Publ. RIMS, Kyoto Univ.
17 (1981), 235-244

Homogenization in Stochastic Differential Geometry[1]

By

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§ 1. Introduction

It is well known that a diffusion process on Euclidean space can be rigorously considered as the limit of a sequence of transport processes. This idea, which dates at least to Rayleigh's problem of random flight [3] has now received a general treatment by modern probabilistic methods [2, 10]. In addition, we have shown that a suitable transport approximation remains valid for the Brownian motion of any complete Riemannian manifold [11].

In another direction several authors [1, 4, 5, 6, 7, 8] have considered "stochastic parallel displacement", i.e. parallel displacement of vectors along Brownian motion curves in a manifold. The purpose of this paper is to show that the stochastic parallel displacement can be rigorously considered as the limit of parallel displacement along the paths of a transport process. The transport approximation introduces an extra velocity variable which disappears in the limit, hence the term *homogenization.*

In addition to its elementary geometric appeal, our approach has the advantage of producing the following coordinate-free definition of the infinitesimal operator of the stochastic parallel displacement:

$$\mathcal{A}f = PZ^2f. \tag{1. 1}$$

$\mathcal{A}$ is a second-order degenerate elliptic operator on the bundle of k-frames of the given manifold. Z is a horizontal vector field on the bundle of

Communicated by H. Araki, Februay 8, 1980.
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1) Work done while the author was Visiting at the University of Chicago. Supported by NSF Grant MCS 78-02144.

$(k+1)$-frames and P is the operator which averages out the extra velocity variable. In case $k=0$, our formula reduces to a multiple of the Laplace-Beltrami operator on functions [11]. In case $k=1$, our coordinate formula for $\mathcal{A}$ agrees with previously obtained formula [1] for stochastic parallel displacement. In case $k=n$ and we work with $O(M)$, the bundle of orthonormal frames, $\mathcal{A}$ is a multiple of the horizontal Laplacian [8] on $O(M)$. Formula (1.1) allows a rapid proof that this process preserves the inner product of tangent vectors.

§ 2a. Transport Process on the Frame Bundle

Let M be a complete Riemannian manifold, $T^{(k+1)}(M)$ the bundle of frames over M;

$$T^{(k-1)}(M) = \{(x, \xi, \eta_1, \cdots, \eta_k) : x \in M, \xi \in M_x, \eta_1 \in M_x, \cdots, \eta_k \in M_x\}.$$

Here $0 \leqq k$, n is the dimension of M, and M_x is the tangent space at x. Let $\gamma_{x,\xi}$ be the geodesic on M with $\gamma(0)=x$, $\dot\gamma(0)=\xi$. Let $\bar\eta_j(t)=\bar\eta(t;\eta_j)$ be the parallel displacement of the tangent vector η_j along γ. The canonical horizontal vector field Z is defined by

$$Zf(x, \xi, \eta_1, \cdots, \eta_k) = \frac{d}{dt} f(\gamma(t), \dot\gamma(t), \bar\eta_1(t), \cdots, \bar\eta_k(t))\big|_{t=0}. \tag{2.1}$$

The projection operator P is defined by

$$Pf(x, \eta_1, \cdots, \eta_k) = \int_{M_x} f(x, \xi, \eta_1, \cdots, \eta_k)\, \mu_x(d\xi) \tag{2.2}$$

where $\mu_x(d\xi)$ is the unique rotationally invariant probability measure on the unit sphere of the tangent space M_x. Note that

$$Pf = f \qquad f \in C(T^{(k)}(M)) \tag{2.3}$$

$$PZf = 0 \qquad f \in C(T^{(k)}(M)) \tag{2.4}$$

where f is independent of ξ.

