u; l'', l')} \tag{3. 3}$$

$$\Psi_{I\,out}(u) = \psi_I(u) e^{2\pi i N_{II}(u; l', l'')}$$

$$\Psi_{II\,in}(u) = \psi_{II}(u) e^{2\pi i N_I(u; l', l'')} \tag{3. 4}$$

$$\Psi_{II\,out}(u) = \psi_{II}(u) e^{2\pi i N_I(u; l'', l')}$$

$$\Psi^*_{I\,in}(u) = \psi^*_I(u) e^{-2\pi i N_{II}(u; l'', l')} \tag{3. 5}$$

$$\Psi^*_{I\,out}(u) = \psi^*_I(u) e^{-2\pi i N_{II}(u; l', l'')}$$

$$\Psi^*_{II\,in}(u) = \psi^*_{II}(u) e^{-2\pi i N_I(u; l', l'')} \tag{3. 6}$$

$$\Psi^*_{II\,out}(u) = \psi^*_{II}(u) e^{-2\pi i N_I(u; l'', l')}.$$

Making use of the equations

$$e^{2\pi i N_\alpha(u; l', l'')} \psi_\alpha(u') e^{-2\pi i N_\alpha(u; l', l'')} = \begin{cases} e^{2\pi i l'} \psi_\alpha(u') & (|u| > |u'|) \\ e^{2\pi i l''} \psi_\alpha(u') & (|u| < |u'|) \end{cases} \tag{3. 7}$$

$$e^{2\pi i N_\alpha(u; l', l'')} \psi^*_\alpha(u') e^{-2\pi i N_\alpha(u; l', l'')} = \begin{cases} e^{-2\pi i l'} \psi^*_\alpha(u') & (|u| > |u'|) \\ e^{-2\pi i l''} \psi^*_\alpha(u') & (|u| < |u'|) \end{cases}$$

we can explicitly verify the canonical anticommutation relations

$$[\Psi_{\alpha \underset{out}{in}}(u), \Psi^*_{\beta \underset{out}{in}}(u')]_+ = \delta_{\alpha\beta} 2\pi |u| \delta(u + u'). \tag{3. 8}$$

Other combinations vanish identically.

Remark. If we start with $\varphi^F_\pm(a; l', l'') = : \psi_\pm(a) e^{\rho(a; l', l'')/2}:$ and $\varphi^{*F}_\pm(a; l', l'') = : \psi^*_\pm(a) e^{\rho(a; -l', -l'')/2}:$ instead of taking the tensor product

(2. 3), we obtain the asymptotic fields (cf. [8])

$$(3.9)\qquad \begin{aligned} \varphi^F_{in}(u;l',l'') &= e^{2\pi i N(u;l'',l')}\psi(u) && (u>0)\\ &= \psi(u)\,e^{2\pi i N(u;l'',l')} && (u<0)\\ \varphi^{*F}_{in}(u;l',l'') &= e^{2\pi i N(u;-l'',-l')}\psi^*(u) && (u>0)\\ &= \psi^*(u)\,e^{2\pi i N(u;-l'',-l')} && (u<0) \end{aligned}$$

(φ^F_{out}, φ^{*F}_{out} are obtained by interchanging l' and l'').

However in this case they do *not* satisfy the canonical anti-commutation relations. For instance

$$(3.10)\qquad \begin{aligned} \varphi^F_{in}(u;l',l'')\varphi^F_{in}(u';l',l'') &= -e^{-2\pi i(l'-l'')}\varepsilon(|u|-|u'|)\\ &\quad\times\varphi^F_{in}(u';l',l'')\varphi^F_{in}(u;l',l''),\\ \varphi^F_{in}(u;l',l'')\varphi^{*F}_{in}(u';l',l'') &= 2\pi|u|\delta(u+u')\\ &\quad - e^{2\pi i(l'-l'')\varepsilon(|u|-|u'|)}\varphi^{*F}_{in}(u';l',l'')\varphi^F_{in}(u;l',l''), \end{aligned}$$

and so forth. Although there seems to be no simple equations of motion for $\varphi^F_\pm$, $\varphi^{*F}_\pm$, they are manifestly covariant fields satisfying the micro-causality. Therefore the canonical anticommutation relation for the asymptotic fields are not a consequence of these properties.

