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# Chaos in $C^0$-Endomorphism of Interval

By

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## § 1. Introduction

The simplest non-trivial dynamical system that exhibits a "chaotic behavior" is the one governed by an endomorphism $F: I \to I$ where $I$ is a closed interval. In this paper, we propose a natural definition of chaos (formal chaos), from which the chaos in the sense of Li and Yorke follows. In the case of $C^0$-endomorphisms, we give necessary and sufficient conditions for the formal chaos (the existence of a periodic orbit with period not equal to any power of 2, the existence of $F^m$-invariant ergodic probability measure for some positive integer $m$). The proof is mainly based on three fundamental lemmas on $C^0$-endomorphisms. They automatically provide a unified exposition of the following known theorems on chaotic behavior of $C^0$-endomorphisms: Li-Yorke's theorem, Šarkovskii's theorem, the estimation of a lower bound of the topological entropy, etc. Furthermore, we get a result that in the case of $F$ with only $2^n$-orbits with $n \leqq M$ for some positive integer $M$, every ergodic invariant probability measure of $F$ is concentrated on some periodic orbit and the topological entropy of $F$ vanishes.

Throughout this paper, $F: I \to I$ is an endomorphism of a closed interval $I$, $N = \{1, 2, \cdots\}$, $N^* = \{0, 1, 2, \cdots\}$ and $p$ denotes some odd integer larger than 2.

We say that $a \in I$ is an $n$-point of $F$ if $F^n a = a$ and $F^i a \neq F^j a$ for $0 \leqq i < j \leqq n-1$, and that $\{a, Fa, \cdots, F^{n-1}a\}$ is an $n$-orbit of $F$. The set of all $n$-points of $F$ is denoted by $\mathrm{Per}(F, n)$.

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## § 2. Fundamental Lemmas on $C^0$-Endomorphisms

In this section we assume that $F: I \to I$ is a continuous endomorphism of a closed interval $I$.

(1) **Lemma.** *If $I$ is compact and if* $\mathrm{Per}(F, 2) = \emptyset$, *then for every point $x \in I$ there exists a point $z \in \mathrm{Per}(F, 1)$ such that* $\lim_{n\to\infty} F^n x = z$.

*Proof.* We may assume $I = [0, 1]$. Let $U = \{x: x \leqq Fx\}$. If there is $M \in N$ such that $F^n x \in U$ (or $F^n x \notin U$) for all $n \geqq M$, then $\{F^n x\}_{n\geqq M}$ is a bounded monotone sequence, so that this sequence has a limit point $z \in \mathrm{Per}(F, 1)$. If we can not find such $M$ as above, then we can choose an increasing sequence of positive integers $n(1), n(2), \cdots$, as follows: $n(1)$ is a positive integer such that $F^{n(1)-1}x \notin U$ and $F^{n(1)}x \in U$. For $k = 1, 2, \cdots$, $n(2k)$ is the least integer larger than $n(2k-1)$ such that $F^{n(2k)-1}x \in U$ and $F^{n(2k)}x \notin U$, and $n(2k+1)$ is the least integer larger than $n(2k)$ such that $F^{n(2k+1)-1}x \notin U$ and $F^{n(2k+1)}x \in U$. Then, $F^{n(1)}x \leqq F^{n(2)-1}x \leqq F^{n(2)}x$ follows from $F^n x \in U$ for $n(1) \leqq n \leqq n(2)-1$. $F^{n(3)}x < F^{n(3)-1}x < \cdots < F^{n(2)}x$ follows from $F^n x \notin U$ for $n(2) \leqq n \leqq n(3)-1$. Assume that there is an integer $n$, $n(2) \leqq n \leqq n(3)-1$ such that $F^{n+1}x < F^{n(2)-1}x \leqq F^n x \leqq F^{n(2)}x$. Then, from $FF^n x < F^{n(2)-1}x < FF^{n(2)-1}x$ and from the continuity of $F$ there is a point $a$ such that $F^{n(2)-1}x < a < F^n x$ and $Fa = F^{n(2)-1}x$. Furthermore, from $FF^{n(2)-1}x > a > Fa$ and the continuity of $F$ there is a point $b$ such that $F^{n(2)-1}x < b < a$ and $Fb = a$. Consider the open set $C = \{(x, y): a < x < 1,\ 0 < y < b\}$. The continuous curve $y = F(x)$ goes into the set $C$ passing through the point $(a, F^{n(2)}x)$ and reaches the line $y = b$ $(a < x < 1)$ or the line $x = 1$ $(0 \leqq y \leqq b)$, and the continuous curve $x = F(y)$ goes into the set $C$ passing through the point $(a, b)$ and reaches the line $x = a$ $(0 < y < F^{n(2)-1}x)$ or the line $y = 0$ $(a \leqq x \leqq 1)$. Then the two curves have at least an intersecting point, which is in $C$ or is the point $(1, 0)$, and then is in $\mathrm{Per}(F, 2)$. This is a contradiction and then $F^{n(2)-1}x \leqq F^{n(3)}x$. By the same way we have

