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On v-Sufficiency and $(\bar{h})$-Regularity

By

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§0. Introduction

In local differential analysis, one of the most fundamental problem is to determine the local topological picture of the variety of C^k-map-germ f: $(\boldsymbol{R}^n, 0)\to(\boldsymbol{R}^p, 0)$, with $n\geqq p$, near $0\in\boldsymbol{R}^n$, where $k=1, 2,\dots, \infty, \omega$, as R. Thom stated in [5]. We may expand f into Taylor's series up to degree k. Then, a natural problem is to find the smallest integer r $(r\leqq k)$ such that all terms of degree $>r$ can be omitted without changing the local topological picture of the set-germ $f^{-1}(0)$ at $0\in\boldsymbol{R}^n$. Thus, T. C. Kuo ([3]) introduced the notion of v-sufficiency of jets.

Let $\mathscr{E}_{[k]}(n, p)$ denote the vector space of germs of C^k-mappings $(\boldsymbol{R}^n, 0)\to(\boldsymbol{R}^p, 0)$, where $k=1, 2,\dots, \infty, \omega$. For a map-germ $f\in\mathscr{E}_{[k]}(n, p)$, $j^r(f)$ denotes an *r-jet* of f, and $J^r(n, p)$ denotes the set of all jets, where $r\leqq k$. For two map-germs $f, g\in\mathscr{E}_{[k]}(n, p)$, they are said to be *v-equivalent* at $0\in\boldsymbol{R}^n$ (where "v" stands for "variety") or $f^{-1}(0)$ and $g^{-1}(0)$ have *the same local topological picture* near $0\in\boldsymbol{R}^n$, if there exists a local homeomorphism σ: $(\boldsymbol{R}^n, 0)\to(\boldsymbol{R}^n, 0)$ such that $\sigma(f^{-1}(0))=g^{-1}(0)$. An r-jet $w\in J^r(n, p)$ is said to be *v-sufficient* in $\mathscr{E}_{[k]}(n, p)$, $k=r, r+1,\dots, \infty, \omega$, if for any two C^k-realizations f and g, they are v-equivalent at $0\in\boldsymbol{R}^n$.

In the case where $k=r, r+1$, an analytic criterion of v-sufficiency for C^k-realizations has been obtained by T. C. Kuo ([3]). But, in the case where $k=r+2, r+3,\dots, \infty, \omega$, no characterization has been known on v-sufficiency for C^k-realizations.

In this paper, we shall introduce the notion of $(\bar{h})$-regularity, and give a geometric characterization in order that an r-jet $w\in J^r(n, p)$ is v-sufficient for C^k-realizations $(k=r+1, r+2,\dots, \infty, \omega)$ in terms of $(\bar{h})$-regularity.

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In introducing the notion of $(\bar{h})$-regularity, we have some hints in T. C. Kuo-Y. C. Lu [4], and D. J. A. Trotman [6].

§1. Statements of the Result

As stated above, concerning v-sufficiency in $\mathscr{E}_{[r]}(n, p)$ or $\mathscr{E}_{[r+1]}(n, p)$, T. C. Kuo has obtained the following result.

Theorem 1 (T. C. Kuo [3]). *For an r-jet $w \in J^r(n, p)$, the following conditions are equivalent.*

(a) *w is v-sufficient in $\mathscr{E}_{[r]}(n, p)$ (resp. in $\mathscr{E}_{[r+1]}(n, p)$).*

(b) *There exists a positive number C (resp. There exist positive numbers C and δ) such that*

$$d(\text{grad } w_1(x), \dots, \text{grad } w_p(x)) \geqq C|x|^{r-1}$$
$$(\textit{resp. } d(\text{grad } w_1(x), \dots, \text{grad } w_p(x)) \geqq C|x|^{r-\delta}),$$

where $x \in H_r(w)$, a horn-neighborhood.

Remark 1 (J. Bochnak and S. Łojasiewicz [1]). Especially, in the case where $p=1$, we can take a neighborhood $|x|<\alpha$ $(\alpha>0)$ instead of a horn-neighborhood.

