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A Generalization of Vanishing Theorems for Weakly 1-Complete Manifolds

By

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§ 1. Introduction

Let X be a connected complex manifold of complex dimension n. X is called weakly 1-complete if there exists an exhaustion function Φ on X which is C^∞ and plurisubharmonic. In [9] S. Nakano established the following.

Theorem 1. *Let B be a positive line bundle on a weakly 1-complete manifold X, then*

$$H^p(X, \Omega^q(B)) = 0 \quad \textit{for} \quad p+q>n.$$

Recently, O. Abdelkader obtained

Theorem 2 (cf. [1]). *Let B be a semi-positive line bundle over a weakly 1-complete Kähler manifold X and assume that the curvature form of B has at least $n-k+1$ positive eigenvalues, then*

$$H^p(X_c, \Omega^q(B)) = 0 \quad \textit{for any real number } c \textit{ with } p+q \geqq n+k,$$

where $X_c = \{x \in X \mid \Phi(x) < c\}$.

In these theorems, the positivity of eigenvalues of the curvature form of B is assumed on the whole space X. In this paper, we shall prove that these vanishing theorems still hold, if the positivity of eigenvalues of the curvature admits a compact exceptional subset $K \subsetneqq X$. We shall prove the following.

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Main Theorem. *Let B be a semi-positive line bundle over a connected weakly 1-complete Kähler manifold X with a metric along the fibres such that its curvature form has at least $n-q+1$ positive eigenvalues on $X\setminus K$, where K is a proper compact subset of X. Then*

$$H^p(X, \mathcal{O}(B\otimes K_X)) = 0 \quad \text{for any } p \geqq q,$$

where K_X is the canonical line bundle of X.

In particular, when $q=1$, we obtain

Corollary. *Let X be a connected weakly 1-complete Kähler manifold and let B be a semi-positive line bundle on X which is positive on $X\setminus K$ for some proper compact subset K of X. Then*

$$H^p(X, \mathcal{O}(B\otimes K_X)) = 0 \quad \text{for any } p \geqq 1.$$

Since a positive line bundle over a complex manifold induces a Kähler metric on it, this is not only a direct generalization of Theorem 1 for $q=n$ but also a generalization of the vanishing theorems for the semi-positive line bundle on 1-convex Kähler manifolds and compact Kähler manifolds by Grauert and Riemenschneider (cf. [4], [11]).

This work is inspired by Ohsawa's article [10] and the author would like to express his hearty thanks to Dr. A. Fujiki and Professor S. Nakano for their kind advices and encouragement during the preparation of this paper.

§ 2. Notations and Definitions

We denote by X a connected paracompact complex manifold of dimension n. Let $\pi: F\to X$ be a holomorphic line bundle over X. Let $\mathcal{U} = \{U_i\}_{i\in I}$ be a covering of X by coordinate neighborhoods such that on each U_i, $F|U_i$ is isomorphic to the trivial line bundle. We denote local coordinates on U_i by $(z_i^1, \cdots, z_i^n)$. If $\Phi_i: U_i\times \boldsymbol{C}\to F|U_i$ $(i\in I)$ are these trivializations of F, we denote by $f_{ij}: U_i\cap U_j\to \boldsymbol{C}^*$ the system of transition functions defined by the conditions:

$$\Phi_j^{-1}\circ\Phi_i(z_i, \xi_i) = (z_i, f_{ij}(z_j)\xi_j)$$

where ξ_i denotes the fibre coordinates over U_i.

An F-valued differential form φ on X is a system $\{\varphi_i\}_{i\in I}$ of differential forms defined on U_i, satisfying $\varphi_i=f_{ij}\varphi_j$ in $U_i\cap U_j$. We denote by $C^{p,q}(X, F)$ the space of F-valued differential forms on X, of class C^∞ and of type (p, q), and by $C_0^{p,q}(X, F)$ the space of the forms in $C^{p,q}(X, F)$ with compact supports.

Let $ds^2=\sum_{\alpha,\beta=1}^{n} g_{i,\alpha\bar\beta}\ dz_i^\alpha\cdot dz_i^{\bar\beta}$ be a hermitian metric on X and let $\{a_i\}$ be a hermitian metric along the fibres of F, that is, a system of positive valued function a_i in U_i satisfying $|f_{ij}|=a_i\cdot a_j^{-1}$ in $U_i\cap U_j$.

Remark. In this paper, we use the notation of a system of metrics along the fibres in the sense of Kodaira [7], page 1268, (1).

For φ, $\psi\in C^{p,q}(X, F)$, we set

$$\langle\varphi, \psi\rangle=a_i^{-1}\sum_{A_p,B_q}\varphi_{iA_p,\bar B_q}\cdot\overline{\psi_i^{\bar A_p,B_q}},$$

where $\varphi_i=\sum_{A_p,B_q}\varphi_{iA_p,\bar B_q}dz_i^{A_p}\wedge dz_i^{\bar B_q}$ and $A_p=(\alpha_1, \cdots, \alpha_p)$ and $B_q=(\beta_1, \cdots, \beta_q)$ run through the sets of multi-indices with $1\leqq\alpha_1<\cdots<\alpha_p\leqq n$ and $1\leqq\beta_1<\cdots<\beta_q\leqq n$ respectively. Then

$$a_i^{-1}\varphi_i\wedge*\overline{\psi_i}=\langle\varphi, \psi\rangle dV$$

where $*$ is the star operator and dV is the volume element with respect to the metric ds^2.

If either φ or $\psi\in C_0^{p,q}(X, F)$, we define

$$(\varphi, \psi)_\Psi=\int_X\langle\varphi, \psi\rangle e^{-\Psi}dV \tag{2.1}$$

for any real-valued C^∞-function Ψ.

In particular we set

$$(\varphi, \psi)=(\varphi, \psi)_0 \tag{2.2}$$

and

$$\|\varphi\|_\Psi^2=(\varphi, \varphi)_\Psi \tag{2.3}$$

$$\|\varphi\|^2=(\varphi, \varphi).$$

We have the operator $\bar\partial: C^{p,q}(X, F)\to C^{p,q+1}(X, F)$ defined by $(\bar\partial\varphi)_i$

$=\bar{\partial}\varphi_i$. With respect to (2. 1) and (2. 2), the formal adjoint operator of $\bar{\partial}$ are defined, we denote them by ϑ_Ψ and ϑ respectively. We denote by $L^{p,q}(X, F, \Psi)$ (resp. $L^{p,q}(X, F)$) the space of the measurable F-valued forms φ of type (p, q), square integrable in the sense that $\|\varphi\|_\Psi^2<\infty$ (resp. $\|\varphi\|^2<\infty$). Then, they are Hilbert spaces with respect to the inner product $(\varphi, \psi)_\Psi$ (resp. (φ, ψ)). We denote again by $\bar{\partial}$ the operator from $L^{p,q}(X, F, \Psi)$ to $L^{p,q+1}(X, F, \Psi)$ extending the original $\bar{\partial}$; thus a form $\varphi\in L^{p,q}(X, F, \Psi)$ is in the domain of $\bar{\partial}$ if and only if $\bar{\partial}\varphi$, defined in the sense of distribution, belongs to $L^{p,q-1}(X, F, \Psi)$. Then $\bar{\partial}$ is a closed, densely defined operator, so the adjoint operator $\bar{\partial}_\Psi^*$ (resp. $\bar{\partial}^*$) can be defined. We denote the domain, range and nullity of $\bar{\partial}$ in $L^{p,q}(X, F, \Psi)$ by $D_{\bar{\partial}}^{p,q}$, $R_{\bar{\partial}}^{p,q}$ and $N_{\bar{\partial}}^{p,q}$ respectively. $D_{\bar{\partial}_\Psi^*}^{p,q}$, $R_{\bar{\partial}_\Psi^*}^{p,q}$ and $N_{\bar{\partial}_\Psi^*}^{p,q}$ are defined similarly.