$$F^{n(1)}x \leqq F^{n(2)-1}x \leqq F^{n(3)}x \leqq F^{n(4)-1}x \leqq F^{n(5)}x \leqq$$
$$\cdots \leqq F^{n(5)-1}x \leqq F^{n(4)}x \leqq F^{n(3)-1}x \leqq F^{n(2)}x\,.$$

Let $\lim_{n\to\infty} F^{n(2k)}x = z$ and $\lim_{n\to\infty} F^{n(2k-1)}x = z'$. Then, since $\lim_{n\to\infty} F^{n(2k-1)-1}x = z$ and $F$ is continuous, $Fz = z'$. Likewise $Fz' = z$. Hence $z = z' \in \mathrm{Per}(F, 1)$.

*Remark* 1. If $I$ is not compact, then the possibility arises that $|F^n x|$ run away toward infinity.

*Remark* 2. This lemma is an extension of Theorem B of Block [2].

It is obvious from (1) and Remark 1 that

(2) *If* $\mathrm{Per}(F, 2) = \emptyset$, *then* $\mathrm{Per}(F, n) = \emptyset$ *for* $n \geqq 2$.

Note a property from the continuity of $F$ that for compact intervals $I_1$ and $I_2$ such that $FI_1 \supset I_2$ there is a compact interval $I_1' \subset I_1$ such that $FI_1' = I_2$. Then, if there is a sequence of compact intervals $I_0, I_1, I_2, \cdots, I_i, I_{i+1}, \cdots$ such that $I_{i+1} \subset FI_i$ $(i \in N^*)$, then we can inductively construct compact intervals $W_F(I_i, I_{i+1}, \cdots, I_{i+k})$ $(i, k \in N^*)$ satisfying the following conditions:

$$W_F(I_i) = I_i,$$

$$W_F(I_i, I_{i+1}, \cdots, I_{i+k}) \subset W_F(I_i, I_{i+1}, \cdots, I_{i+k-1}),$$

$$FW_F(I_i, I_{i+1}, \cdots, I_{i+k}) = W_F(I_{i+1}, \cdots, I_{i+k}), \quad (k \in N,\ i \in N^*).$$

(3) **Lemma**. *Let* $I_0, I_1, \cdots, I_{k-1}$ *be a sequence of compact intervals such that* $I_{i+1} \subset FI_i$ *for* $i = 0, 1, \cdots, k-2$ *and* $I_0 \subset FI_{k-1}$. *Then there exists a point* $x$ *such that* $x \in I_0$, $F^i x \in I_i$ *for* $i = 1, 2, \cdots, k-1$ *and* $F^k x = x$.

*Proof.* Note the fixed point property of a continuous endomorphism $F$ that for a compact interval $I_1$ such that $FI_1 \supset I_1$ there is a point $x \in I_1$ such that $Fx = x$. Then, since $F^k W_F(I_0, I_1, \cdots, I_{k-1}, I_0) = I_0 \supset W_F(I_0, I_1, \cdots, I_{k-1}, I_0)$ and $F^k$ is continuous, there is a point $x \in W_F(I_0, I_1, \cdots, I_{k-1}, I_0)$ such that $F^k x = x$. Furthermore, $F^i x \in F^i W_F(I_0, I_1, \cdots, I_i) = I_i$ for $i = 0, 1, \cdots, k-1$.

(4) **Lemma.** *If* Per $(F, p) \neq \emptyset$, *then* Per $(F, n) \neq \emptyset$ *for any* $n \geqq p-1$.