Definition 1. Let M_1, M_2 be manifolds, $M_1 \supseteqq A_1 \ni a_1$, and $M_2 \supseteqq A_2 \ni a_2$. The germ (A_1, a_1) in M_1 and the germ (A_2, a_2) in M_2 are said to be *topologically equivalent relative to M_1 and M_2*, if there exist a neighborhood U_1 of a_1 in M_1, a neighborhood U_2 of a_2 in M_2, and a homeomorphism $h: (U_1, a_1) \to (U_2, a_2)$ such that $h(A_1 \cap U_1) = A_2 \cap U_2$. Then, we write (A_1, a_1) rel. to $M_1 \cong (A_2, a_2)$ rel. to M_2, and we often omit a_1 and a_2. Especially, in the case where $M_1 = M_2 = \boldsymbol{R}^m$, they are said to be *topologically equivalent*, simply.

Let X, Y be smooth manifolds embedded in $\boldsymbol{R}^m$, and $y \in Y \cap \bar{X}$.

Definition 2. Let S be a submanifold in $\boldsymbol{R}^m$, $\dim S = s = \operatorname{codim} Y$, and $1 \leqq k \leqq \infty$.

(1) X is said to be *(t^k)-regular* over Y at y, if for any C^k-submanifold S which is transversal to Y at y, there exists a neighborhood U of y in $\boldsymbol{R}^m$ such that S is transversal to X in U.

(2) X is said to be *(h^k)-regular* over Y at y, if for any C^k-submanifold S which intersects transversally with Y at y, the topological type of the germ at y

of the intersection of S and X is independent of the choice of S.

(3) X is said to be $(\bar{h}^k)$-*regular* over Y at y, if for any C^k-submanifold S which intersects transversally with Y at y, the topological type relative to S of the germ at y of the intersection of S and X is independent of the choice of S.

By the definition, it is clear that $(\bar{h}^k)$ implies (h^k).

Remark 2. In general, for the case where codim $Y \leqq s \leqq m$, we can think (t)-regularity and (h)-regularity. Then, we say (t_s^k)-*regular* and (h_s^k)-*regular* respectively.

Theorem 2 (D. J. A. Trotman [6]). *For* $1 \leqq k \leqq \infty$,

$$(h_s^k) \text{ implies } (t_s^k), \text{ if } \begin{cases} k=1 \\ \text{or} \\ k>1 \text{ and } s>\text{codim } X. \end{cases}$$

Remark 3. Especially, if dim $X >$ dim Y, (h^k) implies (t^k) $(1 \leqq k \leqq \infty)$.

Now, we introduce the variety V_F, determined by w. Let an r-jet $w \in J^r(n, p)$ be identified as $w=(w_1(x),\dots, w_p(x))$, where $w_i(x)$ are polynomials in $x = (x_1,\dots, x_n)$ of degree r. Consider

$$F(x; \lambda)=(F_1(x; \lambda^{(1)}),\dots, F_p(x; \lambda^{(p)})),$$

where $F_i(x; \lambda^{(i)})=w_i(x)+\sum_{|\alpha|=r} \lambda_\alpha{}^{(i)} x^\alpha$, $1 \leqq i \leqq p$. Here $\alpha=(\alpha_1,\dots, \alpha_n)$ is a multiple index, $|\alpha|=\alpha_1+\cdots+\alpha_n$, $x^\alpha = x_1{}^{\alpha_1}\cdots x_n{}^{\alpha_n}$. The coefficients $(\lambda_\alpha{}^{(i)})$, with a fixed ordering, form an Euclidian space, denoted by Λ. Consider the variety

$$V_F;\ F_1(x; \lambda^{(1)})=0,\dots, F_p(x; \lambda^{(p)})=0$$

in $\boldsymbol{R}^n \times \Lambda$. Then, grad F_i, $1 \leqq i \leqq p$, are linearly independent except the set $\{(x, \lambda) \in \boldsymbol{R}^n \times \Lambda \mid x=0\}$.

For a positive integer s, let π_s: $J^{r+s}(n, p) \to J^r(n, p)$ denote the canonical projection.

Theorem 3 (T. C. Kuo and Y. C. Lu [4]). *The following conditions are equivalent, where* $1 \leqq s < \infty$.

(a) *V_F is (t^s)-regular over Λ at* 0.

(b) *Any jet $z \in \pi_s{}^{-1}(w)$ is v-sufficient in $\mathscr{E}_{[r+s]}(n, p)$.*

(c) *w admits at most a finite number of C^{r+s}-realizations whose germs of varieties at* 0 *are non-homeomorphic.*

Consider the following conditions on a jet $w \in J^r(n, p)$ and a variety V_F:

(S_k) w is v-sufficient in $\mathscr{E}_{[k]}(n, p)$.