The 2-body S-matrices are read off from (3. 7) directly. For $u, u' >0$ we use the notation $|\bar{I}p, IIp'\rangle_{\substack{in\\out}} = \Psi^*_{I\,\substack{in\\out}}(-u)\Psi_{II\,\substack{in\\out}}(-u')|\mathrm{vac}\rangle$ $\left(p=\left(m_I\frac{u+u^{-1}}{2}, m_I\frac{u-u^{-1}}{2}\right), p'=\left(m_{II}\frac{u'+u'^{-1}}{2}, m_{II}\frac{u'-u'^{-1}}{2}\right)\right)$, etc. (Note that $\Psi^*_{\alpha\,\substack{in\\out}}(-u)$ (resp. $\Psi_{\alpha\,\substack{in\\out}}(-u)$) represents a creation operator of particle α (resp. anti-particle $\bar{\alpha}$) with momentum p^1.) Then we find

$$(3.11)\qquad |\alpha p, \alpha' p'\rangle_{out} = S_{2,2}(\alpha p, \alpha' p')\,|\alpha p, \alpha' p'\rangle_{in} \qquad (\alpha, \alpha' = I, II, \bar{I}, \overline{II})$$

where

$$(3.12)\qquad \begin{aligned} &S_{2,2}(\alpha p, \alpha p') = 1 \qquad (\alpha, \alpha' = I, II, \bar{I}, \overline{II})\\ &S_{2,2}(Ip, IIp') = S_{2,2}(\bar{I}p, \overline{II}p') = e^{ig\varepsilon(p'^0p^1-p^0p'^1)}\\ &S_{2,2}(\bar{I}p, IIp') = S_{2,2}(Ip, \overline{II}p') = e^{-ig\varepsilon(p'^0p^1-p^0p'^1)}\,. \end{aligned}$$

That is, the S-matrices are energy independent constants, but they do depend on from which direction the two particles approach each other. This has been first noted in [5]. It is easy to see by using (3. 3) $\sim$(3.6) that a general S-matrix element factorizes into a product of 2-body S-matrices ("factorization property", [5]):

$$|\alpha_1 p_1, \cdots, \alpha_n p_n\rangle_{out} = \prod_{i<j} S_{2,2}(\alpha_i p_i, \alpha_j p_j)\, |\alpha_1 p_1, \cdots, \alpha_n p_n\rangle_{in} \tag{3. 13}$$

where $\alpha_i = I, II, \bar{I}$ or $\overline{II}$ and $S_{2,2}(\alpha p, \alpha' p')$ denotes one of the corresponding factors displayed in (3. 12).

Let us turn to the n-point functions. In general if $\{i_1, \cdots, i_k\} \cup \{j_1, \cdots, j_{n-k}\}$ is a partition of $\{1, \cdots, n\}$, then

$$\begin{aligned} &\langle \Psi^{\sigma_{i_1}}_{I\varepsilon_{i_1}}(x_{i_1}) \cdots \Psi^{\sigma_{j_1}}_{II\varepsilon_{j_1}}(x_{j_1}) \cdots \Psi^{\sigma_{j_{n-k}}}_{II\varepsilon_{j_{n-k}}}(x_{j_{n-k}}) \cdots \Psi^{\sigma_{i_k}}_{I\varepsilon_{i_k}}(x_{i_k}) \rangle \\ &= \langle \psi^{\sigma_{i_1}}_{\varepsilon_{i_1}}(x_{i_1}) \cdots \varphi^{\sigma_{j_1}}_F(x_{j_1}; l'', l') \cdots \varphi^{\sigma_{j_{n-k}}}_F(x_{j_{n-k}}; l'', l') \cdots \psi^{\sigma_{i_k}}_{\varepsilon_{i_k}}(x_{i_k}) \rangle \\ &\times \langle \varphi^{\sigma_{i_1}}_F(x_{i_1}; l', l'') \cdots \psi^{\sigma_{j_1}}_{\varepsilon_{j_1}}(x_{j_1}) \cdots \psi^{\sigma_{j_{n-k}}}_{\varepsilon_{j_{n-k}}}(x_{j_{n-k}}) \cdots \varphi^{\sigma_{i_k}}_F(x_{i_k}; l', l'') \rangle \\ &\qquad\qquad (\varepsilon_i = \pm 1, \sigma_i = \pm 1) \end{aligned} \tag{3. 14}$$