Let $\{e_n\}_1^\infty$ be a sequence of independent random variables on a probability space Ω_1 with the common exponential distribution

$$\text{Prob}\{e_n > t\} = e^{-t} \quad n=1, 2, \cdots, t>0,$$

and let $\tau_n = e_1 + \cdots + e_n$. Define a sequence of $T^{(k+1)}(M)$-valued random variables $(x^{(n)}, \xi^{(n)}, \eta_1^{(n)}, \cdots, \eta_k^{(n)})$ $n=0, 1, 2, \cdots$, as follows:

$$x^{(0)}=x\,, \quad \xi^{(0)}=\xi\,, \quad \eta_1^{(0)}=\eta_1, \cdots, \eta_k^{(0)}=\eta_k\,.$$

If $(x^{(n)}, \xi^{(n)}, \eta_1^{(n)}, \cdots, \eta_k^{(n)})$ have been defined, we let

$$x^{(n+1)}=\gamma_{x^{(n)},\xi^{(n)}}(e_{n+1}), \quad \eta_j^{(n+1)}=\overline{\eta}(e_{n+1}; \eta_j^{(n)}), \quad j=1, \cdots, k.$$

Finally $\xi^{(n+1)}$ is distributed according to $\mu_{x^{(n+1)}}(d\xi)$, independent of $\{x^{(0)}, \cdots, \eta_k^{(n)}\}$. We let

$$x(t)=\gamma_{x^{(n)},\xi^{(n)}}(t-\tau_n) \qquad (\tau_n \leqq t < \tau_{n+1})$$

$$\xi(t)=\dot{\gamma}_{x^{(n)},\xi^{(n)}}(t-\tau_n) \qquad (\tau_n \leqq t < \tau_{n+1})$$

$$\eta_j(t)=\overline{\eta}(t-\tau_n; \eta_j) \qquad (1 \leqq j \leqq k, \ \tau_n \leqq t < \tau_{n+1})$$

$$T_t^0 f(x, \xi, \eta_1, \cdots, \eta_k)=f(\gamma(t), \dot{\gamma}(t), \overline{\eta}_1(t), \cdots, \overline{\eta}_k(t)), \quad f \in C\,,$$

$$R_\lambda^0 f(x, \xi, \eta_1, \cdots, \eta_k)=\int_0^\infty e^{-\lambda t} T_t^0 f(x, \xi, \eta_1, \cdots, \eta_k)\,dt\,, \quad f \in C\,,$$

$$T_t f(x, \xi, \eta_1, \cdots, \eta_k)=Ef(x(t), \xi(t), \eta_1(t), \cdots, \eta_k(t)), \ f \in C\,,$$

$$R_\lambda f(x, \xi, \eta_1, \cdots, \eta_k)=\int_0^\infty e^{-\lambda t} T_t f(x, \xi, \eta_1, \cdots, \eta_k)\,dt\,, \quad f \in C\,,$$

where C is the space of differentiable functions on $T^{(k-1)}(M)$ which vanish at infinity.

Lemma 1. *R_λ^0 maps C into C and $(\lambda-Z)R_\lambda^0 f=f$, $f \in C$.*

Proof. The geodesic flow and parallel displacement depend smoothly on initial conditions, hence the first statement. The second statement is obtained by Laplace transform.

Lemma 2. $R_\lambda f=R_{1+\lambda}^0 f+R_{1+\lambda}^0 PR_\lambda f$, $f \in C$.

Proof. We use the renewal method, applied to τ_1. Thus

$$R_\lambda f=E\left\{\int_0^{\tau_1}+\int_{\tau_1}^\infty\right\} e^{-\lambda t} f(x(t), \xi(t), \eta_1(t), \cdots, \eta_k(t))\,dt\,.$$

The first term is

$$E\int_0^\infty I_{(t<\tau_1)}e^{-\lambda t}f(x(t),\xi(t),\eta_1(t),\cdots,\eta_k(t))dt$$

$$=\int_0^\infty e^{-t}e^{-\lambda t}f(\gamma(t),\dot{\gamma}(t),\bar{\eta}_1(t),\cdots,\bar{\eta}_k(t))dt$$

$$=R^0_{1+\lambda}f\,.$$

The second term is

$$E\int_{\tau_1}^\infty e^{-\lambda t}f(x(t),\xi(t),\eta_1(t),\cdots,\eta_k(t))dt$$