(t^s) V_F is (t^s)-regular over Λ at 0.

(h^s) V_F is (h^s)-regular over Λ at 0.

From Theorem 3 and Remark 3, it is easy to see that the following implications hold:

$$\begin{array}{ccccccc}
(S_{r+1}) & \longrightarrow & (S_{r+2}) & \longrightarrow \cdots \longrightarrow & (S_\infty) & \longrightarrow & (S_\omega) \\
\downarrow & & \downarrow & \cdots\cdots & & & \\
(t^1) & \longrightarrow & (t^2) & \longrightarrow \cdots \longrightarrow & (t^\infty) & \longrightarrow & (t^\omega) \\
\uparrow & & \uparrow & \cdots\cdots & \uparrow & & \\
(h^1) & \longrightarrow & (h^2) & \longrightarrow \cdots \longrightarrow & (h^\infty) & \longrightarrow & (h^\omega) .
\end{array}$$

Remark 4. The sets $V_F - \Lambda$ and Λ are semi-analytic submanifolds in $\boldsymbol{R}^n \times \Lambda$. Therefore, we consider (t^ω)-*regularity* and (h^ω)-*regularity* also.

Our purpose in this paper is to show the following theorem, concerning v-sufficiency and (h), $(\bar{h})$-regularity.

Theorem. *Let w be an r-jet in $J^r(n, p)$.*

(I) *The following conditions* (a), (b) *are equivalent.*

(i) *In the case where $s = 1, 2, \ldots$.*

(a) *w is v-sufficient in $\mathscr{E}_{[r+s]}(n, p)$.*

(b) *V_F is $(\bar{h}^s)$-regular over Λ at* 0.

(ii) *In the case where $k = \infty$, or ω.*

(a) *w is v-sufficient in $\mathscr{E}_{[k]}(n, p)$.*

(b) *V_F is $(\bar{h}^k)$-regular over Λ at* 0.

(II) *Especially, in the case where $p \geqq 2$ and $s = 1, 2, \ldots$, the following condition* (c) *is also equivalent.*

(c) *V_F is (h^s)-regular over Λ at* 0.

§2. Proof of the Theorem

Let M_s^n denote a C^s-submanifold $(s = 1, 2, \ldots, \infty, \omega)$ of dimension n in $\boldsymbol{R}^n \times \Lambda$, which contains 0. If M_s^n is transversal to Λ at 0, then there exists a family of C^s-functions $\lambda_\alpha^{(i)}(x)$, $1 \leqq i \leqq p$, $|\alpha| = r$, $\lambda_\alpha^{(i)}(0) = 0$, and M_s^n is defined, near 0, by

$$\lambda_\alpha^{(i)} - \lambda_\alpha^{(i)}(x) = 0, \quad |\alpha| = r, \quad 1 \leqq i \leqq p,$$

in $\boldsymbol{R}^n \times \Lambda$. We shall identify the set $\boldsymbol{R}^n \times \{0\}$ with $\boldsymbol{R}^n$.

Proof of (I). We shall show (I) in the case where $p=1$, as the arguments of the proof in the case where $p \geqq 2$ are quite parallel except the difference of proofs of Lemma 3 and Lemma 3′.

(b)⇒(a). Let $\phi(x)$ be any C^{r+s}-realization (resp. C^∞, C^ω) of w. Expand ϕ into Taylar's series up to degree r,

$$\phi(x)=w(x)+\sum_{|\alpha|=r}\lambda_\alpha(x)x^\alpha,$$

where $\lambda_\alpha(x)$ are C^s-functions (resp. C^∞, C^ω), and $\lambda_\alpha(0)=0$. Therefore, we have $\phi(x)=F(x;\lambda(x))$.