*Proof.* Let $O_p = \{x_0, x_1, \cdots, x_{p-1}\}$ $(x_0 < x_1 < \cdots < x_{p-1})$ be a $p$-orbit of $F$. There exists a point $x_r \in O_p$ such that $x_r < Fx_r$ and $x_{r+1} > Fx_{r+1}$. Furthermore there exists a point $x_s (\neq x_r) \in O_p$ such that $Fx_s < x_r < x_{r+1} < Fx_{s+1}$. If not, then $F\{x_0, x_1, \cdots, x_r\} \subset \{x_{r+1}, x_{r+2}, \cdots, x_{p-1}\}$ and $F\{x_{r+1}, x_{r+2}, \cdots, x_{p-1}\} \subset \{x_0, x_1, \cdots, x_r\}$, so that $p$ can not be odd, a contradiction. Let $F^*$ be the continuous piecewise linear function which is defined on $[x_0, x_{p-1}]$ and whose breaking points are $(x_i, Fx_i)$ $(x_i \in O_p)$. Let $I_0 = [x_r, x_{r+1}]$. Since $I_0 \subsetneqq F^* I_0$ and the end points of intervals $F^{*n} I_0$ $(n = 0, 1, 2, \cdots)$ are in $O_p$, there is a positive integer $t \leqq p-2$ such that $F^{*n-1} I_0 \subsetneqq F^{*n} I_0$ for $n = 0, 1, \cdots, t-1$ and $F^* I_0 = [x_0, x_{p-1}]$ for $n \geqq t$. Let $k$ be the integer such that $[x_s, x_{s+1}] \subset F^{*k} I_0 \backslash F^{*k-1} I_0$ $(k \leqq t)$ and write $I_k = [x_s, x_{s+1}]$. We can choose a sequence of compact intervals $I_{k-1}, I_{k-2}, \cdots, I_2, I_1$ whose end points are adjacent points in $O_p$ such that $I_i \subset F^{*i} I_0 \backslash F^{*i-1} I_0$ and $I_{i+1} \subset F^* I_i$ $(i = k-1, k-2, \cdots, 2, 1)$. From this choice we have $I_1 \subset F^* I_0$. From the choice of $x_s$, we have $I_0 \subset F^* I_k$. Hence from Lemma (3) there is a point $b$ for $n$ $(\geqq k+1 \geqq p-1) \in N$ such that $b \in I_0$, $Fb \in I_0, \cdots, F^{n-k-1} b \in I_0$, $F^{n-k} b \in I_1, \cdots, F^{n-1} b \in I_k$ and $F^n b = b$. This $b$ is an $n$-point of $F$.

The proof of (4) greatly simplifies the proof of the similar Lemma 15(a) in [11].

(5) (Štefan [11]). *If* Per $(F, p) \neq \emptyset$ *and* Per $(F, p-2) = \emptyset$, *then each $p$-orbit of $F$ has a point $x$ satisfying* $F^{p-2} x < \cdots < Fx < x = F^p x < F^2 x < F^4 x < \cdots < F^{p-1} x$ *or* $F^{p-1} x < \cdots < F^2 x < x = F^p x < Fx < F^3 x < \cdots < F^{p-2} x$.

*Proof.* In the proof of (4) if $k \leqq p-3$, then Per $(F, p-2) \neq \emptyset$, so that $k = p-2$. In this case (5) follows from the choice of $I_k, I_{k-1}, \cdots, I_1$ and $I_0$.

(6) **Lemma** (*Odd Periodicity lemma*). *If* Per $(F, p) \neq \emptyset$, *then there exist two compact intervals $I_0$ and $I_1$ such that* $F^2 I_0 \cap F^2 I_1 \supset I_0 \cup I_1$

*and* $I_0 \cap I_1 = \emptyset$.

*Proof.* We may assume that $p$ satisfies the assumptions of (5) and without loss of generality that there is a point $x$ such that $F^{p-1}x < F^{p-3}x < \cdots < F^2x < x = F^p x < F^1 x < F^3 x < \cdots < F^{p-2}x$. From $Fx < F^2(Fx)$ and the continuity of $F^2$ there is a point $a_4$ such that $x < a_4 < Fx$ and $a_4 < F^2 a_4$. From $F^2(F^{p-1}x) = Fx > a_4$ and the continuity of $F^2$ there is a point $a_1$ such that $F^{p-1}x < a_1 < F^{p-3}x$ and $a_4 < F^2 a_1$. From $F^2(F^{p-3}x) = F^{p-1}x < a_1$ and the continuity of $F^2$ there are points $a_2$ and $a_3$ such that $a_1 < a_2 < F^{p-3}x$, $F^2 a_2 < a_1$, $F^{p-3}x < a_3 < a_4$ and $F^2 a_3 < a_1$. Let $I_0 = [a_1, a_2]$ and $I_1 = [a_3, a_4]$. Then we have $F^2 I_0 \supset [F^2 a_2, F^2 a_1] \supset [a_1, a_4] \supset I_0 \cup I_1$, $F^2 I_1 \supset [F^2 a_3, F^2 a_4] \supset [a_1, a_4] \supset I_0 \cup I_1$ and $I_0 \cap I_1 = \emptyset$.

