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Deformation of Uniruled Manifolds

By

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Let $f: X \to S$ be a proper smooth morphism of complex spaces with connected fibers. Suppose that each point $s \in S$ has a neighborhood $s \in U$ such that there exist a Kähler manifold Y_U, a proper morphism $g: Y_U \to U$ and a generically surjective meromorphic map $h: Y_U \to X_U$ over U. The condition is satisfied, e.g., if each fiber of f is Moishezon or if f is a Kähler morphism in the sense of [1]. Then the main purpose of this note is to show the following (Proposition 2.3): *Under the above condition if X_o is uniruled for some $o \in S$, then X_s is uniruled for all $s \in S$.* (See Theorem 2.4 for a little more general statement.) Thus the class of uniruled manifolds are closed under smooth deformations which is 'weakly polarized' in the sense of the above condition. The method of proof is nothing but a simple adaptation to our situation of the method used by Mori in [8] (cf. Lemma 2.6).

Let R be the set of those points $s \in S$ for which X_s are ruled. Then we also show that, *when restricted to any relatively compact subdomain S' of S, R is a union of at most countably many analytic subsets of S', under the stronger assumption that f is a Moishezon morphism* (Proposition 3.3).

Proposition 2.3 plays an important role in our construction of the coarse moduli space for the polarized family of nonuniruled compact Kähler manifolds in [5].

Convention. In this paper complex spaces are in general assumed to be reduced. Let $f: X \to S$ be a morphism of complex spaces. For any locally closed analytic subspace V of S we set $X_V = X \times_S V$ and $f_V = f|_{X_V}: X_V \to V$. If V consists of a single point $s \in S$ we write X_s instead of $X_{\{s\}}$. For $X = X_\alpha$ we often write $X_{\alpha,s}$ instead of $(X_\alpha)_s$. $\boldsymbol{P}^r$ denotes the complex projective space of complex dimension r.

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§ 1. Preliminaries

The purpose of this section is to fix notations and to summarize some known results on the Douady space of a complex space. The results stated in this section will often be used without further reference in the later sections.

a) Let $h: Y \to S$ be a morphism of complex spaces. Then we denote by $\beta_{Y/S}: D_{Y/S} \to S$ the relative Douady space of Y over S, parametrizing the compact analytic subspaces of X contained in the fibers of h (cf. [1]). Let $\rho_{Y/S}$: $Z_{Y/S} \to D_{Y/S}$ be the corresponding universal family, so that there is a natural embedding $Z_{Y/S} \subseteq D_{Y/S} \times_S Y$ with $\rho_{Y/S}$ induced by the natural projection $D_{Y/S} \times_S Y \to D_{Y/S}$. Recall that $\rho_{Y/S}$ is proper and flat. We denote by $\pi_{Y/S}$ the natural morphism $Z_{Y/S} \to Y$ induced by the other projection $D_{Y/S} \times_S Y \to Y$. Let D_α be any irreducible component of $D_{Y/S,\mathrm{red}}$, the underlying reduced subspace of $D_{Y/S}$. Then we denote by $\rho_\alpha: Z_\alpha \to D_\alpha$ the restriction of $\rho_{Y/S}$ over D_α and by $\pi_\alpha: Z_\alpha \to Y$ the restriction of $\pi_{Y/S}$ to Z_α. Thus we have the following commutative diagram

$$\begin{tikzcd} & Z_\alpha \subseteq D_\alpha \times_S Y \arrow[dl, "\rho_\alpha"'] \arrow[dr, "\pi_\alpha"] & \\ D_\alpha \arrow[dr, "\beta_\alpha"'] & & Y . \arrow[dl, "h"] \\ & S & \end{tikzcd}$$

When S is a point, we shall write $\rho_Y: Z_Y \to D_Y$ and $\pi_Y: Z_Y \to Y$ instead of $\rho_{Y/S}$ and $\pi_{Y/S}$ respectively.

In general for any $s \in S$, $D_{Y/S,s}$ is naturally isomorphic to D_{Y_s}, and we often identify these two spaces.

Let $f: X \to S$ be a morphism of complex spaces. Let $Y = \boldsymbol{P}^1 \times X$ and $h = pf: Y \to S$ where $p: Y \to X$ is the natural projection. Obviously we have $Y_s = \boldsymbol{P}^1 \times X_s$ for all $s \in S$. We set $\bar{\pi}_{Y/S} = p\pi_{Y/S}: Z_{Y/S} \to X$ and $\varphi_{Y/S} = q\pi_{Y/S}: Z_{Y/S} \to \boldsymbol{P}^1$ where $q: Y \to \boldsymbol{P}^1$ is the natural projection. Let D_α be any irreducible component of $D_{Y/S,\mathrm{red}}$ as above. Then we also set $\bar{\pi}_\alpha = p\pi_\alpha: Z_\alpha \to X$ and $\varphi_\alpha = q\pi_\alpha: Z_\alpha \to \boldsymbol{P}^1$.

b) Let X be a compact complex space and $Y = \boldsymbol{P}^1 \times X$. Let $H = \mathrm{Hol}(\boldsymbol{P}^1, X)$ be the set of all holomorphic maps $h: \boldsymbol{P}^1 \to X$ of $\boldsymbol{P}^1$ into X. Associating to each h its graph $\Gamma_h \subseteq \boldsymbol{P}^1 \times X$ we may regard H as a subspace of the Douady space

D_Y of Y. In fact, it is known that H is Zariski open in D_Y with respect to this inclusion. Let H_0 be the subset of H consisting of non-constant holomorphic maps. Then H_0 is again Zariski open in D_Y. Then a point $d \in D_Y$ belongs to H_0 if and only if the induced map $\varphi_{Y,d}: Z_{Y,d} \to \boldsymbol{P}^1$ is isomorphic and $\dim \pi_Y(Z_{Y,d}) = 1$ in X, where $Z_{Y,d} = \rho_Y^{-1}(d)$. This immediately follows from the definition. Let B be an irreducible analytic subspace of $D_{Y,\mathrm{red}}$. Let $\pi_B: Z_B \to B$ and $\varphi_B: Z_B \to \boldsymbol{P}^1$ be induced by π_Y and φ_Y respectively. Let $U_B = B \cap H_0$. Then it follows that if $U_B \neq \emptyset$, Z_B is reduced and irreducible and $\rho_B \times \varphi_B: Z_B \to B \times \boldsymbol{P}^1$ is a bimeromorphic morphism which is isomorphic over $U \times \boldsymbol{P}^1$.

c) Let $h: Y \to S$ and $\beta = \beta_{Y/S}: D_{Y/S} \to S$ be as in a). Let $d \in D_{Y/S}$ and $o = \beta(d) \in S$. Let $Z_d = Z_{Y/S,d} = \rho_{Y/S}^{-1}(d)$. Consider Z_d naturally as a subspace of Y_o by $\pi_{Y/S,o}$. Suppose that h is smooth and Z_d is smooth. Let $N = N_{Z_d/Y_o}$ be the normal sheaf of Z_d in Y_o. Then we recall the following [7]: *β is smooth at d if $H^1(Z_d, N) = 0$.*

