Duality of Numerical Characters of Polar Loci

By

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Contents

§0. Introduction

Let $X \subset P^n$ be an r-dimensional projective variety. For every integer k with $0 \leq k \leq r$ consider an $(n-r+k-2)$-dimensional linear subspace $L_{(k)}$ of P^n. Since the dimension of the tangent space of X at a smooth point $x \in X$ is equal to r, the tangent space necessarily intersects $L_{(k)}$ in a space of at least $k-2$ dimensions. The closure of the set of all smooth points of X where this intersection space has dimension at least $k-1$ is called the k-th polar locus of X and is denoted by $M(L_{(k)})$, following Piene [4].

For a general $L_{(k)}$, every component of $M(L_{(k)})$ has codimension k in X and moreover the rational equivalence class of the cycle defined by $M(L_{(k)})$ is independent of $L_{(k)}$. We denote this equivalence class by $[M_k] = [M(L_{(k)})]$ and call it the k-th polar class. The degree μ_k of $[M_k]$ is called the k-th class. The number μ_0 is nothing but the degree of X. We set $\mu_k = 0$ for any integer k with $k < 0$ or $k > r$.

On the other hand, the dual variety Y in the dual projective space $P^{n\vee}$ is defined as the closure of the set of tangent hyperplanes. A hyperplane is tangent to X if it contains the embedded tangent space of X at a smooth point $x \in X$.

In characteristic 0 the dual variety of Y always coincides with X itself. We know a good sufficient condition in order that the dual varity of Y is again X in the case of positive characteristic.   It is explained in Section 2 (Wallace [6]).   We say that biduality holds for X if this condition is satisfied or the characteristic is 0.

The purpose of this article is to verify the next theorem, which generarizes Proposition (3. 6) in Piene [4].

Main Theorem. *Let $X \subset P^n$ be an r-dimensional projective variety and let Y be its dual variety with dimension s.   Suppose that biduality holds for X.   We denote the k-th class of X by μ_k, and the j-th class of Y by v_j.   Then $\mu_k = v_j$ when $k + j = r + s - n + 1$.*

That is, if we reverse the sequence $..., 0, v_0, v_1, ..., v_s, 0, ...$ and translate it to correspond μ_0 to $v_{r+s-n+1}$, then it coincides with the sequence $..., 0, \mu_0, \mu_1, ..., \mu_r, 0, ...$.

We obtain the following as corollaries.

The dimension of the dual variety is equal to $n - 1 - r + \max \{k \,|\, \mu_k \neq 0\}$.

Set $l = r + s - n + 1$.   Then $l \geqq 0$.   The equality $l = 0$ holds if and only if X is a linear variety, and then Y coincides with the dual space of X as a linear space.

The study of polar loci and dual varieties goes back to Severi and Bertini. It was succeeded by Segre, Todd, Wallace, Porteous, Lascoux, Pohl, Teissier, Lê, Piene and others.   In particular Piene showed in her paper [4] with other results that polar loci and polar classes are invariant under generic projections, and that the intersection X_1 of X with a general hyperplane has the same k-th class as that of X except μ_r, which is 0.   Moreover she showed the special case of above Main Theorem where $r = s = n - 1$.

Our article is a continuation of Piene's work in [4].   I would like to thank Professor H. Hironaka for his interest in this work and Professor T. Shioda to have shown me the interest of the Piene's work.

§1.   Polar Loci and Polar Classes

In this section we will give the precise definition of polar loci and polar classes.

We fix an algebraically closed ground field k.   Every scheme is assumed

to be separated. A reduced irreducible proper scheme of finite type over k is called a variety.

We freely use the theory of the Chow homology group $A.$ and the Chow cohomology group $A^{\cdot}$ developed by Fulton in [1]. For a possibly singular scheme X, $A.X$ is the abelian group of algebraic cycles modulo rational equivalence graded by dimension. $A^{\cdot}X$ is a graded ring where for every vector bundle E on X the Chern class $c(E)=\Sigma c_i(E) \in A^{\cdot}X$ is defined. Every proper morphism $f: X \to Y$ defines graded homomorphisms $f_*: A.X \to A.Y$ and $f^*: A^{\cdot}Y \to A^{\cdot}X$. The cap product $\cap : A^{\cdot}X \otimes A.X \to A.X$ satisfies:

$$\xi \cap t \in A_{j-i}X \quad \text{if} \quad \xi \in A^iX \quad \text{and} \quad t \in A_jX$$

(the projection formula)

$$f_*(f^*\xi \cap t) = \xi \cap f_* t \quad \text{for} \quad \xi \in A^{\cdot}Y \quad \text{and} \quad t \in A.X$$

and

$$\xi \cap (\eta \cap t) = (\xi \eta) \cap t \quad \text{for} \quad \xi, \eta \in A^{\cdot}X \quad \text{and} \quad t \in A.X .$$

