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On the Order of Certain Elements of $J(X)$ and the Adams Conjecture

By

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§1. Introduction

The Adams conjecture [2] was proved by several mathematicians in different methods (cf. [7], [8], [9], [10], [14], [15] and [19]). But in their methods, the localization plays an important role and so we cannot estimate the order of an element

$$J\circ(\psi^k-1)(x).$$

Let η_n be the canonical (complex) line bundle over CP^n and k an integer. Let $m(n, k)$ be the minimal positive integer such that

$$k^{m(n,k)}J\circ(\psi^k-1)(\eta_n)=0,$$

which exists by the Adams conjecture for complex line bundles [2]. We put

$$e(n, k)=m([n/2], k).$$

Then the purpose of this paper is to show

Theorem 1. *If X is an n-dimensional CW complex, then*

$$k^{e(n,k)}J\circ(\psi^k-1)(x)=0$$

for any $x\in K(X)$.

On the other hand let

$$e'(n, k)=\begin{cases} e(n, k) & \text{if } k \text{ is odd} \\ e(n, k)+1 & \text{if } k \text{ is even}. \end{cases}$$

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Then by a quite similar method, we have

Theorem 2. *If X is an n-dimensional CW complex, then*

$$k^{e'(n,k)}J\circ(\psi^k-1)(x)=0$$

for any element $x\in KO(X)$.

To prove the above theorems, we do not use the Adams conjecture for general vector bundles. So as a corollary of Theorem 2, the Adams conjecture is proved. The proof of the above theorems is similar to the proof of the Adams conjecture of Nishida [14] and Hashimoto [10]. But we use relations between the induction homomorphisms and the Adams operations in [12] instead of the localization. We also use the cellular approximation of the Becker-Gottlieb transfer used by Sigrist and Suter in [18] instead of the usual Becker-Gottlieb transfer [8].

The paper is organized as follows: In Section 2 some properties of the Becker-Gottlieb transfer are reviewed. Theorem 1 and Theorem 2 are proved in Section 3 and Section 4 respectively. A property of the real induction homomorphism used in this paper is proved in Appendix.

By a quite similar method to the proof of Theorem 1, we can prove Theorem 1 of Sigrist and Suter [18].

§ 2. Properties of the Becker-Gottlieb Transfer

In this section X is an n-dimensional finite cell complex, $\boldsymbol{G}$ is a compact Lie group and $\boldsymbol{H}$ is a closed subgroup of $\boldsymbol{G}$. Let E be the total space of a principal $\boldsymbol{G}$-bundle over X. Then $p: E/\boldsymbol{H}\to X$ is a fibre bundle whose fibre is a compact smooth manifold $\boldsymbol{G}/\boldsymbol{H}$ and whose structure group is a compact Lie group $\boldsymbol{G}$ acting smoothly on $\boldsymbol{G}/\boldsymbol{H}$. Let $t(p): (E/\boldsymbol{H})_+\to X_+$ be the s-map defined by Becker and Gottlieb in [8]. Since X_+ and $(E/\boldsymbol{H})_+$ are finite complexes, $t(p)$ is represented by a map

$$t: \Sigma^l\wedge X_+\longrightarrow \Sigma^l\wedge(E/\boldsymbol{H})_+$$

for some l. Let $(E/\boldsymbol{H})^{(n)}$ be the n-skelton of $E/\boldsymbol{H}$ (for some cellular decomposition) and $j: (E/\boldsymbol{H})^{(n)}\subset E/\boldsymbol{H}$ be the inclusion. Then by the cellular approximation theorem, there is a map

$$t': \Sigma^l\wedge X_+\longrightarrow \Sigma^l\wedge((E/\boldsymbol{H})^{(n)})_+$$

such that

$$\begin{array}{ccc} \Sigma^l \wedge X_+ & \xrightarrow{t} & \Sigma^l \wedge (E/\boldsymbol{H})_+ \\ & {\scriptstyle t'}\searrow \quad \nearrow{\scriptstyle \Sigma^l \wedge j} & \\ & \Sigma^l \wedge ((E/\boldsymbol{H})^{(n)})_+ & \end{array}$$

commutes up to homotopy. Define $p'_!$ by the commutative diagram:

$$\begin{array}{ccccc} K((E/\boldsymbol{H})^{(n)}) & \xrightarrow{=} & \tilde{K}^0(((E/\boldsymbol{H})^{(n)})_+) & \xrightarrow{\sigma} & \tilde{K}^l(\Sigma^l \wedge ((E/\boldsymbol{H})^{(n)})_+) \\ \downarrow{\scriptstyle p'_!} & & & & \downarrow{\scriptstyle t'^*} \\ K(X) & \xrightarrow{=} & \tilde{K}^0(X_+) & \xrightarrow{\sigma} & \tilde{K}^l(\Sigma^l \wedge X_+) \end{array}$$

where σ is the suspension isomorphism defined by the Bott periodicity theorem ([4]). The Becker-Gottlieb transfer $p_!$: $K(E)\to K(X)$ is defined by a similar way. Then by definitions the following diagram is commutative:

$$\begin{array}{ccc} K((E/\boldsymbol{H})^{(n)}) & \xleftarrow{j^*} & K(E/H) \\ & {\scriptstyle p'_!}\searrow \quad \swarrow{\scriptstyle p_!} & \\ & K(X) & . \end{array}$$

Let V be a complex $\boldsymbol{H}$-module and α: $R(\boldsymbol{H})\to K(E/\boldsymbol{H})$ be a homomorphism defined by $V\to(E\times_{\boldsymbol{H}}V\to E/\boldsymbol{H})$. Define

$$\alpha' : R(\boldsymbol{H}) \longrightarrow K((E/\boldsymbol{H})^{(n)})$$

by $\alpha'=j^*\circ\alpha$. Then we have

Lemma 2.1. *The following diagram is commutative*:

$$\begin{array}{ccc} R(\boldsymbol{H}) & \xrightarrow{\alpha'} & K((E/\boldsymbol{H})^{(n)}) \\ \downarrow{\scriptstyle \mathrm{Ind}_{\boldsymbol{H}}^{\boldsymbol{G}}} & & \downarrow{\scriptstyle p'_!} \\ R(\boldsymbol{G}) & \xrightarrow{\alpha} & K(X)\,, \end{array}$$

where $\mathrm{Ind}_{\boldsymbol{H}}^{\boldsymbol{G}}$ *is the induction homomorphism defined by Segal* [16] (*see also* [10]).

Proof. This is an easy consequence of the commutative diagram

$$\begin{array}{ccc} R(\boldsymbol{H}) & \xrightarrow{\alpha} & K(E/\boldsymbol{H}) \\ \downarrow{\scriptstyle \mathrm{Ind}_{\boldsymbol{H}}^{\boldsymbol{G}}} & & \downarrow{\scriptstyle p_!} \\ R(\boldsymbol{G}) & \xrightarrow{\alpha} & K(X) \end{array}$$

which is Proposition 5.4 of Nishida [14].

Let $\widetilde{Sph}^*(\ )$ be the generalized cohomology theory defined by the stable spherical fibrations and $Sph(X)=\widetilde{Sph}^0(X_+)$. Define

$$p'_*: K((E/\boldsymbol{H})^{(n)}) \longrightarrow K(X)$$

and

$$p'_*: Sph((E/\boldsymbol{H})^{(n)}) \longrightarrow Sph(X)$$

by a similar way to $p'_!$ using the suspension isomorphisms defined by the infinite loop space structures defined by the Γ-structures (cf. Segal [17]). Since J is an infinite loop map with respect to these infinite loop space structures, we have (cf. Nishida [14]).

Lemma 2.2. *The following diagram is commutative*:

$$\begin{array}{ccc} K((E/\boldsymbol{H})^{(n)}) & \xrightarrow{J} & Sph((E/\boldsymbol{H})^{(n)}) \\ \downarrow{\scriptstyle p'_*} & & \downarrow{\scriptstyle p'_*} \\ K(X) & \xrightarrow{J} & Sph(X)\,. \end{array}$$

By May [13], the infinite loop space structure of $\boldsymbol{BU}\times\boldsymbol{Z}$ defined by the Γ-structure is equivalent to that defined by the Bott periodicity theorem. Then $p'_!=p'_*$ and so we have

Theorem 2.3. *The diagram*

$$\begin{array}{ccccc} R(\boldsymbol{H}) & \xrightarrow{\alpha'} & K((E/\boldsymbol{H})^{(n)}) & \xrightarrow{J} & Sph((E/\boldsymbol{H})^{(n)}) \\ \downarrow{\scriptstyle \mathrm{Ind}_{\boldsymbol{H}}^{\boldsymbol{G}}} & & \downarrow{\scriptstyle p'_*} & & \downarrow{\scriptstyle p'_*} \\ R(\boldsymbol{G}) & \xrightarrow{\alpha} & K(X) & \xrightarrow{J} & Sph(X) \end{array}$$

is commutative.