where $\Psi^+_{\alpha\varepsilon}(x) = \Psi^*_{\alpha\varepsilon}(x)$, $\Psi^-_{\alpha\varepsilon}(x) = \Psi_{\alpha\varepsilon}(x)$, $\psi^+_\varepsilon(x) = \psi^*_\varepsilon(x)$, $\psi^-_\varepsilon(x) = \psi_\varepsilon(x)$ and $\varphi^+_F(x; l'', l') = \varphi_F(x; -l'', -l')$, $\varphi^-_F(x; l'', l') = \varphi_F(x; l'', l')$. Using

$$\begin{aligned} &\langle \psi_{i_1} \cdots \varphi_{j_1} \cdots \varphi_{j_{n-k}} \cdots \psi_{i_k} \rangle \\ &= \langle 1 \cdots \varphi_{j_1} \cdots \varphi_{j_{n-k}} \cdots 1 \rangle \\ &\times \mathrm{Pfaffian}\left[\frac{\langle 1 \cdots \overset{\mu}{\breve{\psi_\mu}} \cdots \varphi_{j_1} \cdots \varphi_{j_{n-k}} \cdots \overset{\nu}{\breve{\psi_\nu}} \cdots 1 \rangle}{\langle 1 \cdots 1 \cdots \varphi_{j_1} \cdots \varphi_{j_{n-k}} \cdots 1 \cdots 1 \rangle} \right]_{\mu, \nu = i_1, \cdots, i_k} \end{aligned} \tag{3. 15}$$

(ψ_i, φ_j are shorthands for $\psi^{\sigma_i}_{\varepsilon_i}(x_i)$ and $\varphi^{\sigma_j}_F(x_j; l'', l')$),

we obtain closed expressions as well as infinite series expansions for (3. 14) by the aid of the results of IV [1]. Note that (3. 14) reduces to 0 if either k or $n-k$ is odd. It also vanishes unless positive- and negative-σ_{i_p}'s (resp. σ_{j_p}'s) are equal in number.

Example (2 point functions). The only non-zero combinations for $n=2$ are

$$\begin{aligned} &\langle \Psi^*_{I\varepsilon_1}(x_1) \Psi_{I\varepsilon_2}(x_2) \rangle \\ &= \langle \psi^*_{\varepsilon_1}(x_1) \psi_{\varepsilon_2}(x_2) \rangle \langle \varphi_F(x_1; -l', -l'') \varphi_F(x_2; l', l'') \rangle \end{aligned} \tag{3. 16}$$

$$\langle \Psi_{I\varepsilon_1}(x_1)\Psi^*_{I\varepsilon_2}(x_2)\rangle$$
$$=\langle \psi_{\varepsilon_1}(x_1)\psi^*_{\varepsilon_2}(x_2)\rangle\langle \varphi_F(x_1; l', l'')\varphi_F(x_2; -l', -l'')\rangle$$

and those obtained by interchanging I with II and l' with l''. Here $\langle \varphi_F(x_1; -l', -l'')\varphi_F(x_2; l', l'')\rangle = \langle \varphi_F(x_1; -l'+l''+1/2)\varphi_F(x_2; l'-l''+1/2)\rangle = \tau_{F2}$ is given by (IV [1])

$$\tau_{F2} = \exp\left(-\frac{1}{2}\int_t^\infty ds\left(s\left(\left(\frac{d\psi}{ds}\right)^2 - \sinh^2\psi\right) - \frac{l^2}{s}\tanh^2\psi\right)\right) \tag{3.17}$$

with $t = 2m_2\sqrt{-(x_1-x_2)^+(x_1-x_2)^-}$ and

$$\frac{d}{dt}\left(t\frac{d\psi}{dt}\right) = \frac{l^2}{t}\tanh\psi(1-\tanh^2\psi) + \frac{t}{2}\sinh 2\psi, \tag{3.18}$$
$$l = -2(l'-l'') = -g/\pi.$$

§ 4.