$$=E\int_0^\infty e^{-\lambda(\tau_1+s)}f(x(\tau_1+s),\xi(\tau_1+s),\cdots,\eta_k(\tau_1+s))ds$$

$$=E\left\{e^{-\lambda\tau_1}E\int_0^\infty e^{-\lambda s}f(x(\tau_1+s),\xi(\tau_1+s),\cdots,\eta_k(\tau_1+s))ds\right\}$$

$$=E\left\{e^{-\lambda\tau_1}E\left\{\int_0^\infty e^{-\lambda s}f(x(s;x_1^{(1)}),\cdots,\eta_k(s;\eta_k^{(1)})\,|\,\tau_1,\xi_1\right\}\right\}$$

$$=E\{e^{-\lambda\tau_1}\{R_\lambda f(x^{(1)},\xi^{(1)},\eta_1^{(1)},\cdots,\eta_k^{(1)})\}\}$$

$$=E\{e^{-\lambda\tau_1}(PR_\lambda f)\,(x^{(1)},\xi^{(0)},\eta_1^{(1)},\cdots,\eta_k^{(1)})\}$$

$$=\int_0^\infty e^{-\lambda t}(PR_\lambda f)\,(\gamma(t),\dot{\gamma}(t),\bar{\eta}_1(t),\cdots,\bar{\eta}_k(t))e^{-t}dt$$

$$=R^0_{1+\lambda}PR_\lambda f\,.$$

Lemma 3. $(\lambda-Z-P+I)R_\lambda f=f,\quad f\in C.$

Proof. Applying $(I+\lambda-Z)$ to Lemma 2, we have

$$(I+\lambda-Z)R_\lambda f=f+PR_\lambda f$$

which was to be proved.

Lemma 4. $T_t f-f=\int_0^t(Z+P-I)T_s f ds\,,\quad f\in C\,.$

Proof. The Laplace transform of the left-hand side is $R_\lambda f-\lambda^{-1}f$ while the right-hand side transforms into

$$\int_0^\infty e^{-\lambda t}\int_0^t(Z+P-I)T_s f ds$$

$$=\lambda^{-1}\int_0^\infty e^{-\lambda s}(Z+P-I)T_s f ds$$

$$=\lambda^{-1}(Z+P-I)R_\lambda f .$$

Using Lemma 3, we have proved the result by uniqueness of Laplace transforms.

We now observe that we may interchange the order of the two operators appearing in the right-hand side of Lemma 4. Indeed, from the semi-group property of T_t, it follows that $\lim_{s\to 0} s^{-1}\{T_{t-s}f-T_t f\}=T_t\{\lim_{s\to 0} s^{-1}(T_s f-f)\}=T_t(Z+P-I)f$, from Lemma 4. Applying the fundamental theorem of calculus gives the stated result.

§ 2b. Convergence to a Diffusion Process

Let $\varepsilon>0$ be a small parameter. If we replace Z by εZ in the construction of the previous section, we obtain a process

$$({}^{\varepsilon}x(t), {}^{\varepsilon}\xi(t), {}^{\varepsilon}\eta_1(t), \cdots, {}^{\varepsilon}\eta_k(t)) .$$

Define

$${}^{\varepsilon}T_t f(x, \xi, \eta_1, \cdots, \eta_k)=E\{f({}^{\varepsilon}x(t/\varepsilon^2), {}^{\varepsilon}\xi(t/\varepsilon^2), \cdots, {}^{\varepsilon}\eta_k(t/\varepsilon^2))\}$$

$${}^{\varepsilon}R_\lambda f(x, \xi, \eta_1, \cdots, \eta_k) = \int_0^\infty e^{-\lambda t}({}^{\varepsilon}T_t f)(x, \xi, \eta_1, \cdots, \eta_k) dt .$$