Quite similarly we have (cf. Hashimoto [10])

Theorem 2.4. *The diagram*

$$\begin{array}{ccccc} RO(\boldsymbol{H}) & \xrightarrow{\alpha'} & KO((E/\boldsymbol{H})^{(n)}) & \xrightarrow{J} & Sph((E/\boldsymbol{H})^{(n)}) \\ \downarrow{\scriptstyle \mathrm{Ind}_{\boldsymbol{H}}^{\boldsymbol{G}}} & & \downarrow{\scriptstyle p'_*} & & \downarrow{\scriptstyle p'_*} \\ RO(\boldsymbol{G}) & \xrightarrow{\alpha} & KO(X) & \xrightarrow{J} & Sph(X) \end{array}$$

is commutative where $\mathrm{Ind}_{\boldsymbol{H}}^{\boldsymbol{G}}$ is the induction homomorphism of real representation rings defined by Hashimoto [10].

§3. Proof of Theorem 1

First recall the following lemmas.

Lemma 3.1. *Let $f: Y\to Y'$ be a (continuous) map and $y\in K(Y')$. If*

$k^e J\circ(\psi^k-1)(y)=0$, then $k^e J\circ(\psi^k-1)(f^*(y))=0$.

Proof. This is an easy consequence of the following commutative diagram:

$$\begin{array}{ccc} K(Y') & \xrightarrow{f^*} & K(Y) \\ \downarrow{\scriptstyle J} & & \downarrow{\scriptstyle J} \\ Sph(Y') & \xrightarrow{f^*} & Sph(Y)\,. \end{array}$$

Lemma 3.2. *For any complex line bundle x over an n-dimensional CW complex X,*

$$k^{e(n,k)}J\circ(\psi^k-1)(x)=0\,.$$

Proof. Since $x=f^*(\eta_{[n/2]})$ for some $f: X\to CP^{[n/2]}$, this lemma follows immediately from Lemma 3.1.

To prove Theorem 1, we may assume that X is a finite cell complex by Lemma 3.1, since $BU\times Z$ is skeleton finite (under a suitable cellular decomposition). So from now on X is an n-dimensional finite cell complex.

For any $x\in K(X)$ we may assume that x is an m-dimensional complex vector bundle for some m. Let E be the total space of the associated principal $U(m)$-bundle. Let

$$\beta_m\colon U(1)\times U(m-1)\longrightarrow U(1)$$

be the first projection and

$$\iota_m\colon U(m)\longrightarrow U(m)$$

be the identity map. Put $G=U(m)$ and $H=U(1)\times U(m-1)\subset U(m)$. The following is due to [11] (see also Appendix):

Lemma 3.3. $\mathrm{Ind}_H^G(\beta_m)=\iota_m$.

Note that $\alpha(\iota_m)=x$. Since G is connected we have

Lemma 3.4. *For any integer k, $\psi^k\circ\mathrm{Ind}_H^G=\mathrm{Ind}_H^G\circ\psi^k$.*

A proof is given in [12].

Now we can prove Theorem 1. Note that $\alpha\circ\psi^k=\psi^k\circ\alpha$ and $\alpha'\circ\psi^k=\psi^k\circ\alpha'$ by definitions and

$$\begin{aligned} J\circ(\psi^k-1)(x) &= J\circ(\psi^k-1)(\alpha(\iota_m)) \\ &= J\circ(\psi^k-1)(\alpha(\mathrm{Ind}_H^G(\beta_m))) && \text{(by Lemma 3.3)} \\ &= J\circ\alpha\circ\mathrm{Ind}_H^G\circ(\psi^k-1)(\beta_m) && \text{(by Lemma 3.4)} \end{aligned}$$

$$=p'_{*}\circ J\circ\alpha'\circ(\psi^k-1)(\beta_m) \qquad \text{(by Theorem 2.3)}$$
$$=p'_{*}\circ J\circ(\psi^k-1)\circ\alpha'(\beta_m).$$

