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Pursell-Shanks Type Theorem for Free G-Manifolds

By

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Introduction

Let M and N be connected paracompact C^∞-manifolds and $\mathfrak{X}(M)$ and $\mathfrak{X}(N)$ the Lie algebras of all C^∞-vector fields with compact support on M and N respectively. A well-known theorem of Pursell-Shanks [6] may be stated as follows.

Theorem. *There exists a Lie algebra isomorphism Φ of $\mathfrak{X}(M)$ onto $\mathfrak{X}(N)$ if and only if there exists a C^∞-diffeomorphism φ of M onto N such that $\Phi = d\varphi$.*

The above result still holds for Lie algebras of all infinitesimal automorphisms of several geometric structures on M and N. Indeed, Omori [3] proved the corresponding result in case of volume structures, symplectic structures, contact structures and fibering structures with compact fibers, and Koriyama [2] proved that this is still true for submanifolds regarding a submanifold M' as a geometric structure on M.

Our purpose of this paper is to show that the above result still holds for free G-manifolds under a certain condition.

More precisely, let G be a compact connected semi-simple Lie group and M and N be paracompact connected free G-manifolds without boundary. Let $\mathfrak{X}_G(M)$ and $\mathfrak{X}_G(N)$ be the Lie algebras of all G-invariant C^∞-vector fields with compact support on M and N respectively. Then we obtain the following theorem.

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Theorem 3.2. *Suppose that the automorphism group of the Lie algebra of G is connected. Then* $\mathfrak{X}_G(M)$ *is algebraically isomorphic to* $\mathfrak{X}_G(N)$ *if and only if M is G-equivariantly diffeomorphic to N.*

If Aut$(\mathfrak{g})$ is not connected, our theorem is no longer true (see § 4). Moreover if G is not semi-simple, our theorem does not hold (see § 5). The first part of the proof of our theorem is parallel to that of Pursell-Shanks. In the rest, we discuss equivalences of principal G-bundles.

All manifolds, actions and diffeomorphisms considered here, are differentiable of class C^∞.

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§ 1. Preliminaries

Let M be a connected paracompact C^∞-manifold and G a compact connected Lie group and $\mu: G\times M\to M$ C^∞-free (left-)action. We denote by B_M the orbit space. In our case, B_M is a connected C^∞-manifold.

Then we have the following theorem.

Theorem 1.1 (Bredon [1]). *The orbit map* $\pi: M\to B_M$ *is the projection in a fiber bundle with fiber G and structure group G (acting by right translation on G). Conversely every principal G-bundle comes from such an action.*

Thus, we remark that the notions of a principal G-bundle and of a free G-action are canonically equivalent.

If $g\in G$, $p\in M$, we write $g\cdot p$ to denote the result of letting g act on p. We shall also write g to denote the diffeomorphism $p\to g\cdot p$. An action of G on M induces an action of G on TM, the tangent bundle of M. If $g\in G$, we write $Tg(v)$ for the result of acting on $v\in TM$ by g. The resulting diffeomorphism of TM is $Tg: TM\to TM$ and is just the tangent of $g: M\to M$.

Definition 1.2. A vector field v on M is called *G-invariant* *vector field* (simply, *G-vector field*) if $Tg\circ v=v\circ g$ for all $g\in G$. We denote by $\mathfrak{X}_G(M)$ the Lie algebra of all C^∞-G-vector fields with compact support. The following two lemmas are easily obtained.

Lemma 1. 3. *For any coordinate chart* $(U; (x_1, \cdots, x_n))$ *of* B_M *such that* π *is trivial on* U, *every* $v\in\mathfrak{X}_G(M)$ *is described as follows*:

$$v=\sum_{i=1}^{n} a_i(x_1, \cdots, x_n) \frac{\partial}{\partial x_i}+\sum_{j=1}^{m} b_j(x_1, \cdots, x_n) X_j \quad \text{on } \pi^{-1}(U),$$

where a_i *and* b_j $(1\leqq i\leqq n, 1\leqq j\leqq m)$ *are* C^∞-*functions on* U *and* $\{X_1, \cdots, X_m\}$ *is a basis of the Lie algebra* $\mathfrak{g}$ *of* G.

Lemma 1. 4. *If* $v\in\mathfrak{X}_G(M)$ *satisfies* $d\pi(v)(\pi(p))\neq 0$, *then there is a coordinate chart* $(U; (x_1, \cdots, x_n))$ *at* $\pi(p)$ *such that*

$$v=\frac{\partial}{\partial x_1}+\sum_{j=1}^{m} b_j(x_1, \cdots, x_n) X_j \quad \text{on} \quad \pi^{-1}(U).$$

§ 2. Characterization of Maximal Ideals of $\mathfrak{X}_G(M)$

By Lemma 1. 3, we see that the natural mapping $d\pi: \mathfrak{X}_G(M)\to\mathfrak{X}(B_M)$ is a surjective homomorphism as Lie algebras, where $\mathfrak{X}(B_M)$ denotes the Lie algebra of C^∞-vector fields on B_M with compact support. Let $\mathfrak{a}(M)$ be its kernel. Note that $\mathfrak{a}(M)$ is an ideal of $\mathfrak{X}_G(M)$.

Lemma 2. 1. *Suppose that* $\mathfrak{m}$ *is an ideal of* $\mathfrak{X}_G(M)$ *such that for any point* $p\in M$, *there is* $u\in\mathfrak{m}$ *such that* $(d\pi u)(\pi(p))\neq 0$. *Then* $\mathfrak{m}+\mathfrak{a}(M)=\mathfrak{X}_G(M)$.

Proof. Let v be an arbitrary element of $\mathfrak{X}_G(M)$. From the assumption, for any $p\in\operatorname{supp} v$, there are a vector field $u\in\mathfrak{m}$ and a coordinate chart $(U; (x_1, \cdots, x_n))$ at $\pi(p)$ such that π is trivial on U and $u=\frac{\partial}{\partial x_1}+\sum_{j=1}^{m} b_j(x_1, \cdots, x_n)X_j$ on $\pi^{-1}(U)$. (See Lemma 1. 4.)