$s(E) = c(E^{\vee})^{-1}$ is called the Segre class of the vector bundle E. Here $E^{\vee}$ denotes the dual vector bundle of E. Let $\alpha: \mathbf{P}(E) \to X$ be the projective bundle associated with the vector bundle E. It is well-known that

$$s_k(E) \cap [X] = \alpha_*(c_1(\mathcal{O}_{\mathbf{P}(E)}(1))^{e-1+k} \cap [\mathbf{P}(E)]), \ k=0, 1, 2,\dots .$$

Here $[X]$, $[\mathbf{P}(E)]$ denote the fundamental classes of $X, \mathbf{P}(E)$ respectively and $e = \text{rank } F$.

We write the n-dimensional projective space $\mathbf{P}^n$ in the "coordinate free" way as $\mathbf{P}(V)$ with an $(n+1)$-dimensional vector space V identified with $H^0(\mathbf{P}^n, \mathcal{O}_{\mathbf{P}^n}(1))$. Let $X \subset \mathbf{P}(V)$ be an r-dimensional projective variety. $P_X^1(L)$ denotes the sheaf of principal parts of the line bundle $L = \mathcal{O}_{\mathbf{P}(V)}(1)|_X$, that is,

$$P_X^1(L) = q_{1*}q_2^*L.$$

Here q_i denotes the restriction of the i-th projection $p_i: X \times X \to X$ to the closed subsceme defined by J^2, the square of the ideal of the diagonal.

V_Z denotes $\mathcal{O}_Z \underset{k}{\otimes} V$ for a k-scheme Z. A surjective homomorphism $V_Z \to Q$ for a vector bundle Q with $t = \text{rank } Q$ is called t-quotient. A sheaf homomorphism

$$a: V_X \longrightarrow P_X^1(L)$$

is defined by the equality

$$a_x(\gamma \otimes \phi) = [\gamma \otimes 1] \otimes \phi \in P_X^1(L)_x = ((\mathcal{O}_X \underset{k}{\otimes} \mathcal{O}_X/J^2) \underset{\mathcal{O}_X}{\otimes} L)_{(x,x)}$$

with $x \in X$, $\gamma \in \mathcal{O}_{X,x}$ and $\phi \in V = H^0(\mathbf{P}^n, \mathcal{O}_{\mathbf{P}^n}(1))$.

Let $x \in X$ be a closed smooth point. Set $A = \mathcal{O}_{X,x}$ and fix an isomorphism $L_x \cong A$. Pick a basis of V, say $\phi_0, \phi_1, \ldots, \phi_n$. Their images by the morphism $V_{X,x} \to L_x \cong A$ are denoted by the same letters. Pick a regular system of parameter of A, say $t_1, \ldots, t_r$. $\{1, dt_1, \ldots, dt_r\}$ is an A-free basis of $P_X^1(L)_x$, where dt_i denotes the class of the element $t_i \otimes 1 - 1 \otimes t_i$. Let $\{1, D_1, \ldots, D_r\}$ denotes the dual basis. $D_1, \ldots, D_r$ are regarded as differential operators of A. With the above choice of bases the diagram

$$\begin{array}{ccc} V_{X,x} & \xrightarrow{\;a_x\;} & P_X^1(L)_x \\ \Big\| \wr & & \Big\| \wr \\ A^{n+1} & \xrightarrow{\;M\;} & A^{r+1} \end{array}$$

commutes. Here M is defined by the matrix.

$$M = \begin{pmatrix} \phi_0, & \phi_1, \ldots, \phi_n \\ D_1\phi_0, & D_1\phi_1, \ldots, D_1\phi_n \\ \vdots & \vdots \\ D_r\phi_0, & D_r\phi_1, \ldots, D_r\phi_n \end{pmatrix}.$$

The linear subspace of $\mathbf{P}(V)$ spanned by the row vectors of M evaluated at $x \in X$ is nothing but the embedded tangent space T_x. Therefore for a smooth closed point $x \in X$, the embedding $\mathbf{P}(P_X^1(L)(x)) \hookrightarrow \mathbf{P}(V)$ defined by the surjective homomorphism $a_x \otimes k : V \to P_X^1(L)(x)$ coincides with the inclusion $T_x \hookrightarrow \mathbf{P}(V)$. Let U denote the maximal smooth open subscheme of X. The morphism

$$g : U \longrightarrow G = \mathrm{Grass}_{r+1}(V)$$

defined by the $(r+1)$-quotient

$$a|_U : V_U \longrightarrow P_X^1(L)|_U$$

can be identified with the so-called Gauss map.