Put $M_s^n=\{(x,\lambda)\in \boldsymbol{R}^n\times\Lambda \mid \lambda_\alpha=\lambda_\alpha(x), |\alpha|=r\}$. Then, M_s^n is a C^s-submanifold (resp. C^∞, C^ω), and M_s^n is transversal to Λ at 0, near $0\in\boldsymbol{R}^n\times\Lambda$. Near $0\in\boldsymbol{R}^n\times\Lambda$, we see that

$$\text{(1)}\qquad \begin{array}{ll} M_s^n \cap V_F=\{(x,\lambda)\in\boldsymbol{R}^n\times\Lambda \mid F(x;\lambda(x))=0\} & \text{rel. to } M_s^n \\ \qquad\wr\Vert & \\ \phi^{-1}(0) \;=\{(x,0)\in\boldsymbol{R}^n\times\Lambda \mid F(x;\lambda(x))=0\} & \text{rel. to } \boldsymbol{R}^n\times\{0\}. \end{array}$$

On the other hand, from the fact that $w(x)=F(x;0)$, we see that

$$\text{(2)}\qquad w^{-1}(0)=V_F\cap\boldsymbol{R}^n\times\{0\}.$$

From (1), (2), and (b), we have $\phi^{-1}(0)\cong w^{-1}(0)$, as germs at $0\in\boldsymbol{R}^n$. Therefore, w is v-sufficient in $\mathscr{E}_{[r+s]}(n,1)$ (resp. $\mathscr{E}_{[\infty]}(n,1)$, $\mathscr{E}_{[\omega]}(n,1)$).

(a)⇒(b). It is easy to see the following lemma by simple calculations.

Lemma 1. *For a family of C^1-functions $\lambda_\alpha(x)$, $|\alpha|=r$, $\lambda_\alpha(0)=0$, there exist positive numbers C, d such that*

$$|\mathrm{grad}\,(\sum_{|\alpha|=r}\lambda_\alpha(x)x^\alpha)|\leqq C|x|^r,\ |x|<\mathrm{d}.$$

Let J be an open interval which contains $I=[0,1]$, and let $w\in J^r(n,1)$. Let $\lambda_\alpha(x)$, $|\alpha|=r$, be the same as Lemma 1. Put

$$F_t(x)=w(x)+t\sum_{|\alpha|=r}\lambda_\alpha(x)x^\alpha \qquad \text{for} \quad t\in J.$$

From the calculation of Lemma 1 and Remark 1 (Theorem 1), we see the following lemma.

Lemma 2. *If a jet $w\in J^r(n,1)$ is v-sufficient in $\mathscr{E}_{[r+1]}(n,1)$, then there exist positive numbers C', d', δ, such that*

$$|\mathrm{grad}\, F_t(x)| \geqq C'|x|^{r-\delta}, \quad |x|<d' \qquad \textit{for any} \quad t \in J.$$

Lemma 3. *Let a jet* $w \in J^r(n, 1)$ *be v-sufficient in* $\mathscr{E}_{[r+1]}(n, 1)$, *and* $\lambda_\alpha(x)$ *be a family of* C^1*-functions for* $|\alpha| = r$ *with* $\lambda_\alpha(0)=0$. *Put*

$$\phi(x) = w(x) + \sum_{|\alpha|=r} \lambda_\alpha(x) x^\alpha .$$

Then, we have $\phi^{-1}(0) \cong w^{-1}(0)$, *as germs at* $0 \in \boldsymbol{R}^n$.

Proof. Put

$$G(x, t) = (1-t)w(x) + t\phi(x) \qquad \text{for} \quad t \in J.$$

Consider the vector field,

$$X(x, t) = \begin{cases} -\dfrac{\partial G}{\partial t} |\mathrm{grad}_x G|^{-2} \mathrm{grad}_x G + \dfrac{\partial}{\partial t} & \text{if } \ x \neq 0, \\ \dfrac{\partial}{\partial t} & \text{if } \ x = 0. \end{cases}$$

We write the vector field X as $X_x + (\partial/\partial t)$. From Lemma 2 and the fact that $\partial G/\partial t = \phi(x) - w(x)$, there exist positive numbers C'', d'', such that

$$(3) \qquad |X_x| \leqq C''|x|^{1+\delta} \qquad |x| < d'' \qquad \text{for} \quad t \in J.$$

Recall the proof that $|\mathrm{grad}\, w(x)| \geqq C|x|^{r-\delta}$ implies v-sufficiency (C^0-sufficiency; cf. T. C. Kuo [2]) in $\mathscr{E}_{[r+1]}(n, 1)$. Then X is C^0, and X is $C^1(C^r)$ outside the t-axis. Therefore, the following properties hold:

(P_1) the integral curve of X is unique outside the t-axis;

(P_2) no integral curve of X can enter the t-axis, and no integral curve of X can leave the t-axis (from (3)).

Thus the flow of X gives the local homeomorphism which we demand.