Since $\alpha'(\beta_m)$ is a complex line bundle over an n-dimensional finite cell complex $(E/\boldsymbol{H})^{(n)}$,

$$k^{e(n,k)}J\circ(\psi^k-1)(\alpha'(\beta_m))=0$$

by Lemma 3.2. So

$$k^{e(n,k)}J\circ(\psi^k-1)(x)=k^{e(n,k)}p'_{*}\circ J\circ(\psi^k-1)(\alpha'(\beta_m))=0.$$

This completes the proof.

§4. Proof of Theorem 2

Let $r\colon K(X)\to KO(X)$ be the realization homomorphism defined by forgetting complex structures. Then the following lemmas are well known:

Lemma 4.1. $2KO(X)\subset\operatorname{Im} r$.

Lemma 4.2. *The diagram*

$$\begin{array}{ccc} K(X) & \xrightarrow{r} & KO(X) \\ & {\scriptstyle J}\searrow \quad \swarrow{\scriptstyle J} & \\ & Sph(X) & \end{array}$$

is commutative.

If k is even, then $kx\in\operatorname{Im} r$ for any $x\in KO(X)$. So $k^{e'(n,k)}J\circ(\psi^k-1)(x)$ $=k^{e(n,k)}J\circ(\psi^k-1)(kx)=0$ by Theorem 1.

From now on k is an odd integer. First we prove

Lemma 4.3. *If X is an n-dimensional CW complex and $x\in KO(X)$ is a linear combination of one or two dimensional real vector bundles, then*

$$k^{e(n,k)}J\circ(\psi^k-1)(x)=0.$$

Proof. By Theorem 1, Lemma 4.1 and Lemma 4.2,

$$2k^{e(n,k)}J\circ(\psi^k-1)(x)=k^{e(n,k)}J\circ(\psi^k-1)(2x)=0.$$

But by the Adams conjecture for one or two dimensional real vector bundles [2], $J\circ(\psi^k-1)(x)$ is an odd torsion. This completes the proof. Q.E.D.

Lemma 4.4. *Let $\boldsymbol{G}$ be a compact Lie group and $\boldsymbol{H}$ be its closed subgroup.*

If $(|G/G^0|, k)=1$ *(G^0 denotes the connected component of the identity), then*

$$\psi^k \circ \mathrm{Ind}_H^G = \mathrm{Ind}_H^G \circ \psi^k : \quad RO(H) \longrightarrow RO(G).$$

A proof is given in Appendix.

In particular we have

Corollary 4.5. *If* $G=O(2n+1)$ *and* $H=O(2)\times O(2n-1)\subset O(2n+1)$, *then* $\psi^k \circ \mathrm{Ind}_H^G = \mathrm{Ind}_H^G \circ \psi^k$ *for any odd integer k.*

Let ι be the identity of G, $\nu: H\to O(2)$ be the first projection and $\mu: G\to O(1)$ be the determinant (cf. Hashimoto [10]). Then the following is Proposition 5 of [10]:

Lemma 4.6. $\iota = \mathrm{Ind}_H^G(\nu)+\mu$.

Now using Lemma 4.3, Lemma 4.6 and Theorem 2.4 instead of Lemma 3.2, Lemma 3.3 and Theorem 2.3 respectively, we can prove Theorem 2 by a similar way.

Remark 4.7. We can prove Theorem 1 of Sigrist and Suter [18] by making use of Theorem 2.4 and Lemma 4.6. In the proof of [18], the fact that s-map induces a homomorphism of J'' ([2]) is not clear, since s-map does not commute with the Adams operations. Moreover the Atiyah transfer does not commute with the Adams operations. The fact that the Atiyah transfer coincides with the Becker-Gottlieb transfer, which is an easy consequence of the Atiyah-Singer index theorem for elliptic families ([6]), seems to be necessary.