Now fix an integer k with $0 \leq k \leq r$. Let $L_{(k)} = \mathbf{P}(V/V')$ be an $(n-r+k-2)$-dimensional linear subspace of $\mathbf{P}(V)$. Let $(r+1)$-quotient $V_G \to Q$ denote the tautological sequence of vector bundles on $G = \mathrm{Grass}_{r+1}(V)$. We denote by Σ_k the scheme of zeroes of the morphism

$$\Lambda^{r-k+2} V_G' \longrightarrow \Lambda^{r-k+2} Q$$

induced by the morphism $V_G' \to Q$ which is the composition of the inclusion

$V'_G = V \otimes_k \mathcal{O}_G \hookrightarrow V_G$ with $V_G \to Q$. The scheme Σ_k is one of the Schubert varieties and codim $(\Sigma_k, G) = k$. It is known that the next equality holds in $A.G$ (Porteous [5]).

$$[\Sigma_k] = c_k(Q) \cap [G].$$

The inverse image $g^{-1}\Sigma_k$ is the scheme of zeroes of the morphism

$$\Lambda^{r-k+2}V'_U \longrightarrow \Lambda^{r-k+2}P^1_X(L)|_U$$

denoted by $M_k(U)$. It is easy to see that x belongs to $M_k(U)$ if and only if the dimension of the intersection of the tangent space T_x with $L_{(k)}$ is at least $k-1$. Let M_k be the schematic closure of $M_k(U)$. We call M_k *the k-th polar locus* of X for every integer k with $0 \leq k \leq r$. As a set M_k coincides with the closure of the set

$$\{x \in U \mid \dim (T_x \cap L_{(k)}) \geq k-1\}.$$

Proposition 1.1 (Piene). *For a general $(n-r+k-2)$-dimensional linear space $L_{(k)}$, the class $[M_k]$ of M_k in $A.X$ is independent of $L_{(k)}$. If $\pi: Z \to X$ is any proper birational morphism such that the $(r+1)$-quotient $a|_U$ extends to an $(r+1)$-quotient $b: V_Z \to P$, there is an equality*

$$[M_k] = \pi_*(c_k(P) \cap [Z]).$$

Remark 1.2. We say that the $(r+1)$-quotient $a|_U$ extends to an $(r+1)$-quotient $b: V_Z \to P$ if the following conditions are satisfied. First of all b is surjective and P is a vector bundle over Z. Moreover there exist a non-empty open set $U' \subset U$ and an isomorphism h which makes the next diagram commutative. We set $\tilde{U} = \pi^{-1}(U')$.

$$
\begin{array}{ccc}
V_{\tilde{U}} & \xrightarrow{\pi^*(a)|_{\tilde{U}}} & \pi^* P^1_X(L)|_{\tilde{U}} \\
\| & & \Big\downarrow h \\
V_{\tilde{U}} & \xrightarrow{b|_{\tilde{U}}} & P|_{\tilde{U}}
\end{array}
$$

We can choose such a proper birational morphism $\pi: Z \to X$. For example let Z be the closure in $X \times G$ of the graph of $g: U \to G$, the morphism π be the projection to X and the homomorphism b of vector bundles be the pullback of the tautological sequence on G by the restriction of the projection $X \times G \to G$ to Z.

Proof. See Piene [4].

For every integer k with $0 \leq k \leq r$, the class $[M_k] \in A.X$ which is independent

of the choice of the general $L_{(k)}$ is called the *k-th polar class*. In particular $[M_0] = [X]$.

§2. The Dual Variety

As in the previous section, let $X \subset P(V)$ be an r-dimensional projective variety in an n-dimensional projective space and let U be the maximal smooth open subscheme of X. We will define the dual variety of X.

First of all we take a proper birational morphism $\pi: \overline{X} \to X$ such that the $(r+1)$-quotient $a|_U: V_U \to P_X^1(L)|_U$ has an extension on $\overline{X}$. (See Remark 1.2.) Let $b: V_{\overline{X}} \to P$ be the extension of $a|_U$. We set $K = \mathrm{Ker}\,(b)$, which is a vector bundle on $\overline{X}$. Associated with $(n-r)$-quotient $V_{\overline{X}}^{\vee} \to K^{\vee}$ ($\vee$ denotes the dual bundle), the inclusion $Z = P(K^{\vee}) \subset P(V_{\overline{X}}^{\vee}) = \overline{X} \times P(V^{\vee})$ is defined. Let Y denotes the image of Z by the projection $\overline{X} \times P(V^{\vee}) \to P(V^{\vee})$ equipped with the reduced scheme structure. Y is called *the dual variety* of X.
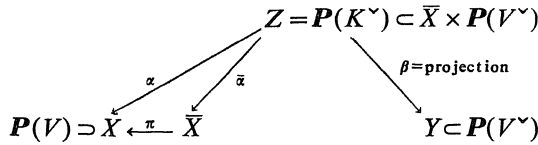

Figure 2.1.