Here we follow the analogue of Section 2 for the (massless) Thirring model defined through the Lagrangian

$$\begin{aligned}\mathcal{L}_{TH} &= \overline{\Psi}(x)i\partial\!\!\!/\Psi(x) + g(\overline{\Psi}(x)\Psi(x))^2 \\ &= -\Psi^*_+(x)\partial_+\Psi_+(x) - \Psi^*_-(x)\partial_-\Psi_-(x) \\ &\quad -2g\Psi^*_+(x)\Psi_+(x)\Psi^*_-(x)\Psi_-(x)\end{aligned} \tag{4.1}$$

with the equations of motion

$$\begin{aligned}&\partial_+\Psi_+(x) + 2ig\Psi^*_-(x)\Psi_-(x)\cdot\Psi_+(x) = 0 \\ &-\partial_-\Psi_-(x) - 2ig\Psi^*_+(x)\Psi_+(x)\cdot\Psi_-(x) = 0.\end{aligned} \tag{4.2}$$

Formally the Federbush model (2.1) reduces in the massless limit to two non-interacting Thirring models ([2]):

$$\begin{aligned}&\mathcal{L}_{FED}|_{m_{I,II}=0} \\ &= -\Psi^*_{I+}(x)\partial_+\Psi_{I+}(x) - \Psi^*_{II-}(x)\partial_-\Psi_{II-}(x) \\ &\quad -2g\Psi^*_{I+}(x)\Psi_{I+}(x)\Psi^*_{II-}(x)\Psi_{II-}(x) \\ &\quad -\Psi^*_{II+}(x)\partial_+\Psi_{II+}(x) - \Psi^*_{I-}(x)\partial_-\Psi_{I-}(x) \\ &\quad +2g\Psi^*_{II+}(x)\Psi_{II+}(x)\Psi^*_{I-}(x)\Psi_{I-}(x).\end{aligned} \tag{4.3}$$

In accordance with this fact, the operator solution for (4. 1) is obtained by modifying the operators constructed in II [7] which are regarded as massless limits of those in IV [1].

Let $\psi^*(x), \psi(x)$ be free fermion operators in one dimension (II [7]) such that

$$(4.4)\qquad \begin{pmatrix} [\psi^*(x), \psi^*(x')]_+ & [\psi^*(x), \psi(x')]_+ \\ [\psi(x), \psi^*(x')]_+ & [\psi(x), \psi(x')]_+ \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \delta(x-x')$$

$$\begin{pmatrix} \langle\psi^*(x)\psi^*(x')\rangle & \langle\psi^*(x)\psi(x')\rangle \\ \langle\psi(x)\psi^*(x')\rangle & \langle\psi(x)\psi(x')\rangle \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \frac{1}{2\pi} \frac{i}{x-x'+i0}.$$

In correspondence with (1. 6) we introduce $\varphi_F(a) = \varphi_F(a; l', l'')$ through

$$(4.5)\qquad \varphi_F(a; l', l'') = :e^{\rho_F(a; l', l'')/2}:$$

$$(4.6)\qquad \rho_F(a; l', l'')/2 = \iint dx dx' R(x-a, x'-a; l', l'') \psi^*(x) \psi(x')$$

$$R(x, x'; l', l'') = (e^{-\pi i l''}(x+i0)^{-l'+l''} - e^{\pi i l''}(x-i0)^{-l'+l''})$$

$$\times \left(\frac{1}{2\pi} \frac{i}{x-x'+i0} e^{-\pi i l''}(x'-i0)^{l'-l''} + \frac{1}{2\pi} \frac{-i}{x-x'-i0} e^{\pi i l''}(x'+i0)^{l'-l''}\right)$$

which enjoys the property

$$(4.7)\qquad \varphi_F(a)\psi(x) = \begin{cases} e^{2\pi i l'}\psi(x)\varphi_F(a) & (x<a) \\ e^{2\pi i l''}\psi(x)\varphi_F(a) & (x>a) \end{cases}$$

$$\varphi_F(a)\psi^*(x) = \begin{cases} e^{-2\pi i l'}\psi^*(x)\varphi_F(a) & (x<a) \\ e^{-2\pi i l''}\psi^*(x)\varphi_F(a) & (x>a). \end{cases}$$