It is readily verified that these correspond to the infinitesimal operator $\varepsilon^{-1}Z+\varepsilon^{-2}(P-I)$ in Lemma 4 above. We now introduce the infinitesimal operator of stochastic parallel displacement on $T^{(k)}(M)$,

$$\mathcal{A}=PZ^2 .$$

Using [5] it can be shown that $\mathcal{A}$ generates a strongly continuous semigroup of contraction operators on $C(T^{(k)}M)$. The resolvent operator is defined by

$$W_\lambda f=(\lambda-PZ^2)^{-1}f .$$

Theorem 1. *If* $g\in C(T^{(k)}(M))$, *then*

$$\lim_{\varepsilon\to 0} {}^{\varepsilon}R_\lambda g=W_\lambda g .$$

We follow the analytic method of Papanicolaou [9]. For this purpose, let $f \in C^3(T^{(k)}(M))$ and let

$$f_\varepsilon = f + \varepsilon f_1 + \varepsilon^2 f_2$$

where

$$f_1 = Zf, \quad f_2 = Z^2 f - PZ^2 f.$$

Note that $f_1, f_2 \notin C(T^{(k)}(M))$ in general.

Lemma 5. $[\varepsilon^{-1}Z + \varepsilon^{-2}(P-I)]f_\varepsilon = PZ^2 f + \varepsilon Z f_2, \quad \varepsilon > 0.$

Proof. Multiply out the six terms involved and collect like powers of ε. The coefficient of ε^{-2} is $Pf - f = 0$, by (2.3). The coefficient of ε^{-1} is $Zf + (P-I)f_1 = Zf + (P-I)Zf = Zf - Zf = 0$, by (2.4). The constant term is $Zf_1 + (P-I)f_2 = Z^2 f + (P-I)(Z^2 f - PZ^2 f) = Z^2 f + PZ^2 f - PZ^2 f - Z^2 f + PZ^2 f = PZ^2 f$. Finally the coefficient of ε is just Zf_2.

Proof of the Theorem. We write Lemma 4, rescaled with ε, in terms of Laplace transforms. Thus

$${}^\varepsilon R_\lambda f_\varepsilon - \lambda^{-1} f_\varepsilon = {}^\varepsilon R_\lambda (\varepsilon^{-1} Z + \varepsilon^{-2}(P-I)) f_\varepsilon .$$

Using Lemma 5 and collecting terms, we have

$${}^\varepsilon R_\lambda (\lambda f - PZ^2 f) = f + \varepsilon F_1 + \varepsilon^2 F_2$$

where

$$F_1 = f_1 - {}^\varepsilon R_\lambda f_1 - {}^\varepsilon R_\lambda Z f_2$$

$$F_2 = f_2 - \lambda^\varepsilon R_\lambda f_2 .$$

Now let $\lambda f - PZ^2 f = g$, $f = W_\lambda g$. Letting $\varepsilon \to 0$, we have proved that $\lim_{\varepsilon \to 0} {}^\varepsilon R_\lambda g = W_\lambda g$, as required.

We now justify the term "stochastic parallel displacement". Let $X(t)$ be the diffusion process on $T^{(k)}(M)$ governed by the differential operator PZ^2. Let $N_{ij} = (\eta_i, \eta_j)$ be the inner product of a pair of tangent vectors, $1 \leqq i, j \leqq k$. N_{ij} is a real valued function on $T^{(k)}(M)$.

Theorem 2. $N_{ij}(X(t)) = N_{ij}(X(0)), \quad 1 \leqq i, j \leqq k, \quad t \geqq 0.$

For the proof we first note the useful

Lemma 6. $ZN_{ij}=0$.

Proof. The classical parallel displacement preserves the inner product of tangent vectors. Thus $(\overline{\eta}_i(t), \overline{\eta}_j(t)) = (\overline{\eta}_i(0), \overline{\eta}_j(0))$. Glancing at (2.1) shows that $ZN_{ij}=0$.

Lemma 7. *$N_{ij}(X(t))$ is a martingale.*

Proof. It suffices to show that $PZ^2(N_{ij})=0$, which is immediate from Lemma 6.

Lemma 8. *The increasing process of $N_{ij}(X(t))$ is zero.*

Proof. By the results of Taylor [12] for example, it suffices to show that $PZ^2(N_{ij}^2)-2N_{ij}PZ^2(N_{ij})=0$. But this also follows immediately from Lemma 6.

§ 3. Explicit Formulas in Local Coordinates

The operator PZ^2 which occurs in the above limit theorem is called the *homogenized transport operator*. It is an invariantly defined second order differential operator on the frame bundle $T^{(k)}(M)$. In case $k=0$, we have shown [11] that the homogenized transport operator is equal to n^{-1} times the Laplace-Beltrami operator of the Riemannian metric. We now obtain an explicit local formula in case $k=1$, i.e. the tangent bundle. For this purpose, we work with the bundle of 2-frames

$$T^{(2)}(M)=\{(x,\xi,\eta): x\in M, \xi\in M_x, \eta\in M_x\}.$$

Let $\gamma(t)$ be the geodesic with $\gamma(0)=x$, $\dot{\gamma}(0)=\xi$. Let $\eta(t)$ be the parallel displacement of η along γ. The canonical horizontal vector field Z is defined by

$$Zf(x,\xi,\eta)=\frac{d}{dt}f(x(t),\xi(t),\eta(t))\big|_{t=0}. \tag{3.1}$$

In a coordinate chart we can compute Z by the formula

$$Z f(x, \xi, \eta)=\xi^{i} \frac{\partial f}{\partial x}-\Gamma_{i j}^{k} \xi^{i} \xi^{j} \frac{\partial f}{\partial \xi^{k}}-\Gamma_{i j}^{k} \xi^{i} \eta^{j} \frac{\partial f}{\partial \eta^{k}} . \tag{3.2}$$

The projection operator P maps from $C(T^{(2)}(M))$ to $C(T(M))$ by the rule

$$P f(x, \xi)=\int_{M_{x}} f(x, \xi, \eta) \mu_{x}(d \xi) \tag{3.3}$$

where μ_x is the rotationally invariant probability measure on the unit sphere of M_x. Of particular relevance are the coordinate formulas [11].

$$P(\xi^{i} \xi^{j})=n^{-1} g^{i j} \tag{3.4}$$

where g^{ij} is the inverse of the metric tensor. The homogenized transport operator can be computed according to the following

Proposition 9. *In a coordinate chart we have the formula*

$$\begin{aligned} P Z^{2} f=n^{-1} g^{i l}\{ & f_{x^{i} x^{l}}+\Gamma_{i j}^{k} \Gamma_{l m}^{n} \eta^{j} \eta^{m} f_{\eta^{k} \eta^{n}}-2 \Gamma_{l m}^{n} \eta^{m} f_{x^{i} \eta^{n}} \\ & -\Gamma_{i l}^{j} f_{x^{j}}+(\Gamma_{i l}^{j} \Gamma_{j m}^{n} \eta^{m}+\Gamma_{i j}^{k} \Gamma_{l k}^{n} \eta^{j}-\Gamma_{l m, i}^{n} \eta^{m}) f_{\eta^{n}}\} \end{aligned} \tag{3.5}$$

where $f=f(x, \eta)$ is a C^2 function.

Proof. This is a straightforward computation, using (3.1), (3.2), (3.4). Except for the sign convention of Γ_{ij}^k, this agrees with the formulas of [1, 4].

We now consider the case $k=n$, specializing to $O(M)$ the bundle of orthonormal frames. Let $(e_1, \cdots, e_n)$ be an orthonormal basis of M_x. Let $\gamma_\beta(t)$ be the geodesic with $\gamma_\beta(0)=x$, $\dot{\gamma}_\beta(0)=e_\beta$. Let $e_{i\beta}(t)$ be the parallel displacement of e_i along γ_β. The horizontal vector field E_β, $1 \leqq \beta \leqq n$, is defined by

$$E_{\beta} f=\frac{d}{d t} f(\gamma_{\beta}(t), e_{1 \beta}(t), \cdots, e_{n \beta}(t))|_{t=0} . \tag{3.6}$$

The horizontal Laplacian is defined by

$$\Delta_{O(M)} f=\sum_{\beta=1}^{n} E_{\beta}^{2} f . \tag{3.7}$$

Proposition 10. $\Delta_{O(M)}f=nPZ^2f|_{O(M)}, \quad f\in C(T^k(M))$.