Let us call each morphism as in Figure 2.1. For a non-empty open subscheme U',

$$Y = \text{the closure of } \beta(\alpha^{-1}(U')).$$

Thus if we take such U' that $\pi|_{\pi^{-1}(U')}: \pi^{-1}(U') \to U'$ is isomorphic, we see that Y is independent of the choice of π. And for such U' and for a closed point $\overline{x} \in \pi^{-1}(U' \cap U)$, the fibre $\overline{\alpha}^{-1}(\overline{x})$ consists of all hyperplanes which contain the tangent space $T_{\pi(\overline{x})}$ of U at $\pi(\overline{x})$. Therefore when we regard all hyperplanes which contain one of the tangent spaces of U as the subset of the dual projective sapce $P(V^{\vee})$, Y is the closure of that set equipped with reduced scheme structure. Let s be the dimension of Y.

Example 2.1. Let X be a linear variety. The tangent space T_x at a closed point $x \in X$ coincides with X itself. Thus Y is the set of all hyperplanes which contain X. Y is nothing but the dual space of X as a linear subspace of $P(V)$. We note in this case that $\dim X + \dim Y = n - 1$.

Example 2.2. Let $X \subset P(V)$ be the hypersurface defined by an irreducible homogeneous polynomial $F(t_0, t_1, \ldots, t_n)$. Then Y is the variety defined by the system of polynomials which we can obtain by eliminating variables $t_0, t_1, \ldots, t_n$ from the equalities

$$F(t_0, t_1, \ldots, t_n) = 0$$

$$u_0 = \frac{\partial F}{\partial t_0}(t_0, t_1, \ldots, t_n), \ldots, u_n = \frac{\partial F}{\partial t_n}(t_0, t_1, \ldots, t_n).$$

Example 2.3. Let t_0, t_1, t_2, t_3 be the homogeneous coordinate of $\mathbb{P}^3$. Pick an irreducible homogeneous polynomial $F(t_0, t_1, t_2)$ with three variables. Consider the surface X defined by the equality $F = 0$. Let X_1 denote the intersection of X with the plane $t_3 = 0$. X is the cone over X_1 with the vertex $p = (0, 0, 0, 1)$.
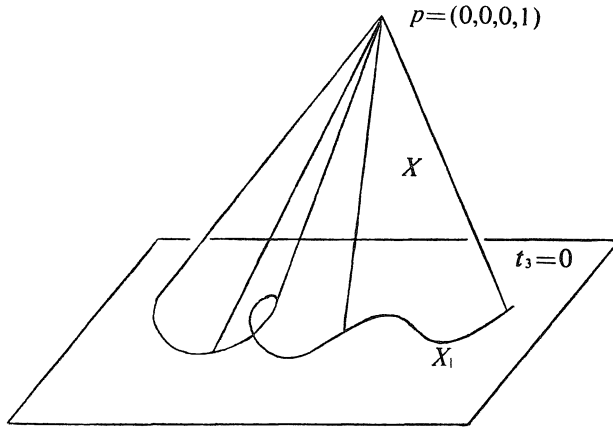

Figure 2.2.

Let u_0, u_1, u_2, u_3 be the dual coordinates of t_0, t_1, t_2, t_3. Then the dual variety Y of X is contained in the plane $u_3 = 0$ which is the dual space of the point p and Y is isomorphic to the dual curve of X_1 in the sense of Plücker. We note that in spite that $\dim X = 2$, Y has dimension at most one.

The objective of this section is to verify Theorem 2.4.

When a non-empty open smooth subscheme $U_1 \subset X$ is given, for every point $y \in P^{n\vee}$ of the dual projective space, we define the *y-contact locus*. It is the closure of the set of all points of U_1 such that the tangent space of X at that point is contained in the hyperplane H_y corresponding to y.
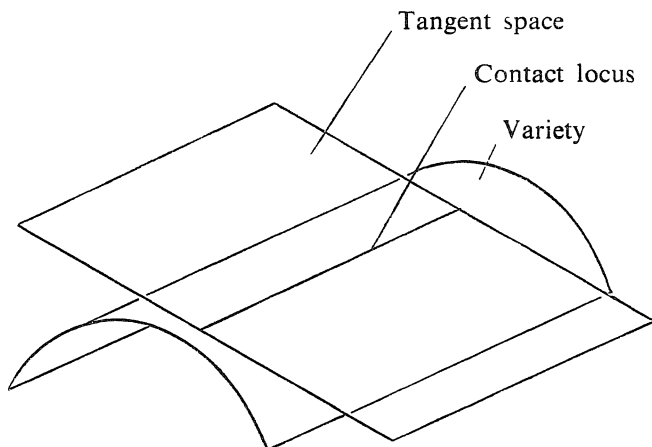
Figure 2.3.

A morphism of schemes $\beta\colon Z \to X$ is called *generically smooth* if the restriction of β to a dense open subscheme of Z is smooth. In characteristic 0 every morphism of varieties is generically smooth.