Setting $j(x) = \psi^*(x)\psi(x)$ we find that

$$(4.8)\qquad j(x)\varphi_F(a; l', l'') = \langle j(x)\varphi_F(a; l', l'')\rangle \varphi_F(a; l', l'') + :\psi^*(x, a; l', l'')\psi(x, a; l', l'') e^{\rho_F(a; l', l'')}:$$

where

$$(4.9)\qquad \langle j(x)\varphi_F(a; l', l'')\rangle = -\frac{l'-l''}{2\pi i} \frac{1}{x-a}$$

$$(4.10)\quad \psi^*(x, a; l', l'')$$
$$= \int dx'\psi^*(x')\Big(e^{-\pi i l''}(x'-a+i0)^{-l'+l''}\frac{1}{2\pi}\frac{i}{x'-x+i0}$$
$$+e^{\pi i l''}(x'-a-i0)^{-l'+l''}\frac{1}{2\pi}\frac{-i}{x'-x-i0}\Big)\cdot e^{\pi i l''}(x-a+i0)^{l'-l''}$$

$$\psi(x, a; l', l'')$$
$$= \int dx'\psi(x')\Big(e^{\pi i l''}(x'-a+i0)^{l'-l''}\frac{1}{2\pi}\frac{i}{x'-x+i0}$$
$$+e^{-\pi i l''}(x'-a-i0)^{l'-l''}\frac{1}{2\pi}\frac{-i}{x'-x-i0}\Big)\cdot e^{-\pi i l''}(x-a+i0)^{-l'+l''}.$$

We have also

$$(4.11)\quad \lim_{\varepsilon\to 0}\frac{1}{2}(j(a+\varepsilon)\varphi_F(a; l', l'')+j(a-\varepsilon)\varphi_F(a; l', l''))$$
$$=-\frac{1}{2\pi i(l'-l'')}\frac{d}{da}\varphi_F(a; l', l'').$$

By a similar argument as in Section 2, the following operators, constructed from copies of ψ^*, ψ and φ_F, are shown to satisfy the microcausality and the equations of motion (4. 2) for the Thirring model.

$$(4.12)\quad \Psi_+(x)=\frac{1}{\sqrt{2}}\psi_I(-x^-)\cdot\varphi_{FII}(x^+; l', l'')$$
$$\Psi_-(x)=\frac{1}{\sqrt{2}}\psi_{II}(x^+)\cdot\varphi_{FI}(-x^-; l'', l')$$
$$\Psi_+^*(x)=\frac{1}{\sqrt{2}}\psi_I^*(-x^-)\cdot\varphi_{FII}(x^+; -l', -l'')$$
$$\Psi_-^*(x)=\frac{1}{\sqrt{2}}\psi_{II}^*(x^+)\cdot\varphi_{FI}(-x^-; -l'', -l').$$

Here the factor $1/\sqrt{2}$ is inserted to adjust the normalization: $[(1/\sqrt{2})\times\psi_\alpha(\mp x^\mp), (1/\sqrt{2})\psi_\alpha^*(\mp x'^\mp)]_+|_{x^0=x'^0}=\delta(x^1-x^{1\prime})$ $(\alpha=I, II)$, and the coupling constant g is given by

$$(4.13)\quad g=2\pi(l'-l'').$$

Since the n-point functions of φ_F are power functions (II, § 4 [7];

the corresponding monodromy is abelian), this expression (4.12) agrees with the known result [9] for n-point functions of the Thirring model.

Appendix

Let $W=V^{\dagger}\oplus V$ be an orthogonal vector space and its holonomic decomposition ([10]). We assume that it is also equipped with a "charge structure"; namely W admits an orthogonal decomposition $W'^{*}\oplus W'$ into isomorphic copies $W'^{*}\cong W'$ of an orthogonal space $W'=V'^{\dagger}\oplus V'$, so that one has the "charge operator" N characterized by the following properties:

(A. 1)
$$[\mathsf{N}, w^{*}]=w^{*},\quad w^{*}\in W'^{*}$$
$$[\mathsf{N}, w]=-w,\quad w\in W'$$
$$\langle\mathsf{N}\rangle=0\,.$$

In other words we have $c^{\mathsf{N}}w^{*}c^{-\mathsf{N}}=cw^{*}\,(w^{*}\in W'^{*})$, $c^{\mathsf{N}}wc^{-\mathsf{N}}=c^{-1}w$ $(w\in W)$ and $\langle c^{\mathsf{N}}\rangle=1$ for any $c\in\boldsymbol{C}$. Choose a basis $\{v_{\mu}\}_{\mu=1,\dots,N}$ for W' and its copy $\{v_{\mu}^{*}\}_{\mu=1,\dots,N}$ for W'^{*}, and set $J=(\langle v_{\mu}, v_{\nu}\rangle)=(\langle v_{\mu}^{*}, v_{\nu}^{*}\rangle)$, $K=(\langle v_{\mu}v_{\nu}\rangle)=(\langle v_{\mu}^{*}v_{\nu}^{*}\rangle)$, $E_{+}=J^{-1}K$ and $E_{-}=J^{-1}\,{}^{t}K$. In terms of these basis the charge operator is given by