In our case, (P_2) also holds from (3), though X is not C^1 even outside the t-axis. And so, we do not know whether the flow of X gives the local homeomorphism, or not. But from Lemma 2, $G^{-1}(0) - \{t\text{-axis}\}$ is a C^1-submanifold of dimension n of $\boldsymbol{R}^n \times \boldsymbol{R}$ in the cylinder around the t-axis (or $G^{-1}(0) - \{t\text{-axis}\}$ is empty, then Lemma 3 is trivial). Similarly, $V_1 = w^{-1}(0) - \{(0, 0)\}$ and $V_2 = \phi^{-1}(0) - \{(0, 1)\}$ are C^1-submanifolds of dimension $n-1$.

Consider the flow of X near the t-axis. From (P_2), the flow carries the points of V_1 to the points of V_2. As X is nearly parallel to the t-axis, the integral curve of X which traverses the plane, $t = \alpha$, does not traverse it again. Therefore, if the flow carries the points of different connected components of V_1 to the same connected components of V_2 (or the contrary holds), $G^{-1}(0)-\{t\text{-axis}\}$ is not a submanifold. Hence the flow of X gives one-to-one correspondence between connected components of V_1 and V_2. Thus we have

(Q_1) near $0 \in \boldsymbol{R}^n$, $w^{-1}(0)$ and $\phi^{-1}(0)$ are homeomorphic, as topological spaces (not germs).

As w is a polynomial, we have

(Q_2) the number of components of V_1 is finite, and so is that of V_2.

From the consideration above, we have

(Q_3) $w^{-1}(0)$ and $\phi^{-1}(0)$ are "in the same position" in the following meaning; two flows which start in different components of V_1 never intersect en route. (For example, in Figure 1, W_1 and W_2 are homeomorphic, as topological spaces, but they are not in the same position.)

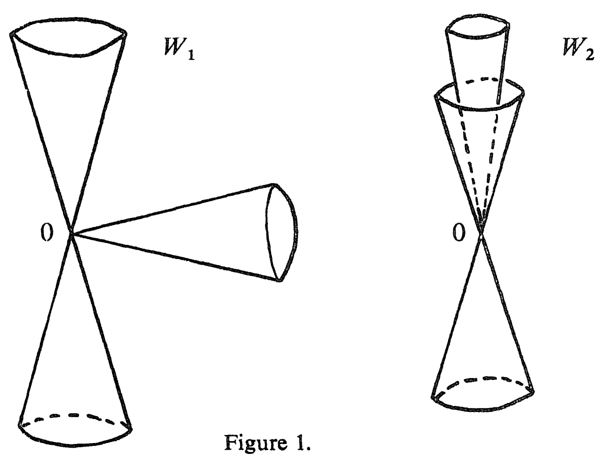

Figure 1.

Put

$$\phi(x)=w(x)+h(x), \quad \text{where} \quad h(x)=\sum_{|\alpha|=r} \lambda_\alpha(x)x^\alpha .$$

From Lemma 1 and Remark 1 (and Lemma 2), we have

$$(4) \qquad \left|\frac{\operatorname{grad} w(x)}{|\operatorname{grad} w(x)|}-\frac{\operatorname{grad} \phi(x)}{|\operatorname{grad} \phi(x)|}\right| \leqq 2\left|\frac{\operatorname{grad} h(x)}{\operatorname{grad} w(x)}\right| \leqq 2\frac{C}{C'}|x|^\delta, \quad 0<|x|<\min(d, d').$$

Suppose that $0<|x|<d''$. Let $(\sigma(x), 1)$ denote the set onto which the flow of X carries $(x, 0)$. For any $y \in \sigma(x)$, put $y=x+\varepsilon_x$. From (3), we have

$$(5) \qquad |\varepsilon_x| \leqq 2C''|x|^{1+\delta}, \quad |x|<d''' .$$

Therefore, we have

$$\left|\frac{x}{|x|}-\frac{y}{|y|}\right|=\left|\frac{x}{|x|}-\frac{x+\varepsilon_x}{|x+\varepsilon_x|}\right|\leqq 2\frac{|\varepsilon_x|}{|x|}\leqq 4C''|x|^{\delta}. \tag{6}$$

Putting $v=\min(d, d', d''')>0$, the inequalities (4), (5) and (6) hold for any x satisfying $0<|x|<v$. Taking v sufficiently small, from (4), (5), and the continuity of $\operatorname{grad}(w+h)(x)/|\operatorname{grad}(w+h)(x)|$, the tangent space $T_x(w^{-1}(0)-\{0\})$ is quite near to $T_y(\phi^{-1}(0)-\{0\})$ (we write $T_x(w^{-1}(0)-\{0\})\approx T_y(\phi^{-1}(0)-\{0\})$) for any $0<|x|<v$.