Definition 2. 1. X is called weakly 1-complete if there exists a C^∞-plurisubharmonic function Φ on X such that for any real number c, $X_c=\{x\in X|\Phi(x)<c\}$ is relatively compact in X.

Remark 2. 1. Let $\lambda(t):(-\infty, \infty)\to(-\infty, \infty)$ be a C^∞-increasing convex function such that $\lambda(t)=0$ for $t\leqq 0$, then the composition $\lambda(\Phi)$ is again C^∞-plurisubharmonic and exhausts X. So we may assume that Φ is non-negative on X. Then, for any $c\in(0, \infty)$, $X_c=\{x\in X|\Phi(x)<c\}$ is weakly 1-complete with respect to the exhaustion function $\dfrac{1}{c-\Phi}$.

Remark 2. 2. Any connected compact complex manifold is weakly 1-complete, any real constant function being taken as the exhaustion function.

Definition 2. 2. A holomorphic line bundle $\pi: F\to X$ is said to be positive (resp. semi-positive) on a subset $Y\subset X$, if there exist a coordinate cover $\mathcal{U}=\{U_i\}_{i\in I}$ of X such that $\pi^{-1}(U_i)$ are trivial and a metric $\{a_i\}$ along the fibres of F such that

$$(2.4)\qquad \left(\frac{\partial^2\log a_i}{\partial z_i^\alpha\partial\bar{z}_i^\beta}\right)>0\ \ (\text{resp. }\geqq 0)\quad \text{on } U_i\cap Y \text{ for every } i\in I\,.$$

Definition 2.3. A holomorphic line bundle $\pi: F\rightarrow X$ is said to be q-semi-positive ($1\leqq q\leqq n$) on a subset $Y\subset X$, if F is semi-positive on Y and the hermitian matrix (2.4) has at least $n-q+1$ positive eigenvalues at each point of Y.

§ 3. A Formulation of L^2-Estimates and Existence Theorems for the $\bar{\partial}$ Operator

Let X be a paracompact complex manifold of dimension n which is not necessarily connected.

Theorem 3.1. *Let F be a holomorphic line bundle over X. If there exist in the degree (p,q)*

(3.1) *a complete hermitian metric ds^2 on X,*

(3.2) *a hermitian metric $\{a_i\}$ along the fibres of F,*

(3.3) *a constant $C_1>0$*

and

(3.4) *a compact subset K of X which does not contain any connected component of X, such that*

(3.5) $\|\varphi\|^2_{X\setminus K}\leqq C_1\{\|\bar{\partial}\varphi\|^2+\|\bar{\partial}^*\varphi\|^2\}$ *for any* $\varphi\in D^{p,q}_{\bar{\partial}}\cap D^{p,q}_{\bar{\partial}^*}$.

Then, there exists a constant $C_2>0$ such that

(3.6) $\|\varphi\|^2\leqq C_2\{\|\bar{\partial}\varphi\|^2+\|\bar{\partial}^*\varphi\|^2\}$ *for any* $\varphi\in D^{p,q}_{\bar{\partial}}\cap D^{p,q}_{\bar{\partial}^*}$.

Proof. Take any sequence $\{\varphi_m\}$ such that $\varphi_m\in D^{p,q}_{\bar{\partial}}\cap D^{p,q}_{\bar{\partial}^*}$, $\|\varphi_m\|^2\leqq 1$, $\lim_{m\rightarrow+\infty}\|\bar{\partial}\varphi_m\|^2=0$ and $\lim_{m\rightarrow+\infty}\|\bar{\partial}^*\varphi_m\|^2=0$. Then we assert that there exists a subsequence $\{\varphi_{m_k}\}$ of $\{\varphi_m\}$ which converges strongly on X. Since ds^2 is complete, $C^{p,q}_0(X,F)$ is dense in $D^{p,q}_{\bar{\partial}}\cap D^{p,q}_{\bar{\partial}^*}$ with respect to the norm

$$(\bar{\partial}\varphi,\bar{\partial}\varphi)+(\bar{\partial}^*\varphi,\bar{\partial}^*\varphi)+(\varphi,\varphi)$$

([12], Theorem 1.1). Hence we may assume $\varphi_m\in C^{p,q}_0(X,F)$. Therefore we obtain that

$$(\bar{\partial}\varphi_m,\bar{\partial}\varphi_m)+(\bar{\partial}^*\varphi_m,\bar{\partial}^*\varphi_m)+(\varphi_m,\varphi_m)$$
$$=((\bar{\partial}\vartheta+\vartheta\bar{\partial})\varphi_m,\varphi_m)+(\varphi_m,\varphi_m)$$

is bounded by the assumption. Since $\bar{\partial}\vartheta+\vartheta\bar{\partial}$ is an elliptic differential operator of order 2, this means that $(\varphi_m)_i$ and their first derivatives with respect to the coordinate of U_i are bounded in the sense of the integral $\| \ , \ \|_{K'}^2$, where K' is a compact subset of X with $K\subset \mathrm{Int}\, K'$ (see for example [3], (2. 2. 1) Theorem). Combining this with Rellich's lemma (see for example [3], Appendix), it follows that $\{\varphi_m\}$ has a subsequence $\{\varphi_{m_k}\}$ which is strongly convergent on compact subsets. By (3. 5), we conclude that $\{\varphi_{m_k}\}$ converges strongly on X. Therefore, by Hörmander [5] Theorem 1. 1. 2 and Theorem 1. 1. 3, there exists a positive constant C_2 such that

$$\|\varphi\|^2 \leqq C_2\{\|\bar{\partial}\varphi\|^2+\|\bar{\partial}^*\varphi\|^2\} \tag{3. 7}$$

for any $\varphi\in D_{\bar{\partial}}^{p,q}\cap D_{\bar{\partial}^*}^{p,q}$ with $\varphi\perp N^{p,q}=N_{\bar{\partial}}^{p,q}\cap N_{\bar{\partial}^*}^{p,q}$.

By the same theorems we obtain the following strong orthogonal decomposition:

$$L^{p,q}(X, F)=R_{\bar{\partial}}^{p,q}\oplus N^{p,q}\oplus R_{\bar{\partial}^*}^{p,q}. \tag{3. 8}$$

Each element φ in $N^{p,q}$ is a solution of the Laplace-Beltrami operator $\square=\bar{\partial}\vartheta+\vartheta\bar{\partial}$ with respect to (3. 1) and (3. 2). Now we refer to the unique continuation theorem for harmonic forms with values in a hermitian vector bundle.