Theorem 2.4 (Wallace [6]). *Assume the same situation as in Figure* 2.1. *First assume that β is generically smooth. Then,*

(1)   *the dual variety of Y, i.e., the twice dual variety of X coincides with X.*

Let U and U^* be the maximal smooth open subschemes of X and Y respectively. U_1 denotes the maximal open subscheme of U such that $\pi|_{\pi^{-1}(U_1)}$ is an isomorphism. Let $U_1^* = U^* \backslash \beta(J)$ where J is the set of ramification points of the morphism β. We set $U_2 = \{x \in U_1 \,|\, \alpha^{-1}(x) \cap \beta^{-1}(U_1^*) \neq \phi\}$, $U_2^* = \{y \in U_1^* \,|\, \beta^{-1}(y) \cap \alpha^{-1}(U_1) \neq \phi\}$. Then,

(2.1)   *for every $y \in U_2^*$, y-contact locus in X is the dual linear space of the tangent space of Y at y;*

(2.2)   *for every $x \in U_2$, x-contact locus in Y is the dual linear space of the tangent space of X at x.*

(3)   *Conversely under the assumption of (1) and (2.1), it follows that β is generically smooth.*

We say biduality holds for X if β is generically smooth. It is easily checked that this definition is independent of the choice of the birational morphism $\pi\colon \overline{X} \to X$.

Corollary 2.5. *Assume that biduality holds for X. Let $r = \dim X$ and $s = \dim Y$.*

(1)   *The y-contact locus for a general point of Y has dimension $n-s-1$.*
The x-contact locus for a general point of X has dimension $n-r-1$.

(2)   *The inequality $r+s \geq n-1$ holds. And the equality $r+s = n-1$
holds if and only if X is an r-dimensional linear variety. In that case Y
coinsides with the dual linear space of X.*

Proof.   (1) is easily deduced from Theorem 2.4 (2).   By Theorem 2.4
a contact locus is a linear subspace of a tangent space.   Thus comparing the
dimension we have the inequality $n-s-1 \leq r$.   If the equality holds, at general
points the contact locus and the tangent space coincide.   Then the tangent space
and the variety also coincide.   As for the latter half of (2), see Example 2.1.

<div align="right">Q. E. D.</div>

Proof of Theorem 2.4.   Pick $y_0 \in U_2^* \neq \phi$.   Pick $z_0 \in \beta^{-1}(y_0) \cap \alpha^{-1}(U_1)$ and
set $x_0 = \bar{\alpha}(z_0)$.   Choose a sufficiently small smooth affine neighbourhood N_1
of x_0.

Pick a basis of V, say $\phi_0, \phi_1,..., \phi_n$ and let $D_1, D_2,..., D_r$ be independent
vector fields on N_1.   We fix an isomorphism of the structure sheaf $\mathcal{O}_X |_{N_1}$ and
the line bundle $\pi^* L |_{N_1}$ with $L = \mathcal{O}_{P(V)}(1) |_X$.   As it is shown in Section 1,
$(r+1)$-quotient

$$a |_U : V_U \longrightarrow P_X^1(L) |_U$$

is represented on N_1 by the matrix

$$[T\phi] = \begin{pmatrix} \phi_0, & \phi_1,..., & \phi_n \\ D_1\phi_0, & D_1\phi_1,..., & D_1\phi_n \\ \vdots & & \vdots \\ D_r\phi_0, & D_r\phi_1,..., & D_r\phi_n \end{pmatrix}.$$

Pick a free basis of the global sections on N_1 of the sheaf $K = \pi^* \text{Ker} (a |_U :
V_U \to P_X^1(L) |_U)$, say $e_1,..., e_{n-r}$.   The morphism $K \to V_{\bar{X}}$ is represented by the
matrix

$$[\psi] = \begin{pmatrix} \psi_{01},..., \psi_{0,n-r} \\ \psi_{11},..., \psi_{1,n-r} \\ \vdots \qquad \vdots \\ \psi_{n1},..., \psi_{n,n-r} \end{pmatrix}$$

with these bases.   Then,

(2.1) $[\phi][\psi] = 0$

with $[\phi] = (\phi_0, \phi_1,..., \phi_n)$ and for every vector field D on N_1

(2.2) $$[D\phi][\psi]=0$$

with $[D\phi]=(D\phi_0, D\phi_1,..., D\phi_n)$. Pick dual bases of $e_1,..., e_{n-r}$, ϕ_0, $\phi_1,...,\phi_n$ for $K^\vee$ and $V_X^\vee$ respectively and morphism $V_X^\vee \to K^\vee$ is represented by the matrix ${}^t[\psi]$. There is an isomorphism

$$\bar{\alpha}^{-1}(N_1) \quad \xrightarrow{\;\sim\;} \quad N_1 \times \boldsymbol{P}^{n-r-1}$$
$$\cup\kern-0.4em\cup \kern12em \cup\kern-0.4em\cup$$
$$(x, [u_1e_1 + \cdots + u_{n-r}e_{n-r}]) \longmapsto (x, (u_1 : u_2 : \cdots : u_{n-r})).$$