Here, we introduce the notion of the tangent cone. For an algebraic set V ($\subseteqq \boldsymbol{R}^N$) which contains p, we define *the tangent cone* at p of V, $C(V, p)$, as follows; we shall say that a vector $v\in\boldsymbol{R}^N$ satisfies condition $(*)$, if there exist a sequence $\{x_n\}\to p$ of points of V and a sequence $\{a_n\}$ of real numbers such that $a_n(x_n-p)\to v$. Let $C(V, p)$ be the set of lines $\bar{v}$ through p in $\boldsymbol{R}^n$, whose direction v satisfies $(*)$.

For the variety $\phi^{-1}(0)$, we define *the tangent cone* $C(\phi^{-1}(0), 0)$ as above. For any $\bar{v}\in C(w^{-1}(0), 0)$, there exists a sequence $\{x_n\}$ of points of $w^{-1}(0)$ such that $x_n/|x_n|\to v$. Taking $y_n\in\sigma(x_n)$, from (6), we see that $y_n/|y_n|\to v$. Therefore, $\bar{v}\in C(\phi^{-1}(0), 0)$, and so $C(w^{-1}(0), 0)\subseteqq C(\phi^{-1}(0), 0)$. Considering the flow of the contrary direction, we see that $C(w^{-1}(0), 0)\supseteqq C(\phi^{-1}(0), 0)$. Thus we have

$$C(w^{-1}(0), 0)=C(\phi^{-1}(0), 0). \tag{7}$$

As (Q_1), (Q_2), and (Q_3) hold, from (7) and the fact that $T_x(w^{-1}(0)-\{0\})\approx T_y(\phi^{-1}(0)-\{0\})$, near $0\in\boldsymbol{R}^n$, we can take a set U whose boundary is a cone of an algebraic set and which contains $w^{-1}(0)$ and $\phi^{-1}(0)$, and we can constract a homeomorphism $h: U\to U$ such that $h(w^{-1}(0))=\phi^{-1}(0)$, by using the normal direction of the tangent cone. (For example, in the case where $n=2$, $w^{-1}(0)$ and $\phi^{-1}(0)$ are graphs from the tangent direction to the normal direction as Figure 2.) From (Q_1), (Q_2), (Q_3), and the form of U, we can extend h to the homeomorphism from a neighborhood of $0\in\boldsymbol{R}^n$ to a neighborhood of $0\in\boldsymbol{R}^n$. Thus we have shown $w^{-1}(0)\cong\phi^{-1}(0)$, as germs at $0\in\boldsymbol{R}^n$.

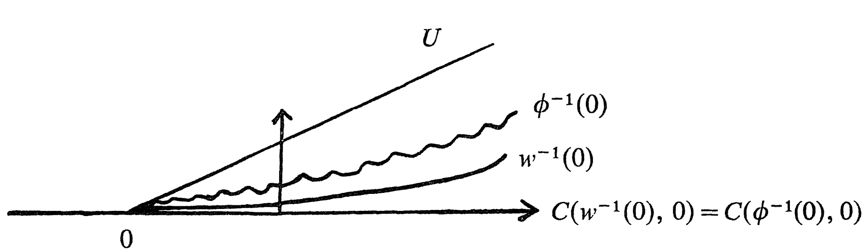

Figure 2.

Lemma 3′. *Let a jet $w \in J^r(n, p)$ be v-sufficient in $\mathscr{E}_{[r+1]}(n, p)$, and $\lambda_\alpha^{(i)}(x)$ $(1 \leqq i \leqq p)$ be a family of C^1-functions for $|\alpha| = r$ with $\lambda_\alpha^{(i)}(0) = 0$. Put*

$$\phi_i(x) = w_i(x) + \sum_{|\alpha|=r} \lambda_\alpha^{(i)}(x) x^\alpha, \quad 1 \leqq i \leqq p,$$

and

$$\phi(x) = (\phi_1(x), \dots, \phi_p(x)).$$

Then, we have $\phi^{-1}(0) \cong w^{-1}(0)$, as germs at $0 \in \mathbf{R}^n$.

Proof. Recall the proof that (b) implies (a) in Theorem 1. In a similar way as Lemma 3, we see that (Q_1) and (Q_2) hold. Here, connected components of $w^{-1}(0) - \{0\}$ and $\phi^{-1}(0) - \{0\}$ are C^1-submanifolds of codimension p in $\mathbf{R}^n$. As $p \geqq 2$, we do not need consider (Q_3). The remainder of the proof follows similarly.

By using Lemma 3, we shall show that (a) implies (b) (it is easy to see that (a) implies $(\bar{h}^{r+s})$).

Let M_s^n, $s = 1, 2, \dots, \infty, \omega$, be a C^s-submanifold transversal to Λ at 0. Then, there exists a family of C^s-functions $\lambda_\alpha(x)$, $|\alpha| = r$, $\lambda_\alpha(0) = 0$, such that near $0 \in \mathbf{R}^n \times \Lambda$,

$$M_s^n = \{(x, \lambda) \in \mathbf{R}^n \times \Lambda \mid \lambda_\alpha = \lambda_\alpha(x) \ (|\alpha| = r)\}.$$

Putting $\phi(x) = F(x; \lambda(x))$, we see that near $0 \in \mathbf{R}^n \times \Lambda$,

$$\begin{aligned} &M_s^n \cap V_F = \{(x, \lambda) \in \mathbf{R}^n \times \Lambda \mid F(x; \lambda(x)) = 0\} &&\text{rel. to } M_s^n \\ (8)\qquad &\qquad \wr\!\| \\ &\phi^{-1}(0) = \{(x, 0) \in \mathbf{R}^n \times \Lambda \mid F(x; \lambda(x)) = 0\} &&\text{rel. to } \mathbf{R}^n \times \{0\}. \end{aligned}$$

Moreover, we have

$$\phi(x) = w(x) + \sum_{|\alpha|=r} \lambda_\alpha(x) x^\alpha.$$

In the case where $s = \infty$ (resp. ω), λ_α is C^∞ (resp. C^ω), and so is ϕ. Therefore, it is clear that (a) implies (b) for $s = \infty, \omega$.

Next, we shall show in the case where $s = 1, 2, \dots$. Expand λ_α into Taylor's series up to degree $s-1$, for $|\alpha| = r$,

$$\lambda_\alpha(x) = v_\alpha(x) + \sum_{|\beta|=s-1} \theta_\beta^\alpha(x) x^\beta,$$

where $v_\alpha(x)$ is a polynomial of degree $s-1$, $\theta_\beta^\alpha(x)$ are C^1-functions, and $\theta_\beta^\alpha(0) = 0$. Therefore, we have

$$\phi(x)=w(x)+\sum_{|\alpha|=r} v_\alpha(x)x^\alpha+\sum_{\substack{|\alpha|=r\\|\beta|=s-1}} \theta^\alpha_\beta(x)x^\alpha x^\beta$$
$$=v(x)+\sum_{|\gamma|=r+s-1}\psi_\gamma(x)x^\gamma,$$

where $v(x)=w(x)+\sum_{|\alpha|=r}v_\alpha(x)x^\alpha$ is a polynomial of degree $r+s-1$, $\psi_\gamma(x)$ are C^1-functions, and $\psi_\gamma(0)=0$.

Remark 5. In the case where $s=1$, (a) implies (b). For, from Lemma 3, we see that

$$\phi^{-1}(0)\cong w^{-1}(0)=\{(x, 0)\in \boldsymbol{R}^n\times\Lambda \mid F(x; 0)=0\}.$$

Hence, V_F is $(\bar{h}^1)$-regular over Λ at 0 from (8).

On the other hand, as w is v-sufficient in $\mathscr{E}_{[r+s]}(n, 1)$, any $z\in\pi_{s-1}^{-1}(w)$ is v-sufficient in $\mathscr{E}_{[r+s]}(n, 1)$, and

(9) $$z^{-1}(0)\cong w^{-1}(0) \quad \text{as germs at} \quad 0\in\boldsymbol{R}^n.$$