Theorem 3. 2 (Aronszajn [2], Riemenschneider [11]). *Let E be a hermitian vector bundle over a connected complex hermitian manifold X. Then a harmonic form $\varphi\in\mathcal{H}^{p,q}(E)$ vanishes identically on X if it vanishes on a non-empty open subset U of X.*

Any form φ in $N^{p,q}$ vanishes on the open subset $X\backslash K$ by (3. 5). Since each connected component of X is not contained in K by the assumption, from Theorem 3. 2, φ vanishes identically on each connected component. Hence φ vanishes identically on X. Therefore $N^{p,q}$ is the null space. Combining this with (3. 7), our theorem follows. q.e.d.

From the above theorem, we obtain (cf. [5], Theorem 1. 1. 4)

Corollary 3. 1. *Let X, F and others be as above. Let $\varphi\in L^{p,q}$*

(X, F) satisfy the equation $\bar{\partial}\varphi=0$, then there exists a $\psi\in L^{p,q-1}(X, F)$ such that $\bar{\partial}\psi=\varphi$. Moreover, if $\varphi\in C^{p,q}(X, F)$, then ψ can be taken from $C^{p,q-1}(X, F)$ (cf. [6], p. 115, Theorem 5. 2. 5).

§ 4. The Basic Estimate

Let X be a connected paracompact complex manifold of dimension n and let $\pi: B\to X$ be a holomorphic line bundle over X. Let $\mathcal{U}=\{U_i\}_{i\in I}$ be a coordinate cover of X such that $\pi^{-1}(U_i)$ are trivial and let $\{a_i\}$ be a hermitian metric along the fibres of B with respect to $\mathcal{U}$. We set

$$\Gamma_{i,\alpha\bar{\beta}}=\frac{\partial^2\log a_i}{\partial z_i^\alpha\partial z_i^{\bar{\beta}}}. \tag{4.1}$$

We assume that X is provided with a Kähler metric

$$ds^2=\sum_{\alpha,\beta=1}^n g_{i,\alpha\bar{\beta}}dz_i^\alpha\cdot dz_i^{\bar{\beta}}. \tag{4.2}$$

The canonical line bundle K_X of X is defined by a system of transition functions $\{K_{X,ij}\}$ on $U_i\cap U_j$, where $K_{X,ij}=\dfrac{\partial(z_j^1,\cdots,z_j^n)}{\partial(z_i^1,\cdots,z_i^n)}$. Then we see that

$$|K_{X,ij}|^2=g_i\cdot g_j^{-1}\quad\text{on } U_i\cap U_j, \tag{4.3}$$

where

$$g_i=\det(g_{i,\alpha\bar{\beta}}). \tag{4.4}$$

Hence $\{g_i\}$ determines a metric along the fibres of K_X. Then $\{A_i\}$ defined by

$$A_i=a_i\cdot g_i \tag{4.5}$$

determines a metric of $B\otimes K_X$.

With the notations (2. 1), (2. 2) and (2. 3), the following inequality has been shown by K. Kodaira (cf. [7], pp. 1269–1270).

$$\int_X\frac{1}{A_i}\sum_{B_{p-1}}\sum_{\beta,\gamma=1}^n\Gamma_{i,\beta\bar{\gamma}}\varphi_{i\bar{B}_{p-1}}^{\beta}\cdot\overline{\varphi_i^{\gamma B_{p-1}}}dV\leqq\|\bar{\partial}\varphi\|^2+\|\bar{\partial}^*\varphi\|^2 \tag{4.6}$$

for any $\varphi\in C_0^{0,p}(X, B\otimes K_X)$ with $p\geqq1$.

From now on, we let X be a connected Kähler manifold, weakly 1-complete with respect to an exhaustion function Φ and let $\pi: B\to X$ be a holomorphic line bundle which is semi-positive on X and q-semi-positive on $X\backslash K$ for some proper compact subset K of X. We fix a constant $c>c_*=\sup_{x\in K}\Phi(x)$. Then $X_c=\{x\in X|\Phi(x)<c\}$ is weakly 1-complete with respect to the exhaustion function $\frac{1}{c-\Phi}$.

We take a Kähler metric

$$(4.7)\qquad ds_0^2=\sum_{\alpha,\beta=1}^{n} g_{i,\alpha\bar{\beta},0}dz_i^\alpha\cdot d\bar{z}_i^\beta$$

on X. We set

$$G_{i,0}=(g_{i,\alpha\bar{\beta},0}).$$

Let $\{a_{i,0}\}$ be a fibre metric of B which corresponds to the assumption and we set

$$(4.8)\qquad \Gamma_{i,0}=(\Gamma_{i,\alpha\bar{\beta},0})\quad\text{where}\quad \Gamma_{i,\alpha\bar{\beta},0}=\frac{\partial^2\log a_{i,0}}{\partial z_i^\alpha\partial\bar{z}_i^\beta}.$$

We can assume that $\inf_{x\in X}\Phi(x)=0$. Then we take a C^∞ increasing convex function $\lambda(t)$ such that

$$(4.9)\qquad \text{i)}\quad \lambda(t):(-\infty,\infty)\to(-\infty,\infty),$$

$$\text{ii)}\quad \lambda(t)=\begin{cases}0 & \text{if } t\leqq\frac{1}{c}\\ >0 & \text{if } t>\frac{1}{c},\end{cases}$$

$$\text{iii)}\quad \int_0^{+\infty}\sqrt{\lambda''(t)}\,dt=+\infty.$$

We replace the metric along the fibers of B by

$$(4.10)\qquad a_i=a_{i,0}\cdot\exp(\Psi)\quad\text{where}\quad \Psi=\lambda\left(\frac{1}{c-\Phi}\right).$$

We set

$$(4.11)\qquad \Gamma_i=(\Gamma_{i,\alpha\bar{\beta}})\quad\text{where}\quad \Gamma_{i,\alpha\bar{\beta}}=\frac{\partial^2\log a_i}{\partial z_i^\alpha\partial\bar{z}_i^\beta}.$$

Then we have

$$(4.12)\qquad \Gamma_i\geqq\Gamma_{i,0}.$$

We define a Kähler metric ds^2 by

(4.13) $$ds^2 = \sum_{\alpha,\beta=1}^{n} (g_{i,\alpha\bar{\beta},0} + \Gamma_{i,\alpha\bar{\beta}}) dz_i^\alpha \cdot d\bar{z}_i^\beta .$$

Remark. By the choice of λ as in (4.9) iii), ds^2 is a complete Kähler metric on X_c (cf. [8], Proposition 1).

