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On Holonomic Systems of Micro-differential Equations. III
—Systems with Regular Singularities—

By

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Introduction

This is the third of the series of the papers dealing with holonomic systems(*). A holonomic system is, by definition, a left coherent $\mathscr{E}$-Module (or $\mathscr{D}$-Modules)(**) whose characteristic variety is Lagrangian. It shares the finiteness theorem with a linear ordinary differential equation, namely, all the cohomology groups associated with its solution sheaf are finite dimensional ([6], [12]). Hence the study of such a system will give us almost complete information concerning the functions which satisfy the system, as in the one-dimensional case. Actually, analyzing special functions by the aid of the theory of ordinary differential equations is one of the most important subjects in the classical analysis. From this point of view, the study of holonomic systems with regular singularities is most important. However, even though the theory of linear ordinary differential equations with regular singularities has been developed quite successfully, the general theory of holonomic systems with regular singularities was not fully developed in the past, especially compared with the fruitful success attained in the one-dimensional case. Still it should be worth doing, and we hope we have established a solid basis for the theory in this paper. For example, we establish several basic results needed for the manipulation of holonomic systems with regular singularities, such as the integration and the restriction of such systems (Chapter V). We also give an analytic character-

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(*) The first one is [6] and the second one is [8].
(**) $\mathscr{E}$ (resp., $\mathscr{D}$) denotes the sheaf of micro-differential (resp. linear differential) operators of finite order. See also the list of notations given at the end of this section.

ization of holonomic $\mathscr{D}$-Modules with regular singularities in terms of a comparison theorem, namely, we show that a holonomic $\mathscr{D}_X$-Module $\mathscr{M}$ is with regular singularities if and only if $\mathscr{E}xt^j_{\mathscr{D}_X}(\mathscr{M}, \mathscr{O}_X)_x \cong \mathscr{E}xt^j_{\mathscr{D}_X}(\mathscr{M}, \hat{\mathscr{O}}_{X,x})$ holds for any point $x \in X$ and for any j, where $\hat{\mathscr{O}}_{X,x}$ denotes the ring of formal power series at x. (Chapter VI.) In developing our theory, we make full use of the technique of micro-local analysis, i.e., the analysis on the cotangent bundle. We use the language of Sato-Kawai-Kashiwara [24], which shall be referred to as S-K-K [24] for brevity. Especially the use of micro-differential operators of infinite order is crucial in our study. Making use of such operators, we establish an important and interesting result to the effect that any holonomic system can be transformed into a holonomic system with regular singularities by micro-differential operators of infinite order (Chapters IV and V). The method of the proof of this result as well as the result itself is efficiently employed for establishing basic properties of a holonomic system with regular singularities mentioned earlier. In the course of our arguments, we also make essential use of the results of Deligne [3]. Since his results are stated in terms of integrable connections, we re-interpret them in terms of $\mathscr{D}$-Modules so that we may apply them to our problems smoothly. (Chapter II. See also Appendix § C.)

Main results of this paper were announced in [15].

Before stating a more detailed plan of this paper, we show one example, which exemplifies the most significant result of this paper (Theorem 5.2.1 in Chapter V, § 2), i.e., the theorem which states that any holonomic system can be transformed into a holonomic system with regular singularities. We hope our explanation of this example will show the reader the essential part of the idea of the proof and help the reader's understanding of our results. We want to emphasize that such a reduction was not known even for ordinary differential equations.

Example. Let us consider the following ordinary differential equation:

$$(x^2D_x - a)u(x) = 0, \qquad (a \in \boldsymbol{C}). \tag{0.1}$$

If $a \neq 0$, (0.1) is clearly an equation with irregular singularities.

Now consider the following correspondences (0.2) and (0.3).

$$(0.2)\quad \begin{pmatrix} u \\ -xD_x u \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{aD_x}} I_1(2\sqrt{aD_x}) & -2\sqrt{aD_x}K_1^*(2\sqrt{aD_x}) \\ I_0(2\sqrt{aD_x}) & 2aD_x K_0^*(2\sqrt{aD_x}) \end{pmatrix} \begin{pmatrix} w_1 \\ w_2 \end{pmatrix},$$

$$(0.3)\quad \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} = \begin{pmatrix} 2aD_x K_0^*(2\sqrt{aD_x}) & 2\sqrt{aD_x}K_1^*(2\sqrt{aD_x}) \\ -I_0(2\sqrt{aD_x}) & \frac{1}{\sqrt{aD_x}} I_1(2\sqrt{aD_x}) \end{pmatrix} \begin{pmatrix} u \\ -xD_x u \end{pmatrix}.$$

Here $I_\nu(\tau) \equiv \left(\frac{\tau}{2}\right)^\nu \sum_{n=0}^{\infty} \frac{(\tau/2)^{2n}}{n!\Gamma(\nu+n+1)}$ and

$$\begin{aligned} K_n^*(\tau) &\equiv (-1)^n I_n(\tau) \log(\tau/2) + K_n(\tau) \\ &= \frac{(-1)^n}{2} \sum_{k=0}^{\infty} \frac{\psi(k+1)+\psi(k+n+1)}{k!(n+k)!} \left(\frac{\tau}{2}\right)^{n+2k} \\ &\qquad + \frac{1}{2}\sum_{r=0}^{n-1} (-1)^r \frac{(n-r-1)!}{r!} \left(\frac{\tau}{2}\right)^{2r-n}, \end{aligned}$$

where $\psi(n) = \sum_{k=0}^{n-1} \frac{1}{k} - \gamma$ with Euler's constant $\gamma = 0.57721\cdots$. Note that operators used in these correspondences are actually linear differential operators (of infinite order).

Then the correspondence (0.2) (resp., the correspondence (0.3)) defines an inverse correspondence of (0.3) (resp., (0.2)), and, furthermore, the equation (0.1) is brought to

$$(0.4)\qquad \begin{pmatrix} x & -a \\ 0 & xD_x \end{pmatrix} \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} = 0.$$

Clearly (0.4) is an equation with regular singularities.

It will be worth mentioning how we have found the transformations (0.2) and (0.3):

We first considered an analytic solution $\exp(-a/x)$ of (0.1) (having $x=0$ as its essential singularity) and a multi-valued holomorphic solution $\exp(-a/x) \cdot \int^x \exp(a/t)dt/t$ of the equation $(x^2D_x - a)u = x$. The last equation implies $(x^2D_x - a)u \equiv 0$ modulo holomorphic functions defined on a neighborhood of the origin. Then we found by direct calculations that these two functions can be obtained by applying operators used in the transformation (0.2) to a/x and 1 in the first case and to a $\log x/x$ and $\log x$ in the second case.

The argument given so far was our starting point and, as a matter of fact, the essential point of the arguments in Chapter IV consists in performing the same manipulation in the general case, namely, we first construct sufficiently many multi-valued holomorphic solutions of the holonomic system in question and next we try to find suitable transformation by operators of infinite order so that these solutions are transformed into functions with moderate growth properties. (See also Chapter IV, § 1 for the idea of the proof.) Needless to say, performing this idea in general case is a very hard task to do as is seen below. Of course, our laborious efforts are rewarded not only by this result itself but also by its fruitful by-products (Chapter V and Chapter VI). Among them, we like to call the reader's attention to the following results which are basic and important in applications:

(i) *For an analytic subset* Y *of* X *and a holonomic* $\mathscr{D}_X$*-Module* $\mathscr{M}$ *with R.S.,* $\mathscr{H}^k_{[Y]}(\mathscr{M})$ *has R.S. and* $\mathscr{D}^\infty_X \underset{\mathscr{D}_X}{\otimes} (\mathscr{H}^k_{[Y]}(\mathscr{M})) = \mathscr{H}^k_Y(\mathscr{D}^\infty_X \underset{\mathscr{D}_X}{\otimes} \mathscr{M})$ *holds for any* k. (Chapter V, § 4.)

(ii) *For holonomic* $\mathscr{E}_X$*-Modules* $\mathscr{M}$ *and* $\mathscr{N}$ *with R.S.,*

$$\mathbf{R}\mathscr{H}om_{\mathscr{E}_X}(\mathscr{M},\mathscr{N}) \cong \mathbf{R}\mathscr{H}om_{\mathscr{E}_X}(\mathscr{M}, \mathscr{E}^\infty_X \underset{\mathscr{E}_X}{\otimes} \mathscr{N})$$

holds. (Chapter VI, § 1.)

(iii) *For a projective map* $F: X \to Y$ *and a holonomic* $\mathscr{D}_X$*-Module* $\mathscr{M}$ *with R.S.,* $R^k F_*(\mathscr{D}_{Y \leftarrow X} \underset{\mathscr{D}_X}{\otimes} \mathscr{M})$ *is a holonomic* $\mathscr{D}_Y$*-Module with R.S.* (Chapter VI, § 2.)

The plan of this paper is as follows.

Chapter I. Basic Properties of Holonomic Systems

In Section 1, after an algebraic preparation, we give the definition of a holonomic system with R.S., which is an abbreviation of regular singularities (Definition 1.1.16). Some elementary results on such systems are also given. Note that we define the notion "with R.S." as a property of the system at generic points of its characteristic variety. However, we prove that a holonomic system with R.S. has regular singularities along any involutory variety containing the characteristic veriety of the system (Chapter V, § 1, Corollary 5.1.7). Also the validity of the comparison theorems (Chapter VI, § 3, Theorem 6.3.1.

and § 4, Theorem 6.4.1) will justify our usage of the terminology "with R.S.". After defining a holonomic system with R.S., we introduce the notion of regular part $\mathscr{M}_{\mathrm{reg}}$ of a holonomic system $\mathscr{M}$ (Definition 1.1.19). It is an $\mathscr{E}$-sub-Module of $\mathscr{M}^{\infty} \underset{\mathrm{def}}{=} \mathscr{E}^{\infty} \underset{\mathscr{E}}{\otimes} \mathscr{M}$ (Proposition 1.1.20). We later (Chapter I, § 3) analyze the structure of $\mathscr{M}$ on the non-singular locus of its characteristic variety and find that $\mathscr{M}_{\mathrm{reg}}$ is a holonomic $\mathscr{E}$-Module with R.S. there. We eventually (Chapter V, § 2) prove that $\mathscr{M}_{\mathrm{reg}}$ is actually a holonomic $\mathscr{E}$-Module with R.S. The most important result of this article is to prove that $\mathscr{E}^{\infty} \underset{\mathscr{E}}{\otimes} \mathscr{M}_{\mathrm{reg}} = \mathscr{E}^{\infty} \underset{\mathscr{E}}{\otimes} \mathscr{M}$ holds for any holonomic $\mathscr{E}$-Module $\mathscr{M}$ (Chapter V, § 2, Theorem 5.2.1). This is the precise meaning of the statement "any holonomic system can be transformed into a holonomic system with regular singularities".

In Section 2 we prove several Hartogs' type theorems for $\mathscr{E}_X$-Modules, namely, the vanishing of $\mathscr{E}xt^j_Z(\mathscr{M}, \mathscr{N})$, $\mathscr{E}xt^j_Z(\mathscr{M}, \mathscr{N}^{\infty})$ and $\mathscr{E}xt^j_Z(\mathscr{M}, \mathscr{N}^{\infty}/\mathscr{N})$ for $j < \mathrm{codim}_{T^*X} Z - \mathrm{projdim}\, \mathscr{N}$ for coherent $\mathscr{E}$-Modules $\mathscr{M}$ and $\mathscr{N}$ (Theorems 1.2.1 and 1.2.2). Here and in what follows, for an $\mathscr{E}$-Module $\mathscr{N}$, $\mathscr{N}^{\infty}$ denotes $\mathscr{E}^{\infty} \underset{\mathscr{E}}{\otimes} \mathscr{N}$. These results will play important roles in our subsequent arguments. For example, we often use these results in the following manner (Corollary 1.2.3): Let $\mathscr{M}$ be a holonomic $\mathscr{E}$-Module. If a section s of $\mathscr{M}^{\infty}$ belongs to $\mathscr{M}$ at generic points of Supp $\mathscr{M}$, then s belongs to $\mathscr{M}$ everywhere. (See also Proposition 1.3.8 in the next section, where we find that supp s is an analytic set.)

In Section 3 we determine the structure of $\mathscr{M}^{\infty}$ for a holonomic $\mathscr{E}$-Module $\mathscr{M}$ with non-singular characteristic variety (Lemma 1.3.4). After a quantized contact transformation which brings Supp $\mathscr{M}$ to a conormal bundle of a non-singular hypersurface $\{x \in X;\, x_1 = 0\}$, $\mathscr{M}^{\infty}$ has the form $\underset{\text{finite}}{\otimes} \mathscr{M}^{\infty}_{\lambda_j, m_j}$ with $\mathscr{M}_{\lambda,m} = \mathscr{E}/(\mathscr{E}(x_1 D_1 - \lambda)^m + \mathscr{E} D_2 + \cdots + \mathscr{E} D_n)$. Several basic properties of $\mathscr{M}_{\mathrm{reg}}$ follows from this structure theorem (Propositions 1.3.5 and 1.3.6). For example: $\mathscr{M}_{\mathrm{reg}}$ is a holonomic $\mathscr{E}$-Module with R.S. on the non-singular locus of the support of $\mathscr{M}$.

We also use the structure of $\mathscr{M}^{\infty}$ studied in this section to show that, for a coherent $\mathscr{E}$-Module $\mathscr{M}$ such that $\mathscr{E}xt^j_{\mathscr{E}}(\mathscr{M}, \mathscr{E}) = 0$ for $j \neq r$, the support of a section s of $\mathscr{M}^{\infty}$ is an analytic set (Proposition 1.3.8). This result often plays an important role when we want to use the results in Section 2.

In Section 4 we first recall several elementary results on the structure of $\mathscr{E}xt^j_{\mathscr{D}}(\mathscr{M}, \mathscr{O})$ for a holonomic $\mathscr{D}$-Module $\mathscr{M}$. One important property of $\mathscr{E}xt^j_{\mathscr{D}}(\mathscr{M}, \mathscr{O})$ is that it is a constructible sheaf. A naturally raised question is how much the structure of $\mathcal{M}$ is determined by these solution sheaves. Theorem 1.4.9 gives a clear answer to this question: The structure of $\mathcal{M}^\infty \underset{\text{def}}{=} \mathcal{D}^\infty \underset{\mathcal{D}}{\otimes} \mathcal{M}$ is completely determined by $\mathbf{R}\mathscr{H}om_{\mathcal{D}}(\mathcal{M}, \mathcal{O})$. We often refer to this result as "Reconstruction Theorem", because it asserts that $\mathcal{M}^\infty$ is reconstructed from $\mathbf{R}\mathscr{H}om_{\mathcal{D}}(\mathcal{M}, \mathcal{O})$ $(=\mathbf{R}\mathscr{H}om_{\mathcal{D}^\infty}(\mathcal{M}^\infty, \mathcal{O}))$. We emphasize that the use of linear differential operators of infinite order is crucial in getting such an isomorphism.

In Section 5 we recall the definition of principal symbols for a system of micro-differential equations with regular singularities, which was given in [18]. Then we discuss more precisely this notion applied to a holonomic system $\mathcal{M}$ with regular singularities along a Lagrangian submanifold. In this case we can define a kind of indicial equations (§ 5.2). The order of a section u of $\mathcal{M}$ is, by definition, the set of the roots of the indicial equations introduced here. Then using this notion of the order, we see that there exists a subset Z of $\boldsymbol{C}$ such that, for any holonomic $\mathcal{E}$-Module $\mathcal{M}$ with R.S., $\mathcal{N} \underset{\text{def}}{=} \{u \in \mathcal{M};\ \text{ord}\, u \subset Z\}$ is a coherent $\mathcal{E}_\Lambda$-Module, where $\Lambda = \text{Supp}\, \mathcal{M}$. (Proposition 1.5.8.)

In Section 6 we prepare some elementary results in symplectic geometry which we shall need in later sections. The main result is Corollary 1.6.4 which guarantees that any Lagrangian variety Λ can be brought to a generic position in the sense of Definition 1.6.3 by a homogeneous canonical transformation.

Chapter II. Holonomic Systems of D-Type

In Section 1 we explain how the notion of integrable connections is reinterpreted by the language of $\mathcal{D}$-Modules.

In Section 2 we first recall the definition of (strict) Nilsson class functions (associated with a locally constant sheaf L of finite rank on $X-Y$ for a hypersurface Y). We denote by $\mathcal{L}$ (resp., $\mathcal{L}_0$) the subsheaf of $j_*(L \underset{\boldsymbol{C}}{\otimes} \mathcal{O}_{X-Y})$ consisting of sections in the Nilsson (resp., strict Nilsson) class. Here j is the embedding map from $X-Y$ into X. Note that $j_*(L \underset{\boldsymbol{C}}{\otimes} \mathcal{O}_{X-Y})$ acquires a structure of $\mathcal{D}_X^\infty$-Module canonically. Then the results of Deligne [3] assert that $\mathcal{L}_0$ is coherent over $\mathcal{O}_X$. Hence $\mathcal{L}$ is coherent over $\mathcal{D}_X$. Furthermore $\mathcal{L}$ is a holonomic $\mathcal{D}$-Module with R.S. on T_Y^*X and $\mathscr{H}^k_{[Y]}(\mathcal{L}) = 0$ holds for any k (Theorems 2.2.1 and 2.2.2). It also follows from [3] that a Hartogs' type result holds for $\mathcal{L}$ and $\mathcal{L}_0$ (Theorem 2.2.1 (iii)). Since the results proved in [3] are stated in a different manner, we give in Appendix C some supplementary arguments which are intended to fill the apparent gap between the results in [3] and our statement of the results. When we introduce the notion of a holonomic $\mathscr{D}$-Module of D-type in the next section, the properties of $\mathscr{L}$ stated in Theorems 2.2.1 and 2.2.2 are used as the defining properties of such a system. Here "D-type" is an abbreviation of "Deligne-type". Using another result essentially given in [3] (see also Appendix C) and "Reconstruction Theorem" proved in Chapter I, Section 4, we find in Theorem 2.2.4.

$$\mathscr{D}^{\infty} \underset{\mathscr{D}}{\otimes} \mathscr{L} = j_*(L \underset{\boldsymbol{C}}{\otimes} \mathcal{O}_{X-Y}). \tag{0.5}$$

This result implies that any multi-valued section of L over $X-Y$ can be obtained by applying a linear differential operator of infinite order to a section in the Nilsson class. This result will play an important role in Chapter IV (through the results in Chapter III, § 4).

In Section 3 we introduce the notion of a holonomic system of D-type along a hypersurface $Y \subset X$ (Definition 2.1.1). It immediately follows from this definition and the results obtained in the preceding section that the category of holonomic systems of D-type is isomorphic to the category of locally constant sheaves of finite rank on $X-Y$ (Theorem 2.3.2.(i)). We also prove several basic results on a holonomic system of D-type (Propositions 2.3.3 and 2.3.4). Among them, the following two results are particularly important.

(0.6) *For a holonomic system $\mathscr{L}$ of D-type along $Y \subset X$ and a hypersurface $S \subset X$, we have $\mathscr{D}^{\infty} \underset{\mathscr{D}}{\otimes} (\mathscr{H}^k_{[S]}(\mathscr{L})) = \mathscr{H}^k_S(\mathscr{D}^{\infty} \underset{\mathscr{D}}{\otimes} \mathscr{L})$.*

(0.7) *Let Z be a hypersurface of X. Let $\mathscr{M}$ be a holonomic $\mathscr{D}_X$-Module with R.S. on T^*X such that $\mathrm{SS}(\mathscr{M}) \subset \pi^{-1}(Z) \cup T^*_X X$. Then $\mathscr{L} \underset{\mathrm{def}}{=} \mathscr{H}^0_{[X|Z]}(\mathscr{M})$ is of D-type.*

Actually, (0.6) is the most essential ingredient of the proof of the results in Chapter V, Section 4. The result (0.7) gives an important link between D-type equations and general holonomic $\mathscr{D}$-Modules with R.S. We also prove a result (Proposition 2.3.7) which characterizes the strict Nilsson class function in terms of the notion of the order introduced in Chapter I, