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I^*-Algebras and their Applications

By

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Abstract

We analyze the algebraic, topological, and order properties of I^*-algebras: complex unital topological *-algebras for which $\sum_J x_j^* x_j=0$ implies $x_j=0$ $(j\in \boldsymbol{J})$, $\boldsymbol{J}\subset \boldsymbol{N}$ any finite subset. We consider the ergodic properties of states on an I^*-algebra with a distinguished group of automorphisms. Particular attention is given to I^*-algebras of the form $\underline{E}=\sum_{\boldsymbol{N}}^{\oplus}\overline{\otimes}^n E$ where E is a nuclear LF-space. When $E=\mathscr{S}(\boldsymbol{R}^4)$ $(\mathscr{D}(\boldsymbol{R}^3)\oplus\mathscr{D}(\boldsymbol{R}^3)$ respectively) then $\underline{E}$ has applications to relativistic quantum field theory (the canonical anti-commutation or commutation relations, respectively).

Introduction

In this paper we study complex topological *-algebras that are nuclear LF-spaces in which $\sum_J x_j^* x_j=0$ implies $x_j=0$ $(j\in \boldsymbol{J})$ for any finite subset $\boldsymbol{J}\subset \boldsymbol{N}$. We call such algebras I^*-algebras.

The characteristic example of an I^*-algebra is the completed tensor algebra $\underline{E}=\sum_{\boldsymbol{N}}^{\oplus}\overline{\otimes}^n E$ over a nuclear LF-space E, equipped with the inductive topology. The systematic study of these tensor algebras was initiated independently by Borchers [5] and Uhlmann [6]. We say that $\underline{E}$ is the BU-algebra over E. Obviously nuclearity precludes any infinite dimensional C^*-algebra being an I^*-algebra.

In the first section we derive some general algebraic and topological consequences of the definition of I^*-algebras. An interesting result here is that the multiplication on a BU-algebra $\underline{E}$ is jointly continuous if and only if E is an LB-space.

The second section considers the order properties of proper I^*-

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algebras. These are I^*-algebras whose completed positive cone is proper. A result of Dubois-Violette [40] implies that all BU-algebras are proper. An important notion here is that of a state on an I^*-algebra, defined to be a normalized linear functional, positive on the complete positive cone. The GNS construction (Gel'fand, Naimark; Segal) associates with each state a canonical strongly cyclic representation of the algebra by a family of symmetric unbounded operators with a common dense domain in a Hilbert space, i.e. by a *-operator family. We indicate that for proper I^*-algebras the states separate points (Proposition 2. 3).

I^*-algebras with the best order properties are those satisfying a certain technical condition on convergent nets (Condition (N)). In such algebras, e.g., the initial topology is determined by the states (Proposition 2. 4). For a BU-algebra $\underline{E}$ with E a nuclear F-space, Yngvason [41] has shown that a necessary and sufficient condition for $\underline{E}$ to satisfy (N) is for E to be isomorphic to a closed subspace of the sequence space s. More generally $\underline{E}$ cannot satisfy (N) if $E \otimes_{\iota} E \neq E \otimes_{\pi} E$. Thus, e.g. the states on $\underline{\mathcal{D}(\boldsymbol{R}^n)}$ do not determine the initial topology, whereas they do for $\underline{\mathcal{S}(\boldsymbol{R}^n)}$. We note that the proof of Proposition 2. 8 implies that the states on $\underline{E}$ are the same as the states on $\sum_N^{\oplus} \hat{\otimes}^n E$.

The third section is concerned with an application. Certain representations of the BU-algebra with $E = \mathcal{D}(\boldsymbol{R}^n) \oplus \mathcal{D}(\boldsymbol{R}^n)$ comprise the continuous representations of the canonical anticommutation and commutation relations of quantum field theory. This is considered in some detail.

In the final section we adjoin an automorphism group to an I^*-algebra. The ergodic structure of such a system is more restrictive than for C^*-algebras. For example, G-ergodicity of a state does not imply that the GNS cyclic vector is the only normalized G-invariant vector (Proposition 4. 1). The notion of asymptotic abelianness, so important in the ergodic theory of C^*-algebras [2, 3, 35] seems to take its natural form as weak asymptotic abelianness (Definition 4. 2) for I^*-algebras. By considering the closure of the GNS representation for weakly asymptotically abelian states we are able to generalize a result of Borchers [42] (Proposition 4. 4).

§ 1. I^*-Algebras

The fields describing quantum systems with infinitely many degrees of freedom are families of unbounded operator-valued distributions on various representation Hilbert spaces [1]. In order to bypass the difficulties inherent in working with such *-operator families, it is conventional in Statistical Mechanics to consider various bounded functions of the fields. This leads to the C^*- or W^*-models. Such models are also available for relativistic systems [2–4].

Clearly it is also of interest to consider the fields directly. For relativistic systems a model of the fields as representations of a topological *-algebra has been proposed by Borchers [5] and Uhlmann [6] and considered by various authors since: [7–10] are review articles. We have considered similar models, for nonrelativistic systems and for current algebras. In the course of our work it became clear that much of the analysis depended only upon the algebras having certain algebraic and topological properties. The class of algebras so defined we shall call I^*-algebras.

Definition 1. 1. An I^*-algebra is a unital topological complex *-algebra which is a nuclear LF-space. Furthermore if $\sum_J x_i^* x_i = 0$, then $x_i = 0$ $(i \in \boldsymbol{J})$, $\boldsymbol{J}$ any finite subset of $\boldsymbol{N}$.

Note that for a topological *-algebra, the product is separately continuous and the involution continuous.

Lemma 1. 2. *An I^*-algebra is barrelled, bornological, and complete. It is Montel and reflexive. Its strong dual is nuclear and complete, barrelled, bornological, and Montel.*

Proof. As an I^*-algebra $\mathcal{A}$ is an LF-space, it is barrelled, bornological, and complete: [13], Cor 2 (p. 61); Cor 2 (p. 62); and Cor (p. 60). Being complete, barrelled, and nuclear implies that $\mathcal{A}$ is a Montel space: [14], Cor 3 (p. 520); consequently it is reflexive: [13] (p. 147). As $\mathcal{A}$ is nuclear LF its dual, $\mathcal{A}'$, is nuclear: [13], Thm 9. 6 (p.

172); Ex 2 (p. 173). $\mathcal{A}'$ is complete because $\mathcal{A}$ is bornological: [13], Thm 6. 1 (p. 148). As $\mathcal{A}$ is nuclear, it is a Schwartz space ([15]) and the dual of a complete Schwartz space is barrelled and bornological: [16], Prob 9 (p. 287). Finally, $\mathcal{A}'$ is Montel because $\mathcal{A}$ is: [13], Thm 5. 9 (p. 147).

The question of the existence of non-trivial I^*-algebras is answered affirmatively by the following paradigmatic example of a Borchers-Uhlmann algebra, or BU-algebra.

Definition 1. 3. Let $E_{\boldsymbol{R}}$ be a real nuclear LF-space and $E=E_{\boldsymbol{R}}\otimes\boldsymbol{C}$ its complexification. The BU-algebra over E is the locally convex direct sum tvs

(1. 1) $$\underline{E}=\sum_{n\geq 0}^{\oplus}\overline{\otimes}^{n}E$$

where $n=0$ corresponds to $\boldsymbol{C}$ by convention, and $\overline{\otimes}$ indicates the completion of the tensor product in the inductive tensor product topology [17]. The product with respect to which $\underline{E}$ is an algebra follows from the graded structure: if $x=(x_n)_n$, $y=(y_n)_n\in\underline{E}$, then

(1. 2) $$(xy)_n=\sum_{0\leq p\leq n}x_{n-p}\otimes y_p\,.$$

It is further assumed that a continuous involution, J, is defined on $E_{\boldsymbol{R}}$. In an obvious was this extends linearly to an involution $x\rightsquigarrow x^*$ on $\underline{E}$, with $(xy)^*=y^*x^*$.

Theorem 1. 4. *A BU-algebra is an I^*-algebra with no divisors of zero whose invertible elements are $\boldsymbol{C}-\{0\}$ and whose centre is $\boldsymbol{C}$.*

Proof. For the algebraic properties we may modify Lemma 1. 2. 4 of [7] slightly. The identity is $\mathbf{1}=(1,0,0,\cdots)$. For the divisors of zero we come to $x_h\otimes y_i=0$ with $x_h\in\overline{\otimes}^hE$ and $y_i\in\overline{\otimes}^iE$, $y_i\neq 0$ (c.f. [7] ibid.). Then $x_h=0$ by the linear disjointness of tensor products: [14] (p. 403). If x is in the centre of the algebra, choosing elements y such that $y_n=0$ unless $n=h$, where $x_h\neq 0$, $x_i=0$ for all $i>h$, then $x_h\otimes y_h=y_h\otimes x_h$ for all $y_h\in\overline{\otimes}^hE$. The case $h=0$ gives $x\in\boldsymbol{C}$ so assume $h>0$ and $x_h\neq 0$. By linear disjointness it follows that $x_h=\lambda y_h (\lambda\in\boldsymbol{C})$ for all $y_h\in\overline{\otimes}^h E$, implying $x_h=0$. This proves the algebraic properties.

We shall prove below that if F, G are nuclear LF-spaces, then so is $F\overline{\otimes}G$; and the locally convex direct sum of LF-spaces is LF. Granting these results, $\overline{\otimes}^n E$ is nuclear LF for all $n\geqq 0$; and $\sum^{\oplus}\overline{\otimes}^n E$ is a nuclear LF-space, which is the required topological result.

First of all, let $F=\uparrow\lim_i F_i$, $G=\uparrow\lim_i G_i$. Then ([18], App 2) $F\overline{\otimes}G=\uparrow\lim_i F_i\overline{\otimes}G_i$. But as F_i, G_i are nuclear F-spaces, $F_i\overline{\otimes}G_i =F_i\hat{\otimes}G_i$ are nuclear F-spaces. Then $F\overline{\otimes}G$ is LF, and as the countable inductive limit of nuclear spaces is nuclear ([14], Prop 50.1), so is $F\overline{\otimes}G$. That $\overline{\otimes}=\hat{\otimes}$ for F-spaces is found, e.g. in [17], I. 5.1 (p. 74).

Next let $F=\sum^{\oplus}F_i$ be a locally convex direct sum, with $F_i=\uparrow\lim_j F_{ij}$ an LF-space. Define $G_k=\sum^{\oplus}_{1\leq j\leq k}F_{jk}$. Then G_k is closed in G_{k+1} and $\cup G_k=F$. Moreover the G_k are F-spaces and $\uparrow\lim_k G_k$ is the set F equipped with a bornological topology which has the same bounded sets as the original topology on F. Then these two topologies are equal: [13] II. 8.3, so $F=\uparrow\lim_k G_k$ and we are done.

It remains to show the continuity of the product and the involution, and the order axiom. The multiplication is separately left continuous iff for every net in $\overline{\otimes}^n E$ which converges to zero, $x_\nu\to 0$, the net $y\otimes x_\nu\to 0$ in $\overline{\otimes}^{n+m}E$ for all $y\in\overline{\otimes}^m E$, and this for all n, $m\geqq 0$. But $(y, x_\nu)\to(y, 0)$ in $\overline{\otimes}^m E\times\overline{\otimes}^n E$, so $y\otimes x_\nu\to 0$ in the inductive tensor product topology. Similarly for right continuity.

The involution is continuous iff it is continuous on each $\overline{\otimes}^n E$ $(n\geqq 0)$. We need only consider $n\geqq 2$ and prove that $(x_1, \cdots, x_n)\leadsto x_n^*\otimes\cdots\otimes x_1^*$ $(x_i\in E)$ is separately continuous. As above, let $(x_{n\nu})_\nu$ be a net converging to zero in E. This implies $\lim_\nu x_{n\nu}^*\otimes\cdots\otimes x_1^*=0$ and similarly for each factor separately. This gives continuity for the involution on $\otimes^n E$ which extends to $\overline{\otimes}^n E$ by continuity.

Assume that $\sum^N x_i^* x_i=0$ with $(x_i)_i$ nonzero. Let $(x_i)_{1\leq i\leq s}$ be the subfamily whose greatest non-vanishing components $(x_{ir})_{i\leq s}$ contribute to the greatest order component of $\sum^N x_i^* x_i$. Assume the $(x_{ir})_i$ are linearly independent, so $\sum^s x_{ir}^*\otimes x_{ir}=0$ implies $x_{ir}^*=0$ $(1\leqq i\leqq s)$ by linear disjointness. We now consider the case where the $(x_{ir})_{i\leq s}$ are linearly dependent, say $x_{ir}=\sum_{i\leq j\leq k}\lambda_{ji}x_{jr}$, where $(x_{jr})_{i\leq j\leq k}$ is a maximally linearly independent subset. After some manipulation we find that

$$x_{ir}+\sum_{j=k+1}^{s}|\lambda_{ij}|^2x_{ir}+\sum_{j=k+1}^{s}\sum_{\substack{e=1\\e\neq i}}^{r}\bar{\lambda}_{ij}\lambda_{ej}x_{er}=0\,,$$

for $1\leqq i\leqq k$. By linear independence, $1+\sum_{j=k+1}^{s}|\lambda_{ij}|^2=0$, contradicting the linear dependence.

Corollary 1.5. *In a BU-algebra the only idempotents are zero and the identity. There are no proper minimal ideals, and the Jacobson radical is* $\{0\}$: [7], [11].

We wish to point out that the inductive tensor product topology is the natural topology to use in BU-algebras. First of all, $E\overline{\otimes}F$ is, but $E\hat{\otimes}F$ generally is not, barrelled when E and F are. Secondly, $E\overline{\otimes}F$ has transitive properties for inductive limits. Note that $\mathscr{D}(\boldsymbol{R})\hat{\otimes}\mathscr{D}(\boldsymbol{R})$ is the set $\mathscr{D}(\boldsymbol{R}^2)$ but with a strictly coarser topology than the canonical LF-topology.

Examples 1.6. (a) $E=\mathscr{S}(\boldsymbol{R}^4)$ and $E=\mathscr{D}(\boldsymbol{R}^4)$ are the original BU-algebras due to Borchers [1] and Uhlmann [2] respectively. Here J is the identity. (b) For applications to the canonical anticommutation relations (CAR) and the canonical commutation relations (CCR) we take $E_{\boldsymbol{R}}=\mathscr{D}_r(M)\oplus\mathscr{D}_r(M)$ and $E=\mathscr{D}(M)\oplus\mathscr{D}(M)$, where the configuration space M is a paracompact C^∞-manifold, countable at infinity and Hausdorff. The case $M=\boldsymbol{R}^d$ is typical. Here and hereafter $\mathscr{D}$ will be equipped with its canonical LF-topology. For ease of notation we shall write $E_{\boldsymbol{R}}={}^2\mathscr{D}_r(M)$ and $E={}^2\mathscr{D}(M)$, so that $\underline{E}=\underline{{}^2\mathscr{D}(M)}$. Note that ${}^2\mathscr{D}(M)\simeq\mathscr{D}(M,\boldsymbol{C}^2)\simeq\mathscr{D}(M)\otimes\boldsymbol{C}^2$. The involution J on $E_{\boldsymbol{R}}$ is taken to be $J(x\oplus y)=y\oplus x$. (c) For any real finite dimensional Lie algebra $\mathfrak{g}$, the BU-algebra formed from $E_{\boldsymbol{R}}=\mathscr{D}_r(M,\mathfrak{g})\simeq\mathscr{D}_r(M)\otimes\mathfrak{g}$ will be shown to be pertinent to the description of the current commutation relations for $\mathfrak{g}$: [19].

As I^*-algebras are barrelled, the product law is hypocontinuous. Hence if $x_n\to x$ and $y_n\to y$ are two convergent sequences, $x_ny_n\to xy$. In general the product is not jointly continuous.

Proposition 1.7. *(a) Let* $\mathcal{A}$ *be an* I^**-algebra,* τ *its initial topology and* ν *the finest locally convex topology for which the linear map* $(x\otimes y)\rightsquigarrow xy$ *of* $\mathcal{A}(\tau)\otimes_\pi\mathcal{A}(\tau)\to\mathcal{A}(\nu)$ *is continuous. Denote the kernel of this map by K. Then*

$$\mathcal{A}(\tau)\otimes_\pi\mathcal{A}(\tau)/K\simeq\mathcal{A}(\nu) \tag{1.3}$$

$$\mathcal{A}(\tau)\otimes_\iota\mathcal{A}(\tau)/K\simeq\mathcal{A}(\tau). \tag{1.4}$$

(b) For a BU-algebra E, $\nu=\tau$ *if and only if* $E_{\mathbf{R}}$ *is an LB-space.*

Proof. (a) Let μ be the finest convex topology on $\mathcal{A}$ such that $M\colon\mathcal{A}(\tau)\otimes_\iota\mathcal{A}(\tau)\to\mathcal{A}(\mu)$; $x\otimes y\rightsquigarrow xy$ is continuous. Then $m\colon\mathcal{A}(\tau)\times\mathcal{A}(\tau)\to\mathcal{A}(\mu)$; $(x,y)\rightsquigarrow xy$ is separately continuous, and so the maps $L_x\colon y\rightsquigarrow xy$ are continuous from $\mathcal{A}(\tau)\to\mathcal{A}(\mu)$. Taking $x=\mathbf{1}$ implies that τ is finer than μ; but by definition μ is finer than τ, so $\mu=\tau$.

Going to the quotient, it is obvious that

$$\widetilde{M}:\mathcal{A}(\tau)\otimes_\iota\mathcal{A}(\tau)/K\to\mathcal{A}(\tau)$$

is continuous and one-to-one.

Now $M^{-1}\circ M$ is the canonical projection of $\mathcal{A}(\tau)\otimes_\iota\mathcal{A}(\tau)$ onto $\mathcal{A}(\tau)\otimes_\iota\mathcal{A}(\tau)/K$, which is continuous. But as $\tau=\mu$, $\widetilde{M}^{-1}\circ M$ is continuous iff $\widetilde{M}^{-1}$ is. Thus $\widetilde{M}$ furnishes the indicated isomorphism. The projective case is similar and we omit the proof.

For part (b) we need the following result: let F be a non-normable metrizable lch space. Then there exists a neighbourhood basis $(W_k)_{k\geq1}$ and a family $(\phi_k)_{k\geq1}$ of linear forms such that $\sup\{|\phi_k(x)|: x\in W_{k+1}\}<\infty$ and $\sup\{|\phi_k(x)|: x\in W_k\}=+\infty$. To show this we consider a family of increasing mutually inequivalent seminorms $p_1(x)\leqq p_2(x)\leqq\cdots$ defining the topology of F. As F is non-normable, the open unit ball of p_1, W_1, is unbounded. Hence there is a continuous linear functional ϕ_1 on F such that $\sup\{|\phi_1(x)|: x\in W_1\}=+\infty$. As ϕ_1 is continuous, there exists an index $n_1\geqq2$ and a constant $c_1>0$ such that $|\phi_1(x)|\leqq c_1p_{n_1}(x)$ for all $x\in F$. We may proceed to the result inductively.

Now consider F, with F not an LB-space; we assert that the bilinear functional $(x, y) \rightsquigarrow \phi(xy)$, where $\phi = (1, \phi_1, \phi_2 \otimes \phi_2, \cdots)$, is not jointly continuous, and hence that $(x, y) \rightsquigarrow xy$ is not jointly continuous. Assuming the product jointly continuous, and choosing $x = (0, x_1, 0, \cdots)$, $y = (0, \cdots, y_n, 0, \cdots)$ we compute $\phi(xy)$ and use $|\phi(xy)| \leqq p(x)p(y)$ for some continuous seminorm. This leads to the dominance of all the ϕ_n by a single continuous seminorm $p \upharpoonright F$ which is a contradiction. For the general case we consider such a space F as one of the defining sequence of E. The proof is an exercise in extension and we omit the details: [20].

It remains to show that if F is LB, then $\nu = \tau$. As F is nuclear, $\bar{\otimes}^n F$ $(n \geqq 2)$ are LB, and so is $\underline{F}$ therefore: Theorem 1. 5 with B replacing Frechet. From [36] (p. 316) it follows that $\underline{F} \otimes_\iota \underline{F} = \underline{F} \otimes_\pi \underline{F}$, and $\nu = \tau$ by Proposition 1. 8.

§ 2. Order Properties

The order properties of $\underline{\mathscr{S}(\boldsymbol{R}^4)}$ are well known [7, 21, 22]. In this section we consider the corresponding properties for a certain class of I^*-algebras. Note that by a cone we mean a convex cone which contains its vertex 0.

Definition 2. 1. (a) For an I^*-algebra the set of hermitian elements is

$$\mathscr{A}_h = \{x \in \mathscr{A} : x^* = x\}, \tag{2. 1. a}$$

the set of positive elements is

$$\mathscr{A}_+ = \{\sum^N x_i^* x_i : x_i \in \mathscr{A},\ N = 1, 2, \cdots\}, \tag{2. 1. b}$$

and the closure of $\mathscr{A}_+$ is written $\bar{\mathscr{A}}_+$. An I^*-algebra will be termed proper if $\bar{\mathscr{A}}_+$ is a proper cone.

(b) The hermitian functionals on $\mathscr{A}$ comprise the $\mathscr{A}'$ subset

$$\mathscr{A}'_h = \{\psi \in \mathscr{A}' : \psi(x^*) = \overline{\psi(x)},\ x \in \mathscr{A}\}, \tag{2. 2. a}$$

the positive functionals are

(2.2.b) $\mathcal{A}'_+ = \{\psi \in \mathcal{A}' : \psi(x) \geqq 0,\ x \in \mathcal{A}_+\}$,

and the states

(2.2.c) $\boldsymbol{E}(\mathcal{A}) = \{\psi \in \mathcal{A}'_+ : \psi(\mathbf{1}) = 1\}$.

In this section we consider only proper I^*-algebras and so will not indicate this explicitly. Let us note that all BU-algebras are proper I^*-algebras [40].

Lemma 2.2. *A positive functional obeys the Cauchy-Schwarz inequality*:

(2.3) $|\phi(x^*y)|^2 \leqq \phi(x^*x)\phi(y^*y)$

$(x, y \in \mathcal{A})$: [7, 21].

Proposition 2.3. *The hermitian part, $\mathcal{A}_h$, of an I^*-algebra is a complete real vector space whose complexification is $\mathcal{A}$. The cones $\mathcal{A}_+$, $\overline{\mathcal{A}}_+$ are proper strict b-cones which are generating for $\mathcal{A}_h$.*

The hermitian functionals $\mathcal{A}'_h$ constitute a complete real vector space whose complexification is $\mathcal{A}'$, with $\mathcal{A}'_h = (\mathcal{A}_h)'$. The cone $\mathcal{A}'_+$ is a complete proper normal cone with base $\boldsymbol{E}(\mathcal{A})$. The set $\mathcal{A}'_+ - \mathcal{A}'_+$ is dense in $\mathcal{A}'_h$.

Proof. The analysis for $\underline{\mathcal{S}(\boldsymbol{R}^4)}$ in [7, 21] suffices to prove most of this. We need only show that $\overline{\mathcal{A}}_+$ is a proper strict b-cone and $\mathcal{A}'_+$ a proper normal cone. The former follows because $\mathcal{A}_+$ is a strict b-cone and $\overline{\mathcal{A}}_+$ is proper. Together with the reflexivity of $\mathcal{A}$, this implies the latter: [23], 1.26 (p. 75).

For a proper I^*-algebra, therefore, the states separate points of the algebra.

We have found that the following condition on the ν topology enables us to prove a number of further order properties.

(N) An I^*-algebra $\mathcal{A}$ has property (N) if for any net $(x_\nu)_\nu$ in the cone $(\mathcal{A}(\tau) \otimes_\pi \mathcal{A}(\tau))_+$, the convergence of the net $\lim M(x_\nu) = 0$

implies $\lim_\nu x_\nu=0$. Here $M(x\otimes y)=xy$ as in Propostion 1. 7, and

$$(\mathcal{A}(\tau)\otimes_\pi\mathcal{A}(\tau))_+=\{\sum^N x_i^*\otimes x_i: x_i\in\mathcal{A},\ N=1,2,\cdots\}.$$

Proposition 2. 4. *The topology of an I^*-algebra $\mathcal{A}$ is given by its states, i.e, by the family $\{x\rightsquigarrow\phi(x^*x)^{1/2}: \phi\in\boldsymbol{E}(\mathcal{A})\}$, iff it has property (N).*

Proof. Let $(p_\alpha)_{\alpha\in A}$ be a family of $*$-symmetric seminorms defining the initial topology τ on $\mathcal{A}$. For any zero neighbourhoods $U, V\subset\mathcal{A}$ let $\varepsilon_{U,V}$ be the seminorm on $\mathcal{A}\otimes\mathcal{A}$

$$\varepsilon_{U,V}(x)=\sup\{|\sum^N\phi(y_i)\psi(z_i)|:\phi\in U^\circ,\psi\in V^\circ\}$$

associated with the $\otimes_\varepsilon$ topology. Here $x=\sum^N y_i\otimes z_i$. For p_α in the defining family we write ε_α for $U=V$ the open unit ball of p_α. Now let p be any continuous seminorm on $\mathcal{A}(\tau)$. By nuclearity there is a sequence of positive numbers $\lambda=(\lambda_n)_{n\geq1}\in l^1$ and a τ-equicontinuous sequence $(T_n)_{n\geq1}$ of linear functionals such that [15]

$$p(x)^2\leqq\sum_{n\geq1}\lambda_n|T_n(x)|^2. \tag{2.4}$$

There exists an $\alpha\in A$ such that

$$\sum^N p(x_i)^2\leqq\|\lambda\|_1\varepsilon_\alpha(\sum^N x_i^*\otimes x_i). \tag{2.5}$$

To see this we note that by $*$-symmetry and Cauchy-Schwarz,

$$\varepsilon_\alpha(y^*\otimes y)=\sup\{|\phi(y)|^2:\phi\in U_\alpha^\circ\}$$

and hence

$$\varepsilon_\alpha(\sum^N y_i^*\otimes y_i)=\sup\{\sum^N|\phi(y_i)|^2:\phi\in U_\alpha^\circ\}.$$

As $T_n\in U_\alpha^\circ$, $|T_n(x)|^2\leqq\varepsilon_\alpha(x^*\otimes x)$, giving the desired inequality.

By a theorem of Ky Fan ([24], Thm 1), to any continuous seminorm p there is a dominating positive functional ω,

$$\omega(x^*x)\geqq p(x)^2,$$

if whenever $\lim_\nu(\sum^{n(\nu)}x_{i\nu}^*x_{i\nu})=0$ and $\lim_\nu\sum^{n(\nu)}p(x_{i\nu})^2=\xi$, then $\xi=0$. For a zero-convergent net $\sum x_{i\nu}^*x_{i\nu}$, condition (N) implies that the net

$\sum x_{i\nu}^*\otimes x_{i\nu}$ converges to zero. The inequality (2. 5) then gives $\xi=0$ and the existence of a dominating state. As $\mathcal{A}(\tau)$ is barrelled, $x\rightsquigarrow\phi(x^*x)^{1/2}$ is continuous: [25].

Now assume τ generated by the states. For any seminorm p in a defining family for τ let $\pi(z)=\inf\{\sum p(x_i)p(y_i)\}$ be the indicated seminorm on $\mathcal{A}\otimes\mathcal{A}$. The infimum is over all product representations $z=\sum x_i\otimes y_i$. Let ϕ be a state dominating p. Then $\pi(\sum x_i^*\otimes x_i)\leqq\phi(\sum x_i^*x_i)$.

Corollary 2. 5. *Let $\mathcal{A}$ have property (N). Then $\bar{\mathcal{A}}_+$ has a base iff there is a continuous norm on $\mathcal{A}(\tau)$.*

Proof. Now $\bar{\mathcal{A}}_+$ has a base iff there is a strictly positive linear functional on $\mathcal{A}$: [23] Prop 3. 6 (p. 26). Let $x\rightsquigarrow\|x\|$ be the hypothesized norm. By Proposition 2. 4 there is a dominating state, so $0<\|x\|\leqq\omega(x^*x)^{1/2}$.

Proposition 2. 6. *For an I^*-algebra $\mathcal{A}$, $\mathcal{A}(\nu)$ is Hausdorff and nuclear.*

Proof. From Equation (1. 3) it follows that $\mathcal{A}'_+\subset(\mathcal{A}(\nu)_h)'$, as in [21] (p. 324). Since $\mathcal{A}'_+-\mathcal{A}'_+$ is dense in $\mathcal{A}'_h$, then $\mathcal{A}'_+$ separates points in $\mathcal{A}$. Therefore $\mathcal{A}(\nu)$ is Hausdorff. It follows from this that K is closed (Prop 1. 8) and from Equation (1. 3) that $\mathcal{A}(\nu)$ is nuclear.

Proposition 2. 7. *If $\mathcal{A}$ has property (N), $\mathcal{A}_+$ is normal in $\mathcal{A}(\nu)$. If $\tau\neq\nu$, ν is not barrelled. If ν is also complete, it is not bornological, and $\bar{\mathcal{A}}_+$ is generated by its extreme rays.*

Proof. From [21] follows the ν-continuity of $x\rightsquigarrow\omega(x)$ for any state. If ϕ is a state, $x\rightsquigarrow\phi(y^*xy)$ is again a state, so a polarization argument shows that $x\rightsquigarrow\omega(yx)$ is ν-continuous for any state, all $y\in\mathcal{A}$.

From Cauchy-Schwarz it follows that for any state ω

$$\omega(x^*x)=\sup_y\{|\omega(y^*x)|:\omega(y^*y)=1\}.$$

We may write, therefore,

$$\{x\in\mathcal{A}: \omega(x^*x)^{1/2}\leqq 1\} = \cap_y\{x\in\mathcal{A}: |\omega(y^*x)|\leqq 1, \omega(y^*y)=1\}.$$

From the continuity of $x\rightsquigarrow\omega(yx)$ it follows that this set is closed, and hence a ν-barrel. As ν is strictly coarser than τ, there is a state ω for which the above set is a τ-neighbourhood of zero (Prop 2. 4) but not a ν-neighbourhood of zero; hence ν is not barrelled. The normality of $\overline{\mathcal{A}}_+$ in $\mathcal{A}(\nu)$ follows as in [21], Theorem 4.

As a complete bornological space is barrelled: [13], II. 8. 4, if ν complete it is not bornological.

For $x\in\overline{\mathcal{A}}_+$, let $[0, x]$ be the order interval $\overline{\mathcal{A}}_+\cap(x-\overline{\mathcal{A}}_+)$. As $\mathcal{A}'$ is reflexive, barrelled and nuclear, $\mathcal{A}$ is quasicomplete and conuclear: [26] (p. 46). Therefore if the order intervals $[0, x]\,(x\in\mathcal{A}_+)$ are compact, $\overline{\mathcal{A}}_+$ is the closed convex hull of its extreme rays ([27] Thm 1). But $\mathcal{A}$ is Montel so we need only prove boundedness of these order intervals. As $\overline{\mathcal{A}}_+$ is normal in $\mathcal{A}(\nu)$, $[0, x]$ is bounded in $\mathcal{A}(\nu)$: [13], V. 3. 1, Cor 2. Hence every ν-barrel absorbs $[0, x]$: [14], Lemma 36. 2; [16] (p. 109). We have shown above that there is a fundamental system of τ-neighbourhoods of zero consisting of ν-barrels. Hence $[0, x]$ is τ-compact, giving the proposition.

The following proposition characterizes the BU-algebras that have poperty (N).

Proposition 2. 8. *(a) If E is a nuclear Fréchet space then the BU-algebra $\underline{E}$ has property (N) iff E is isomorphic to a closed subspace of s, the Fréchet space of rapidly decreasing sequences.*

(b) If E is a nuclear LF-space such that $E\otimes_\iota E\neq E\otimes_\pi E$ then the BU-algebra $\underline{E}$ does not have property (N).

Proof. (a) See [41] Satz 4. 8.

(b) If T is a state on $\underline{E}$ it is also a state on $\sum_{n\geq 0}^{\oplus}\hat{\otimes}^n E$, because $(\underline{f}, \underline{g})\rightsquigarrow T(\underline{f}\times\underline{g})$ is a jointly continuous bilinear form. By [41], Lemma 4. 2 $\underline{f}\rightsquigarrow T(\underline{f}^*\times\underline{f})^{1/2}$ is a continuous seminorm on $\sum_{n\geq 0}^{\oplus}\hat{\otimes}^n E$ and the conclusion follows immediately from this and the fact that $E\otimes_\iota E\neq E\otimes_\pi E$.

Corollary 2.9. *The BU-algebras* $\underline{\mathcal{D}(\boldsymbol{R}^n)}$, $\underline{{}^2\mathcal{D}(\boldsymbol{R}^n)}$ *and* $\underline{\mathcal{D}(\boldsymbol{R}^n, \mathfrak{g})}$ *do not have property (N).*

Proposition 2.10. *Every bounded set in* $\mathcal{A}(\tau)_h$ *is order-bounded.*

Proof. Mokobodzki has shown ([28], Thm 3) that as $\mathcal{A}'_+$ is normal in $\mathcal{A}(\tau)'_h$, and as the continuous seminorms p_B on $\mathcal{A}(\tau)'_h$ are given by $p_B(L)=\sup\{|L(x)|: x\in B\}$, where $B\subset\mathcal{A}(\tau)_h$ is bounded,

$$p_B(L)\leqq\sum\lambda_n|L(x_n)| \quad (L\in\mathcal{A}'_h) \tag{2.6}$$

for some sequence $\lambda=(\lambda_n)\in l^1$ with $\lambda_n\geqq 0$, and (x_n) a positive equicontinuous sequence in $\overline{\mathcal{A}}_+$. Now for $L\in\mathcal{A}'_+$ and $x\in B$, where $B\subset\overline{\mathcal{A}}_+$, $L(\sum_n\lambda_n x_n - x)\geqq 0$. Thus B is contained in the order interval $[0, \sum_n\lambda_n x_n]$, i.e. it is order-bounded.

Now if B_1 is an arbitrary bounded subset of $\mathcal{A}_h$, there is a bounded $B\subset\overline{\mathcal{A}}_+$ such that $B_1=B-B$, since $\overline{\mathcal{A}}_+$ is a strict b-cone. Then $B_1\subset[-y, y]$, $y\in\overline{\mathcal{A}}_+$.

§ 3. Representations of the Commutation Relations

Every state of an I^*-algebra determines a strongly cyclic representation of the algebra as a *-operator family. For BU-algebras, this corresponds to the Wightman reconstruction [1]; in all cases it is the GNS construction [12].

The construction and notation is standard: e.g. [7], Thm I. 4. 5. We remind the reader that for every state ω, $L(\omega)=\{x\in\mathcal{A}: \omega(x^*x)=0\}$ is a closed left ideal, $\mathfrak{D}_\omega=\mathcal{A}/L(\omega)$ is a pre-Hilbert space with canonical projection $x\rightsquigarrow[x]_\omega$ and $\langle[x]_\omega, [y]_\omega\rangle=\omega(x^*y)$. The GNS representation $\pi_\omega(x)\,[y]_\omega=[xy]_\omega$ is strongly cyclic and *-symmetric, with $\pi_\omega(x)^*\supset\pi_\omega(x^*)$ and cyclic vector $\Omega_\omega=[\mathbf{1}]_\omega$.

The canonical relations CAR, CCR are introduced for the algebra $\underline{{}^2\mathcal{D}(M)}$ by means of the fields (a, a^*).

Proposition 3.1. *To any *-representation* $(\pi, \mathfrak{D})$ *of* $\underline{{}^2\mathcal{D}(M)}$ *associate*

(3.1) $$a^*(f)=\pi(f,0), \quad a(f)=\pi(0,f)$$

for $f\in\mathcal{D}(M)$. *Then*

(3.2) $$a(f)^*\supset a^*(\bar{f}), \quad a^*(f)^*\supset a(\bar{f}).$$

Conversely, let there be given a pair (a, a^*) *of complex linear mappings from* $\mathcal{D}(M)$ *into operators on a dense domain* $\mathfrak{D}$ *of some Hilbert space such that* (i) $\mathrm{dom}[a^\#(f)]\supset\mathfrak{D}$, (ii) $\mathfrak{D}$ *is stable,* (iii) *Equation* (3.2), (iv) $f\rightsquigarrow(\Psi, a^\#(f)\Phi)$ *is in* $\mathcal{D}(M)'$ *for all* Ψ, Φ *in* $\mathfrak{D}$. *Then every normalized vector* $\Phi\in\mathfrak{D}$ *defines a state on* $\underline{{}^2\mathcal{D}(M)}$ *by extension from*

(3.3) $$\omega_n(F_1\otimes\cdots\otimes F_n)=\langle\Phi, \prod_{j=1}^{n}[a^*(f_j)+a(g_j)]\Phi\rangle$$

with $F_j=f_j\oplus g_j$ *in* $\mathcal{D}(M)\oplus\mathcal{D}(M)$. (*Cf. Thm* I. 4. 5 *of* [7].)

Proposition 3.2. *Let there be given a jointly continuous symmetric bilinear form* $\langle|\rangle$ *from* $\mathcal{D}_r(M)$ *to* $\mathbf{R}$ *which is non-degenerate and vanishes between functions of disjoint support. The CAR resp. CCR ideal for this form is the smallest closed *-ideal* I_ε ($\varepsilon=+1$ *resp.* -1) *generated by*

(3.4)
$$(0, 0, (f, 0)\otimes(g, 0)+\varepsilon(g, 0)\otimes(f, 0), 0, \cdots)$$
$$(0, 0, (0, f)\otimes(0, g)+\varepsilon(0, g)\otimes(0, f), 0, \cdots) \quad (f, g\in\mathcal{D}_r(M))$$
$$(-\langle g|f\rangle, 0, \varepsilon(f, 0)\otimes(0, g)+(0, g)\otimes(f, 0), 0, \cdots).$$

Any state ω *on* $\underline{{}^2\mathcal{D}(M)}$ *which annihilates* I_ε *gives rise to fields* (a, a^*) *satisfying*

(3.5)
$$[a^\#(f), a^\#(g)]_\varepsilon\Phi=0$$
$$[a(f), a^*(g)]_\varepsilon\Phi=\langle f|g\rangle\Phi$$

for all $\Phi\in\mathfrak{D}_\omega(f, g\in\mathcal{D}_r(M))$.

Conversely, if fields are given satisfying the hypotheses of Proposition 3.3 *and the relations Equation* (3.5), *then the* ω_Φ *annihilate* I_ε.

For a proof see [7], Theorem I. 5. 6, mutatis mutandis.

For obvious reasons, a state annihilating I_+ will be termed a CAR state, and one which annihilates I_- will be termed CCR state.

By a well-known theorem: e.g. [2], Thm 1 (p. 270), the CAR fields $a^{\#}(f)$ are bounded. It follows immediately that

$$a(f)^* = a^*(\bar{f}), \quad a^*(f)^* = a(\bar{f}). \tag{3.6}$$

Thus a CAR state on $\underline{{}^2\mathscr{D}(M)}$ leads to a representation of the CAR in the usual sense [2, 29].

The CCR fields are not bounded. Hegerfeldt [30] has shown that if the Weyl fields are continuous from $\mathscr{D}_r(M)$ into $\boldsymbol{B}(\mathscr{H})$, equipped with the weak operator topology, then the fields Q, P exist as self-adjoint operators on a dense Garding domain. This may be adapted trivially to (a, a^*) and so a CCR state on $\underline{{}^2\mathscr{D}(M)}$. Conversely, the same diagonalization of the fields may be used to show that (a, a^*) determines a Weyl representation iff it is essentially self-adjoint on all finite dimensional subspaces of $\mathscr{D}_r(M)$ and if the two-point function is continuous (private communication: G. Hegerfeldt, cf. [30]).

§ 4. Symmetries

By a symmetry we mean a *-automorphism of the algebra in question; automorphisms which are *-antisymmetric may be treated analogously. By a symmetry group we mean a group of such automorphisms. Continuity will always be assumed. Much of what follows is actually true for arbitrary locally convex Hausdorff topological *-algebras, but for our purposes consideration of I^*-algebras suffices.

In the same way we can consider fairly general groups, but in view of our applications we shall take G to be locally compact, non-compact, II-countable, and amenable.

We proceed as for C^*-algebras, terming G-ergodic the extreme points of the set of G-invariant states. As in the bounded case, $\alpha(G)$ is unitarily implemented in the GNS representation of every G-invariant state and conversely

Proposition 4. 1. *Let ω be a G-invariant state on an I^*-algebra. Consider the following conditions:*

(i) *ω is G-ergodic.*

(ii) *Let $\pi_\omega(\mathcal{A}) \cup U^\omega(G) = \mathcal{B}_\omega$; then $(\mathcal{B}_\omega, \mathfrak{D}_\omega)'_w = \boldsymbol{C}$.*

(iii) *The range of P, the orthogonal projection onto the G-invariant vectors, is one dimensional, spanned by Ω_ω.*

Then (i) $\Leftrightarrow$ (ii) $\Leftarrow$ (iii).

Proof. To show (i) $\Leftrightarrow$ (ii) we note that in the proof of Theorem 6.3 of [33], assuming the states $\omega = \lambda\omega_1 + (1-\lambda)\omega_2$, ω_1 G-invariant implies that the operator $C \in U^\omega(G)'$ (ibid.), and the result holds.

Next assume (iii) and let $C \in U^\omega(G)'$, then $C\Omega_\omega$ is G-invariant and so $C\Omega_\omega = \lambda\Omega_\omega$. Now if $C \in (\pi_\omega(\mathcal{A}))'_w$ also, then $C = \lambda$, because Ω_ω is separating for $(\pi_\omega(\mathcal{A}))'_w$.

For C^*-algebras, (i) $\Rightarrow$ (iii) follows from certain G-abelian conditions [2, 35]. For I^*-algebras this result is not implied by the following definition of G-abelianess: see [37] (p. 250) for a counterexample involving a local field, hence w.a.a. with respect to the group of space translations.

Definition 4.2. A state ω is $\alpha(G)$-weakly asymptotically abelian, or waa, if for every pair $\Phi, \Psi \in \mathfrak{D}_\omega$, every pair $x, y \in \mathcal{A}$, and every $\varepsilon > 0$, there exists a compact $\Delta \subset G$ such that

$$|\langle \Phi, [\pi_\omega(\alpha_g(x)), \pi_\omega(y)]_- \Psi \rangle| < \varepsilon \tag{4.1}$$

for all $g \in G \backslash \Delta$.

Even though this definition is not good enough to derive the implication (i) $\Rightarrow$ (iii) we have given it because in Proposition 4.4 we have a generalization of Theorems 1 and 2 in [42].

Lemma 4.3. *For a waa state ω on $\mathcal{A}$, for every $x \in \mathcal{A}$* and *$\Phi \in \mathfrak{D}_\omega$, the function*

$$f(g) = \|\pi_\omega(x) U_g^\omega \Phi\| \tag{4.3}$$

is continuous and bounded, $f \in CB(G)$.

Proof. From Equation (4.1), for every $x, y \in \mathcal{A}$, $\varepsilon > 0$, there is a compact $\Delta\subset G$ such that for every $g\in G\backslash\Delta^{-1}$,

$$f(g)^2\leqq|\langle[y]_\omega, [\pi_\omega(\alpha_{g^{-1}}(x^*x)), \pi_\omega(y)]_-\Omega_\omega\rangle| \\ +\langle[y]_\omega, \pi_\omega(y)\pi_\omega(\alpha_{g^{-1}}(x^*x))\Omega_\omega| \\ \leqq\varepsilon+\|\pi_\omega(y^*)[y]_\omega\|\cdot\|\pi_\omega(x^*x)\Omega_\omega\|,$$

where it suffices to consider $\Phi=[y]_\omega$. Thus f is bounded in the complement of a compacta. Now as $\mathcal{A}$ is barrelled, π_ω is strongly continuous, i.e., $x\rightsquigarrow\|\pi_\omega(x)\Phi\|$ is continuous $\mathcal{A}\to\boldsymbol{C}$ for all $\Phi\in\mathfrak{D}_\omega$. Therefore, the map f is a continuous function of G. Thus f must be bounded on Δ^{-1}, or $f\in CB(G)$.

Let P be the orthogonal projection from $\mathcal{H}_\omega$ into the G-invariant vectors. We shall show that P behaves well with regard to operator domains.

Proposition 4.4. *(a) For a G-invariant waa-state ω, we have $P:\mathfrak{D}_\omega\to\mathfrak{D}_\omega^{**}$, where*

$$\mathfrak{D}_\omega^{**}=\bigcap_{x\in\mathcal{A}}\mathrm{dom}[\pi(x)^{**}]. \tag{4.4}$$

*(b) For a G-invariant waa-state ω, the reduced family $\{P\pi_\omega^{**}(x)P:x\in\mathcal{A}\}$ is a strongly abelian *-operator family with domain $\mathfrak{D}_\omega^{**}$.*

Proof. (a) First we note that Størmer [31] has shown that there exists a net $\{A_a\in\Gamma U^\omega(G):a\in I\}$ in the convex hull of $U^\omega(G)$, which converges to P in the strong operator topology. Now as $\mathcal{H}_\omega$ is separable and $\Gamma U^\omega(G)$ is a bounded subset of $\boldsymbol{B}(\mathcal{H}_\omega)$ in the uniform operator topology, the strong operator topology on $\Gamma U^\omega(G)$ may be described by a norm: [32], Prop (2.4.2). Therefore the above net contains a subsequence $\{A_n: n=1, 2, \cdots\}$ converging to P in the strong operator topology.

Taking $A\in\Gamma U^\omega(G)$, it is clear from the previous lemma that $\|\pi_\omega(x)A\Phi\|\leqq M/2$ where the constant M depends upon $x\in\mathcal{A}$, $\Phi\in\mathfrak{D}_\omega$, but not on A. Then for the above sequence.

$$\|\pi_\omega(x)(A_n - A_m)\Phi\| \leqq M .$$

For any $\Psi \in \mathrm{dom}[\pi_\omega(x)^*]$, $\Phi \in \mathfrak{D}_\omega$,

$$\lim_{n\to\infty} \langle \pi_\omega(x)^*\Psi, A_n\Phi\rangle = \langle \pi_\omega(x)^*\Psi, P\Phi\rangle .$$

Now let $\Theta \in \mathcal{H}_\omega$ and $\varepsilon > 0$: there is a sequence $\Psi_n \in \mathrm{dom}[\pi_\omega(x)^*]$ such that for all $n > N(\varepsilon)$, $\|\Theta - \Psi_n\| < \varepsilon$. Then

$$|\langle \Theta, \pi_\omega(x)(A_n - A_m)\Phi\rangle|$$
$$\leqq \|\Theta - \Psi_p\| \cdot \|\pi_\omega(x)(A_n - A_m)\Phi\| + |\langle \Psi_p, \pi_\omega(x)(A_n - A_m)\Phi\rangle| .$$

For n, m large enough, the second term is small because

$$|\langle \Psi_p, \pi_\omega(x)(A_n - A_m)\Phi\rangle| \leqq \|\pi_\omega(x)^*\Psi_p\| \cdot \|(A_n - A_m)\Phi\|.$$

The first term is bounded by εM, and so $\pi_\omega(x)(A_n - A_m)\Phi$ converges to zero weakly. As every Hilbert space is sequentially weakly complete, there exists some vector $\Xi \in \mathcal{H}_\omega$ such that

$$\lim_{n\to\infty} (\Theta, \pi_\omega(x) A_n\Phi) = (\Theta, \Xi)$$

for all $\Theta \in \mathcal{H}_\omega$.

Taking $\Psi \in \mathrm{dom}[\pi_\omega(x)^*]$, it is clear from this that

$$\langle \Psi, \Xi\rangle = \langle \pi_\omega(x)^*\Psi, P\Phi\rangle .$$

Then $P\Phi \in \mathrm{dom}\,[\pi_\omega(x)^{**}]$ and we are done.

(b) In the same way as we proved that $P\mathfrak{D}_\omega \subset \mathfrak{D}_\omega^{**}$, we can easily show that $P\mathfrak{D}_\omega^{**} \subset \mathfrak{D}_\omega^{**}$, since $\mathfrak{D}_\omega^{****} = \mathfrak{D}_\omega^{**}$ ([34]).

Consider

$$F_1(g) = \langle \pi_\omega^{**}(x)^* P\Phi, U_g^* \pi_\omega^{**}(y) P\Psi\rangle$$
$$F_2(g) = \langle \pi_\omega^{**}(y)^* P\Phi, U_g \pi_\omega^{**}(x) P\Psi\rangle$$

for $x, y \in \mathcal{A}$, $\Phi \in \mathfrak{D}_\omega^*$ and $\Psi \in \mathfrak{D}_\omega^{**}$. For ω weakly asymtotically abelian, $F_1, F_2 \in CB(G)$ and their difference is as small as is wanted in the complement of a G-compacta. Hence $F_1 - F_2 \in CB(G)$, with $|(F_1 - F_2)(g)| < \varepsilon$ for $g \in G \backslash \Delta(\varepsilon)$. Applying the invariant mean η, $\eta(F_1 - F_2) = 0$. Using the mean ergodic theorem, [2], p. 177,

$$0 = \eta(F_1) - \eta(F_2)$$
$$= \langle \Phi, [P\pi_\omega^{**}(x)P, P\pi_\omega^{**}(y)P]_- \Psi\rangle .$$

As $\mathfrak{D}_\omega^*$ is dense in $\mathcal{H}_\omega$, the result follows.

When $\alpha(G)$ is equicontinuous of class $\mathscr{C}_0$ [39], and this includes all compact groups, $g \rightsquigarrow \omega(\alpha_g(x))$ is in $CB(G)$. Then if η is the invariant mean, $\eta \circ \omega$ is a G-invariant state, just as for C^*-algebras. In general, this function is only in $C(G)$ and the existence of a mean on this space is still an open question.

Now for BU-algebras $(1, 0, 0, \cdots)$ is always a G-invariant state. It would be interesting to have conditions to ensure that the subspace generated by the G-invariant states is dense in the annihilator $L_G^\perp$ of the real subspace L_G. Here L_G is generated by elements of the form $\langle x - \alpha_g x : x \in \mathcal{A}_h, g \in G \rangle$, cf. [10, 22]. Yngvason has shown ([41]) that this density condition holds for the translation group on $\underline{\mathscr{S}(\boldsymbol{R}^n)}$.

For BU-algebras one often starts from a linear representation of a group G on E, $\alpha(G) \subset L(E)$.

Proposition 4. 5. *Let $\alpha(G) \subset L(E)$ be given. Then $\{\underline{\alpha}_g = \sum_{n\geq 0}^{\oplus} \overline{\otimes}^n \alpha_g : g \in G\}$ is a continuous representation $\underline{\alpha}(G) \subset L(E)$.* [20]

The standard symmetries of the field algebra ${}^2\underline{\mathscr{D}(\boldsymbol{R}^d)}$ are space translations and gauge invariant.

Definition 4. 6. (a) For $f \in \mathscr{D}(\boldsymbol{R}^d)$, let $f_a \in \mathscr{D}(\boldsymbol{R}^d)$ be the function $x \rightsquigarrow f(x-a)$, for every $a \in \boldsymbol{R}^d$. In an obvious way, this map induces maps on $\underline{\mathscr{D}(\boldsymbol{R}^d)}$ and ${}^2\underline{\mathscr{D}(\boldsymbol{R}^d)}$, denoted by $\underline{\sigma}_a$, ${}^2\underline{\sigma}_a$ respectively. The automorphism groups $\underline{\sigma}(\boldsymbol{R}^d)$ and ${}^2\underline{\sigma}(\boldsymbol{R}^d)$ are termed space translations.

(b) For $f \in \mathscr{D}(\boldsymbol{R}^d)$ respectively $(f, g) \in \mathscr{D}(\boldsymbol{R}^d) \oplus \mathscr{D}(\boldsymbol{R}^d)$ consider the mappings $f \rightsquigarrow e^{i\theta} f$ respectively $(f, g) \rightsquigarrow (e^{i\theta} f, e^{-i\theta} g)$, $\theta \in T^1$, the gauge group. The corresponding automorphism groups of $\underline{\mathscr{D}(\boldsymbol{R}^d)}$ and ${}^2\underline{\mathscr{D}(\boldsymbol{R}^d)}$ are written $\underline{\gamma}$ and ${}^2\underline{\gamma}$ respectively.

Proposition 4. 7. *The symmetries ${}^2\underline{\sigma}(\boldsymbol{R}^d)$, ${}^2\underline{\gamma}(T^1)$ are locally equicontinuous with generators*

$${}^2\underline{\partial}_i = (0, \partial_i \oplus \partial_i, (\partial_i \oplus \partial_i) \otimes I + I \otimes (\partial_i \oplus \partial_i), \cdots)$$

$$^{2}\underline{N}=(0, 1\oplus -1, (1\oplus -1)\otimes I+I\otimes(1\oplus -1), \cdots)$$

respectively. The generators are implemented by essentially self-adjoint operators on the translationally invariant and gauge invariant states, respectively. For invariant CAR or CCR states we have the field equations $(x\in \boldsymbol{R}^d, \Phi\in\mathfrak{D}_\omega)$

$$a^\#_\omega(\underline{\partial}_i x)\Phi = i[P^\omega_i, a^\#_\omega(x)]\Phi$$

$$a^\#_\omega(\underline{N}x)\Phi = i[N_\omega, a^\#_\omega(x)]\Phi$$

respectively.

The proof is obvious.

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