Let $Y = \boldsymbol{P}^1 \times X$ as in b) with X nonsingular. Identifying $D_{Y/S,o}$ with D_{Y_o} as in a), consider $d \in D_{Y_o}$. Then suppose that $d \in H = \mathrm{Hol}(\boldsymbol{P}^1, X_o) \subseteq D_{Y_o}$ in the notation of b) (with $X = X_o$) so that d corresponds to a holomorphic map $h_d: \boldsymbol{P}^1 \to X_o$. Then $Z_d \cong \Gamma_h \cong \boldsymbol{P}^1$ and we have $N_{Z_d/Y_o} \cong h_d^* \Theta_{X_o}$ where Θ_{X_o} is the sheaf of germs of holomorphic vector fields on X_o. Thus we obtain the following: *β is smooth at $d \in H_0$ if $H^1(\boldsymbol{P}^1, h_d^* \Theta_{X_o}) = 0$.*

d) Let $f: X \to S$ be a proper morphism of complex spaces. Then we shall write $f \in \mathscr{C}/S$ if there exist a proper and locally Kähler morphism $g: Y \to S$ (e.g. this is the case when Y is a Kähler manifold) (cf. [3, Section 2]), and a generically surjective meromorphic S-map $h: Y \to X$. For instance $f \in \mathscr{C}/S$ if f itself is a proper Kähler morphism [1], or if f is a Moishezon morphism, i.e., bimeromorphic over S to a projective morphism. Note that in the latter case each fiber of f is a Moishezon space. Conversely, if f is smooth and each fiber is Moishezon, then for every $s \in S$ there exists a neighborhood $s \in U$ such that f_U is Moishezon in the sense mentioned above (cf. [6]). We write $f \in \text{loc-}\mathscr{C}/S$ if for every $s \in S$ there exists a neighborhood $s \in U$ such that the induced morphism $f_U: X_U \to U \in \mathscr{C}/S$. Let X be a compact complex space. Then we write $X \in \mathscr{C}$ if there exist a compact Kähler manifold Y and a surjective meromorphic map $h: Y \to X$. If $f \in \mathscr{C}/S$, then each fiber X_s of f belongs to $\mathscr{C}$.

Recall from [1] [2] the following facts which explains our assumptions on f in the results below. *If $f \in \mathscr{C}/S$, then for any relatively compact subdomain*

S' of S and for any irreducible component D_α of $D_{X'/S',\mathrm{red}}$ such that Z_α is reduced, the natural morphism $D_\alpha \to S'$ is proper and belongs to $\mathscr{C}/S'$, where $X'=f^{-1}(S')$. Similarly if f is Moishezon, then $D_\alpha \to S'$ is again proper and Moishezon.

§2. Deformations of Uniruled Manifolds

Let N be a complex manifold and $\varepsilon: E \to N$ a holomorpihc vector bundle of finite rank r. Identifying N with the zero section of E we set $\boldsymbol{P}(E)=(E-N)/\boldsymbol{C}^*$, $\boldsymbol{C}^*$ acting naturally on each fiber of ε. Then ε induces a natural projection $\bar{\varepsilon}: \boldsymbol{P}(E) \to N$ making $\boldsymbol{P}(E)$ a holomorphic fiber bundle over N with typical fiber $\boldsymbol{P}^{r-1}$.

Definition 2.1. Let X be an irreducible compact complex space. 1) X is called *ruled* if there exist a compact complex manifold N and a holomorphic vector bundle E of rank 2 over N as above such that X is bimeromorphic to $\boldsymbol{P}(E)$. 2) X is called *uniruled* if there exist N and E as above such that there exists a generically surjective meromorphic map $\lambda: \boldsymbol{P}(E) \to X$ which is not factored by $\bar{\varepsilon}$ and a meromorphic map $N \to X$.

Remark. If X is Moishezon and uniruled, then we can take the above λ to be generically finite.

Let X be a compact complex space. Let $g: Z \to T$ and $\psi: Z \to X$ be morphisms of compact irreducible complex spaces. Then we call the pair $(g: Z \to T, \psi: Z \to X)$ a *covering family of rational curves* on X if the following conditions are satisfied; 1) ψ is surjective and 2) there exists a Zariski open subset $U \subseteqq T$ such that for all $t \in U$, Z_t are isomorphic to $\boldsymbol{P}^1$ and $\dim \psi(Z_t)=1$.

Lemma 2.2. *Let X be an irreducible compact complex space in $\mathscr{C}$. Then the following conditions are equivalent.* 1) *X is uniruled.* 2) *X admits a covering family of rational curves.* 3) *Let $Y=\boldsymbol{P}^1 \times X$. Then there exists an irreducible component D_α of $D_{Y,\mathrm{red}}$ such that $D_\alpha \cap H_0 \neq \emptyset$ (cf. Section* 1, *b)) and that $\bar{\pi}_\alpha: Z_\alpha \to X$ is surjective.*

Proof. 1)$\to$2). Let $\bar{\varepsilon}: \boldsymbol{P}(E) \to N$ and $\lambda: \boldsymbol{P}(E) \to X$ be as in 2) of Definition 2.1. Let $\boldsymbol{P}=\boldsymbol{P}(E)$. Eliminating the indeterminacy of λ by some bimeromorphic morphism $\tilde{\boldsymbol{P}} \to \boldsymbol{P}$, we obtain the induced surjective morphism $\tilde{\lambda}: \tilde{\boldsymbol{P}} \to X$ and $\tilde{\varepsilon}: \tilde{\boldsymbol{P}} \to N$. Clearly the general fiber of $\tilde{\varepsilon}$ is isomorphic to $\boldsymbol{P}^1$. The pair $(\tilde{\lambda}: \tilde{\boldsymbol{P}} \to X, \tilde{\varepsilon}: \tilde{\boldsymbol{P}} \to N)$ then gives the covering family of rational curves on X. 2)$\to$3).

Let $(g: Z \to T, \psi: Z \to X)$ be a covering family of rational curves on X. Let $U \subseteqq T$ be a Zariski open subset such that g is smooth over U and $Z_t \cong \boldsymbol{P}^1$ for $t \in U$. Then $g_U: Z_U \to U$ is a holomorphic $\boldsymbol{P}^1$-bundle. Let $\nu: P \to U$ be the principal $PGL(2)$-bundle to which g_U is associated. Then the induced bundle $P \times_U Z_U \to P$ is trivial so that we have a P-isomorphism $\delta: P \times \boldsymbol{P}^1 \cong P \times_U Z_U$. Let $\mu = \psi p_2 \delta: P \times \boldsymbol{P}^1 \to X$, where $p_2: P \times Z_U \to Z_U$ is the natural projection. Since ψ is surjective and hence is smooth at general point of Z_U, the image of μ contains an open subset of X and $\dim \mu(\{p\} \times \boldsymbol{P}^1) = 1$ for all $p \in P$.

