Publ. RIMS, Kyoto Univ.
17 (1981), 553-556

Induced Representations of Compact Lie Groups and the Adams Operations

By

Akira KONO*

§1. Introduction

Let $\boldsymbol{G}$ be a compact Lie group and $R(\boldsymbol{G})$ its (complex) representation ring. Then for any element $x \in R(\boldsymbol{G})$ there are complex representations V and W such that $x = V - W$. Then the virtual character $\chi_x(g)$ $(g \in \boldsymbol{G})$ is defined by

$$\chi_x(g) = \chi_V(g) - \chi_W(g).$$

As is well known $x = y$ in $R(\boldsymbol{G})$ if and only if

$$\chi_x(g) = \chi_y(g)$$

for any element $g \in \boldsymbol{G}$.

The Adams operation

$$\psi^k : R(\boldsymbol{G}) \longrightarrow R(\boldsymbol{G})$$

is a ring homomorphism such that

$$\chi_{\psi^k(x)}(g) = \chi_x(g^k)$$

for any $g \in \boldsymbol{G}$ (cf. Adams [1]). On the other hand let $\boldsymbol{H}$ be a closed subgroup of $\boldsymbol{G}$. Then Segal defined in [8] the induction homomorphism

$$\mathrm{Ind}_{\boldsymbol{H}}^{\boldsymbol{G}} : R(\boldsymbol{H}) \longrightarrow R(\boldsymbol{G})$$

which is characterized by

$$\chi_{\mathrm{Ind}_{\boldsymbol{H}}^{\boldsymbol{G}}(x)}(g) = \sum_{a\boldsymbol{H} \in (\boldsymbol{G}/\boldsymbol{H})^g} \chi_x(a^{-1}ga)$$

for generic g where $(\boldsymbol{G}/\boldsymbol{H})^g = \{a\boldsymbol{H} \in \boldsymbol{G}/\boldsymbol{H};\ ga\boldsymbol{H} = a\boldsymbol{H}\}$ (cf. §2). (In [8], $\mathrm{Ind}_{\boldsymbol{H}}^{\boldsymbol{G}}$ is denoted by $i_!$).

Communicated by N. Shimada, August 2, 1980.
* Department of Mathematics, Kyoto University, Kyoto 606, Japan.
This research was partially supported by the Grant-in-Aid for Scientific Research, (No. 574023), Ministry of Education.

In general $\psi^k \circ \mathrm{Ind}_H^G \neq \mathrm{Ind}_H^G \circ \psi^k$. The purpose of this paper is to show

Theorem 1. *If* $(|G/G^0|, k)=1$ *then*

$$\psi^k \circ \mathrm{Ind}_H^G = \mathrm{Ind}_H^G \circ \psi^k$$

for any closed subgroup H *of* G *where* G^0 *is the connected component of the identity of* G.

The above theorem plays an important role in the proof of the Adams conjecture in [6].

Let G be a compact connected Lie group such that $\pi_1(G)$ is torsion free and U and U' be its closed subgroups such that $U' \subset U$. Consider the Becker-Gottlieb transfer

$$p_!: K(G/U') \longrightarrow K(G/U)$$

for the fibre bundle $U/U' \to G/U' \xrightarrow{p} G/U$ (cf. [5], [7]). Then as a corollary of the above theorem, we have

Corollary 2. *If* U *and* U' *are connected and of maximal rank, then* $p_! \circ \psi^k = \psi^k \circ p_!$.

§2. Proof of Theorem 1

In this section G is a compact Lie group and k is an integer such that $(|G/G^0|, k)=1$. A closed cyclic subgroup C of G is called a Cartan subgroup if and only if $N_G(C)/C$ is a finite group where $N_G(C)$ is the normalizer of C in G. An element $g \in G$ is called generic if and only if it is a generator of a Cartan subgroup of G. Then generic elements are dense in G (for details, see Segal [8]). Moreover the following is remarked in Section 1 of Segal [8].

Lemma 2.1. $|C/C^0|$ *divides* $|G/G^0|^2$.

Let H be a closed subgroup of G then the following is also due to Segal [8]:

Lemma 2.2. *If* g *is generic then*

$$\chi_{\mathrm{Ind}_H^G(x)}(g) = \sum_{aH \in (G/H)^g} \chi_x(a^{-1}ga),$$

where $(G/H)^g = \{aH \in G/H;\ gaH = aH\}$.

Using Lemma 2.1, we can easily prove the following:

Lemma 2.3. *If* g *is a generator of a Cartan subgroup* C *of* G, *then* g^k

is also a generator of C.

Let g be a generic element. Then $a\boldsymbol{H}\in(\boldsymbol{G}/\boldsymbol{H})^g$ if and only if $a^{-1}ga\in\boldsymbol{H}$ which is equivalent to $a^{-1}Ca\subset\boldsymbol{H}$. Quite similarly $a\boldsymbol{H}\in(\boldsymbol{G}/\boldsymbol{H})^{g^k}$ is equivalent to $a^{-1}Ca\subset\boldsymbol{H}$ by Lemma 2.3. So we have

Lemma 2.4. *If g is generic then*

$$(\boldsymbol{G}/\boldsymbol{H})^g=(\boldsymbol{G}/\boldsymbol{H})^{g^k}$$

for any closed subgroup $\boldsymbol{H}$ of $\boldsymbol{G}$.

Now we can prove Theorem 1. If g is generic then by Lemma 2.2,

$$\begin{aligned}\chi_{\mathrm{Ind}_{\boldsymbol{H}}^{\boldsymbol{G}}\circ\psi^k(x)}(g)&=\sum_{a\boldsymbol{H}\in(\boldsymbol{G}/\boldsymbol{H})^g}\chi_{\psi^k(x)}(a^{-1}ga)\\&=\sum_{a\boldsymbol{H}\in(\boldsymbol{G}/\boldsymbol{H})^g}\chi_x(a^{-1}g^ka)\end{aligned}$$

and

$$\begin{aligned}\chi_{\psi^k\circ\mathrm{Ind}_{\boldsymbol{H}}^{\boldsymbol{G}}(x)}(g)&=\chi_{\mathrm{Ind}_{\boldsymbol{H}}^{\boldsymbol{G}}(x)}(g^k)\\&=\sum_{a\boldsymbol{H}\in(\boldsymbol{G}/\boldsymbol{H})^{g^k}}\chi_x(a^{-1}g^ka)\,,\end{aligned}$$

since g^k is also generic by Lemma 2.3. Then applying Lemma 2.4, we have

$$\chi_{\mathrm{Ind}_{\boldsymbol{H}}^{\boldsymbol{G}}\circ\psi^k(x)}(g)=\chi_{\psi^k\circ\mathrm{Ind}_{\boldsymbol{H}}^{\boldsymbol{G}}(x)}(g)$$

for generic g. But since virtual characters are continuous and generic elements are dense in $\boldsymbol{G}$, we have

$$\chi_{\mathrm{Ind}_{\boldsymbol{H}}^{\boldsymbol{G}}\circ\psi^k(x)}(g)=\chi_{\psi^k\circ\mathrm{Ind}_{\boldsymbol{H}}^{\boldsymbol{G}}(x)}(g)$$

for any $g\in\boldsymbol{G}$ and Theorem 1 is proved.

§3. Proof of Corollary 2

In this section $\boldsymbol{G}$ is a compact connected Lie group such that $\pi_1(\boldsymbol{G})$ is torsion free. $\boldsymbol{U}$ and $\boldsymbol{U}'$ are its closed connected subgroups of maximal rank such that $\boldsymbol{U}'\subset\boldsymbol{U}$.

Let V be a complex representation of $\boldsymbol{U}$. The correspondence $V\to\boldsymbol{G}\times_{\boldsymbol{U}}V$ defines a homomorphism

$$\alpha=\alpha(\boldsymbol{G};\,\boldsymbol{U})\colon R(\boldsymbol{U})\longrightarrow K(\boldsymbol{G}/\boldsymbol{U})\,.$$

Since $\boldsymbol{G}$ is a compact free $\boldsymbol{U}$-space, the following diagram commutes by Proposition 5.4 of [7]:

$$\begin{array}{ccc} R(U') & \xrightarrow{\alpha(G;U')} & K(G/U') \\ \big\downarrow \mathrm{Ind}_{U'}^{U} & & \big\downarrow p_! \\ R(U) & \xrightarrow{\alpha(G;U)} & K(G/U)\,, \end{array}$$

where $p_!$ is the Becker-Gottlieb transfer for $p\colon G/U' \to G/U$.

On the other hand if U' is connected and of maximal rank then $\alpha(G;\,U')$ is surjective by the Atiyah-Hirzebruch conjecture (cf. Snaith [9] or Pittie [10]). So to prove $p_! \circ \psi^k = \psi^k \circ p_!$, we need only show $p_! \circ \psi^k \circ \alpha(G;\,U') = \psi^k \circ p_! \circ \alpha(G;\,U')$. Since $\alpha(G;\,U') \circ \psi^k = \psi^k \circ \alpha(G;\,U)$ by definitions and $\mathrm{Ind}_{U'}^{U} \circ \psi^k = \psi^k \circ \mathrm{Ind}_{U'}^{U}$, by Theorem 1, we have

$$\begin{aligned} p_! \circ \psi^k \circ \alpha(G;\,U') &= p_! \circ \alpha(G;\,U') \circ \psi^k \\ &= \alpha(G;\,U) \circ \mathrm{Ind}_{U'}^{U} \circ \psi^k \\ &= \alpha(G;\,U) \circ \psi^k \circ \mathrm{Ind}_{U'}^{U} \\ &= \psi^k \circ \alpha(G;\,U) \circ \mathrm{Ind}_{U'}^{U} \\ &= \psi^k \circ p_! \circ \alpha(G;\,U') \end{aligned}$$

and so Corollary 2 is obtained.

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