*Remark* 3. This lemma is a stronger version of Theorem A of Block [4] and Lemma B of Oono [8]; $n$ necessary for $F^n I_0 \cap F^n I_1 \supset I_0 \cup I_1$ is given explicitly.

(7) *If* $\mathrm{Per}(F, p) \neq \emptyset$, *then* $\mathrm{Per}(F, 6) \neq \emptyset$.

*Proof.* From (6) and (3), there exists a point $b$ such that $b \in I_0$, $F^2 b \in I_0$, $F^{2\cdot 2} b \in I_1$ and $F^{2\cdot 3} b = b$. Clearly $b \in \mathrm{Per}(F^2, 3)$. Hence $b \in \mathrm{Per}(F, 6) \cup \mathrm{Per}(F, 3)$. $\mathrm{Per}(F, 3) \neq \emptyset$ implies $\mathrm{Per}(F, 6) \neq \emptyset$ by (4), so that $\mathrm{Per}(F, 6) \neq \emptyset$.

## § 3. A Simple Proof of Šarkovskii's Theorem

(8) **Theorem** (Šarkovskii [10]). *Let* $F: I \to I$ *be a continuous endomorphism of a closed interval I. By* $n \prec m$ *we mean that* $\mathrm{Per}(F, m) \neq \emptyset$ *follows from* $\mathrm{Per}(F, n) \neq \emptyset$. *Then*

$$3 \prec 5 \prec 7 \prec 9 \prec \cdots \prec 2\cdot 3 \prec 2\cdot 5 \prec \cdots \prec 2^2\cdot 3 \prec 2^2 \cdot 5 \prec \cdots$$
$$\cdots \prec 2^3 \prec 2^2 \prec 1 .$$

This theorem is equivalent to the collection of the following propositions ($n \in N^*$):

(9) $2^n p \prec 2^n(p+2)$,

(10) $2^n p \prec 2^{n+1} \cdot 3$,

(11) *If* $2^k \geqq p$, *then* $2^n p \lessdot 2^{n+k}$,

(12) $2^{n+1} \lessdot 2^n$.

*Proof of* (9). For $n=0$, (9) follows from (4). For $n=1$, $\mathrm{Per}(F^2, p) \neq \emptyset$ follows from $\mathrm{Per}(F, 2p) \neq \emptyset$, so that $\mathrm{Per}(F^2, p+2) \neq \emptyset$ follows from (9) $n=0$. Therefore $\mathrm{Per}(F, 2+p) \cup \mathrm{Per}(F, 2(p+2)) \neq \emptyset$, but $\mathrm{Per}(F, p+2) \neq \emptyset$ implies $\mathrm{Per}(F, 2(p+2)) \neq \emptyset$ because of (4). Assume (9) holds for $n=k$. Then $\mathrm{Per}(F, 2^{k+1}p) \neq \emptyset$ implies $\mathrm{Per}(F^2, 2^k(p+2)) \neq \emptyset$. This, in turn, implies $\mathrm{Per}(F, 2^{k+1}(p+2)) \neq \emptyset$.

*Proof of* (10). $\mathrm{Per}(F, 2^n p) \neq \emptyset$ implies that $\mathrm{Per}(F^{2^n}, p) \neq \emptyset$. Hence $\mathrm{Per}(F^{2^n}, 6) \neq \emptyset$ follows from (7), so that $\mathrm{Per}(F, 2^{n+1} \cdot 3) \neq \emptyset$.

*Proof of* (11). If $\mathrm{Per}(F, 2^n p) \neq \emptyset$, then $\mathrm{Per}(F^{2^n}, p) \neq \emptyset$. From (4) $\mathrm{Per}(F^{2^n}, 2^k) \neq \emptyset$ $(k \geqq 1)$ follows, so that $\mathrm{Per}(F, 2^{n+k}) \neq \emptyset$.

*Proof of* (12). For $n=0$ this is obvious. For $n=1$ this follows from (2). Assume (12) holds for $n=k(\geqq 1)$. If $\mathrm{Per}(F, 2^{k+2}) \neq \emptyset$, then we have $\mathrm{Per}(F^2, 2^{k+1}) \neq \emptyset$, so that $\mathrm{Per}(F^2, 2^k) \neq \emptyset$ follows from the assumption. Since $2^k$ is even, $\mathrm{Per}(F, 2^{k+1}) \neq \emptyset$.