Put

$$G(x; \psi)=v(x)+\sum_{|\gamma|=r+s-1}\psi_\gamma x^\gamma,$$

where the coefficients (ψ_γ) form a Euclidean space Γ. For any $z\in J^{r+s-1}(n, 1)$, we define the variety V_{F_z} in a similar way as V_F (cf. § 1). Then, V_{F_z} is $(\bar{h}^1)$-regular over Γ at $0\in\boldsymbol{R}^n\times\Gamma$ from Remark 5. Put

$$'M_1^n=\{(x, \psi)\in\boldsymbol{R}^n\times\Gamma \mid \psi_\gamma=\psi_\gamma(x)\ (|\gamma|=r+s-1)\}.$$

Then, near $0\in\boldsymbol{R}^n\times\Gamma$, $'M_1^n$ is a C^1-submanifold, and $'M_1^n$ is transversal to Γ at 0. Therefore, we have

(10)
$$\begin{array}{ll} v^{-1}(0)=\{(x, 0)\in\boldsymbol{R}^n\times\Gamma \mid G(x; 0)=0\} & \text{rel. to } \boldsymbol{R}^n\times\{0\}\\ \wr\| & \\ 'M_1^n\cap V_{F_z}=\{(x, \psi_\gamma)\in\boldsymbol{R}^n\times\Gamma \mid G(x; \psi_\gamma(x))=0\} & \text{rel. to } 'M_1^n\\ \wr\| & \\ \phi^{-1}(0)=\{(x, 0)\in\boldsymbol{R}^n\times\Gamma \mid G(x; \psi_\gamma(x))=0\} & \text{rel. to } \boldsymbol{R}^n\times\{0\}.\end{array}$$

From (8), (9) and (10), we see that

$$\begin{array}{ll} \phi^{-1}(0)\cong M_s^n\cap V_F & \text{rel. to } M_s^n\\ \wr\| & \\ w^{-1}(0)=\{(x, 0)\in\boldsymbol{R}^n\times\Gamma \mid F(x; 0)=0\} & \text{rel. to } \boldsymbol{R}^n\times\{0\}.\end{array}$$

Thus V_F is $(\bar{h}^s)$-regular over Λ at 0.

Proof of (II). As (b) implies (c), we shall show that (c) implies (a).

Let V_F be (h^s)-regular over Λ at 0. Any $z\in\pi_s^{-1}(w)$ is v-sufficient in $\mathscr{E}_{[r+s]}(n, p)$ from Remark 3 and Theorem 3. Therefore, it is enough to show that

$w^{-1}(0) \cong z^{-1}(0)$, as germs at $0 \in \mathbf{R}^n$ for any $z \in \pi_s^{-1}(w)$. As $z \in \pi_s^{-1}(w)$, we have

$$z_i(x) = w_i(x) + \sum_{|\alpha|=r} \lambda_\alpha^{(i)}(x) x^\alpha, \quad 1 \leqq i \leqq p,$$

where $\lambda_\alpha^{(i)}(x)$ are polynomials of degree s, and $\lambda_\alpha^{(i)}(0)=0$. Put

$$M_1 = \{(x, 0) \in \mathbf{R}^n \times \Lambda\} \text{ and } M_2 = \{(x, \lambda(x)) \in \mathbf{R}^n \times \Lambda\}.$$

M_1 and M_2 are C^s-submanifolds, and they are transversal to Λ at 0. From (c), we have $M_1 \cap V_F \cong M_2 \cap V_F$, as germs at $0 \in \mathbf{R}^n \times \Lambda$. Therefore, we have

$$(11) \qquad M_1 \cap V_F \cong M_2 \cap V_F, \quad \text{as topological spaces.}$$

On the other hand, we see that

$$(12) \qquad \begin{cases} M_1 \cap V_F = w^{-1}(0) \\ M_2 \cap V_F \text{ rel. to } M_2 \quad \cong \quad z^{-1}(0) \quad \text{rel. to } \mathbf{R}^n \times \{0\}. \end{cases}$$

From (11) and (12), we have $w^{-1}(0) \cong z^{-1}(0)$, as topological spaces. And from Theorem 1, $w^{-1}(0) - \{0\}$ and $z^{-1}(0) - \{0\}$ are C^∞-submanifolds of codimension $p \geqq 2$ (or $w^{-1}(0) - \{0\}$ and $z^{-1}(0) - \{0\}$ are empty, then (II) is trivial). Further, $w^{-1}(0)$ and $z^{-1}(0)$ are algebraic sets. Thus, we see that $w^{-1}(0) \cong z^{-1}(0)$, as germs at $0 \in \mathbf{R}^n$.

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