Appendix

Let G be a compact Real Lie group and $RR(G)$ be the Real representation ring. If we forget involutions, a homomorphism $r: RR(G)\to R(G)$ is defined. As is well known r is a monomorphism (cf. Atiyah-Segal [5]). Moreover we know the diagram

$$\begin{array}{ccc} RR(G) & \xrightarrow{r} & R(G) \\ \downarrow{\scriptstyle \psi^k} & & \downarrow{\scriptstyle \psi^k} \\ RR(G) & \xrightarrow{r} & R(G) \end{array}$$

is commutative. Let H be a Real subgroup of G and Ind_H^G be the induction homomorphism defined by Hashimoto [10]. Then the diagram

$$\begin{array}{ccc} RR(\boldsymbol{H}) & \xrightarrow{r} & R(\boldsymbol{H}) \\ \downarrow{\scriptstyle \mathrm{Ind}_{\boldsymbol{H}}^{\boldsymbol{G}}} & & \downarrow{\scriptstyle \mathrm{Ind}_{\boldsymbol{H}}^{\boldsymbol{G}}} \\ RR(\boldsymbol{G}) & \xrightarrow{r} & R(\boldsymbol{G}) \end{array}$$

is commutative (cf. [10]). Now applying Theorem 1 of [12], we have

Lemma A.1. *If* $(|\boldsymbol{G}/\boldsymbol{G}^0|, k)=1$, *then*

$$\psi^k \circ \mathrm{Ind}_{\boldsymbol{H}}^{\boldsymbol{G}} = \mathrm{Ind}_{\boldsymbol{H}}^{\boldsymbol{G}} \circ \psi^k \colon RR(\boldsymbol{H}) \longrightarrow RR(\boldsymbol{G}).$$

If the involution of $\boldsymbol{G}$ is trivial, then $RR(\boldsymbol{G})=RO(\boldsymbol{G})$ and ψ^k and $\mathrm{Ind}_{\boldsymbol{H}}^{\boldsymbol{G}}$ on $RO(\ )$ coincide with those on $RR(\ )$. So Lemma 4.4 is proved.

References

[1] Adams, J. F., Vector fields on spheres, *Ann Math.*, **75** (1962), 603–632.
[2] ———, On the groups $J(X)$ I, *Topology*, **2** (1963), 181–195.
[3] ———, *Lecture on Lie groups*, Benjamin, 1969.
[4] Atiyah, M. F., *K-theory*, Benjamin, 1967.
[5] Atiyah, M. F, and Segal, G. B., Equivariant K-theory and completions, *J. Differential Geometry*, **3** (1969), 1–18.
[6] Atiyah, M. F. and Singer, I. M., The index of elliptic operators IV, *Ann. Math.*, **93** (1971), 119–138.
[7] Becker, J. C., Characteristic classes and K-theory, *Lecture Notes in Math.*, **428**, Springer, 132–143.
[8] Becker, J. C. and Gottlieb, G. H., The transfer maps and fibre bundles, *Topology*, **14** (1975), 1–12.
[9] Friedlander, E., Fibrations in etale homotopy theory, *Publ. I.H.E.S.*, **42** (1972), 281–322.
[10] Hashimoto, S., The transfer map in KR_G-theory (to appear in *Osaka J. Math.*).
[11] Kono, A., Segal-Becker theorem for KR-theory, *Japan J. Math.*, **7** (1981), 195–199.
[12] ———, Induced representations of compact Lie groups and the Adams operations, *Publ. RIMS, Kyoto Univ.*, **17** (1981), 553–556.
[13] May, J. P., E_∞ ring spaces and E_∞ ring spectra, *Lecture Notes in Math.*, **577**, Springer.
[14] Nishida. G. The tranfer homomorphism in equivariant generalized cohomology theories, *J. Math. Kyoto Univ.*, **18** (1978), 435–451.
[15] Quillen, D., The Adams conjecture, *Topology*, **10** (1971), 67–80.
[16] Segal, G. B., The representation ring of a compact Lie group, *Publ. I.H.E.S.*, **34** (1968), 113–128.
[17] ———, Categories and cohomology theories, *Topology*, **13** (1974), 293–312.
[18] Sigrist, F. and Suter, U., On the exponent and the order of the group $\tilde{J}(X)$, *Lecture Notes in Math.*, **673**, Springer, 116–122.
[19] Sullivan, D., Genetics of homotopy theory and the Adams conjecture, *Ann. Math.*, **100** (1974), 1–79.