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The Projectivity of Y-Games

By

Yōhei YAMASAKI*

Introduction

The first approach to the theory of regular Nim-type games was made by Grundy [2] and Sprague [5], where it was clarified that the Grundy number plays an important role. Guy and Smith [3] introduced the games parametrized by Grundy value and studied the periodicity of the series of Grundy numbers of games of fixed Grundy value with respect to the heap size. Examples of Grundy values whose series of Grundy numbers are non-periodic were found by Yoneda (see [4]).

On the other hand Yamasaki [6] studied the misère Nim-type games, remarking the singular D-schemes, where a D-scheme $\boldsymbol{D}$ indicates a Nim-type game without decision of winners and $\boldsymbol{D}$ is said to be singular if the first player has a winning strategy in strictly one of the regular game and the misère game played over $\boldsymbol{D}$. It is shown that the flat D-schemes form a unique maximal class of D-schemes satisfying the end game modification theory.

The flatness, however, of a D-scheme $\boldsymbol{D}$ is proved when the whole restrictions of $\boldsymbol{D}$ of Grundy number 0 or 1 are known. A sufficient condition 'projectivity' of 'flatness' was introduced so that one can see the projectivity of a D-scheme $\boldsymbol{D}$ when he finds a set $\mathfrak{F}$ of restrictions of $\boldsymbol{D}$ satisfying several conditions, where $\mathfrak{F}$ becomes clear to coincide with the set of singular restrictions of $\boldsymbol{D}$. It was clarified that Nim, Restricted Nim, Block Nim, Tsyan-shizi (Chinese Nim, Wythoff's Nim) and Sato's Maya game (Welter's game) are projective and that Keyles is not flat. In this paper we shall see the projectivity of the generalized Yoneda's games which are equivalent to the games with Grundy value 0.*****·····

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* Department of Mathematics, Osaka University, Toyonaka, Osaka 560, Japan.

where every $*$ is 0 or 3. The readers are supposed to be acquainted with [6]. We adopt the definitions and notions in [6]. Here we give a table of those appear in this paper.

D-scheme	p. 461
Nim-type game	p. 461
ω^-, restriction	p. 462
Grundy number	p. 463
singular	p. 465
$\mathcal{T}, \mathcal{S}, \mathcal{I}, \mathcal{N}, \mathcal{F}^g$	p. 466
projective, $\mathcal{F}^{(g)}$	p. 471

§ 1. Y-Games

We fix a set P of positive integers throughout this paper. Let X be a finite set. Then we define a D-scheme $\boldsymbol{D}=(X, \rho_P)$ by

$$\rho_P(A, B)=1 \quad \text{iff} \quad |A-B| \in P.$$

We call $\boldsymbol{D}^{=}$ the $\begin{Bmatrix}\text{regular}\\ \text{misère}\end{Bmatrix}$ Y-game of heap size $|X|$. This game is equivalent to the game whose Grundy value $0.*****\cdots$ is taken so that the i-th $*$ is 3 if $i \in P$ and 0 if $i \notin P$. Our purpose is to prove that the relevant D-scheme is projective. If P is empty, then $\boldsymbol{D}$ is projective. From now on we assume that P is non-empty.

We put $p_0=\min P$ and

$$a-P=\{n \in \bar{\boldsymbol{N}} \mid a-n \in P\}$$

for a non-negative integer a where $\bar{\boldsymbol{N}}$ is the set of all non-negative integers. We define a mapping $g: \bar{\boldsymbol{N}} \rightarrow \bar{\boldsymbol{N}}$ by

$$g(n)=G(\boldsymbol{D})$$

for a set X of n points.

§ 2. The Projectivity

Lemma 1. i) $g(n)=0$ *if* $n<p_0$,

ii) $g(n)=1$ *iff* $g(n-p_0)=0$.

Proof. The first assertion is obvious. We shall show the second assertion by induction on n.

Step 1. By i), ii) holds if $n<p_0$.