## § 4. Formal Chaos

The following definition of chaos was proposed by Li and Yorke [5, 7]: an endomorphism $F: I \to I$ of a closed interval $I$ is chaotic if, first, there are points $x \in I$ of arbitrarily large period and, second, there is an uncountable set $R \subset I$ such that no point in $R$ is even asymptotically periodic. More precisely the Li-Yorke chaos is defined as follows:

**Definition.** An endomorphism $F: I \to I$ of a closed interval $I$ shows a Li-Yorke chaos if

(13) there are points in $I$ of arbitrarily large period, and there is an uncountable set $R \subset I$ of non-periodic points satisfying

(14) for every $x, y \in R$ with $x \neq y$, $\limsup_{n \to \infty} |F^n x - F^n y| > 0$,

(15) for every $x, y \in R$ with $x \neq y$, $\liminf_{n \to \infty} |F^n x - F^n y| = 0$, and

(16) for every $x \in R$ and every periodic point $y \in I$,

$$\limsup_{n\to\infty} |F^n x - F^n y| > 0.$$

However, the definition above is not totally satisfactory for physicists, since non-periodicity does not necessarily imply "chaos". The dynamical systems which are "chaotic" in the intuitive sense of physicists such as the baker's transformation, Smale's horseshoe map ([9]) etc. are intimately connected with shift dynamical systems. We propose an intuitively satisfactory definition of chaos which implies Li-Yorke chaos as is shown in the next section.

Let $\Omega = \{0, 1\}^N$ be the set of all one-sided sequences of two symbols and $\omega_k$ be the $k$-th coordinate of $\omega \in \Omega$. The shift $T$ on $\Omega$ is defined by $(T\omega)_k = \omega_{k+1}$, $k \in N$. We call $[a_1, a_2, \cdots, a_k] = \{\omega : \omega_i = a_i, i = 1, 2, \cdots, k, \omega \in \Omega\}$, where $a_i = 0$ or $1$ $(i = 1, 2, \cdots, k)$, a $k$-cylinder. The following definition is a 1-dimensional version of one given in [8], which can be readily extended to many dimensional cases.

**Definition.** We say that an endomorphism $F: I \to I$ is *formally chaotic* (or shows a *formal chaos*) if there exist $M \in N$, an $F^M$-invariant set $K \subset I$ and one-to-one map $\phi: \Omega \to K$ having the following properties:

(A) $F^M \phi\omega = \phi T\omega$ $(\omega \in \Omega)$,

(B) Let $V[a_1, a_2, \cdots, a_k]$ be the smallest closed interval containing $\phi[a_1, a_2, \cdots, a_k]$. $V[a_1, a_2, \cdots, a_k]$ is compact for every $k$-cylinder $[a_1, a_2, \cdots, a_k]$ $(k \in N)$ and for mutually distinct $k$-cylinders $[a_1, a_2, \cdots, a_k]$ and $[b_1, b_2, \cdots, b_k]$ $(k \in N)$, $V[a_1, a_2, \cdots, a_k] \cap V[b_1, b_2, \cdots, b_k] = \emptyset$.

The meaning of the adjective "formal" is explained in the end of this section.

(17) **Theorem.** *Let $F: I \to I$ be a continuous endomorphism of a closed interval $I$. Then $F$ is formally chaotic if and only if $\mathrm{Per}(F, r) \neq \emptyset$ for some $r \neq 2^n$ $(n \in N^*)$.*

*Proof.* Assume that $\mathrm{Per}(F, 2^n p) \neq \emptyset$ for some $n \in N^*$. Then $\mathrm{Per}(F^{2^n}, p) \neq \emptyset$, so that by the odd periodicity Lemma (6) there are compact intervals $I_0$ and $I_1$ in $I$ such that $F^M I_0 \supset I_0 \cup I_1$, $F^M I_1 \supset I_0 \cup I_1$ and $I_0 \cap I_1 = \emptyset$, where $M = 2^{n+1}$. For $\omega \in \Omega$, define $W_\omega$ by $W_\omega = \bigcap_{k=2}^{\infty} W_F M(I_{\omega_1}, I_{\omega_2}, \cdots, I_{\omega_k})$. Then $W_\omega$ is a point or a compact interval. If $\omega \neq \omega'$, then $W_\omega \cap W_{\omega'} = \emptyset$. Since $F^M$ is continuous, $F^M W_\omega = \bigcap_{k=2}^{\infty} F^M W_F M(I_{\omega_1}, I_{\omega_2}, \cdots, I_{\omega_k}) = \bigcap_{k=2}^{\infty} W_F M(I_{\omega_2}, \cdots, I_{\omega_k}) = W_{T\omega}$. We have only to construct $\phi$ and $K$ such that $\phi\omega \in W_\omega$ and $F^M \phi\omega = \phi T\omega$ for any $\omega \in \Omega$.