Let $\Gamma \subseteqq P \times \boldsymbol{P}^1 \times X$ be the graph of μ. Let $\rho: \Gamma \to P$, $\beta: \Gamma \to P \times \boldsymbol{P}^1$, $\tilde{\pi}: \Gamma \to X$ be the morphisms induced by the natural projections from $P \times \boldsymbol{P}^1 \times X$. Clearly $\tilde{\pi} = \mu \cdot \beta$. Further since β is an isomorphism over P (Γ is over P by ρ), ρ is naturally regarded as a smooth family of subspaces of $Y = \boldsymbol{P}^1 \times X$. Thus there exists a unique morphism $\tau: P \to D_Y$ such that ρ is induced from the universal family $\rho_Y: Z_Y \to D_Y$ by τ. We have the associated commutative diagram

$$\begin{tikzcd}
\Gamma \arrow[rr, "\gamma"] \arrow[dr, "\tilde{\pi}"'] \arrow[dd, "\rho"'] & & Z_Y \arrow[dl, "\bar{\pi}_Y"] \arrow[dd, "\rho_Y"] \\
& X & \\
P \arrow[rr, "\tau"'] & & D_Y
\end{tikzcd} .$$

Let D_α be any irreducible component of $D_{Y,\mathrm{red}}$ containing $\tau(P)$. Then we claim that this D_α enjoys the desired property. Note first that for all $d \in \tau(P)$ we have $Z_{\alpha,d} \cong \Gamma_p \cong Z_p \cong \boldsymbol{P}^1$ and $\bar{\pi}_\alpha(Z_{\alpha,d}) = \bar{\pi}_\alpha \gamma(\Gamma_p) = \tilde{\pi}(\Gamma_p) = \mu(Z_p)$ for any $p \in P$ with $\tau(p) = d$. Hence for such d, $Z_{\alpha,d} \cong \boldsymbol{P}^1$ and $\dim \bar{\pi}_\alpha(Z_{\alpha,d}) = 1$, i.e., $D_\alpha \cap H_0 \neq \emptyset$ (cf. Section 1, b)). Moreover we have $\bar{\pi}_\alpha(Z_\alpha) = X$. In fact, Z_α is compact since $X \in \mathscr{C}$, so that $\bar{\pi}_\alpha(Z_\alpha)$ is an analytic set. Further $\bar{\pi}_\alpha(Z_\alpha) \supseteqq \tilde{\pi}(\Gamma) = \mu(P \times \boldsymbol{P}^1)$ and $\mu(P \times \boldsymbol{P}^1)$ contains an open subset of X. Hence $X = \bar{\pi}_\alpha(Z_\alpha)$. 3)$\to$1) Let D_α be as in 3). Then $\rho_\alpha \times \varphi_\alpha: Z_\alpha \to D_\alpha \times \boldsymbol{P}^1$ is bimeromorphic (cf. Section 1, b)). Thus there exists a surjective meromorphic map $\tilde{D}_\alpha \times \boldsymbol{P}^1 \to X$ which is not factored by the projection $\tilde{D}_\alpha \times \boldsymbol{P}^1 \to \tilde{D}_\alpha$ where $\tilde{D}_\alpha$ is any resolution of D_α. Q.E.D.

Remark. If X is uniruled, then the Kodaira dimension $\kappa(X) = -\infty$. *Proof*: In 3) of the above lemma take any smooth subspace $N \subseteqq H_0 \cap D_\alpha$ with $\dim N = \dim X - 1$ such that the associated holomorphic map $h_N: \boldsymbol{P}^1 \times N \to X$ is locally biholomorphic at some point. Then any non-zero element of $\Gamma(X, K_X^m)$, $m > 0$, would induce a nonzero element of $\Gamma(\boldsymbol{P}^1 \times N, K_N^m)$ which in turn gives rise to a nonzero element of $\Gamma(\boldsymbol{P}_v^1, K_{\boldsymbol{P}_v^1}^m) = 0$ for general $v \in N$ (where $\boldsymbol{P}_v^1 = \boldsymbol{P}^1 \times v$), leading to a contradiction.

The following proposition shows the stability of uniruled complex manifolds under smooth deformations which is 'polarized' in a weak sense there.

Proposition 2.3. *Let $f: X \to S$ be a proper smooth morphism of complex spaces with connected fibers. Suppose that $f \in$ loc-$\mathscr{C}/S$ (cf. Section 1, d)) and that S is connected. Then if X_o is uniruled for some $o \in S$, X_s are uniruled for all $s \in S$.*

For the proof we may clearly assume that S is irreducible. Hence the proposition is a special case of the following:

Theorem 2.4. *Let $f: X \to S$ be a proper morphism of irreducible complex spaces with connected fibers. Suppose that $f \in$ loc-$\mathscr{C}/S$ (cf. Section 1, d)) and that there exists an open subset U of S such that $f|_{f^{-1}(U)}: f^{-1}(U) \to U$ is smooth. Then if X_o is uniruled for some $o \in U$, every irreducible component of X_s is again uniruled for all $s \in S$.*

First we need a lemma. Consider a commutative diagram

$$\begin{tikzcd} & Z \arrow[dl, "\bar{\pi}"'] \arrow[dr, "\rho"] \arrow[dd, "\gamma"] & \\ X \arrow[dr, "f"'] & & D \arrow[dl, "\beta"] \\ & S & \end{tikzcd}$$

of irreducible complex spaces where β and f are proper. Let $o \in S$ be a fixed point. We assume that ρ is proper, flat and surjective and that in a neighborhood of X_o, f is smooth and has connected fibers.

Lemma 2.5. *Let $d_o \in D$ with $\beta(d_o) = o$. Assume the following conditions; 1) β is smooth at d_o, 2) $Z_{d_o} = \rho^{-1}(d_o)$ is irreducible, smooth and of dimension 1 and $\dim \bar{\pi}(Z_{d_o}) = 1$, and 3) $\bar{\pi}(Z'_o) = X_o$ for some irreducible component Z'_o of Z_o containing Z_{d_o}. Then for each $s \in S$ and for each irreducible component $X_{s,k}$ of $X_{s,\mathrm{red}}$ there exists an irreducible component $Z_{s,k}$ of $Z_{s,\mathrm{red}}$ such that $\bar{\pi}(Z_{s,k}) = X_{s,k}$ and that $\dim \bar{\pi}((Z_{s,k})_d) = 1$ for all $d \in D_{s,k}$ where $(Z_{s,k})_d = Z_{s,k} \cap Z_d$ and $D_{s,k} = \rho(Z_{s,k})$.*