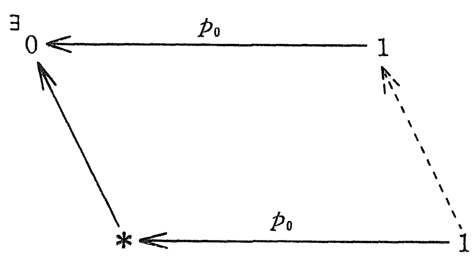
Step 2. Let $n\geqq p_0$ and assume that ii) holds while the heap size is less than n. First suppose $g(n)=1$ and $g(n-p_0)\neq 0$. Then there exists $m\in n-P$ such that

$$g(m-p_0)=0 .$$

Therefore by the induction hypothesis,

$$g(m)=1 ,$$

which contradicts the facts $n-m\in P$ and $g(n)=1$.



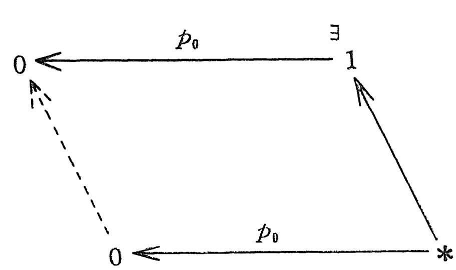
Next suppose $g(n)\neq 1$ and $g(n-p_0)=0$. Then $g(n)\geqq 2$, since $g(n)\neq 0$. Therefore there exists $m\in n-P$ such that

$$g(m)=1 .$$

Hence by the induction hypothesis,

$$g(m-p_0)=0 ,$$

which contradicts the facts $(n-p_0)-(m-p_0)\in P$ and $g(n-p_0)=0$. Now our lemma is verified.



Lemma 2. *Let $\boldsymbol{D}=(X,\rho)$ be a $\boldsymbol{D}$-scheme satisfying*

$$\mathscr{S}=\mathscr{J}.$$

Then $\boldsymbol{D}$ is projective.

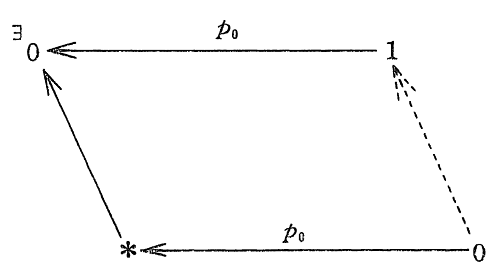
Proof. We put $\mathfrak{F}^{(0)}=\mathscr{J}^0$ and $\mathfrak{F}^{(1)}=\mathscr{J}^1$. We must show only that there exists a successor $I'\in\mathscr{J}^1$ of I for $I\in\mathscr{J}^0-\mathfrak{T}$. Since $I\in(\mathscr{J}^0-\mathfrak{T})\cap\mathscr{S}$, $\omega^-(\boldsymbol{D}_I)=1$. Then there exists a successor $I'\in 2^X$ of I such that $\omega^-(\boldsymbol{D}_{I'})=0$. We have $I'\notin\mathscr{J}^0$. Thus $I'\in\mathscr{S}=\mathscr{J}$, hence $I'\in\mathscr{J}^1$. Now our lemma is verified.

Theorem. *Let X be a finite set and $\boldsymbol{D}=(X,\rho_P)$. Then $\mathscr{S}=\mathscr{J}$ and $\boldsymbol{D}$ is projective.*

Proof. We have to prove only $\mathscr{S}=\mathscr{J}$, namely,

$$\omega^-(\boldsymbol{D}_Y)=0 \quad \text{iff} \quad Y\in\mathscr{J}^1.$$

We have $\mathscr{J}^1\cap\mathfrak{T}=\emptyset$ and $\rho_P(I,I')=0$ if $I,I'\in\mathscr{J}^1$. We have to find a successor $I'\in\mathscr{J}^1$ of I for $I\in 2^X-(\mathscr{J}^1\cup\mathfrak{T})$, especially for $I\in\mathscr{J}^0-\mathfrak{T}$. Let $I\in\mathscr{J}^0-\mathfrak{T}$. Then $|I|\geqq p_0$ and $g(|I|-p_0)\geqq 1$. We choose $m\in(|I|-p_0)-P$ such that $g(m)=0$ and have $g(m+p_0)=1$. Therefore we obtain a successor $I'\in\mathscr{J}^1$ of I removing $|I|-p_0-m$ points.



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