(i) For $\omega$ such that $W_\omega$ consists of a point $a_\omega$, then $\phi\omega = a_\omega$.

(ii) If $\phi\omega$ is defined for $\omega$, then $\phi T\omega = F^M \phi\omega$.

(iii) If $\phi T\omega$ is defined for $\omega$, then $\phi\omega$ is defined such that $F^M \phi\omega = \phi T\omega$.

(iv) If $\phi$ is not yet defined on $\{T^i\omega : i \in N^*\}$ and all $T^i\omega$ are not periodic points, then $\phi\omega$ is chosen as an arbitrary point in $W_\omega$.

(v) If $\phi$ is not yet defined on $\{T^i\omega : i \in N^*\}$ and $T^n\omega = \omega$, $T^i\omega \neq T^j\omega$ for $0 \leqq i < j \leqq n-1$, for some $n \in N$, then we can choose $\phi\omega \in W_\omega$ such that $F^{Mn}\phi\omega = \phi\omega$, since $F^{Mn} W_\omega = W_\omega$.

Thus desired $\phi$ and $K = \{\phi\omega : \omega \in \Omega\}$ are simultaneously constructed.

Conversely, there is $n \in N$ such that $\mathrm{Per}(F^n, 3) \neq \emptyset$, because $\mathrm{Per}(T, 3) \neq \emptyset$. Hence there is a divisor $n'$ of $n$ such that $\mathrm{Per}(F, 3n') \neq \emptyset$.

Since $F^M$ restricted on $K$ is isomorphic to the diadic shift $T$, we have

(18) **Corollary** (Štefan [11]). *Let $F: I \to I$ be a continuous endomorphism of a compact interval $I$. If* $\mathrm{Per}(F, 2^n p) \neq \emptyset$ *for some $n \in N^*$, then* $\mathrm{ent}(F) \geqq \log 2/2^{n+1}$, *where* $\mathrm{ent}(F)$ *is the topological entropy of $F$* ([1]).

For determining whether $F$ is formally chaotic or not, the following version of Theorem (17) is practically more useful.

(19) *If there exist compact intervals $I_0$ and $I_1$ in $I$ sharing at most one point and $m, n \in N$ such that $F^n I_0 \cap F^m I_1 \supset I_0 \cup I_1$, then $F$ is formally chaotic. (Take $M =$* (L.C.M. *of $m$ and $n$*) $\times 2$.)

By (17) a continuous endomorphism $F: I \to I$ of a closed interval $I$

is formally chaotic if and only if $F$ satisfies the condition of the formal chaos with only (A) but without (B). However, there are endomorphisms which are not continuous but formally chaotic, e.g. $\beta$-transformation. In this case (B) does not follow from (A).

Even if $F$ is formally chaotic, it is not guaranteed that we can observe a chaotic behavior. The chaotic behavior is "experimentally" observable when the set of initial points which eventually behave chaotically has positive Lebesgue measure. This is why the adjective "formal" is added to the definition.

## § 5. Formal Chaos Implies Li-Yorke Chaos

(20) **Theorem.** *If an endomorphism $F: I\to I$ of a closed interval $I$ is formally chaotic, then $F$ shows Li-Yorke chaos.*

*Proof.* Note that there are uncountable sequences $\omega\in\Omega$ (called transitive sequences) such that, for every cylinder set $Q$ in $\Omega$, $T^n\omega\in Q$ for infinitely many $n\in N$. We can choose a transitive sequence $E=e_1e_2e_3\cdots\in\Omega$ such that $W_E$ consists of one point (the transitivity is unnecessary for this theorem, but we need it in (21)). For any point $B=b_1b_2b_3\cdots\in\Omega$ define $X_B\in\Omega$ by

$$X_B=e_1b_1e_1e_2b_1b_2e_1e_2e_3b_1b_2b_3\cdots e_1e_2\cdots e_nb_1b_2\cdots b_n\cdots,$$

and $R=\{\phi X_B : B\in\Omega\}$. (14), (15) and (16) hold for this $R$.

*Proof of* (14). Let $B, B'\in\Omega$, $B\neq B'$ and $B=b_1b_2b_3\cdots$, $B'=b_1'b_2'b_3'\cdots$. Then there exists $k\in N$ such that $b_k\neq b_k'$. Hence there is a positive $\eta$ such that for any $n>k$,

$$|F^{M(n^2+k-1)}\phi X_B-F^{M(n^2+k-1)}\phi X_{B'}|>\eta,$$

because of (B).