We set

$$G_i = (g_{i,\alpha\bar{\beta}}) \quad \text{where} \quad g_{i,\alpha\bar{\beta}} = g_{i,\alpha\bar{\beta},0} + \Gamma_{i,\alpha\bar{\beta}} .$$

We replace the metric along the fibres of $B \otimes K_X$ by

(4.14) $$A_i = a_i \cdot g_i \quad \text{where} \quad g_i = \det G_i .$$

We replace (4.1), (4.2) and (4.5) by (4.11), (4.13) and (4.14), then from (4.6) we obtain

(4.15) $$\int_{X_c} \frac{1}{A_i} \sum_{B_{p-1}} \sum_{\beta,\gamma=1}^{n} \Gamma_{i,\beta\bar{\gamma}} \varphi_{i\bar{B}_{p-1}}^{\beta} \cdot \overline{\varphi_i^{\gamma \bar{B}_{p-1}}} dV \leqq \|\bar{\partial}\varphi\|^2 + \|\bar{\partial}^*\varphi\|^2$$

for any $\varphi \in C_0^{0,p}(X_c, B \otimes K_X)$ with $p \geqq 1$.

We rewrite the left hand side as

(4.16) $$\int_{X_c} \frac{1}{A_i} \sum_{B_{p-1}} \sum_{\alpha,\gamma=1}^{n} \Big(\sum_{\beta=1}^{n} g_i^{\bar{\alpha}\beta} \cdot \Gamma_{i,\beta\bar{\gamma}}\Big) \varphi_{i,\bar{\alpha}\bar{B}_{p-1}} \cdot \overline{\varphi_i^{\gamma B_{p-1}}} dV .$$

We can choose a matrix T_i which depends, together with T_i^{-1}, differentiably on $x \in U_i$, satisfying $G_{i,0} = {}^tT_i \cdot \overline{T}_i$. Since $G_i = G_{i,0} + \Gamma_i$, we have $G_i = {}^tT_i \{E + {}^tT_i^{-1} \cdot \Gamma_i \cdot \overline{T}_i^{-1}\} \overline{T}_i$. The eigenvalues of the hermitian matrix ${}^tT_i^{-1} \cdot \Gamma_i \cdot \overline{T}_i^{-1}$ (resp. ${}^tT_i^{-1} \cdot \Gamma_{i,0} \cdot \overline{T}_i^{-1}$) are continuous functions on X_c (resp. X). From (4.12), we have

(4.17) $${}^tT_i^{-1} \cdot \Gamma_i \cdot \overline{T}_i^{-1} \geqq {}^tT_i^{-1} \cdot \Gamma_{i,0} \cdot \overline{T}_i^{-1} \quad \text{on } X_c \cap U_i .$$

Let K' be a compact subset of X_c with $K \subset \operatorname{Int} K' \subset K' \subsetneqq X_c$. Since the closure of X_c is compact, (4.17) implies that the first $n-q+1$ eigenvalues of the matrix ${}^tT_i^{-1} \cdot \Gamma_i \cdot \overline{T}_i^{-1}$ taken in the order of decreasing magnitude, are positive and stay away from zero on $X_c \backslash K'$. Let $x_0 \in X_c \backslash K'$ and choose a system of local coordinates $(z_i^1, \cdots, z_i^n)$ around x_0 as follows:

(4.18) $$G_{i,0}(x_0) = (\delta_{\alpha\beta}) \quad \text{and} \quad \Gamma_i(x_0) = (v_\alpha \cdot \delta_{\alpha\beta}) ,$$

where $\{v_\alpha\}_{1 \leqq \alpha \leqq n}$ are eigenvalues of the matrix ${}^tT_i^{-1} \cdot \Gamma_i \cdot \overline{T}_i^{-1}$ at x_0 and satisfy $v_1 \geqq v_2 \geqq \cdots \geqq v_{n-q+1} > 0$ and $v_{n-q+2} \geqq \cdots \geqq v_n \geqq 0$. Then there exists a positive constant ε, independent of the choice of $x_0 \in X_c \backslash K'$, such that $v_{n-q+1} \geqq \varepsilon > 0$. Therefore we have

$$(4.19) \qquad G_i(x_0)^{-1} \cdot \Gamma_i(x_0) = \begin{pmatrix} \frac{v_1}{1+v_1}, & & \\ & \ddots & \\ & & , \frac{v_n}{1+v_n} \end{pmatrix} \geqq \varepsilon' \begin{pmatrix} E_{n-q+1} & 0 \\ 0 & 0 \end{pmatrix}$$

where $\varepsilon' = \frac{\varepsilon}{1+\varepsilon}$ and E_{n-q+1} is the $(n-q+1, n-q+1)$ unit matrix.

We apply (4.19) to (4.16), then at x_0

$$(4.20) \qquad \sum_{B_{p-1}} \sum_{\alpha,\gamma=1}^{n} \Big(\sum_{\beta=1}^{n} g_i^{\bar{\alpha}\beta} \cdot \Gamma_{i,\bar{\beta}\gamma}\Big) \varphi_{i,\bar{\alpha}\bar{B}_{p-1}} \cdot \overline{\varphi_i^{\bar{\gamma}\bar{B}_{p-1}}}$$
$$\geqq \varepsilon' \sum_{B_{p-1}} \sum_{\beta=1}^{n-q+1} \varphi_{i,\bar{\beta}\bar{B}_{p-1}} \cdot \overline{\varphi_i^{\bar{\beta}\bar{B}_{p-1}}}.$$

If $p \geqq q$, then $p+n-q+1 \geqq n+1$, thus any block B_p of p indices taken from $\{1, 2, \cdots, n\}$ must contain one of the indices $\{1, 2, \cdots, n-q+1\}$, i.e. one of the indices corresponding to the positive eigenvalues $v_1, v_2, \cdots, v_{n-q+1}$. It follows that

$$(4.21) \qquad \sum_{B_{p-1}} \sum_{\beta=1}^{n-q+1} \varphi_{i,\bar{\beta}\bar{B}_{p-1}} \cdot \overline{\varphi_i^{\bar{\gamma}\bar{B}_{p-1}}} \geqq \sum_{\beta_1 < \cdots < \beta_p} \varphi_{i,\bar{B}_p} \cdot \varphi_i^{\bar{B}_p}.$$

Since the matrix $G_i^{-1} \cdot \Gamma_i$ is positive semi-definite on X_c, from (4.15), (4.16), (4.20) and (4.21) we have

$$(4.22) \qquad \|\varphi\|^2_{X_c \backslash K'} \leqq C_1 \{\|\bar{\partial}\varphi\|^2 + \|\bar{\partial}^*\varphi\|^2\} \qquad \left(C_1 = \frac{1+\varepsilon}{\varepsilon}\right)$$

for any $\varphi \in C_0^{0,p}(X, B \otimes K_X)$ with $p \geqq q$.

§ 5. Proof of the Main Theorem

Step 1. Vanishing Theorems on Each Sublevel Set X_c. Let the situations be as above. By Remark in Section 4, our base metric ds^2 is complete. Hence, by the same argument as in the proof of Theorem 3.1 and (4.22), we have

$$(5.1) \qquad \|\varphi\|^2_{X_c \backslash K'} \leqq C_1 \{\|\bar{\partial}\varphi\|^2 + \|\bar{\partial}^*\varphi\|^2\}$$

for any $\varphi\in D_{\bar\partial}^{0,p}\cap D_{\bar\partial^*}^{0,p}$ with $p\geqq q$.