Proof. Note that γ and $\bar{\pi}$ also are proper. First we show that $\bar{\pi}$ is surjective. Since ρ is flat and Z_{d_o} is smooth, ρ is smooth along Z_{d_o}. Together with the smoothness of β at d_o, this implies that γ is smooth in a neighborhood V of Z_{d_o} in Z. In particular $\gamma(V)$ is open. Moreover for each $s \in \gamma(V)$ and for each connected component of $Z_{si}(V)$ of $Z_s \cap V$ there is a unique irreducible component

Z'_{si} of $Z_{s,\mathrm{red}}$ such that $Z'_{si} \cap V = Z_{si}(V)$. (For $s=o$ we may assume that $Z_o \cap V$ is connected and the corresponding component is Z'_o. Hence the notations are compatible.) Let $k=\dim(Z_s \cap V)$ and $n=\dim X_s$, both of which we may assume to be independent of $s \in \gamma(V)$ in view of the smoothness of f and γ near Z_{d_o}. For $z \in Z$ let $b(z)=\dim_z \bar{\pi}^{-1}\bar{\pi}(z)$ and $b_{si}(z)=b(z)|_{Z'_{si}}$. Since $b(z)$ is upper semicontinuous with respect to the Zariski topology and Z'_{si} is irreducible, there is a dense Zariski open subset W'_{si} of Z'_{si} such that $b_{si}(z)$ has the minimum value, say b_{si}, on W_{si}. Then if s is sufficiently near to o, we have $b_{si} \leqq b_o$, $k-b_{si} = \dim \bar{\pi}(Z'_{si}) \leqq n$ and further $k-b_o=n$ since $\bar{\pi}(Z'_o)=X_o$ by our assumption. It follows that $\dim \bar{\pi}(Z'_{si})=n$, i.e., $\bar{\pi}_{si}=\bar{\pi}|_{Z'_{i}}$ is surjective for all $s \in \gamma(V)$ since X_s is irreducible if V is sufficiently small as we may assume. Hence the image under $\bar{\pi}$ of Z contains the open set $f^{-1}\gamma(V)$ and hence $\bar{\pi}$ is surjective by the irreducibility of X and the properness of $\bar{\pi}$.

Next, let $F=\rho \times_S \bar{\pi}\colon Z \to D \times_S X$, $\bar{Z}=F(Z)$ and let $\rho_1\colon \bar{Z} \to D$ and $\pi_1\colon \bar{Z} \to X$ the natural projections. Let $A=\{z \in Z;\ \dim_z F^{-1}F(z) \geqq 1\}$ and $\bar{A}=\rho_1(A)$. Then by Remmert A and $\bar{A}$ are analytic subsets of Z and D respectively. On the other hand, since Z_{d_o} is irreducible, from our assumption that $\dim \bar{\pi}(Z_{d_o})=1$ it follows that $d_o \notin \bar{A}$ and therefore $\bar{A}$ is a proper analytic subset of D. This then implies that every fiber of ρ_1 is again of dimension 1 as well as ρ. Hence for every $d \in D$, $\dim \bar{\pi}(Z_d)=\dim \pi_1(\bar{Z}_d)=1$. Since $\bar{\pi}=\pi_1 F$ and $\bar{\pi}(Z)=X$, we have $\pi_1(\bar{Z}_s)=X_s$ for every $s \in S$. Hence for every irreducible component $X_{s,k}$ of $X_{s,\mathrm{red}}$ there is an irreducible component $\bar{Z}_{s,k}$ of $\bar{Z}_{s,\mathrm{red}}$ such that $\pi_1(\bar{Z}_{s,k})=X_{s,k}$. Note that since fibers of ρ_1 are connected as well as ρ the dimension of the general fiber of $\rho_{1,s,k}\colon \bar{Z}_{s,k} \to X_{s,k}$ cannot be zero, and hence each fiber of $\rho_{1,s,k}$ has dimension one. Then it suffices to take any irreducible component $Z_{s,k}$ of $Z_{s,\mathrm{red}}$ which are mapped surjectively onto $\bar{Z}_{s,k}$ by F. Q.E.D.

Now we shall prove Theorem 3.4. We first reduce the proof to the case where $f \in \mathscr{C}/S$: Suppose that the theorem is true if $f \in \mathscr{C}/S$. In general let U be any dense Zariski open subset of S such that $f_U\colon X_U \to U$ is smooth. Let $V=\{u \in U;\ X_u \text{ is uniruled}\}$. Then in view of the definition of loc-$\mathscr{C}/S$ together with our assumption it follows immediately that V is open and closed in U. Hence $V=U$ since $o \in V$. Then since U is dense in S, again by the definition of loc-$\mathscr{C}/S$ the theorem follows from the case $f \in \mathscr{C}/S$. So *in what follows we assume that* $f \in \mathscr{C}/S$.

Set $Y=\boldsymbol{P}^1 \times X$ and let $h=pf\colon Y \to S$ be as in Section 1, a). Since X_o is uniruled, by Lemma 2.2 there exists an irreducible component $D_{o,\alpha}$ of $D_{Y_o,\mathrm{red}}$ such that $D_{o,\alpha} \cap H_0 \neq \emptyset$ and that the natural morphism $\pi_{o,\alpha}: Z_{o,\alpha} \to X_o$ is surjective where $Z_{o,\alpha} = Z_{Y_o} \times_{D_{Y_0}} D_{o,\alpha}$ and $\pi_{o,\alpha}$ is the restriction of $\pi_{Y,o}$ to $Z_{o,\alpha}$. Identifying D_{Y_o} with $D_{Y/S,o}$ as in Section 1, a) we regard $D_{o,\alpha}$ as an irreducible component of $(D_{Y/S,o})_{\mathrm{red}}$. Then take any irreducible component D_α of $D_{Y/S,\mathrm{red}}$ which contains $D_{o,\alpha}$. Let $\beta_\alpha: D_\alpha \to S$ be the natural morphism. Let $U_\alpha = D_{o,\alpha} \cap H_0$ and U'_α the set of smooth points of U_α.

Lemma 2.6. *There exists a Zariski open subset V of $D_{o,\alpha}$ with $V \subseteqq U'_\alpha$ such that β_α is smooth at each point of V.*

Proof. Let $n = \dim X_o$. Let W be a nonempty Zariski open subset of $Z_{o,\alpha}$ on which $\pi_{o,\alpha}$ is smooth. This is possible since $\pi_{o,\alpha}$ is surjective. Set $V = U'_\alpha \cap \rho_{o,\alpha}(W)$. Then it is immediate to see that V is Zariski open in $D_{o,\alpha}$. We show that this V has the desired property. Let $d \in V$ be any point. Then $\pi_{o,\alpha}$ is smooth at points of $Z_d \cap W \neq \emptyset$ where $Z_d = Z_{o,d} = (Z_{o,\alpha})_d$. Moreover we see readily that there exist a neighborhood $V(d)$ of d in V and an irreducible and smooth subspace N of $V(d)$ of dimension $n-1$ with $d \in N$ such that if we put $M = \rho_{o,\alpha}^{-1}(N)$, then $h = \pi_{o,\alpha}|_M: M \to X_o$ is locally biholomorphic at some point of Z_d.

Now in general let Θ_B be the sheaf of germs of holomorphic vector fields on B. Then we have the natural homomorphism of $\mathcal{O}_M$-modules $\lambda: \Theta_M \to h^*\Theta_{X_o}$. Since h is locally biholomorphic at a point of Z_d, λ is injective and, after restricting to $C = Z_d$, we obtain the following exact sequence of $\mathcal{O}_C$-modules

$$0 \longrightarrow \Theta_M|_C \longrightarrow h^*\Theta_{X_0}|_C \longrightarrow \mathcal{Q} \longrightarrow 0$$

where $\mathcal{Q}$ is a torsion $\mathcal{O}_C$-module. Since $H^1(C, \mathcal{Q}) = 0$ and $h^*\Theta_{X_o}|_C = h_d^*\Theta_{X_o}$ where h_d is the holomorphic map $\boldsymbol{P}^1 \to X_o$ corresponding to $d \in U_\alpha$, for the lemma we have only to show that $H^1(M, \Theta_M|_C) = 0$, in view of Section 1, c). On the other hand, since the normal bundle of C in M is trivial, we have the following exact sequence of $\mathcal{O}_C$-modules

$$0 \longrightarrow \Theta_C \longrightarrow \Theta_M|_C \longrightarrow \mathcal{O}_C^{\oplus(n-1)} \longrightarrow 0\,.$$

Then from the long exact sequence associated to this sequence we get the desired vanishing of $H^1(C, \Theta_M|_C)$ since $C \cong \boldsymbol{P}^1$. Q.E.D.

Remark 2.7. Tensoring $\mathcal{O}_C(-1)$ with the two short exact sequences appearing in the above proof and taking the long exact sequences of cohomology we even cbtain the vanishing: $H^1(C, h_d^*\Theta_{X_o}(-1))=0$. In fact this also has some geometric significance which will be discussed at the end of this section.

Now fix any relatively compact subdomain $S'\ni o$. It suffices to show the theorem for $f_{S'}\colon X_{S'}\to S'$ since S' is arbitrary. For notational simplicity we denote S' again by S and hence $X_{S'}$ by X and $f_{S'}$ by f. We then consider the following commutative diagram

$$\begin{tikzcd} & Z \arrow[dl, "\bar{\pi}_\alpha"'] \arrow[dr, "\rho_\alpha"] \arrow[dd] & \\ X \arrow[dr, "f"'] & & D_\alpha \arrow[dl, "\beta_\alpha"] \\ & S & \end{tikzcd}$$

and show that this satisfies the conditions of Lemma 2.5. First of all by our restriction of S as above, β_α and $\bar{\pi}_\alpha$ are proper (cf. Section 1, d)). Further by our definition ρ_α is flat and surjective, and in a neighborhood of X_o, f is smooth and has connected fibers by assumption. We take $d_o\in D_{\alpha,o}$ to be any point from $V\subseteqq U_\alpha$ in Lemma 2.6. Then β_α is smooth at d_o, so that 1) of Lemma 2.5 is satisfied. Moreover since $d_o\in U_\alpha$, 2) also is satisfied in view of Section 1, b). Finally taking $Z'_o=Z_{o,\alpha}$, 3) also is true. Thus by that lemma for every $s\in S$ and for every irreducible component $X_{s,k}$ of $X_{s,\mathrm{red}}$ there exists an irreducible component $Z^k_{\alpha,s}$ of $(Z_{\alpha,s})_{\mathrm{red}}$ such that $\bar{\pi}_\alpha(Z^k_{\alpha,s})=X_{s,k}$ and $\dim\bar{\pi}_\alpha((Z^k_{\alpha,s})_d)=1$ for all $d\in D_{\alpha,s}=\rho_\alpha(Z^k_{\alpha,s})$. Let $\mu\colon \tilde{Z}^k_{\alpha,s}\to Z^k_{\alpha,s}$ be the normalization of $Z^k_{\alpha,s}$ and $\tilde{g}^k_s\colon \tilde{Z}^k_{\alpha,s}\to D^k_{\alpha,s}$ be μ followed by the natural map $\rho^k_{\alpha,s}\colon Z^k_{\alpha,s}\to D^k_{\alpha,s}$. Let $g^k_s\colon \tilde{Z}^k_{\alpha,s}\to T^k_{\alpha,s}$, $T^k_{\alpha,s}\to D^k_{\alpha,s}$ be the Stein factorization of $\tilde{g}^k_s$. Then g^k_s is a proper morphism of irreducible complex spaces with connected fibers. Further, every fiber of ρ_α, and hence that of $\rho^k_{\alpha,s}$ also, is union of rational curves as a specialization of $\boldsymbol{P}^1$. Hence the general fiber of g^k_s is isomorphic to $\boldsymbol{P}^1$ being a normalization of that of $\rho^k_{\alpha,s}$. Therefore g^k_s together with $\bar{\pi}_\alpha\mu\colon \tilde{Z}^k_{\alpha,s}\to X_{s,k}$ gives a covering family of rational curves on $X_{s,k}$. Hence $X_{s,k}$ is uniruled by Lemma 2.2. Q.E.D.

In the rest of this section we shall give a geometric implication of the vanishing result mentioned in Remark 2.7. Since the method of proof is essentially the same as above (cf. Mori [8]), we omit the proof and give the statements only.

Definition 2.8. Let X be an irreducible compact complex space. We call X *rationally connected* (resp. *to a point* $x\in X$) if there exists a covering family of rational curves ($g\colon Z\to T$, $\psi\colon Z\to X$) on X such that $\psi\times_T\psi\colon Z\times_T Z\to X\times X$ is surjective (resp. $g\psi^{-1}(x)=T$).

Remark. It is immediate to see that if X is rationally connected, then it is rationally connected to any general point of X. Further, X is rationally connected if and only if there exists a Zariski open subset $V \subseteq X$ such that any two points $x_1, x_2 \in V$ can be connected by an irreducible rational curve. In this case X is Moishezon. If, further, X is normal, then X is simply connected. The proof of the last two facts will be given in the Appendix.

Proposition 2.9. *Let $f: X \to Y$ be a proper smooth morphism of irreducible complex spaces with connected fibers. Suppose that $f \in \mathscr{C}/S$ and X_o is rationally connected to a point $x \in X_o$ for some $o \in S$. Then there exists a dense open subset $o \in U \subseteq S$ such that X_s is rationally connected for every $s \in U$.*

Corollary. *A compact complex manifold X which is rationally connected to some of its points is rationally connected.*

Corollary is false if X has a singularity. (Consider the projective cone over a non-uniruled projective variety.) The above proposition shows that general quartic threefolds in $\boldsymbol{P}^4$ are rationally connected in view of the result of Segre-Iskovskih-Manin.