Proof. In any coordinate chart, we have

$$Zf=\xi^i\frac{\partial f}{\partial x^i}-\Gamma^k_{ij}\xi^i\xi^j\frac{\partial f}{\partial \xi^k}-\Gamma^k_{ij}\xi^i\eta^j_\alpha\frac{\partial f}{\partial \eta^k_\alpha}. \tag{3.8}$$

Assuming a normal chart centered at x, we have for $f\in C(T^{(k)}(M))$

$$\begin{aligned}Z^2f&=\xi^l\frac{\partial}{\partial x^l}\left\{\xi^i\frac{\partial f}{\partial x^i}-\Gamma^k_{ij}\xi^i\eta^j_\alpha\frac{\partial f}{\partial \eta^k_\alpha}\right\}\\&=\xi^i\xi^l\frac{\partial^2 f}{\partial x^i\partial x^l}-\Gamma^k_{ij,l}\xi^l\xi^i\eta^j_\alpha\frac{\partial f}{\partial \eta^k_\alpha}.\end{aligned}$$

Using (3.4)

$$PZ^2f=n^{-1}\left[\frac{\partial^2 f}{\partial x^i\partial x^i}-\Gamma^k_{ij,i}\eta^j_\alpha\frac{\partial f}{\partial \eta^k_\alpha}\right]. \tag{3.9}$$

To compare with (3.7), we recall the coordinate form of (3.6) [5]:

$$E_\beta f=e^i_\beta\frac{\partial f}{\partial x^i}-e^i_\beta e^j_\alpha\Gamma^k_{ij}\frac{\partial f}{\partial (e^k_\alpha)} \tag{3.10}$$

where $(x^1, \cdots, x^n, e^1_1, e^2_1, \cdots, e^n_n)$ is any coordinate chart on $O(M)$. Assuming a normal chart centered at x, we have

$$\begin{aligned}E^2_\beta f&=e^l_\beta\frac{\partial}{\partial x^l}\left\{e^i_\beta\frac{\partial f}{\partial x^i}-e^i_\beta e^j_\alpha\Gamma^k_{ij}\frac{\partial f}{\partial (e^k_\alpha)}\right\}\\&=e^l_\beta e^i_\beta\frac{\partial^2 f}{\partial x^i\partial x^l}-e^i_\beta e^l_\beta e^j_\alpha\Gamma^k_{ij,l}\frac{\partial f}{\partial (e^k_\alpha)}.\end{aligned}$$

Summing on β and using the orthonormality, we have

$$\sum_{\beta=1}^n E^2_\beta f=\frac{\partial^2 f}{\partial x^i\partial x^i}-e^j_\alpha\Gamma^k_{ij,i}\frac{\partial f}{\partial (e^k_\alpha)}$$

which agrees with (3.9) to within the factor n^{-1}.

Finally, we note that the orthonormal frame $m=(x, e_1, \cdots, e_n)$ has a unique stochastic parallel displacement along the Brownian motion path. For this purpose, recall the Stratanovich equation [5] for horizontal diffusion on $O(M)$:

$$dm=\sum_{\beta=1}^n E_\beta\cdot dw^\beta.$$

Using the coordinate representation (3. 10), we have

$$dx^i = e^i_\beta \cdot dw^\beta$$

$$de^k_\alpha = -e^i_\beta e^j_\alpha \Gamma^k_{ij} \cdot dw^\beta$$

$$= -e^i_\alpha \Gamma^k_{ij} \cdot dx^i.$$

Thus, given the Brownian motion path $\{x(t), t \geqq 0\}$, the vector $e_\alpha(t)$ is uniquely determined by solving a *linear* system of stochastic differential equations. Thus we have proved

Proposition 11. *The mapping* $(x(t), e_1(t), \cdots, e_n(t)) \to x(t)$ *is a measure-preserving bijection from the path space of the horizontal diffusion on* $O(M)$ *to the path space of the Brownian motion on* M.

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