The affine coordinate u_j/u_1 with $j = 2, 3,..., n - r$ on the open set $\tilde{N} \subset \boldsymbol{P}^{n-r-1}$ defined by $u_1 \neq 0$ is also denoted by the same letter. Here we set $u_1 = 1$. We may assume that $z_0 \in N_1 \times \tilde{N}$. 1-quotient $V_Z \to \mathcal{O}_Z(1)$ is represented on $N_1 \times \tilde{N}$ by the matrix

(2.3) $$[\Psi] = \begin{pmatrix} \Psi_0 \\ \Psi_1 \\ \vdots \\ \Psi_n \end{pmatrix} = [\psi][u] \quad \text{with} \quad [u] = \begin{pmatrix} u_1 \\ u_2 \\ \vdots \\ u_{n-r} \end{pmatrix}.$$

The morphism β is defined on $N_1 \times \tilde{N}$ by

$$z \longmapsto (\Psi_0(z) : \Psi_1(z) : \cdots : \Psi_n(z)) \in \boldsymbol{P}(V^\vee).$$

Choose a sufficiently small smooth affine neighbourhood N_1^* of $y_0 \in Y$. Let $D_1^*, D_2^*,..., D_s^*$ be independent vector fields on N_1^*. The tangent space of Y at $y_0 \in N_1^*$ is nothing but the space spanned by the column vectors of the value at y_0 of the next matrix

$$[T^*\Psi] = \begin{pmatrix} \Psi_0, D_1^*\Psi_0,..., D_s^*\Psi_0 \\ \Psi_1, D_1^*\Psi_1,..., D_s^*\Psi_1 \\ \vdots \kern5em \vdots \\ \Psi_n, D_1^*\Psi_n,..., D_s^*\Psi_n \end{pmatrix}.$$

Next we show that on a neighbourhood $\tilde{N}_1$ of $z_0 \in Z$,

(2.4) $$[\phi][T^*\Psi]=0$$

holds. First the equalities (2.1) and (2.3) imply that

(2.5) $$[\phi][\Psi]=[\phi][\psi][u]=0.$$

Since β is smooth at $z_0 \in Z$, D_j^* can be extended to the vector field $\tilde{D}_j^*$ on $\tilde{N}_1$ for every $j = 1, 2,..., s$. In view of the fact $(\partial\phi_i/\partial u_j) = 0$ $(i = 0, 1,..., n, j = 2, 3,..., n - r)$ and the equality (2.2), we see that for every vector field $\tilde{D}$ on $\tilde{N}_1$

$$[\tilde{D}\phi][\psi]=0$$

with $[\tilde{D}\phi] = (\tilde{D}\phi_0, \tilde{D}\phi_1, \ldots, \tilde{D}\phi_n)$. Thus in particular

(2.6) $\qquad [\tilde{D}_j^* \phi][\Psi] = [\tilde{D}_j^* \phi][\psi][u] = 0, \qquad j = 1, 2, \ldots, s$.

Differentiating the equality (2.5) by $\tilde{D}_j^*$, we obtain that

$$[\tilde{D}_j^* \phi][\Psi] + [\phi][\tilde{D}_j^* \Psi] = 0 .$$

This equality and (2.6) yield

$$[\phi][\tilde{D}_j^* \Psi] = 0, \qquad j = 1, 2, \ldots, s .$$

The last equality and (2.5) conclude the equality (2.4).

By (2.4) the value at $z \in \beta^{-1}(y_0) \cap \tilde{N}_1$ satisfies

(2.7) $\qquad [\phi(\alpha(z))][(T^*\Psi)(y_0)] = 0 .$

Let C be the image of the morphism

$$\alpha|_{\beta^{-1}(y_0) \cap \tilde{N}_1} : \beta^{-1}(y_0) \cap \tilde{N}_1 \longrightarrow X$$

which may be assumed to be isomorphic. Let S be the linear subspace defined by the $(s+1)$ linear equalities whose coefficients are one of the column vectors of $[(T^*\Psi)(y_0)]$. The equality (2.7) shows $C \subset S$.