*Proof of* (15). This follows from that $T^{n(n+1)}X_B\in[e_1, e_2, \cdots, e_{n+1}]$ and that $V[e_1, e_2, \cdots, e_{n+1}]$ converges to a point as $n\to\infty$ because of the choice of $E$.

*Proof of* (16). If (16) does not hold, then for some periodic point $q \in I$ of $F$ such that $\lim_{n \leftarrow \infty} |F^{Mn(n+1)}\phi X_B - F^{Mn(n+1)}q| = 0$. Hence we can choose a subsequence $\{n_k\} \subset N$ so that $F^{Mn_k(n_k+1)}\phi X_B$ converges to some point $r$ in the periodic orbit containing $q$, but this implies $\phi E = r$, an impossibility.

For dynamical systems which show chaotic behaviors such as the horseshoe map and the Lorenz model [6], non-periodic points or orbits are closely mingled with periodic points. L. Block [3] showed that the non-wandering set of a continuous endomorphism $F: I \to I$ is in the closure of the set of periodic points and eventually periodic points (i.e., points mapped into periodic points by a finite-time operation of $F$). We have the following proposition.

(21) **Proposition.** *If a continuous endomorphism $F: I \to I$ of a closed interval $I$ is formally chaotic, then for any $\varepsilon > 0$ there are infinitely many periodic points $q$ of $F$ such that*

$$\liminf_{n \to \infty} |F^n x - q| < \varepsilon, \quad \textit{for any point } x \in R,$$

*where $R$ is the same set appearing in* (20).

*Proof.* There is a cylinder $Q$ such that the length of the interval $VQ$ is less than $\varepsilon$. There are infinitely many periodic points in $VQ$. From the transitivity of $E$, $F^n x \in VQ$ for infinitely many $n \in N$.

## § 6. Mixing Invariant Probability Measure and Formal Chaos

Among the properties which physicists consider as the characteristics of chaos, the existence of a mixing invariant probability measure is appearing. The following theorem shows that the mixing property and formal chaos of a continuous endomorphism $F: I \to I$ is intimately related. This shows that formal chaos is an intuitively satisfactory concept.

A probability measure $\mu$ on $I$ is $F$-invarint if $\mu(F^{-1}A) = \mu(A)$ for every Borel set $A$. $\mu$ is mixing if $\lim_{n \to \infty} \mu(F^{-n}A \cap B) = \mu(A)\mu(B)$ for every pair of Borel sets $A$ and $B$.

(22) **Theorem.** *A continuous endomorphism $F: I \to I$ of a closed interval $I$ is formally chaotic if and only if there is a positive integer $m$ such that $F^m$ has a mixing invariant probability measure.*

*Proof.* It is obvious that formal chaos implies the existence of an $F^m$-invariant mixing probability measure for some $m \in N$.

Assume that there is an $F^m$ $(m \in N)$-invariant mixing probability measure $\mu$. Take six disjoint subintervals $I_1, I_2, I_3, I_4, I_5, I_6$ of $I$ from left to right such that $\mu(I_j) > 0$ for $j \in \{1, 2, \cdots, 6\}$. Then there is $n \in N$ such that $I_2 \cap F^{-nm} I_1 \neq \emptyset$, $I_3 \cap F^{-nm} I_6 \neq \emptyset$, $I_4 \cap F^{-nm} I_1 \neq \emptyset$ and $I_5 \cap F^{-nm} I_6 \neq \emptyset$. If not, then for some pair $i$ and $j$ in $\{1, 2, \cdots, 6\}$ there are infinitely many $n \in N$ such that $I_i \cap F^{-nm} I_j = \emptyset$, i.e., $\mu(I_i \cap F^{-nm} I_j) = 0$, but this contradicts with the mixing property of $\mu$. Choose four points $a_1$, $a_2$, $a_3$ and $a_4$ such that $a_1 \in I_2$, $a_2 \in I_3$, $a_3 \in I_4$, $a_4 \in I_5$ and $F^{nm} a_1 \in I_1$, $F^{nm} a_3 \in I_1$, $F^{nm} a_2 \in I_6$, $F^{nm} a_4 \in I_6$. Put $I_0 = [a_1, a_2]$ and $I_0' = [a_3, a_4]$. Then $F^{nm} I_0 \cap F^{nm} I_0' \supset I_0 \cup I_0'$ and $I_0 \cap I_0' = \emptyset$, so that $F$ is formally chaotic by (19).