(A. 2)
$$\mathsf{N}=\sum_{\mu,\nu=1}^{N}(J^{-1})_{\mu\nu}v_{\mu}^{*}v_{\nu}-\frac{1}{2}N=\sum_{\mu,\nu=1}^{N}(J^{-1})_{\mu\nu}:v_{\mu}^{*}v_{\nu}:\,.$$

Now let $g\in G(W)$ be an element that commutes with N:

(A. 3)
$$\mathrm{Nr}\,(g)=\langle g\rangle e^{\rho/2}\,,$$
$$\rho/2=\sum_{\mu,\nu=1}^{N}r_{\mu\nu}v_{\mu}^{*}v_{\nu}=v^{*}R^{t}v$$
$$R=(r_{\mu\nu})\,,\ v^{*}=(v_{1}^{*},\cdots,v_{N}^{*})\,,\ v=(v_{1},\cdots,v_{N})\,.$$

We have then the following formulas:

(A. 4)
$$\mathrm{Nr}\,(c^{\mathsf{N}}g)=\langle g\rangle e^{\rho_{c}/2}=\mathrm{Nr}\,(gc^{\mathsf{N}})$$
$$\rho_{c}/2=v^{*}R_{c}{}^{t}v\,,$$
$$R_{c}J=(1-c^{-1})\,E_{c}+c^{-1}E_{c}\cdot RJ\cdot E_{c}$$

Here we have set $E_{c}=cE_{+}+E_{-}$. In particular we have $\mathrm{Nr}\,(c^{\mathsf{N}})=\exp(v^{*}\cdot(1-c^{-1})\,E_{c}J^{-1}\cdot{}^{t}v)$.

(A. 5) $$\mathrm{Nr}\,(c^{\mathsf{N}}\colon (w_1+w_2^*)\cdots(w_{2k-1}+w_{2k}^*)\,e^{\rho/2}\colon)$$
$$=(E_{c^{-1}}(w_1)+E_c(w_2^*))\cdots(E_{c^{-1}}(w_{2k-1})+E_c(w_{2k}^*))\,e^{\rho_c/2}\,.$$

In (A.5) $E_c(w)$ is understood to be $cw^{(+)}+w^{(-)}$ for $w=w^{(+)}+w^{(-)}$ $\in W'$ $(w^{(+)}\in V'^{\dagger},\ w^{(-)}\in V')$, and similarly for $w^*\in W'^*$.

All these formulas are valid in the symplectic case, provided that we replace J, E_+, E_-, N by $H=K-{}^tK$, $E_+=H^{-1}K$, $E_-=-H^{-1\,t}K$ and $\mathsf{N}=v^*H^{-1\,t}v+N/2$, respectively.

References

[1] Sato, M., Miwa, T. and Jimbo, M., *Publ. RIMS, Kyoto Univ.*, **15** (1979), 871-972.
[2] Federbush, P. G., *Phys. Rev.*, **121** (1961), 1247-1249, *Progr. Theoret. Phys.*, **26** (1961), 148-150.
[3] Thirring, W., *Ann. Phys.*, **3** (1958), 91-112.
[4] Lehmann, H. and Stehr, J., The bose field structure associated with a free massive Dirac field in one space dimension, *Hamburg preprint*, DESY 76/29 (1976).
[5] Schroer, B., Truong, T. T. and Weisz, P., *Ann Phys.*, **102**, (1976), 156-169.
[6] Wightman, A. S., in *High Energy Interactions and Field Theory, Cargèse Lectures in Theoretical Physics*, Edited by Maurice Lévy, Gordon and Breach, New York, 1967.
[7] Sato, M., Miwa, T. and Jimbo, M., *Publ. RIMS, Kyoto Univ.*, **15** (1979), 201-278.
[8] ———, *Proc. Japan Acad.*, **54A** (1978), 36-41.
[9] Klaiber, B., *Helv. Phys. Acta*, **37** (1964), 554-562.
[10] Sato, M., Miwa, T. and Jimbo, M., *Publ. RIMS, Kyoto Univ.*, **14** (1978), 223-267.