Since $\operatorname{supp} v$ is compact, there are $u_i\in\mathfrak{m}$, $v_i\in\mathfrak{X}_G(M)$ and coordinate charts $(U_i; (x_1^i, \cdots, x_n^i))$, $i=1, 2, \cdots, r$ such that

$$\bigcup_{i=1}^{r} U_i\supset\operatorname{supp} v,\ v=v_1+\cdots+v_r,\ \operatorname{supp} v_i\subset\pi^{-1}(U_i)$$

and

$$u_i=\frac{\partial}{\partial x_1^i}+\sum_{j=1}^{m} b_j^i(x_1^i,\cdots,x_n^i)X_j^i\,,$$

$$v_i=\sum_{k=1}^{n}\xi_k^i(x_1^i,\cdots,x_n^i)\frac{\partial}{\partial x_k^i}+\sum_{l=1}^{m}\mu_l^i(x_1^i,\cdots,x_n^i)X_l^i$$

on $\pi^{-1}(U_i)$. If we want to prove that $v\in\mathfrak{m}+\mathfrak{a}(M)$, it suffices to prove that $v_i\in\mathfrak{m}+\mathfrak{a}(M)$ for each i. Hence we may assume that there is a coordinate chart $(U;\,(x_1,\cdots,x_n))$ (of B_M) such that v is written as

$$v=\sum_{i=1}^{n}\xi_i(x_1,\cdots,x_n)\frac{\partial}{\partial x_i}+\sum_{j=1}^{m}\mu_j(x_1,\cdots,x_n)X_j\quad\text{on }\pi^{-1}(U)$$

with $\operatorname{supp}\xi_i\subset U$, $\operatorname{supp}\mu_j\subset U$ for $i=1,2,\cdots,n$, $j=1,2,\cdots,m$ and a suitable extension of $\dfrac{\partial}{\partial x_1}+\sum_{j=1}^{m}b_j(x_1,\cdots,x_n)X_j$ is contained in $\mathfrak{m}$. Thus a suitable extension of $\dfrac{\partial}{\partial x_1}$ is contained in $\mathfrak{m}+\mathfrak{a}(M)$. We use the same notation for the extended vector fields because all argument is local. Since $\dfrac{\partial}{\partial x_1}\in\mathfrak{m}+\mathfrak{a}(M)$ and $\dfrac{1}{2}\left[\dfrac{\partial}{\partial x_1},\,(x_1)^2\dfrac{\partial}{\partial x_1}\right]=x_1\dfrac{\partial}{\partial x_1}$, $x_1\dfrac{\partial}{\partial x_1}\in\mathfrak{m}+\mathfrak{a}(M)$. For $\xi_1(x)\dfrac{\partial}{\partial x_1}$, we have the following relations:

$$\left[\frac{\partial}{\partial x_1},\,x_1\xi_1(x)\frac{\partial}{\partial x_1}\right]=\left(\xi_1(x)+x_1\frac{\partial\xi_1}{\partial x_1}\right)\frac{\partial}{\partial x_1}\in\mathfrak{m}+\mathfrak{a}(M)$$

and

$$\left[x_1\frac{\partial}{\partial x_1},\,\xi_1(x)\frac{\partial}{\partial x_1}\right]=\left(x_1\frac{\partial\xi_1}{\partial x_1}-\xi_1(x)\right)\frac{\partial}{\partial x_1}\in\mathfrak{m}+\mathfrak{a}(M)\,.$$

Hence we have

$$\frac{1}{2}\left(\left[\frac{\partial}{\partial x_1},\,x_1\xi_1(x)\frac{\partial}{\partial x_1}\right]-\left[x_1\frac{\partial}{\partial x_1},\,\xi_1(x)\frac{\partial}{\partial x_1}\right]\right)=\xi_1(x)\frac{\partial}{\partial x_1}\in\mathfrak{m}+\mathfrak{a}(M)\,.$$

On the other hand, for $\xi_i\dfrac{\partial}{\partial x_i}$, $i\geqq 2$, we have the following relations:

$$\left[\frac{\partial}{\partial x_1},\,x_1\xi_i\frac{\partial}{\partial x_i}\right]=\left(\xi_i+x_1\frac{\partial\xi_i}{\partial x_1}\right)\frac{\partial}{\partial x_i}\in\mathfrak{m}+\mathfrak{a}(M)$$

and

$$\left[x_1\frac{\partial}{\partial x_1}, \xi_i\frac{\partial}{\partial x_i}\right]=x_1\frac{\partial \xi_i}{\partial x_i}\frac{\partial}{\partial x_i}\in\mathfrak{m}+\mathfrak{a}(M).$$

Hence we have $\xi_i\frac{\partial}{\partial x_1}\in\mathfrak{m}+\mathfrak{a}(M)$. Therefore we have

$$v=\sum_{i=1}^{n}\xi_i\frac{\partial}{\partial x_i}+\sum_{j=1}^{m}\mu_j X_j\in\mathfrak{m}+\mathfrak{a}(M).$$

This completes the proof.

Lemma 2.2. *Suppose that* $\mathfrak{m}$ *is an ideal of* $\mathfrak{X}_G(M)$ *such that for any point* $p\in M$, *there is* $u\in\mathfrak{m}$ *such that* $(d\pi u)(\pi(p))\neq 0$. *Then* $\mathfrak{m}\supset\mathfrak{a}(M)$.

Proof. Let v be an arbitrary element of $\mathfrak{a}(M)$ and $(U; (x_1, \cdots, x_n))$ be a coordinate chart of B_M such that π is trivial on U. Let V, W be open subsets of U such that $W\subset\overline{W}\subset V\subset\overline{V}\subset U$. We may assume that supp v is contained in W and that $v=\sum_{j=1}^{m}\xi_j(x_1, \cdots, x_n)X_j$ on U. From the assumption, there is a vector field $u\in\mathfrak{m}$ such that $u\equiv\frac{\partial}{\partial x_1}+\sum_{j=1}^{m}b_j(x_1, \cdots, x_n)X_j$ on $\pi^{-1}(U)$. Let $f, g:U\to\boldsymbol{R}$ be C^∞-non-negative functions on U such that $f\equiv 1$ on W, $f\equiv 0$ outside of V and $g\equiv 1$ on V, $g\equiv 0$ outside of U. Since

$$\frac{1}{2}\left[\frac{\partial}{\partial x_1}+\sum_{j=1}^{n}b_j(x_1, \cdots, x_n)X_j, (x_1)^2\frac{\partial}{\partial x_1}\right]$$

$$=x_1\frac{\partial}{\partial x_1}+\sum_{j=1}^{m}(x_1)^2\frac{\partial b_j}{\partial x_1}X_j,$$