*Remark* 4. A necessary and sufficient condition for a continuous endomorphism $F: I \to I$ having a mixing invariant probability measure is $\mathrm{Per}(F, p) \neq \emptyset$ for some odd $p \geqq 3$. The necessity follows from that there are an odd integer $n$ and disjoint intervals $I_1, I_2, I_3, I_4, I_5, I_6$ arranging from left to right such that $I_2 \cap F^{-n} I_1 \neq \emptyset$, $I_3 \cap F^{-n} I_6 \neq \emptyset$, $I_4 \cap F^{-n} I_1 \neq \emptyset$ and $I_5 \cap F^{-n} I_6 \neq \emptyset$, and from that $\mathrm{Per}(F^n, p) \neq \emptyset$ for an odd $p \geqq 3$. The sufficiency follows from (5) and that the Markov subshift with the following structure matrix has a mixing invariant Markov measure;

$$
\begin{pmatrix}
0 & 0 & \cdots\cdots\cdots & & & & 0 & 1 \\
0 & 0 & \cdots\cdots\cdots & & & & 1 & 0 \\
 & & & & & \cdot^{\cdot^{\cdot}} & & \\
0 & 0 \cdots 0 & 0 & 0 & 1 & \cdots & 0 & 0 \\
0 & 0 \cdots 0 & 1 & 1 & 0 & \cdots & 0 & 0 \\
0 & 0 \cdots 1 & 0 & 0 & 0 & \cdots & 0 & 0 \\
 & \cdot^{\cdot^{\cdot}} & & & & & & \\
0 & 1 \cdots\cdots 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\
1 & 0 \cdots\cdots 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\
1 & 1 \cdots\cdots 1 & 0 & 0 & 0 & \cdots & 0 & 0
\end{pmatrix}.
$$

## § 7. Concluding Note

If only $2^n$-orbits ($n \leqq M$ for some $M \in N$) appear for a continuous endomorphism $F: I \to I$, then (1) implies that every $F$-invarinat ergodic probability measure is on a single periodic orbit and then $\mathrm{ent}(F)=0$. On the other hand, (21) and (22) show that if $\mathrm{Per}(F, 2^n p) \neq \emptyset$ for some $n \in N^*$ and odd $p \geqq 3$, $F^m$ for some $m \in N$ has a mixing invariant probability measure and $\mathrm{ent}(F)>0$. What happens when all $2^n$-orbits but no other orbits appear for a continuous $F: I \to I$? (22) also shows that in this case $F^m$ for every $m \in N$ has no mixing invariant probability measure. The following example suggests that the case is marginal.

(23) **Example.** $F: [0,1] \to [0,1]$ is defined as follows:

$$Fx = -7(x-2/3^n)/3 + (1-1/3^{n-1}) \quad \text{for} \quad x \in (1/3^n, 2/3^n]$$
$$= (x-2/3^n) + (1-1/3^{n-1}) \quad \text{for} \quad x \in (2/3^n, 1/3^{n-1}],$$

for $n \in N$, and $F0=1$. From its construction, $F$ has all $2^n$-orbits ($7/5 \cdot 3^n$ is a $2^n$-point) but has no other orbits. Any point in $[0,1]$ is eventually attracted to the periodic orbits or the classical Cantor set $D = \{w : w = \sum_{i=1}^{\infty} w_i/3^i,$ $w_i = 0$ or $2$ for $i \in N\}$. The restriction $F|_D$ is isomorphic to the adding machine $S$ on $\Omega$: for any $k \in N$ and $\omega \in \Omega$ satisfying $\omega_1 = \omega_2 = \cdots = \omega_{k-1} = 1$ and $\omega_k = 0$, $S\omega = \omega'$ where $\omega'_1 = \omega'_2 = \cdots = \omega'_{k-1} = 0$, $\omega'_k = 1$ and $\omega'_j = \omega_j$ for $j > k+1$. In fact, $\psi F w = S \psi w$ for $w \in D$, where $\psi: D \to \Omega$ is defined by $\psi w = \omega$ with $w = \sum_{i=1}^{\infty} w_i/3^i \in D$ and $\omega_i = (2-w_i)/2$ for $i \in N$. Hence $\mathrm{ent}(F)=0$ and $F$ has a continuous ergodic but not mixing invariant probability measure.

The example above and (18) suggest that a continuous endomorphism $F: I \to I$ of a compact interval $I$ is formally chaotic if and only if $\mathrm{ent}(F)>0$. Indeed, Y. Takahashi communicated that he proved this conjecture.

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