Any connected compact complex manifold X is weakly 1-complete with respect to the real constant functions. Then we have $X_c=X$. If X is non-compact, $X_c=\{x\in X|\Phi(x)<c\}$ has countable connected components. If one of them is contained in the compact subset K', it must be a compact connected component (or manifold) of X. Since X is connected, this is a contradiction. Therefore, in our situation, the conditions of Theorem 3.1 are satisfied. Hence there exists a constant $C_2>0$ such that

$$(5.2)\qquad \|\varphi\|^2\leqq C_2\{\|\bar\partial\varphi\|^2+\|\bar\partial^*\varphi\|^2\}$$

for any $\varphi\in D_{\bar\partial}^{0,p}\cap D_{\bar\partial^*}^{0,p}$ with $p\geqq q$.

Take any $\varphi\in C^{0,p}(X_c, B\otimes K_X)$ with $\bar\partial\varphi=0$. We can choose a C^∞-function λ with the condition (4.9) such that

i) the Kähler metric ds^2 induced by (4.13) is complete,

ii) $(\varphi,\varphi)<+\infty$ (cf. [9], § 2, Proof of Theorem 1).

Hence, by Corollary 3.1, we have $\varphi=\bar\partial\psi$ for some $\psi\in C^{0,p-1}(X_c, B\otimes K_X)$. Therefore we have proved that for any $c>c_*=\sup_{x\in K}\Phi(x)$,

$$(5.3)\qquad H^p(X_c, \mathcal{O}(B\otimes K_X))=0\quad \text{for any } p\geqq q.$$

Step 2. Approximation Lemmas. We fix two constants d and e such that

(5.4) i) $d>e>c_*$,

ii) the boundary ∂X_e of $\{x\in X|\Phi(x)\leqq e\}$ is smooth.

We take a C^∞-increasing convex function $\tau(t)$ such that

(5.5) i) $\tau(t):(-\infty,\infty)\to(-\infty,\infty)$,

$$\text{ii)}\quad \tau(t)=\begin{cases} 0 & \text{if } t\leqq \dfrac{1}{d-e}\\[2mm] >0 & \text{if } t>\dfrac{1}{d-e},\end{cases}$$

$$\text{iii)}\quad \int_0^{+\infty}\sqrt{\tau''(t)}\,dt=+\infty.$$

We set

$$\Psi=\tau\left(\frac{1}{d-\Phi}\right).$$

We define the metrics of B on X_d by

(5. 6) i) $a_i=a_{i,0}\cdot\exp(\Psi)$,

ii) $a_{m,i}=a_i\cdot\exp(m\Psi)$ for any $m\geqq 0$.

We set

i) $\Gamma_i=(\Gamma_{i,\alpha\bar{\beta}})$ where $\Gamma_{i,\alpha\bar{\beta}}=\dfrac{\partial^2\log a_i}{\partial z_i^\alpha\partial\bar{z}_i^\beta}$,

ii) $\Gamma_{m,i}=(\Gamma_{m,i,\alpha\bar{\beta}})$ where $\Gamma_{m,i,\alpha\bar{\beta}}=\dfrac{\partial^2\log a_{m,i}}{\partial z_i^\alpha\partial\bar{z}_i^\beta}$ for any $m\geqq 0$.

We define a Kähler metric ds^2 on X_d by

$$ds^2=\sum_{\alpha,\beta=1}^{n}(g_{i,\alpha\bar{\beta},0}+\Gamma_{i,\alpha\bar{\beta}})\,dz_i^\alpha\cdot d\bar{z}_i^\beta\,. \tag{5.7}$$

Remark. By the choice (5. 5), ds^2 is a complete Kähler metric as in Remark in Section 4.

We set

$$G_i=(g_{i,\alpha\bar{\beta}}) \quad\text{where}\quad g_{i,\alpha\bar{\beta}}=g_{i,\alpha\bar{\beta},0}+\Gamma_{i,\alpha\bar{\beta}}\,.$$

Using (5. 6), we define the metrics of $B\otimes K_X$ on X_d:

(5. 8) i) $A_i=a_i\cdot g_i$,

ii) $A_{m,i}=a_{m,i}\cdot g_i$ for any $m\geqq 0$, where $g_i=\det G_i$.

For any integer $m\geqq 0$, we define

$$(\varphi,\psi)_m=(\varphi,\psi)_{m\Psi} \tag{5.9}$$

$$\|\varphi\|_m^2=(\varphi,\varphi)_m$$

for any $\varphi,\psi\in L^{0,p}(X_d,B\otimes K_X,m\Psi)$. We denote the formal adjoint of $\bar{\partial}$ with respect to the inner product $(\varphi,\psi)_m$ by ϑ_m and the adjoint operator in $L^{0,\cdot}(X_d,B\otimes K_X,m\Psi)$ by $\bar{\partial}_m^*$.

Now we have

$$G_i^{-1}\cdot\Gamma_i \leqq G_i^{-1}\cdot\Gamma_{m,i} \quad \text{for any } m\geqq 0\,.$$

Hence by the same argument as in Section 4, we have, for any $m\geqq 0$,

$$\|\varphi\|^2_{m\,X_d\setminus K'} \leqq C_1\{\|\bar\partial\varphi\|^2_m + \|\bar\partial^*_m\varphi\|^2_m\} \tag{5. 10}$$

for any $\varphi\in D^{0,p}_{\bar\partial}\cap D^{0,p}_{\bar\partial^*_m}$ with $p\geqq q$, where $C_1>0$ is independent of m and K' is a compact subset with $K\subset \operatorname{Int} K'\subset K'\subsetneqq X_d$. Then, for each m, we have a positive constant such that (5. 2) holds. In general, this constant depends on m. The basic idea of the following lemma is due to Hörmander [5]. (Compare with [10], Proposition 4. 2.)

Lemma 5. 1. *There exists m_0 and $C_0>0$ such that for any $m\geqq m_0$ and $p\geqq q$,*

$$\|\varphi\|^2_m \leqq C_0\{\|\bar\partial\varphi\|^2_m + \|\bar\partial^*_m\varphi\|^2_m\}\,,$$

provided $\varphi\in D^{0,p}_{\bar\partial}\cap D^{0,p}_{\bar\partial^*_m}\subset L^{0,p}(X_d, B\otimes K_X, m\Psi)$.

Proof. Assume that the assertion is false. There would be a sequence $\{\varphi_k\}$ such that

(5. 11) i) $\varphi_k\in D^{0,p}_{\bar\partial}\cap D^{0,p}_{\bar\partial^*_k}\subset L^{0,p}(X_d, B\otimes K_X, k\Psi)$,

ii) $\|\varphi_k\|^2_k=1$,

iii) $\|\bar\partial\varphi_k\|^2_k,\ \|\bar\partial^*_k\varphi_k\|^2_k\to 0$ as $k\to+\infty$.

Let $g_k=e^{-k\Psi}\cdot\varphi_k$, then we have

(5. 12) i) $\bar\partial^* g_k = e^{-k\Psi}\bar\partial^*_k\varphi_k$,

ii) $\|\bar\partial^* g_k\|_{-k} = \|\bar\partial^*_k\varphi_k\|_k$.

By (5. 11), we have

$$\|g_k\|\leqq\|g_k\|_{-k}=\|\varphi_k\|_k=1\,.$$

Therefore choosing a subsequence if necessary, we may assume that $\{g_k\}$ has a weak limit g in $L^{0,p}(X, B\otimes K_X)$. On the other hand, it follows that

$$\|g_k\|^2_{X_d\setminus K'}\leqq\|\varphi_k\|^2_{k\,X_d\setminus K'}\leqq C_1\{\|\bar\partial\varphi_k\|^2_k+\|\bar\partial^*_k\varphi_k\|^2_k\}\,.$$

By (5. 11)

$$\lim_{k\to+\infty} \|g_k\|^2_{X_d\setminus K'}=0 .$$

Hence we have $g|_{X_d\setminus K'}\equiv 0$. Then it follows that

(5.13) $$\operatorname{supp} g\subseteqq K' .$$

From (5.11), (5.12) and (5.13), we have $\bar\partial g=0$ and $\bar\partial^* g=0$ in $L^{0,p+1}(X_e, B\otimes K_X)$ and $L^{0,p-1}(X_e, B\otimes K_X)$ respectively. Since any connected component of X_e is not contained in K', by Theorem 3.2, we have

(5.14) $$g=0 .$$

By (5.11), we may assume that $\{g_k\}$ is strongly convergent on K'. (5.14) implies that the limit is zero on K'. From (5.10) and (5.11), we obtain a contradiction. q.e.d.

Lemma 5.2. *If $\psi\in L^{0,p}(X_e, B\otimes K_X)$ with $p\geqq q-1$ and $\bar\partial\psi=0$, then for any $\varepsilon>0$, there exists $\tilde\psi\in L^{0,p}(X_d, B\otimes K_X)$ such that $\|\tilde\psi|_{X_e}-\psi\|^2_{X_e}<\varepsilon$ and $\bar\partial\tilde\psi=0$.*

Proof. It suffices to show that if $u\in L^{0,p}(X_e, B\otimes K_X)$ and

(5.15) $$\int_{X_e}\langle f, u\rangle dV=0$$

for any $f\in L^{0,p}(X_d, B\otimes K_X)$ with $\bar\partial f=0$, then we have

(5.16) $$\int_{X_e}\langle g, u\rangle dV=0$$

if $g\in L^{0,p}(X_e, B\otimes K_X)$ and $\bar\partial g=0$.

Extend the definition of u by setting $u=0$ on $X_d\setminus X_e$. We denote it by u'. Then (5.15) implies that u' is orthogonal to $N^{0,p}_{\bar\partial}\subset L^{0,p}(X_d, B\otimes K_X, m\Psi)$ for any m, we have $u'\in\overline{R^{0,p}_{\bar\partial^*_m}}\subset L^{0,p}(X_d, B\otimes K_X, m\Psi)$. The condition $R^{0,p}_{\bar\partial^*_m}=\overline{R^{0,p}_{\bar\partial^*_m}}$ is equivalent to $R^{0,p+1}_{\bar\partial}=\overline{R^{0,p+1}_{\bar\partial}}$ (cf. [5], Theorem 1.1.1). By (5.10), we have $R^{0,p+1}_{\bar\partial}=\overline{R^{0,p+1}_{\bar\partial}}\subset L^{0,p+1}(X_d, B\otimes K_X, m\Psi)$ for $m\geqq 0$ and $p\geqq q-1$. Hence, from Lemma 5.1, for any $m\geqq m_0$ we have

(5.17) $$u'=\bar\partial^*_m v_m$$

for some $v_m\in L^{0,p-1}(X_d, B\otimes K_X, m\Psi)$ with $\|v_m\|^2_m\leqq C_0\cdot\|u'\|^2$.

We set

$$w_m = e^{-m\Psi}\cdot v_m \qquad \text{for } m \geqq m_0,$$

then

$$\|w_m\|^2 \leqq \|w_m\|^2_{-m} = \|v_m\|^2_m \leqq C_0\cdot\|u'\|^2.$$

Hence $\{w_m\}$ has a subsequence which is weakly convergent in $L^{0,p-1}(X_d, B \otimes K_X)$. Let the weak limit be w. On the other hand, for every $\varepsilon>0$

$$\int_{\{x\in X_d \mid \Psi(x)>\varepsilon\}} e^{m\Psi}\langle w_m, w_m\rangle dV \leqq C_0\|u'\|$$

and we have

$$e^{m\varepsilon}\int_{\{x\in X_d \mid \Psi(x)\geqq\varepsilon\}} \langle w_m, w_m\rangle dV \leqq C_0\|u'\| .$$

It follows that $\int_{\{x\in X_d \mid \Psi(x)\geqq\varepsilon\}} \langle w_m, w_m\rangle dV$ tends to zero, and hence $w_m \to 0$ almost everywhere in $\{x\in X_d \mid \Psi(x)\geqq\varepsilon\}$. Hence $w=0$ on $\{x\in X_d \mid \Psi(x)\geqq\varepsilon\}$ for every $\varepsilon>0$. Therefore we have

(5. 18) $$\operatorname{supp} w \subseteqq \overline{X}_e \quad\text{and}\quad \bar{\partial}^* w = u'.$$

Since $\overline{X}_e$ is compact and ∂X_e is smooth, from [5] Proposition 1. 2. 3, there exists a sequence $\{w^k\}$ such that $\{w^k\} \subset C_0^{0,p+1}(X_e, B\otimes K_X)$ and $\|w^k - w\|^2_{X_e}$, $\|\bar{\partial}^* w^k - \bar{\partial}^* w\|^2_{X_e} \to 0$ as $k\to+\infty$.

We have, for any $v \in D_{\bar{\partial}}^{0,p} \subset L^{0,p}(X_e, B\otimes K_X)$,

$$(\bar{\partial}v, w|_{X_e})_{X_e} = \lim_{k\to+\infty}(\bar{\partial}v, w^k)_{X_e} = \lim_{k\to+\infty}(v, \bar{\partial}^* w^k)_{X_e}$$

$$= (v, \bar{\partial}^*(w|_{X_e}))_{X_e} .$$

Hence

(5. 19) $$\bar{\partial}^*(w|_{X_e}) = u .$$

Therefore, if $g \in L^{0,p}(X_e, B\otimes K_X)$ and $\bar{\partial} g = 0$, we have

$$\int_{X_e} \langle g, u\rangle dV = \int_{X_e} \langle \bar{\partial} g, w\rangle dV = 0 . \qquad \text{q.e.d.}$$

If in particular $q=1$, replacing $L^{0,p}(X_d, B\otimes K_X)$ (resp. $L^{0,p}(X_c, B\otimes K_X)$) by $\Gamma(X_d, \mathcal{O}(B\otimes K_X))$ (resp. $\Gamma(\overline{X}_e, \mathcal{O}(B\otimes K_X))$), we can prove the following in the same way as we proved Lemma 5. 2.

Lemma 5. 3. *Let X_d and X_e be as above and let a holomorphic* *line bundle B be positive on $X\backslash K$ and semi-positive on X. Then for any holomorphic section $\varphi\in\Gamma(\overline{X}_e, \mathcal{O}(B\otimes K_X))$, $\overline{X}_e$ being the closure of X_e in X, and for any $\varepsilon>0$, there exists a section $\tilde{\varphi}\in\Gamma(X_d, \mathcal{O}(B\otimes K_X))$ such that $\|\tilde{\varphi}-\varphi\|^2_{X_e}<\varepsilon$.*

Let C be a compact subset of X_d. We set $|\varphi|_C=\sup_{x\in C}\sqrt{\langle\varphi, \varphi\rangle(x)}$ for $\varphi\in\Gamma(X_d, \mathcal{O}(B\otimes K_X))$, where $\langle\varphi, \varphi\rangle=A_i^{-1}|\varphi_i|^2$ (see (5.8)). Then, using Cauchy's integral formula in each local coordinate U_i with $U_i\cap C\neq\emptyset$, we can find a positive constant M such that

$$|\varphi|_C\leqq M\|\varphi\|_C.$$

Hence we obtain the following.

Lemma 5.4. *Let X_d and X_e be as above. Let a holomorphic line bundle B be positive on $X\backslash K$ and semi-positive on X. Then for any holomorphic section $\varphi\in\Gamma(\overline{X}_e, \mathcal{O}(B\otimes K_X))$ and for any $\varepsilon>0$, there exists a section $\tilde{\varphi}\in\Gamma(X_d, \mathcal{O}(B\otimes K_X))$ such that $|\tilde{\varphi}-\varphi|_{X_e}<\varepsilon$.*

Step 3. Global Vanishing Theorems. By Sard's theorem, we can choose a sequence $\{c_\nu\}_{\nu=0,1,\cdots,}$ of real numbers such that

(5.20) i) $c_0>c_*$,

ii) $c_{\nu+1}>c_\nu$ and $c_\nu\to\infty$ as $\nu\to+\infty$,

iii) the boundary ∂X_{c_ν} of $\{x\in X|\Phi(x)\leqq c_\nu\}$ is smooth for any $\nu\geqq0$.

For any pair $(c_{\nu+2}, c_\nu)$ $(\nu\geqq0)$, we choose a C^∞ increasing convex function $\tau_{\nu+2}$ such that

(5.21) i) $\tau_{\nu+2}(t)\colon(-\infty, \infty)\to(-\infty, \infty)$,

ii) $$\tau_{\nu+2}(t)=\begin{cases}0 & \text{if } t\leqq\dfrac{1}{c_{\nu+2}-c_\nu}\\ >0 & \text{if } t>\dfrac{1}{c_{\nu+2}-c_\nu},\end{cases}$$

iii) $$\int_0^{+\infty}\sqrt{\tau''_{\nu+2}(t)}\,dt=+\infty.$$

We set

$$X_\nu=\{x\in X\,|\,\Phi(x)<c_\nu\},$$
$$\Psi_{\nu+2}=\tau_{\nu+2}\left(\frac{1}{c_{\nu+2}-\Phi}\right)$$

for any $\nu\geqq 0$. Then, for any pair $(c_{\nu+2}, c_\nu)$, Lemma 5. 2 and Lemma 5. 4 hold.

The case $q=1$. $\mathcal{X}=\{X_\nu\}_{\nu\geqq 0}$ is a covering of X. For any $\nu\geqq 1$, we set $\mathcal{X}_\nu=\{X_\mu\}_{\mu\leqq\nu}$, then $\mathcal{X}_\nu$ is a covering of X_ν. By (5. 3), $\mathcal{X}$ (resp. $\mathcal{X}_\nu$) is a Leray covering for the sheaf $\mathcal{O}(B\otimes K_X)$ on X (resp. X_ν). Then we have, for any $i\geqq 1$ and $\nu\geqq 1$,

$$H^i(X, \mathcal{O}(B\otimes K_X))=H^i(\mathcal{X}, \mathcal{O}(B\otimes K_X))$$

and

$$H^i(\mathcal{X}_\nu, \mathcal{O}(B\otimes K_X))=H^i(X_\nu, \mathcal{O}(B\otimes K_X))=0\,.$$

Let $\sigma\in Z^i(\mathcal{X}, \mathcal{O}(B\otimes K_X))$, $i\geqq 1$. Let σ_ν be the restriction of σ to X_ν. Then $\sigma_\nu\in Z^i(\mathcal{X}_\nu, \mathcal{O}(B\otimes K_X))$ so there is an $\alpha_\nu\in C^{i-1}(\mathcal{X}_\nu, \mathcal{O}(B\otimes K_X))$ such that $\delta\alpha_\nu=\sigma_\nu$. As an element of $C^{i-1}(\mathcal{X}_{\nu-1}, \mathcal{O}(B\otimes K_X))$, $\delta\alpha_\nu=\delta\alpha_{\nu-1}$, and thus $\alpha_\nu-\alpha_{\nu-1}\in Z^{i-1}(\mathcal{X}_{\nu-1}, \mathcal{O}(B\otimes K_X))$.

When $i>1$. Since $\alpha_\nu-\alpha_{\nu-1}\in Z^{i-1}(\mathcal{X}_{\nu-1}, \mathcal{O}(B\otimes K_X))$, there is a $\beta_{\nu-1}\in C^{i-2}(\mathcal{X}_{\nu-1}, \mathcal{O}(B\otimes K_X))$ such that $\delta\beta_{\nu-1}=\alpha_\nu-\alpha_{\nu-1}$ on $X_{\nu-1}$. Define $\alpha\in C^{i-1}(\mathcal{X}, \mathcal{O}(B\otimes K_X))$ as follows:

$$\alpha=\alpha_\nu-\delta(\sum_{\mu<\nu}\beta_\mu)\quad \text{on } X_\nu.$$

It is easily verified that α is well defined. Finally, for any ν, $\delta\alpha=\delta\alpha_\nu-\delta\delta(\sum_{\mu<\nu}\beta_\mu)=\delta\alpha_\nu=\sigma_\nu$. Hence we have $\delta\alpha=\sigma$.

When $i=1$. Since $\alpha_\nu-\alpha_{\nu-1}\in\Gamma(X_{\nu-1}, \mathcal{O}(B\otimes K_X))$, by Lemma 5. 4 we can find, for any $\varepsilon>0$, a $\gamma\in\Gamma(X_\nu, \mathcal{O}(B\otimes K_X))$ such that $|\alpha_\nu-\alpha_{\nu-1}-\gamma|_{\bar{X}_{\nu-2}}<\varepsilon$. Therefore, inductively, we have a sequence $\{\lambda_\nu\}_{\nu\geqq 1}$ so that

(5. 22) i) $\lambda_\nu\in C^0(\mathcal{X}_\nu, \mathcal{O}(B\otimes K_X))$ and $\lambda_1=\alpha_1$,

ii) $\delta\lambda_\nu=\sigma_\nu$,

iii) $|\lambda_{\nu+1}-\lambda_\nu|_{\bar{X}_{\nu-1}}<2^{-\nu}$.

For any ν, $\lim_{\mu\geqq\nu}\lambda_\mu$ defines an element of $C^0(\mathfrak{X}_\nu, \mathcal{O}(B\otimes K_X))$ and clearly this limit is the same as the restriction of $\lim_{\mu\geqq\eta}\lambda_\mu$ for any $\eta\geqq\nu+1$. Thus we can define an element λ of $C^0(\mathfrak{X}, \mathcal{O}(B\otimes K_X))$ by $\lambda=\lim_{\nu\to+\infty}\lambda_\nu$. For any ν, $\delta(\lim_{\mu\geqq\nu}\lambda_\mu)=\lim_{\mu\geqq\nu}\delta\lambda_\mu=\sigma_\nu$. Hence we have $\delta\lambda=\sigma$.

The case $q>1$. We denote by $L^{0,p}_{\mathrm{loc}}(X, B\otimes K_X)$ the set of the locally square integrable $(0,p)$ forms on X with values in $B\otimes K_X$. For $p\geqq 1$, there is a natural isomorphism

(5.23) $H^p(X, \mathcal{O}(B\otimes K_X))$

$$\cong\frac{\{f\in L^{0,p}_{\mathrm{loc}}(X, B\otimes K_X);\ \bar\partial f=0\}}{\{f\in L^{0,p}_{\mathrm{loc}}(X, B\otimes K_X);\ f=\bar\partial g \text{ for some } g\in L^{0,p-1}_{\mathrm{loc}}(X, B\otimes K_X)\}}.$$

Therefore, for $p\geqq q$, it suffices to show that for any $\varphi\in L^{0,p}_{\mathrm{loc}}(X, B\otimes K_X)$ with $\bar\partial\varphi=0$, there exists a $\psi\in L^{0,p-1}_{\mathrm{loc}}(X, B\otimes K_X)$ such that $\bar\partial\psi=\varphi$.

In this proof, for any $\nu\geqq 0$, we set

(5.24) i) $\varphi_\nu=\varphi|_{X_\nu}$,

ii) $L^{0,p}(X_{\nu+2}, B\otimes K_X, \Psi_{\nu+2})=L^{0,p}(X_{\nu+2}, B\otimes K_X)$,

iii) $L^{0,p}(X_\nu, B\otimes K_X, 0)=L^{0,p}(X_\nu, B\otimes K_X, \Psi_{\nu+2})$,

iv) $\|f\|^2_{\nu+2}=\int_{X_{\nu+2}}\langle f, f\rangle e^{-\Psi_{\nu+2}}dV$

for $f\in L^{0,p}(X_{\nu+2}, B\otimes K_X, \Psi_{\nu+2})$

where $\langle f, f\rangle=(a_{i,0}\cdot g_i)^{-1}\sum_{B_p} f_{i,\bar B_p}\cdot f_i^{\bar B_p}$.

Then $\varphi_\nu\in L^{0,p}(X_\nu, B\otimes K_X, \Psi_\nu)$ and $\bar\partial\varphi_\nu=0$ $(\nu\geqq 2)$. Hence there exists a $\psi'_\nu\in L^{0,p-1}(X_\nu, B\otimes K_X, \Psi_\nu)$ such that $\bar\partial\psi'_\nu=\varphi_\nu$ for any $\nu\geqq 2$. We now choose, by induction, a sequence $\{\psi_\nu\}_{\nu\geqq 1}$ so that

(5.25) i) $\psi_\nu\in L^{0,p}_{\mathrm{loc}}(X, B\otimes K_X)$

ii) $\bar\partial\psi_\nu=\varphi_\nu$ on X_ν

iii) $\|\psi_{\nu+1}-\psi_\nu\|^2_{\nu+2, X_\nu}<2^{-\nu}$.

We set

$$\psi_1=\begin{cases}\psi'_2|_{X_1} & \text{on } X_1\\ 0 & \text{on } X\backslash X_1.\end{cases}$$

Since $\psi_2' \in D_{\bar\partial}^{0,p-1} \subset L^{0,p-1}$ $(X_2, B \otimes K_X, \Psi_2)$, we have $\psi_1 \in D_{\bar\partial}^{0,p-1} \subset L^{0,p-1}$ $(X_1, B \otimes K_X, 0)$ and $\bar\partial\psi_1 = \varphi_1$. Suppose $\psi_1, \cdots, \psi_{\nu-1}$ are chosen. Then

$$(\psi_{\nu+1}' - \psi_{\nu-1})|_{X_{\nu-1}} \in L^{0,p-1}(X_{\nu-1}, B \otimes K_X, 0)$$

and

$$\bar\partial(\psi_{\nu+1}' - \psi_{\nu-1})|_{X_{\nu-1}} = 0 .$$

By Lemma 5.2, there exists a $g \in L^{0,p-1}(X_{\nu+1}, B \otimes K_X, \Psi_{\nu+1})$ such that $\|g - (\psi_{\nu+1}' - \psi_{\nu-1})\|_{\nu+1, X_{\nu-1}}^2 < 2^{-(\nu-1)}$ and $\bar\partial g = 0$.

We set

$$\psi_\nu = \begin{cases} \psi_{\nu+1}'|_{X_\nu} - g|_{X_\nu} & \text{on } X_\nu \\ 0 & \text{on } X \backslash X_\nu . \end{cases}$$

Then we have

(5.26) i) $\psi_\nu \in D_{\bar\partial}^{0,p-1} \subset L^{0,p-1}(X_\nu, B \otimes K_X, 0)$

ii) $\bar\partial\psi_\nu = \varphi_\nu$

iii) $\|\psi_\nu - \psi_{\nu-1}\|_{\nu+1, X_{\nu-1}}^2 < 2^{-(\nu-1)}$.

From (5.26), for any ν, $\{\psi_\mu\}_{\mu \geqq \nu}$ converges with respect to the norm $\| \quad \|_\nu$, and clearly the limit is the same as the restriction of $\lim_{\mu \geqq \eta} \psi_\mu$ for any $\eta \geqq \nu + 1$. Thus we can define an element ψ of $L_{\mathrm{loc}}^{0,p-1}(X, B \otimes K_X)$ by $\psi = \lim_{\nu \to +\infty} \psi_\nu$.

For every $\nu \geqq 1$,

(5.27) i) $\lim_{\mu \geqq \nu} \psi_\mu = \psi$ in $L^{0,p-1}(X_\nu, B \otimes K_X, 0)$,

ii) $\lim_{\mu \geqq \nu} \bar\partial\psi_\mu|_{X_\nu} = \varphi_\nu$ in $L^{0,p}(X_\nu, B \otimes K_X, 0)$.

Since $\bar\partial$ is a closed operator in $L^{0,p-1}(X_\nu, B \otimes K_X, 0)$ for every $\nu \geqq 1$, we have, for any $\nu \geqq 1$,

$$\bar\partial\psi = \varphi_\nu \quad \text{in } L^{0,p}(X_\nu, B \otimes K_X, 0) .$$

Hence we have

$$\bar\partial\psi = \varphi \quad \text{in } L_{\mathrm{loc}}^{0,p}(X, B \otimes K_X) .$$

q.e.d.

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Added in proof: The author and T. Ohsawa have proved that the global vanishing thheorem of Theorem 2 holds i.e. $H^p(X, \Omega^q(B))=0$ for $p+q\geqq n+k$. See "A vanishing theorem for $H^p(X, \Omega^q(B))$ on weakly 1-complete manifolds", forthcoming.