$x_1\frac{\partial}{\partial x_1}+\sum_{j=1}^{m}(x_1)^2\frac{\partial b_j}{\partial x_1}X_j$ is contained in $\mathfrak{m}$. We have following relations:

$$\left[\frac{\partial}{\partial x_1}+\sum_{j=1}^{m}b_j(x_1, \cdots, x_n)X_j, x_1 f(x)\frac{\partial}{\partial x_1}\right]$$

$$=\left(f(x)+x_1\frac{\partial f}{\partial x_1}\right)\frac{\partial}{\partial x_1}+x_1 f(x)\sum_{j=1}^{m}\frac{\partial b_j}{\partial x_1}X_j$$

and

$$\left[x_1\frac{\partial}{\partial x_1}+\sum_{j=1}^{m}(x_1)^2\frac{\partial b_j}{\partial x_1}X_j,\ f(x)\frac{\partial}{\partial x_1}\right]$$

$$=\left(x_1\frac{\partial f}{\partial x_1}-f(x)\right)\frac{\partial}{\partial x_1}+f(x)\sum_{j=1}^{m}\frac{\partial}{\partial x_1}\left((x_1)^2\frac{\partial b_j}{\partial x_1}\right)X_j\in\mathfrak{m}\,.$$

Thus we have

$$f(x)\frac{\partial}{\partial x_1}+\sum_{j=1}^{m}\frac{1}{2}f(x)\left(x_1\frac{\partial b_j}{\partial x_1}-\frac{\partial}{\partial x_1}\left((x_1)^2\frac{\partial b_j}{\partial x_1}\right)\right)X_j\in\mathfrak{m}\,.$$

Put

$$\mu_j(x)=\frac{1}{2}\left(x_1\frac{\partial b_j}{\partial x_1}-\frac{\partial}{\partial x_1}\left((x_1)^2\frac{\partial b_j}{\partial x_1}\right)\right),\quad j=1,2,\cdots,m\,.$$

We consider the following system of differential equations on U:

$$\frac{\partial \lambda_j}{\partial x_1}+\sum_{i,k=1}^{n}\mu_i(x)\lambda_k(x)c_{i,k}^j=\xi_j(x),\quad j=1,2,\cdots,m\,,$$

where $\{c_{i,k}^j\}$ denote the structure constants with respect to the basis $\{X_1,\cdots,X_m\}$. This system of differential equations can be solved on U. Indeed,

$$\boldsymbol{\lambda}=\exp\left(-\int_{-\infty}^{x_1}\boldsymbol{\mu}dt\right)\left\{\int_{-\infty}^{x_1}\left(\exp\left(\int_{-\infty}^{t}\boldsymbol{\mu}ds\right)\boldsymbol{\xi}\right)dt+\boldsymbol{C}\right\},$$

where

$$\boldsymbol{\lambda}(x)=\begin{pmatrix}\lambda_1(x)\\ \vdots\\ \lambda_n(x)\end{pmatrix},\quad \boldsymbol{\xi}(x)=\begin{pmatrix}\xi_1(x)\\ \vdots\\ \xi_n(x)\end{pmatrix},$$

$$\boldsymbol{\mu}(x)\text{ is an } n\times n\text{-matrix } \left(\sum_{i=1}^{n}\mu_i(x)c_{i,k}^j\right),$$

and $\boldsymbol{C}=C(x_2,\cdots,x_n)$ denotes a vector.

Put $I(v)=g(x)\sum_{j=1}^{n}\lambda_j(x)X_j$. Then we have

$$\left[f(x)\left(\frac{\partial}{\partial x_1}+\sum_{j=1}^{m}\mu_j(x)X_j\right),\ I(v)\right]=v$$

on V. Since $\operatorname{supp} f\subset V$, the above equality holds on U. This completes the proof.

Proposition 2. 3. *Let* $\mathfrak{m}$ *be a proper ideal of* $\mathfrak{X}_G(M)$. *Then there is a point* $\bar{p}\in B_M$ *such that every element* $d\pi u$ *of* $d\pi\mathfrak{m}$ *vanishes at* $\bar{p}$ *with all of its derivatives.*

Proof. From Lemma 2.1 and 2.2, it is easy to see that there is a point $\bar{p}\in B_M$ such that every element $d\pi u$ of $d\pi\mathfrak{m}$ vanishes at $\bar{p}$. Let $(U; (x_1, \cdots, x_n))$ be a coordinate chart at $\bar{p}$ such that π is trivial on U and $d\pi u\equiv\sum_{i=1}^{n} a_i(x)\frac{\partial}{\partial x_i}$ on U. Suppose that for some function a_i and some integer $r>0$, $\frac{\partial^r a_i}{\partial x_{i_1}\cdots\partial x_{i_r}}(\bar{p})\neq 0$. Then we have the following relation:

$$\left[d\pi u, \frac{\partial}{\partial x_{i_r}}\right]=-\sum_{j=1}^{n}\frac{\partial a_j}{\partial x_{i_r}}(x)\frac{\partial}{\partial x_j}\in d\pi\mathfrak{m}.$$

Therefore $\left[d\pi u, \frac{\partial}{\partial x_{i_r}}\right]$ has at least one component with non-vanishing derivatives of order $r-1$ at $\bar{p}$. This procedure leads to give an element of $d\pi\mathfrak{m}$ not all of whose components vanish at $\bar{p}$. This completes the proof.

Proposition 2. 4. *Every maximal ideal of* $\mathfrak{X}_G(M)$ *contains the ideal* $\mathfrak{a}(M)$. *Moreover, this must be equal to* $d\pi^{-1}\mathfrak{m}_{\bar{p}}$ *for a point* $\bar{p}\in B_M$, *where* $\mathfrak{m}_{\bar{p}}$ *is the maximal ideal of* $\mathfrak{X}(B_M)$ *corresponding to the point* $\bar{p}$.

Proof. The first part of this proposition follows from Lemmas 2.1 and 2.2. Let $\mathfrak{m}$ be a maximal ideal of $\mathfrak{X}_G(M)$. It is easy to see that $d\pi\mathfrak{m}$ is a maximal ideal of $\mathfrak{X}(B_M)$. Hence there exists a unique point $\bar{p}\in B_M$ corresponding to $d\pi\mathfrak{m}$ which is denoted by $\mathfrak{m}_{\bar{p}}$ (for the proof, see [3] and [6]). This completes the proof.

Remark 2. 5. The ideal $\mathfrak{a}(M)$ is given by the intersection of all maximal ideals of $\mathfrak{X}_G(M)$.

Theorem 2. 6. *Let* G *be a compact connected Lie group and* M, M' *be free* G-*manifolds. If* $\mathfrak{X}_G(M)$ *is algebraically isomorphic* *to* $\mathfrak{X}_G(M')$, *then the orbit manifold* B_M *is diffeomorphic to* $B_{M'}$.

Proof. Since $\mathfrak{a}(M)$ and $\mathfrak{a}(M')$ are ideals of $\mathfrak{X}_G(M)$ and $\mathfrak{X}_G(M')$ respectively, the quotient Lie algebras $\mathfrak{X}_G(M)/\mathfrak{a}(M)$ and $\mathfrak{X}_G(M')/\mathfrak{a}(M')$ are isomorphic to $\mathfrak{X}(B_M)$ and $\mathfrak{X}(B_{M'})$ respectively. Let $\Phi: \mathfrak{X}_G(M) \to \mathfrak{X}_G(M')$ be a Lie algebra isomorphism. We have

$$\Phi(\mathfrak{a}(M)) = \Phi(\bigcap_{\bar{p}\in B_M} d\pi^{-1}\mathfrak{m}_{\bar{p}}) = \bigcap \{\Phi(d\pi^{-1}\mathfrak{m}_{\bar{p}}); \Phi(d\pi^{-1}\mathfrak{m}_{\bar{p}}): \text{maximal ideal of } \mathfrak{X}_G(M')\} = \mathfrak{a}(M').$$

Thus we can see that $\mathfrak{X}_G(M)/\mathfrak{a}(M)$ is isomorphic to $\mathfrak{X}_G(M')/\mathfrak{a}(M')$. Hence $\mathfrak{X}(B_M)$ is isomorphic to $\mathfrak{X}(B_{M'})$. By Pursell-Shanks theorem ([3], [6]), there is a diffeomorphism $\varphi: B_M \to B_{M'}$ such that $d\varphi$ is the above isomorphism of $\mathfrak{X}(B_M)$ to $\mathfrak{X}(B_{M'})$. This completes the proof.

§ 3. Proof of Main Theorem

Let $\pi: M \to B_M$ be the principal G-bundle over B_M. By a system of coordinate transformations in B_M with values in G is meant an indexed covering $\{U_i\}_{i\in I}$ of B_M by coordinate charts and a collection of continuous maps $g_{ji}: U_i \cap U_j \to G$ such that

$$g_{kj}(x) \cdot g_{ji}(x) = g_{ki}(x) \qquad \text{for } x \in U_i \cap U_j \cap U_k\,.$$

Any principal G-bundle over B_M determines such a set of coordinate transformations. In case of our principal G-bundles, each coordinate transformation $g_{ji}(x): G \to G\ (x \in U_i \cap U_j)$ is given by right translation and $g_{ji}: U_i \cap U_j \to G$ is differentiable of class C^∞. Let $\mathfrak{g}$ denote the Lie algebra of G consisting of left invariant vector fields on G. Consider the adjoint of $g_{ji}(x)$, $\mathrm{ad}(g_{ji}(x)): \mathfrak{g} \to \mathfrak{g}$. Then the map $\mathrm{ad}(g_{ji}): U_i \cap U_j \to \mathrm{Aut}(\mathfrak{g})$ is differentiable of class C^∞, where $\mathrm{Aut}(\mathfrak{g})$ denotes the group of all automorphisms of $\mathfrak{g}$ which is a subgroup of $GL(m, \boldsymbol{R})$, $m = \dim G$. Since G is connected, the image of $\mathrm{ad}(g_{ji})$ is contained in the identity component of $\mathrm{Aut}(\mathfrak{g})$ which is denoted by $\mathrm{Aut}_0(\mathfrak{g})$. Furthermore we may see that for any i, j, k in I, $\mathrm{ad}(g_{kj}(x)) \cdot \mathrm{ad}(g_{ji}(x)) = \mathrm{ad}(g_{ki}(x))$ for $x \in U_i \cap U_j \cap U_k$. Hence from N. Steenrod ([7, Existence theorem]), there exists a bundle $E(M, \mathfrak{g})$ with base space B_M, fiber $\mathfrak{g}$, group $\mathrm{Aut}_0(\mathfrak{g})$ and the coordinate transformations $\{\mathrm{ad}(g_{ji})\}$. Since each fiber of $E(M, \mathfrak{g})$

has the Lie algebra structure, the space of global C^∞-sections of $E(M, \mathfrak{g})$, $\Gamma(E(M, \mathfrak{g}))$, has a natural Lie algebra structure. Then we may identify the space $\Gamma(E(M, \mathfrak{g}))$ with the ideal $\mathfrak{a}(M)$ as Lie algebra. Let $E(M', \mathfrak{g})$ denote the bundle corresponding to the principal G-bundle $\pi': M' \to B_{M'}$. Suppose that $\mathfrak{X}_G(M)$ is algebraically isomorphic to $\mathfrak{X}_G(M')$. From Theorem 2. 6, the space B_M is diffeomorphic to $B_{M'}$, via φ. Since the pull-back of φ, φ^*M', is equivalent to M' via a bundle isomorphism $\tilde{\varphi}$ and the Lie algebra $\mathfrak{X}_G(\varphi^*M')$ (resp. $\mathfrak{a}(\varphi^*M')$) is isomorphic to $\mathfrak{X}_G(M')$ (resp. $\mathfrak{a}(M')$) via the differential of $\tilde{\varphi}$, it is sufficient to consider for $B_M = B_{M'}$, $\varphi =$ identity.

Proposition 3. 1. *If the Lie algebras $\Gamma(E(M, \mathfrak{g}))$ and $\Gamma(E(M', \mathfrak{g}))$ are isomorphic, then the vector bundle $E(M, \mathfrak{g})$ is isomorphic in* $\mathrm{Aut}(\mathfrak{g})$ *to $E(M', \mathfrak{g})$.*

Proof. Let $\Psi: \Gamma(E(M, \mathfrak{g})) \to \Gamma(E(M', \mathfrak{g}))$ be the Lie algebra isomorphism and $\{U_i\}$ be the common covering of B_M by coordinate charts with respect to $E(M, \mathfrak{g})$ and $E(M', \mathfrak{g})$.

For each U_i, the local constant sections $\sigma_k^{(i)}: U_i \to X_k^{(i)}$ $(k=1, 2, \cdots, m)$ are bases for $\Gamma(E(M, \mathfrak{g}) | U_i)$ and $\Gamma(E(M', \mathfrak{g}) | U_i)$, where $\{X_1^{(i)}, \cdots, X_m^{(i)}\}$ denotes the appointed basis of $\mathfrak{g}$ with respect to U_i. Then we have the following formulae:

$$\{(\Psi |_{U_i})(\sigma_k^{(i)})\}(x) = \sum_{l=1}^{m} b_{k,l}^{(i)}(x) \sigma_l^{(i)} \quad (x \in U_i,\ k=1, 2, \cdots, m).$$

Since for each $x \in B_M$, $\Psi|_{\pi^{-1}(x)}$ is a Lie algebra isomorphism of $\mathfrak{g}$, $(b_{k,l}^{(i)})(x)$ is contained in $\mathrm{Aut}(\mathfrak{g})$. Furthermore the map $(b_{k,l}^{(i)}): U_i \to \mathrm{Aut}(\mathfrak{g})$ is differentiable of class C^∞. For $x \in U_i \cap U_j$, we have the following formulae:

$$(b_{k,l}^{(i)}(x)) \cdot \mathrm{ad}(g'_{ji}(x)) = \mathrm{ad}(g_{ji}(x)) \cdot (b_{k,l}^{(j)}(x)), \quad k=1, 2, \cdots, m .$$

From these formulae and Lemma 3. 2 of [7], we obtain that $E(M, \mathfrak{g})$ is isomorphic in $\mathrm{Aut}(\mathfrak{g})$ to $E(M', \mathfrak{g})$. This completes the proof.

Theorem 3. 2. *Supppose that G is a compact connected semi-*

simple Lie group with $\mathrm{Aut}(\mathfrak{g})=\mathrm{Aut}_0(\mathfrak{g})$. *If* $\mathfrak{X}_G(M)$ *is algebraically isomorphic to* $\mathfrak{X}_G(M')$, *then* M *is G-equivariantly diffeomorphic to* M'.

Proof. Let $\Phi: \mathfrak{X}_G(M)\to\mathfrak{X}_G(M')$ be a Lie algebra isomorphism. From Theorem 2.7, the orbit manifolds B_M and $B_{M'}$ are diffeomorphic, via φ. And by the above argument, we may assume $B_M=B_{M'}$ and $\varphi=$identity. From Proposition 3.1 and $\mathrm{Aut}(\mathfrak{g})=\mathrm{Aut}_0(\mathfrak{g})$, that $\mathfrak{a}(M)$ is isomorphic to $\mathfrak{a}(M')$, via Φ, implies that $E(M, \mathfrak{g})$ is isomorphic in $\mathrm{Aut}_0(\mathfrak{g})$ to $E(M', \mathfrak{g})$. From Lemma 3.2 of [7], there exist C^∞-maps $\tilde{\lambda}_j: U_j\to\mathrm{Aut}_0(\mathfrak{g})$ such that

$$\mathrm{ad}(g'_{ji}(x))=\tilde{\lambda}_j^{-1}(x)\cdot\mathrm{ad}(g_{ji}(x))\cdot\tilde{\lambda}_i(x) \quad \text{for } x\in U_i\cap U_j$$

where $\{\mathrm{ad}(g_{ji})\}$ and $\{\mathrm{ad}(g'_{ji})\}$ denote the coordinate transformations for $E(M, \mathfrak{g})$ and $E(M', \mathfrak{g})$ respectively.

On the other hand, it is known that if G is semi-simple, then $\mathrm{Aut}_0(\mathfrak{g})$ is equal to the adjoint group $\mathrm{Ad}(G)$, which is isomorphic to G/Z, where Z is the centor of G. Therefore there exist C^∞-maps $\bar{\lambda}_j: U_j\to G/Z$ such that

$$g'_{ji}(x)=\bar{\lambda}_j^{-1}(x)\cdot g_{ji}(x)\cdot\bar{\lambda}_i(x) \quad (x\in U_i\cap U_j) \quad \text{in} \quad G/Z\,.$$

We can easily lift the maps $\bar{\lambda}_j$ to maps $\lambda_j: U_j\to G$ such that

$$g'_{ji}(x)=\lambda_j^{-1}(x)\cdot g_{ji}(x)\cdot\lambda_i(x) \quad (x\in U_i\cap U_j).$$

Hence using Lemma 3.2 of [7] again, we may see that two principal G-bundles $\pi: M\to B_M$ and $\pi': M'\to B_{M'}$ are G-equivalent. This completes the proof.

Let G_i be compact connected simple Lie groups ($1\leqq i\leqq k$) and G be the product $G_1\times\cdots\times G_k$. Let $\mathfrak{g}_i$ denote the Lie algebra of G_i. Then the Lie algebra $\mathfrak{g}$ of G is isomorphic to $\mathfrak{g}_1\oplus\cdots\oplus\mathfrak{g}_k$ (direct sum). We remark that any principal G-bundle M is decomposed as a sum of principal G_i-bundles, $M(G_1)\oplus\cdots\oplus M(G_k)$.

Definition 3.3. Two principal G-bundles M and M' are essentially isomorphic if there are a diffeomorphism $f: M\to M'$ and an automorphism $\mu: G\to G$ such that $f(g\cdot x)=\mu(g)\cdot f(x)$ for $x\in M, g\in G$.

Then we have the following theorem.

Theorem 3.4. *Let G_i be compact connected simple Lie groups $(1 \leqq i \leqq k)$ such that* $\mathrm{Aut}(\mathfrak{g}_i)$ *is connected. Let G be the product $G_1 \times \cdots \times G_k$ and M, M' be free G-manifolds. If $\mathfrak{X}_G(M)$ is algebraically isomorphic to $\mathfrak{X}_G(M')$, then M is essentially isomorphic to M'.*

Proof. By the same argument as in the first part of the proof of Theorem 3.2, we may see that the vector bundle $E(M, \mathfrak{g})$ is isomorphic in $\mathrm{Aut}(\mathfrak{g})$ to $E(M', \mathfrak{g})$. We denote this isomorphism by Ψ. Note that $E(M, \mathfrak{g})$ and $E(M', \mathfrak{g})$ are isomorphic to the Whitney sum $\bigoplus_{i=1}^{k} E(M, \mathfrak{g}_i)$ and $\bigoplus_{i=1}^{k} E(M', \mathfrak{g}_i)$, where $E(M, \mathfrak{g}_i)$ and $E(M', \mathfrak{g}_i)$ denote vector bundles over B_M with fiber $\mathfrak{g}_i$ $(1 \leqq i \leqq k)$. Thus we can split the isomorphism Ψ into $\bigoplus \Psi_i : \bigoplus_{i=1}^{k} E(M, \mathfrak{g}_i) \to \bigoplus_{i=1}^{k} E(M', \mathfrak{g}_{j_i})$. From the assumption, $E(M, \mathfrak{g}_i)$ is isomorphic in $\mathrm{Aut}_0(\mathfrak{g}_i)$ to $E(M', \mathfrak{g}_{j_i})$ for each i. Thus by the same argument as in the last part of the proof of Theorem 3.2, $M(G_i)$ is G_i-equivariantly diffeomorphic to $M'(G_{j_i})$. Hence, M is G-equivariantly diffeomorphic to $M'(G_{j_1}) \oplus \cdots \oplus M'(G_{j_k})$ which is essentially isomorphic to M'. This completes the proof.

Remark 3.5. Let A_k $(k \geqq 1)$ denote the local isomorphism class of $SU(k+1)$; B_k $(k \geqq 2)$, that of $SO(2k+1)$; C_k $(k \geqq 3)$, that of $Sp(k)$; D_k $(k \geqq 4)$, that of $SO(2k)$, and G_2, F_4, E_6, E_7 and E_8 denote "exceptional" simple Lie groups. These form a complete list, without repetition, of the local isomorphism classes of the compact simple Lie group. Let $\mathfrak{g}$ be the Lie algebra of G. The orders of the quotient groups $\mathrm{Aut}(\mathfrak{g}) / \mathrm{Aut}_0(\mathfrak{g})$ are given by the following table.

G	A_k	B_k	C_k	D_4	D_k $(k \geqq 5)$	G_2	F_4	E_6	E_7	E_8
Order of $\mathrm{Aut}(\mathfrak{g})/\mathrm{Aut}_0(\mathfrak{g})$	2	1	1	6	2	1	1	2	1	1

§ 4. Counter Example

In this section, we show that if $\mathrm{Aut}(\mathfrak{g})$ is not connected, then

Theorem 3. 2 does not hold.

Let $U(n)$ and $PU(n)$ be the n-dimensional unitary group and the projective group and $i: U(1) \to U(n)$ be the canonical inclusion, that is,

$$i(e^{2\pi it}) = \begin{pmatrix} e^{2\pi it} & & & \\ & 1 & & \\ & & \ddots & \\ & & & 1 \end{pmatrix} \quad \text{for } e^{2\pi it} \in U(1)$$

and $j: U(n) \to PU(n)$ be the projection map and $c: U(n) \to U(n)$ be the conjugation map, that is, $c(A) = \bar{A}$ for $A \in U(n)$, where $\bar{A}$ denotes the conjugate matrix of A. Let $L(n) = S^{2m-1}/\boldsymbol{Z}_n$ $(n \geqq 3)$ be the Lens space and $h: \pi_1(L(n)) \cong \boldsymbol{Z}_n \to U(1)$ be a homomorphism defined by $h(1) = e^{2\pi i/n}$ where 1 denotes the generator of $\boldsymbol{Z}_n$. Then the homomorphisms $j \circ i \circ h$ and $c \circ j \circ i \circ h$ induce principal $PU(n)$-bundles ξ and $\bar{\xi}$ over $L(n)$ which possess foliatins $\mathcal{F}$ and $\bar{\mathcal{F}}$ transverse to the fibers. (See Milnor [4, Lemma 1].)

Proposition 4. 1. *The principal* $PU(n)$*-bundles* ξ *and* $\bar{\xi}$ *over* $L(n)$ $(n \geqq 3)$ *are not trivial and not isomorphic in* $PU(n)$*-bundles.*

Proof. Let η and $\bar{\eta}$ denote principal $U(1)$-bundles over $L(n)$ induced by the homomorphisms h and $c \circ h$. Then using Lemma 2 of [4] we can compute the Euler classes $c_1(\eta)$, $c_1(\bar{\eta}) \in H^2(L(n); \boldsymbol{Z}) \cong \boldsymbol{Z}_n$ and prove that $c_1(\eta) = -c_1(\bar{\eta}) \neq 0$. Since $\xi = \eta \times_{U(1)} PU(n)$ and $\bar{\xi} = \bar{\eta} \times_{U(1)} PU(n)$, ξ and $\bar{\xi}$ are not trivial. Furthermore the mod n characteristic classes $X(\xi)$ $(\equiv c_1(\eta) \bmod n)$ and $X(\bar{\xi})$ $(\equiv c_1(\bar{\eta}) \bmod n)$ are not equal in $H^2(L(n); \boldsymbol{Z}_n) \cong \boldsymbol{Z}_n$. More precisely, let f and $c \circ f: L(n) \to BPU(n)$ be the classifying map representing ξ and $\bar{\xi}$ respectively, where the map $c: BPU(n) \to BPU(n)$ is the map induced from the conjugation map c. Let 1 be a generator of $H^2(BPU(n); \boldsymbol{Z}_n) \cong \boldsymbol{Z}_n$. Then we have the relations: $X(\xi) = f^*(1)$ and $X(\bar{\xi}) = (c \circ f)^*(1) = -f^*(1)$. Hence this completes the proof.

Remark 4. 2. Let $\mathfrak{su}(n)$ denotes the Lie algebra of $SU(n)$ and $\mathrm{Aut}(\mathfrak{su}(n))$ its automorphism group. We know that the connected component of $\mathrm{Aut}(\mathfrak{su}(n))$, which is denoted by $\mathrm{Aut}_0(\mathfrak{su}(n))$, is isomorphic to $PU(n)$. So ξ and $\bar{\xi}$ are the principal $\mathrm{Aut}_0(\mathfrak{su}(n))$ bundles. Then using the conjugation map c, we can construct a bundle isomorphism $C:\xi\to\bar{\xi}$ as follows: for $(p, v)\in\xi, p\in L(n), v\in\xi_p\cong PU(n)$, $C(p, v) = (p, c(v))\in\bar{\xi}$. Hence ξ is isomorphic in $\mathrm{Aut}(\mathfrak{su}(n))$ to $\bar{\xi}$, via the map C.

Remark 4.3. Let M_1 and M_2 be the total space of ξ and $\bar{\xi}$ respectively. Then we have already known that M_1 and M_2 possess foliations $\mathcal{F}$ and $\bar{\mathcal{F}}$ transeverse to the fibers. Then the isomorphism C maps $\mathcal{F}$ to $\bar{\mathcal{F}}$.

Remark 4.4. Let $E(\xi)$ and $E(\bar{\xi})$ denote associated bundles with fiber $\mathfrak{su}(n)$ of ξ and $\bar{\xi}$ respectively and $\Gamma(E(\xi))$ and $\Gamma(E(\bar{\xi}))$ denote the C^∞-section space of $E(\xi)$ and $E(\bar{\xi})$ which have natural Lie algebra structures. Then the isomorphism $C:\xi\to\bar{\xi}$ induces the isomorphism $C:E(\xi)\to E(\bar{\xi})$ and so the isomorphism $C:\Gamma(E(\xi))\to\Gamma(E(\bar{\xi}))$ as Lie algebras.

Theorem 4.5. *$\mathfrak{X}_{PU(n)}(M_1)$ is algebraically isomorphic to $\mathfrak{X}_{PU(n)}(M_2)$.*

Proof. We have exact sequences of Lie algebras:

$$0\to\mathfrak{a}(M_i)\to\mathfrak{X}_{PU(n)}(M_i)\to\mathfrak{X}(L(n))\to 0 \quad (i=1, 2).$$

Then we define Lie algebra splittings $s_i:\mathfrak{X}(L(n))\to\mathfrak{X}_{PU(n)}(M_i)$ using the foliation as follows: for each $u\in\mathfrak{X}(L(n))$, the vector field $s_i(u)$ is tangent to the leaves of the foliation $\mathcal{F}$ (resp. $\bar{\mathcal{F}}$) and $d\pi\circ s_i=\text{identity}$. Thus using these splittings s_i $(i=1, 2)$, we see that $\mathfrak{X}_{PU(n)}(M_i)$ is isomorphic to $\mathfrak{a}(M_i)\oplus\mathfrak{X}(L(n))$ (direct sum) and for (X, u), $(Y, v)\in\mathfrak{a}(M_i)\oplus\mathfrak{X}(L(n))$,

$$[(X, u), (Y, v)] = ([X, Y] + [u, Y] + [X, v], [u, v])$$

where $[\,,\,]$ denotes the Lie bracket. Since the ideals $\mathfrak{a}(M_1)$ and $\mathfrak{a}(M_2)$ are identified with $\Gamma(E(\xi))$ and $\Gamma(E(\bar{\xi}))$ respectively, we define a map $\Psi:\mathfrak{X}_{PU(n)}(M_1)\to\mathfrak{X}_{PU(n)}(M_2)$ by $\Psi((X, u)) = (C(X), u)$ for any $(X, u)\in\Gamma(E(\xi))\oplus\mathfrak{X}(L(n))$. It is easy to see that this map is an isomorphism. And moreover by easy computations, we have the relation: $C([X, u])$

$=[C(X), u]$. Therefore for any $(X, u), (Y, v) \in \Gamma(E(\xi)) \oplus \mathfrak{X}(L(n))$,

$$\begin{aligned}\Psi([(X,u),(Y,v)]) &= \Psi([X,Y]+[u,Y]+[X,v],[u,v]) \\ &= (C([X,Y])+C([u,Y])+C([X,v]),[u,v]) \\ &= ([C(X),C(Y)]+[u,C(Y)]+[C(X),v],[u,v]) \\ &= [\Psi(X,u),\Psi(Y,v)].\end{aligned}$$

This completes the proof.

Remark 4. 6. These free $PU(n)$-manifolds M_1 and M_2 are essentially isomorphic via the map C.

Theorem 4. 7. *Let* $G=SU(n)$ $(n\geqq 3)$, $\mathrm{Spin}(2n)$ $(n\geqq 2)$, E_6 *or* $SO(2n)$ $(n\geqq 3)$ *and* M_1 *and* M_2 *be free G-manifolds. If* $\mathfrak{X}_G(M_1)$ *is algebraically isomorphic to* $\mathfrak{X}_G(M_2)$, *then* M_1 *is essentially isomorphic to* M_2.

Proof. First we consider the case $G=SU(n)$. Let $E(M_i)$ be associated bundles with fiber $\mathfrak{su}(n)$ of $M_i (i=1, 2)$. Then Proposition 3. 1 says that $E(M_1)$ is isomorphic in $\mathrm{Aut}(\mathfrak{su}(n))$ to $E(M_2)$. Using the conjugation map $c: SU(n) \to SU(n)$, we can construct another principal $SU(n)$-bundle $\overline{M}_1$ which is essentially isomorphic to M_1 (see Remark 4. 6). Furthermore $E(M_1)$ is isomorphic in $\mathrm{Aut}(\mathfrak{su}(n))$ but not in $\mathrm{Aut}_0(\mathfrak{su}(n))$ to $E(\overline{M}_1)$. Since $\mathrm{Aut}(\mathfrak{su}(n))$ is not connected and the order of the quotient group $\mathrm{Aut}(\mathfrak{su}(n))/\mathrm{Aut}_0(\mathfrak{su}(n))$ is equal to 2, if $E(M_1)$ is not isomorphic in $\mathrm{Aut}_0(\mathfrak{su}(n))$ to $E(M_2)$, then $E(\overline{M}_1)$ is isomorphic in $\mathrm{Aut}_0(\mathfrak{su}(n))$ to $E(M_2)$. Thus by the same way as in the proof of Theorem 3. 2, we can prove that $\overline{M}_1$ is $SU(n)$-equivariantly diffeomorphic to M_2.

For the cases $G=\mathrm{Spin}(2n)$ $(n\geqq 3)$ or E_6, the order of $\mathrm{Aut}(G)/\mathrm{Aut}_0(G)$ is equal to 2. So there exists an automorphism $\tau: G \to G$ such that $\tau \notin \mathrm{Aut}_0(G)$. For the case $G=\mathrm{Spin}(4)$, the order of $\mathrm{Aut}(\mathrm{Spin}(4))/\mathrm{Aut}_0(\mathrm{Spin}(4))$ is equal to 6. So there exist five automorphisms τ_i: $\mathrm{Spin}(4) \to \mathrm{Spin}(4)$ $(i=1, 2, \cdots, 5)$ such that each τ_i and each $\tau_i \circ \tau_j^{-1}$ $(i \neq j)$ are not contained in $\mathrm{Aut}_0(\mathrm{Spin}(4))$. For the case $G=SO(2n)$, we define an automorphism $\mu: SO(2n) \to SO(2n)$ by $\mu(A) = TAT^{-1}$ for $A \in SO(2n)$, where $T = \begin{pmatrix} -1 & & & \\ & 1 & & \\ & & \ddots & \\ & & & 1 \end{pmatrix}$. Then the differential $d\mu: \mathfrak{so}(2n) \to \mathfrak{so}(2n)$ is not contained in $\mathrm{Aut}_0(\mathfrak{so}(2n))$, where $\mathfrak{so}(2n)$ denotes the Lie algebra of $SO(2n)$. Using $\tau, \tau_i (i=1, 2, \cdots, 5)$ or μ instead of c, we can prove our theorem for $G = \mathrm{Spin}(2n)$ $(n \geqq 2)$, E_6 or $SO(2n)$ $(n \geqq 3)$ similarly. This completes the proof.

§ 5. Concluding Remarks

In this section, we show that if G is not semi-simple, Pursell-Shanks type theorem for free G-manifolds is no longer true.

Let B be an oriented closed surface of negative Euler characteristic, $\chi(B) < 0$. Let $M_1 = S^1 \times B$ and $\pi: M_2 \to B$ be a non-trivial principal S^1-bundle with a foliation transverse to the fibers. J. Wood proved the following theorem ([8]).

Theorem 5.1. *Such a bundle ξ exists iff $|\chi(\xi)| \leqq |\chi(B)|$, where $\chi(\xi)$ denotes the Euler class of ξ.*

Then there are canonical free (left-) S^1-actions on M_1 and M_2.

Proposition 5.2. *$\mathfrak{X}_{S^1}(M_1)$ is isomorphic to $\mathfrak{X}_{S^1}(M_2)$.*

Proof. We have an exact sequence of Lie algebras;

$$0 \to \mathfrak{a}(M_2) \to \mathfrak{X}_{S^1}(M_2) \to \mathfrak{X}(B) \to 0.$$

Then we define a Lie algebra splitting $s: \mathfrak{X}(B) \to \mathfrak{X}_{S^1}(M_2)$ using the foliation as follows; for each $u \in \mathfrak{X}(B)$, the vector field $s(u)$ is tangent to leaves of the foliation and $d\pi \circ s = \text{identity}$. Thus using this splitting s, we see that $\mathfrak{X}_{S^1}(M_2)$ is isomorphic to $\mathfrak{a}(M_2) \oplus \mathfrak{X}(B)$ (direct sum) and for $(X, u), (Y, v) \in \mathfrak{a}(M_2) \oplus \mathfrak{X}(B)$,

$$[(X, u), (Y, v)] = ([X, Y] + u \cdot Y - v \cdot X, [u, v])$$

where $[\ ,\ ]$ denotes the Lie bracket. Since the adjoint group $\mathrm{Ad}(S^1)$ is trivial, the vector bundles $E(M_1, \boldsymbol{R}^1)$ and $E(M_2, \boldsymbol{R}^1)$ are trivial. Thus $\mathfrak{a}(M_1)$ is algebraically isomorphic to $\mathfrak{a}(M_2)$. Let $\psi: \mathfrak{a}(M_1) \to \mathfrak{a}(M_2)$ be its isomorphism. Then we define a map $\Psi: \mathfrak{X}_{S^1}(M_1) \to \mathfrak{X}_{S^1}(M_2)$ by $\Psi((X, u)) = (\psi(X), u)$ for any $(X, u) \in \mathfrak{a}(M_1) \oplus \mathfrak{X}(B)$. This map is an isomorphism. For any $(X, u), (Y, v) \in \mathfrak{X}_{S^1}(M_1)$,

$$\begin{aligned}\Psi([(X, u), (Y, v)]) &= \Psi([X, Y] + u\cdot Y - v\cdot X, [u, v]) \\ &= (\psi([X, Y]) + \psi(u\cdot Y) - \psi(v\cdot X), [u, v]) \\ &= ([\psi(X), \psi(Y)] + u\cdot\psi(Y) - v\cdot\psi(X), [u, v]) \\ &= [(\psi(X), u), (\psi(Y), v)] \\ &= [\Psi(X, u), \Psi(Y, v)].\end{aligned}$$

Hence Ψ is a Lie algebra isomorphism. This completes the proof.

Let T^2 be a 2-torus. Express a point p of T^2 as (x, y), where x and y are contained in $S^1 = \boldsymbol{R}/\boldsymbol{Z}$. Define a linear action of T^2, $\varphi_\alpha: \boldsymbol{R} \times T^2 \to T^2$ by $\varphi_\alpha(t, (x, y)) = (x + t, y + \alpha\cdot t)$, where α is an irrational number. Note that φ_α is a free action and each orbit of this action is everywhere dense.

Proposition 5.3. *For any irrational number α, $\mathfrak{X}_{(\boldsymbol{R}, \varphi_\alpha)}(T^2)$ is isomorphic to $\boldsymbol{R}^2$ as Lie algebra.*

Indeed, let $\mathrm{Diff}_{(\boldsymbol{R}, \varphi_\alpha)}(T^2)$ denote the group of equivariant diffeomorphisms of T^2 which are equivariantly isotopic to the identity. Then we have that $\mathrm{Diff}_{(\boldsymbol{R}, \varphi_\alpha)}(T^2)$ is isomorphic to T^2 as Lie group.

J. Milnor proved the following proposition ([5, Assertion 8.1]).

Proposition 5.4. *Let f^s denote the codimension 1 foliation with constant slope s on T^2. If $s \neq s'$, then the foliation f^s is not C^r-integrably homotopic to $f^{s'}$ for any $r \geqq 2$.*

Using Proposition 5.4 and the fact that the quotient group $\mathrm{Diff}(T^2)/\mathrm{Diff}_0(T^2)$ is isomorphic to $SL(2, \boldsymbol{Z})$ (where $\mathrm{Diff}_0(T^2)$ denotes the identity connected component of $\mathrm{Diff}(T^2)$), we can prove the following proposition.

Proposition 5.5. *There are irrational numbers α and β such that the action φ_α is not equivariantly diffeomorphic to the action φ_β.*

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