§3. Locus of Ruled Manifolds

We begin with the following lemma analogous to Lemma 2.2.

Lemma 3.1. *Let X be an irreducible compact Moishezon space of dimension n. Then the following conditions are equivalent.* 1) *X is ruled.* 2) *Let $Y = \boldsymbol{P}^1 \times X$. Then there exists an irreducible complex subspace B of $D_{Y,\mathrm{red}}$ of dimension $n-1$ such that* a) *$B \cap H_0 \neq \emptyset$ (cf. Section 1, b)) and* b) *$\pi_B: Z_B \to X$ is bimeromorphic, where $Z_B = Z_Y \times_{D_Y} B$ and $\pi_B = \pi_Y|_{Z_B}$.*

Proof. 1)$\to$2). Let $\bar{\varepsilon}: \boldsymbol{P}(E) \to N$ be a $\boldsymbol{P}^1$-bundle as in 1) of Definition 2.1. Since N is Moishezon, $\bar{\varepsilon}$ is actually bimeromorphic to a product bundle $N \times \boldsymbol{P}^1 \to N$. Therefore we may assume that $\boldsymbol{P}(E) = N \times \boldsymbol{P}^1$. Let $\lambda: N \times \boldsymbol{P}^1 \to X$ be the given bimeromorphic map. Then there exists a Zariski open subset $U \subseteq N$ such that λ is holomorphic on $U \times \boldsymbol{P}^1$. Let $\Gamma \subseteq U \times \boldsymbol{P}^1 \times X$ be the graph of this holomorphic map;

$$\begin{array}{c} U \times \boldsymbol{P}^1 = \Gamma \subseteq U \times \boldsymbol{P}^1 \times X = U \times Y \\ \searrow \quad \downarrow \quad \swarrow \\ U \end{array} .$$

Considering the above diagram as a smooth family of subspaces of Y parametrized by U we get a holomorphic map $\tau\colon U\to D_Y$ such that the family is induced from the corresponding universal family. Let B be the closure of $\tau(U)$ in D_Y, which is an analytic subset of D_Y (cf. [1]). We shall see that this B satisfies all the desired properties. In fact we have the following commutative diagram of meromorphic maps

$$\begin{tikzcd} N\times \boldsymbol{P}^1 \arrow[rr, "\tilde{\tau}^*"] \arrow[dr, "\lambda"'] \arrow[d] & & Z_B \arrow[dl, "\bar{\pi}_B"] \arrow[d] \\ N \arrow[rr, "\tau^*"'] & X & B \end{tikzcd}$$

where τ^* is the meromorphic map extending τ and $\tilde{\tau}^*$ is induced by τ^*. Since $\tilde{\tau}^*$ is surjective, $\bar{\pi}_B$ must be bimeromorphic as well as λ. (This is b).) In particular $\dim Z_B=n$. Hence $\dim B=n-1$ since the general fiber of $\rho_B\colon Z_B\to B$ is $\boldsymbol{P}^1$. In fact, we have $H_0\cap B\supseteq\tau(U)\neq\emptyset$, which is a).

2)$\to$1). $\varphi_Y\times\rho_Y$ induces a bimeromorphic morphism $Z_B\to\boldsymbol{P}^1\times B$. Thus X is bimeromorphic to $\boldsymbol{P}^1\times B$ as desired. Q.E.D.

We consider the following commutative diagram of irreducible complex spaces

$$\begin{tikzcd} & (W\times_E X)\times\boldsymbol{P}^1\supseteq Z \arrow[r, "\varphi"] \arrow[dl, "\bar{\pi}"'] \arrow[dr, "\rho"] \arrow[dd] & \boldsymbol{P}^1 \\ X \arrow[dr, "f"'] & & W \arrow[dl, "\xi"] \\ & E & \end{tikzcd}$$

where f is proper, smooth and has connected fibers, ρ and ξ are proper, flat and surjective, and where $\bar{\pi}$, ρ and φ are induced by the projections from $(W\times_E X)\times\boldsymbol{P}^1$ onto X, W and $\boldsymbol{P}^1$ respectively.

Lemma 3.2. *Suppose further that the following conditions are satisfied*: 1) *The general fiber of* ξ *is irreducible*, 2) $\varphi_{w_0}\colon Z_{w_0}\to\boldsymbol{P}^1$ *is isomorphic for some* $w_0\in W$, *and* 3) $\pi_{e_0}\colon Z_{e_0}\to X_{e_0}$ *is bimeromorphic for some* $e_0\in E$. *Then* X_e *is ruled for all* $e\in E$.

Proof. α) We show that X_e is ruled for general $e\in E$. Define $U=\{w\in W;\ \varphi_w\colon Z_w\to\boldsymbol{P}^1$ is isomorphic$\}$, and $M=\{e\in E;\ \pi_e\colon Z_e\to X_e$ is bimeromorphic$\}$. Then U and M are Zariski open in W and E respectively (cf. [1, Lemma 5.5]). Moreover since $w_0\in U$ and $e_0\in M$, they are nonempty. Let $\bar{U}=\xi(U)$. Then it is easy to see that $\bar{U}$ contains a Zariski open subset, say V_0, of E. Restricting V_0 if necessary, we may assume that W_e are reduced and irreducible for all

$e \in V_0$ by 1) and [1, Lemma 1.4]. Then we show that X_e is ruled for any $e \in V = V_0 \cap M$. Since X_e is bimeromorphic to Z_e, it is enough to show that Z_e is ruled for any $e \in V_0$. First since $U \cap W_e$ is dense in W_e, the general fiber of $\rho_e \colon Z_e \to W_e$ is isomorphic to $\boldsymbol{P}^1$ via φ_e. Then it follows that $\rho_e \times \varphi_e \colon Z_e \to W_e \times \boldsymbol{P}^1$ is bimeromorphic (cf. Section 1, b)). Thus Z_e is ruled. (By the flatness of ρ, Z_e is actually reduced and irreducible.)