Since the rank of the matrix $[T^*\Psi]$ is $s+1$, $\dim S = n - 1 - s$. The dimension of C is not less than $n - 1 - s = \dim Z - \dim Y$ because β is surjective and $\alpha|_{\beta^{-1}(y_0) \cap \tilde{N}_1}$ is assumed to be isomorphic. Thus for a sufficiently small nieghbourhood N_3 of $\pi(x_0)$, $C \cap N_3 = S \cap N_3$. Since z_0 is an arbitrary point of $\beta^{-1}(y_0) \cap \alpha^{-1}(U_1)$, this concludes that the y_0-contact locus C_{y_0} i.e., the closure of the set of points of U_1 whose tangent spaces are contained in the hyperplane corresponding to y_0 is the dual linear space of the tangent space of Y at y_0.

Consider the restriction of the fibre space $\beta: Z \rightarrow Y$ to the open set $\tilde{U}_1 = \alpha^{-1}(U_1) \cap \beta^{-1}(U_1^*)$. Its fibre at a point $y \in U_2^*$ is a dense subset of all hyperplanes containing the tangent space T_y of Y at y. Hence the relation between X and the fibre space $\alpha|_{\tilde{U}_1} : \tilde{U}_1 \rightarrow X$ is the same as between Y and the fibre space $\beta|_{\tilde{U}_1} : \tilde{U}_1 \rightarrow Y$, which conclude (1), X is the dual variety of Y. Since the relation between α and β still hold only under assumption (1) and (2.1) and α is obviously generically smooth, we can conclude (3).

Applying for (Y, α) what we have verified for (X, β) in the above, we can also conclude (2.2). $\qquad\qquad$ Q. E. D.

§3. The Main Theorem

Let $X \subset P(V)$ be a variety of dimension r in a projective space of dimension n. For every integer k with $0 \leq k \leq r$, let

$$[M_k] \in A_{r-k}X$$

be the k-th polar class of X. In particular recall $[M_0] = [X]$.

Notation 3.1. $\int \alpha$ denotes the degree of the 0-dimensional component of the element α of the Chow homology group. That is,

$$\int \alpha = s_* \alpha \in A_0 \operatorname{Spec}(k) = \mathbf{Z}$$

where $s: X \to \operatorname{Spec}(k)$ denotes the structure morphism.

Definition 3.2. The k-th class μ_k of X is defined as the degree of the k-th polar class $[M_k]$, that is, $\mu_k = \int c_1(L)^{r-k} \cap [M_k]$ with $L = \mathcal{O}_{P(V)}(1)|_X$ for an integer k with $0 \leq k \leq r$, and $\mu_k = 0$ if $k < 0$ or $k > r$. We note that $\mu_0 = $ degree of X.

Main Theorem 3.3. *Let $X \subset P^n$ be an r-dimensional projective variety in an n-dimensional projective space. Assume that biduality holds for X. Let Y be the dual variety of X, and let s be its dimension. Let μ_k denote the k-th class of X and let v_j denote the j-th class of Y. Then the inequality $r + s \geq n - 1$ holds and if $k + j = r + s - n + 1$, then $\mu_k = v_j$.*

Whole this section is devoted to verify Main Theorem 3.3.

Following Section 2, we construct $\bar{X}$ the closure of the graph of the Gauss map, Z and morphisms α, $\bar{\alpha}$, π, β. Applying the same procedure to Y, we obtain $\bar{Y}$, W and morphisms γ, $\bar{\gamma}$, ρ, δ.
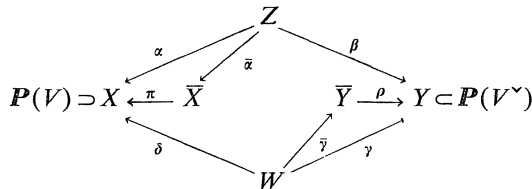

Figure 3.1.

Closely reexaming the proof of Theorem 2.4, we know that the open set $\tilde{U}_1 = \alpha^{-1}(U_1) \cap \beta^{-1}(U_1^*) \subset Z$ is isomorphic to the open set $\hat{U}_1 = \delta^{-1}(U_1) \cap \gamma^{-1}(U_1^*) \subset W$. (Of course $U_1 = U$, if $\bar{X}$ is the closure of the graph of the Gauss

map.) And moreover there exists an isomorphism $\theta: \tilde{U}_1 \to \hat{U}_1$ such that $\delta\theta = \alpha$, $\gamma\theta = \beta$.

Set $L = \mathcal{O}_{P(V)}(1)|_X$, $L^* = \mathcal{O}_{P(V^\vee)}(1)|_Y$. Let P, K be the same as in the beginning of Section 2. By Proposition 1.1, for every integer k with $0 \leq k \leq r$,

$$[M_k] = \pi_*(c_k(P) \cap [\bar{X}]).$$

Since $c(P) = c(K)^{-1} = s(K^\vee)$,

$$\begin{aligned} c_k(P) \cap [\bar{X}] &= s_k(K^\vee) \cap [\bar{X}] \\ &= \bar{\alpha}_*(c_1(\mathcal{O}_{P(K)}(1))^{n-r-1+k} \cap [Z]) \\ &= \bar{\alpha}_*(\beta^* c_1(L^*)^{n-r-1+k} \cap [Z]). \end{aligned}$$

Hence

$$(3.1) \qquad \begin{aligned} \mu_k &= \int c_1(L)^{r-k} \cap [M_k] \\ &= \int c_1(L)^{r-k} \cap \pi_*(c_k(P) \cap [\bar{X}]) \\ &= \int c_1(L)^{r-k} \cap \pi_* \bar{\alpha}_*(\beta^* c_1(L^*)^{n-r-1+k} \cap [Z]) \\ &= \int \alpha^* c_1(L)^{r-k} \cdot \beta^* c_1(L^*)^{n-r-1+k} \cap [Z]. \end{aligned}$$

For an integer k with $k < 0$, since

$$\alpha_*(\beta^* c_1(L^*)^{n-r-1+k} \cap [Z]) \in A_{r+(-k)} X = 0,$$

we have

$$\begin{aligned} \int \alpha^* c_1(L)^{r-k} \cdot \beta^* c_1(L^*)^{n-r-1+k} \cap [Z] \\ = \int c_1(L)^{r-k} \cap \alpha_*(\beta^* c_1(L^*)^{n-r-1+k} \cap [Z]) \\ = 0 = \mu_k. \end{aligned}$$

Set $l = r + s - n + 1$. We know similarly that for an integer k with $r - n + 1 \leq k \leq r$,

$$(3.2) \qquad \nu_{l-k} = \int \delta^* c_1(L)^{r-k} \cdot \gamma^* c_1(L^*)^{n-r-1+k} \cap [W].$$

Consider

$$\begin{aligned} \zeta &= (\alpha, \beta): Z \longrightarrow \mathbb{P}(V) \times \mathbb{P}(V^\vee), \\ \eta &= (\delta, \gamma): W \longrightarrow \mathbb{P}(V) \times \mathbb{P}(V^\vee). \end{aligned}$$

Let S_1 be a linear subspace of $\mathbb{P}(V)$ of dimension $n - r + k$, and S_2 be a linear

subspace of $\boldsymbol{P}(V)$ of dimension $r+1-k$.   The equalities (3.1) and (3.2) show

$$\mu_k = \int \zeta^*[S_1 \times S_2],$$

$$\nu_{l-k} = \int \eta^*[S_1 \times S_2].$$

By Kleiman's Transversality Lemma (Kleiman [2]), we may assume that for general S_1, S_2, the supports of $M = \zeta^{-1}(S_1 \times S_2)$ and $N = \eta^{-1}(S_1 \times S_2)$ (M and N are regarded as schemes,) have dimension 0 and $M \subset \tilde{U}_1$, $N \subset \hat{U}_1$ since for example the intersection $(Z \backslash \tilde{U}_1) \cap M$ is proper and necessarily empty.   By Corollary 11 in Kleiman [2], we can assume moreover that every point in their supports has multiplicity 1, which implies that the regular sequence generating the defining ideal of $S_1 \times S_2$ in $\mathcal{O}_{\boldsymbol{P}(V) \times \boldsymbol{P}(V^\vee)}$ is still a regular sequence on Z or W.   Thus all higher Tor-modules appearing in the definition of $\zeta^*[S_1 \times S_2]$ and $\eta^*[S_1 \times S_2]$ vanish.   We have

$$\mu_k = \text{the number of points in} \quad \zeta^{-1}(S_1 \times S_2),$$
$$\nu_{l-k} = \text{the number of points in} \quad \eta^{-1}(S_1 \times S_2).$$

Since $\tilde{U}_1$ and $\hat{U}_1$ are isomorphic through θ, we conclude $\mu_k = \nu_{l-k}$.   As for the inequality $r+s \geq n-1$, see Corollary 2.5.   Q. E. D.

Corollary 3.4.   *Under the same assumption as in Theorem 4.2,*

$$\dim Y = n - 1 - r + \max\{k \mid \mu_k \neq 0\}.$$

Proof.   Since $\mu_{r+s-n+1} = \nu_0 \neq 0$ and   since $\mu_k = 0$ if $k > r+s-n+1$ by Theorem 3.3, it is obvious.   Q. E. D.

Remark 3.5.   The assertion in Wallace [6] Section 2.1 is false.   It asserts that for a generic point $x \in X$ and a generic hyperplane H such that H contains the tangent space T_x of X at x, $\{x' \in U \mid T_{x'} = T_x\} = \{x' \in U \mid T_{x'} \subset H\}$.   Here U is the smooth part of X.   As a counter-example, choose $X = \boldsymbol{P}^1 \times \boldsymbol{P}^2 \hookrightarrow \boldsymbol{P}^5$ (the Veronese embedding).   Calculation shows that the left-hand side is a single point $\{x\}$, but the right-hand side is a line.

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