β) Let $E' = \{e \in E;\, X_e \text{ is not ruled}\}$. Supposing that $E' \neq \emptyset$ we shall derive a contradiction. Let $H = \{t \in \boldsymbol{C};\, |t| < 1\}$. Take a morphism $q \colon H \to E$ in such a way that $q^{-1}(V) = H' \colon = H - \{0\}$ and $q(0) \in E'$. Then taking the base change of everything to H by q we may assume from the beginning that $E = H$, $E' = \{0\}$ and $V = H'$. (Recall that f, ρ, ξ are all flat.) In particular we shall denote the point of E by t. Recall from α) that for each $t \neq 0$, $(\rho_t \times \varphi_t)\bar{\pi}_t^{-1}$ gives a bimeromorphic map of X_t and $W_t \times \boldsymbol{P}^1$. This in turn implies that $h = (\rho \times \varphi)\bar{\pi}^{-1}$ gives a bimeromorphic map of X and $W \times \boldsymbol{P}^1$ over E. Let $r \colon \tilde{W} \to W$ be any resolution of W. $\tilde{W}$ is a complex space over E by ξr. Let $\tilde{h} \colon X \to \tilde{W} \times \boldsymbol{P}^1$ be the bimeromorphic map over E induced by h. Now if X_0 corresponds to an irreducible component $\tilde{W}'_0$ of $\tilde{W}_0 \times \boldsymbol{P}^1$ by $\tilde{h}$, then X_0 is ruled since $\tilde{W}'_0$ is necessarily of the form $\tilde{W}_{0i} \times \boldsymbol{P}^1$ for some irreducible component of $\tilde{W}_{0i}$ of $\tilde{W}_0$. This is impossible by our assumption that $0 \in E'$. Thus X_0 must be an exceptional divisor for the map h, i.e., must correspond to a lower dimensional subspace of $\tilde{W} \times \boldsymbol{P}^1$. Then since $\tilde{W} \times \boldsymbol{P}^1$ is nonsingular, X_0 is again ruled (cf. [4, Lemma 4.1]). Thus we get a desired contradiction. Q.E.D.

Proposition 3.3. *Let $f \colon X \to S$ be a proper smooth morphism of irreducible complex spaces with connected fibers. Let $R = \{s \in S;\, X_s \text{ is ruled}\}$. Suppose that f is Moishezon (cf. Section 1, d)). Then for any relatively compact open subset S' of S, $R \cap S'$ is a union of at most countably many analytic subsets of S'.*

Proof. Fix any relatively compact open subset $S' \subseteqq S$. Replacing S by S' we denote S' again by S. Let $n = \dim X_s$, which is independent of s since f is smooth and X and S are irreducible. We first construct the following commutative diagram of irreducible complex spaces

$$(*)\qquad \begin{tikzcd}
 & (W_\mu \times_{E_\mu} X_\mu) \times \boldsymbol{P}^1 \supseteqq Z_\mu \arrow[r, "\varphi_\mu"] \arrow[dl, "\bar{\pi}_\mu"'] \arrow[dd] \arrow[dr, "\rho_\mu"] & \boldsymbol{P}^1 \\
X_\mu \arrow[dr, "f_\mu"'] & & W_\mu \arrow[dl, "\xi_\mu"] \\
 & E_\mu &
\end{tikzcd} .$$

Fix any irreducible component D_α of $D_{Y/S,\mathrm{red}}$ where $Y=X\times \boldsymbol{P}^1$. Let E_μ be any irreducible component of $D_{D_\alpha/S,\mathrm{red}}$. Let ξ_μ: $W_\mu\to E_\mu$ be the morphism induced from the universal family $\rho_{D_\alpha/S}$: $W_{D_\alpha/S}\to D_{D_\alpha/S}$ by restriction so that we have the following commutative diagram corresponding to the one in Section 1, a)

$$\begin{tikzcd} & W_\mu\subseteq E_\mu\times_S D_\alpha \arrow[dl, "\xi_\mu"'] \arrow[dr, "\delta_\mu"] \arrow[d] & \\ E_\mu \arrow[r, "\beta_\mu"'] & S & D_\alpha \arrow[l] \end{tikzcd} .$$

Let $Z_\mu=W_\mu\times_{D_\alpha}Z_\alpha$ and let ρ_μ: $Z_\mu\to W_\mu$ be induced from ρ_α: $Z_\alpha\to D_\alpha$. Let $X_\mu=E_\mu\times_S X$ and let f_μ: $X_\mu\to E_\mu$ be induced from f. On the other hand, π_α: $Z_\alpha\to X$ (resp. φ_α: $Z_\alpha\to\boldsymbol{P}^1$) (cf. Section 1, a)) induces the morphism $\bar{\pi}_\mu$: $Z_\mu\to X_\mu$ (resp. φ_μ: $Z_\mu\to\boldsymbol{P}^1$), and the natural inclusion $Z_\alpha\subseteq(D_\alpha\times_S X)\times\boldsymbol{P}^1$ induces the natural inclusion $Z_\mu\subseteq(W_\mu\times_S X)\times\boldsymbol{P}^1=(W_\mu\times_{E_\mu}X_\mu)\times\boldsymbol{P}^1$ such that π_μ, ρ_μ and φ_μ are induced by the natural projections from $(W_\mu\times_{E_\mu}X_\mu)\times\boldsymbol{P}^1$ onto the corresponding factors. This completes the construction of (*). Now we impose some additional conditions on our choice of D_α and E_μ, so that (*) satisfies the conditions of Lemma 3.2. 1) There exists a dense Zariski open subset V_μ of E_μ such that $W_{\mu,e}$ are reduced and irreducible of dimension $n-1$ for all $e\in V_\mu$. 2) Let $U_\alpha=\{d\in D_\alpha;\ \varphi_{\alpha,d}: Z_{\alpha,d}\to\boldsymbol{P}^1$ is isomorphic$\}$. Then $\delta_\mu(W_\mu)\cap U_\alpha\neq\emptyset$. 3) $\bar{\pi}_{\mu,e'}$: $Z_{\mu,e'}\to X_{\mu,e'}$ is a bimeromorphic morphism for some $e'\in E_\mu$. Note that from 2) it follows that W_μ is reduced and irreducible and from 3) that $Z_{\mu,e}$ is irreducible as well as $X_{\mu,e}$. Then we see readily that (*) satisfies all the conditions of Lemma 3.2. Therefore by that lemma it follows that $X_{\mu,e}=X_{\beta_\mu(e)}$ is ruled for all $e\in E$.

Now let $S_\mu=\beta_\mu(E_\mu)$. We show that S_μ is an analytic subset of S. In fact, since f is Moishezon, $D_\alpha\to S$ is again proper and Moishezon (cf. Section 1, d)). Hence $E_\mu\to S$ also is proper and Moishezon. Thus by Remmert S_μ is an analytic subset of S. Let $\mathfrak{A}$ be the set of indices (α,μ) such that the pair (D_α, E_μ) satisfies the conditions 1)–3) above. Then by [2] $\mathfrak{A}$ is at most a countable set. We have thus constructed at most countably many analytic subsets $S_\mu=S_{\alpha,\mu}$, $(\alpha,\mu)\in\mathfrak{A}$, of S such that $\bigcup_{(\alpha,\mu)\in\mathfrak{A}}S_{\alpha,\mu}\subseteq R$.

It remains to show that $R\subseteq\bigcup_{\mathfrak{A}}S_{\alpha,\mu}$ for some $(\alpha,\mu)\in\mathfrak{A}$. Namely we have to show that if X_s is ruled for some $s\in S$, then $s\in S_{\alpha,\mu}$ for some $(\alpha,\mu)\in\mathfrak{A}$. Let $Y_s=\boldsymbol{P}^1\times X_s$. Then by Lemma 3.1 there exists an irreducible analytic subset B_s of D_{Y_s} of dimension $n-1$ such that $B_s\cap H_0\neq\emptyset$ and that π_{B_s}: $Z_{B_s}\to X_s$ is bimeromorphic (cf. Section 1, b)). Identifying D_{Y_s} with $D_{Y/S,s}$, let D_α be any irreducible component of $D_{Y/S,\mathrm{red}}$ containing B_s. Then B_s defines a point e_s of $D_{D_\alpha/S,s}$. Let E_μ be any irreducible component of $D_{D_\alpha/S,\mathrm{red}}$ containing e_s. Then we claim that the pair (D_α, E_μ) satisfies 1)–3), i.e., $(\alpha, \mu) \in \mathfrak{A}$. This would prove the proposition since $s = \beta_\mu(e_s) \in \beta_\mu(E_\mu) = S_{\alpha,\mu}$. Now we prove 1)–3). 1) $\delta_\mu(W_\mu) \cap U_\alpha \supseteqq B_s \cap U_\alpha = B_s \cap H_0 \neq \emptyset$. 2) $W_{\mu,e_s} = B_s$ and hence is reduced, irreducible and of dimension $n-1$. By the flatness of ξ_μ it follows readily that the same is true for any $W_{\mu,e'}$ where e' is from some Zariski open subset of E_μ (cf. [1, Lemma 1.4]). 3) It suffices to take $e = e_s$ by our construction. Q. E. D.

Proposition 3.4. *Let $f: X \to S$ be a proper smooth morphism of irreducible complex spaces with connected fibers. Suppose that there exist compact complex spaces X^* and S^* such that* 1) *X and S are Zariski open subsets of X^* and S^* respectively, and* 2) *f extends to a Moishezon morphism $f^*: X^* \to S^*$. Then the set $R = \{s \in S;\ X_s$ is ruled$\}$ is a union of at most countably many analytic subsets of S which extend to analytic subsets of S^*.*

The proof is the same as that of the above proposition if we take D_α to be irreducible components of D_{X^*/S^*} instead of those of $D_{X/S}$.

Appendix

Let X be an irreducible compact complex space. Suppose that X is rationally connected to $x \in X$ (cf. Definition 2.8). First we show that *X is Moishezon*. In fact, in the notation of Definition 2.8 $\psi^{-1}(x)$ is Moishezon by [3] and hence T also is Moishezon. Then since $g: Z \to T$ is a Moishezon morphism, being of fiber dimension 1, Z also is Moishezon. Hence X is Moishezon as a surjective image of Z. Next we show the following:

Proposition A. *Suppose that X is normal. Then X is simply connected.*

Proof. Let $\alpha: \tilde{X} \to X$ be the universal covering of X. Suppose that there exist an unramified covering $\alpha_1: \tilde{X}_1 \to X_1$ of complex spaces and surjective morphisms $\varphi: X_1 \to X$ and $\tilde{\varphi}: \tilde{X}_1 \to \tilde{X}$ such that $\varphi\alpha_1 = \alpha\tilde{\varphi}$ and that $(\varphi\alpha_1)^{-1}(x)$ is connected. Then it follows that $\tilde{\varphi}^{-1}(\alpha^{-1}(x))$, and hence $\alpha^{-1}(x)$ also, is connected. This implies that α is isomorphic and hence X is simply connected. Thus it suffices to show the existence of α_1, φ, $\tilde{\varphi}$ as above. By our assumption it follows that there exists an irreducible component D_α of $D_{X,\mathrm{red}}$ such that

$Z_{\alpha,d}=\rho_\alpha^{-1}(d)$ is an irreducible and rational curve, and that $D_\alpha \times \{x\} \subseteqq Z_\alpha$. Then as in the proof of Lemma 2.2 after a suitable base change $T \to D_\alpha$ which is proper and surjective, the induced map $Z_T = Z_{\alpha,T} \to T$ is bimeromorphic to the projection $T \times \boldsymbol{P}^1 \to T$. In particular if we take T to be nonsingular as we may, then for any resolution $Z \to Z_T$ of Z_T the natural map $Z \to T$ induces an isomorphism $\pi_1(Z) \cong \pi_1(T)$ of the fundamental groups. Then we set $X_1 = Z_T$ and define φ by the natural morphism $Z_T \to X$ which is obviously surjective. Next we construct α_1. Let $\tilde{Z}$ be an irreducible component of $Z \times_X \tilde{X}$. Let $\tilde{X}_1$ be the image of Z under the natural proper bimeromorphic morphism $Z \times_X \tilde{X} \to X_1 \times_X \tilde{X}$. Then $\tilde{X}_1$ is an irreducible component of $X_1 \times_X \tilde{X}$. Let $\alpha_1 \colon \tilde{X}_1 \to X_1$ be the natural projection $\tilde{Z} \to Z$. We show that the other projection $\tilde{\varphi} \colon \tilde{X}_1 \to \tilde{X}$ is surjective. In fact, $\tilde{\varphi}$ is proper as well as φ. Therefore $\tilde{\varphi}(\tilde{X}_1)$ is an analytic subset of $\tilde{X}$. On the other hand, we have $\dim \tilde{\varphi}(\tilde{X}_1) = \dim \tilde{X}$. Since $\tilde{X}$ is normal as well as X, this implies that $\tilde{\varphi}(\tilde{X}_1) = \tilde{X}$. Thus it remains only to show that $(\varphi\alpha_1)^{-1}(x)$ is connected. First, since $\pi_1(Z) \cong \pi_1(T)$, there exists an unramified covering $u \colon \tilde{T} \to T$ such that $\tilde{Z} = Z \times_T \tilde{T}$. Let $X_1' = X_1 \times_T \tilde{T}$ and let $\alpha_1' \colon X_1' \to X_1$ be the natural projection. X_1' is naturally bimeromorphic to Z. Then we get the following commutative diagram

$$\begin{tikzcd} X_1' \arrow[rr, "\beta"] \arrow[dr, "\alpha_1'"'] & & X_1 \arrow[dl, "\alpha_1"] \\ & X_1 & \end{tikzcd}$$

where β is a bimeromorphic map. Since α_1 and α_1' are unramified, it is easy to show that β is actually isomorphic. Then $(\varphi\alpha_1)^{-1}(x) = \alpha_1'^{-1}(T \times \{x\}) = \tilde{T} \times \{x\}$, which is connected. Q. E. D.

Note finally that there is no holomorphic p-form on X for any $p>0$ by Roytman [9][1]. (In fact, in our special case the proof becomes quite simple.)

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Added in proof: After submitting this paper the author received a preprint: Levine, M., Deformations of uni-ruled varieties, which contains essentially the same result as the present paper.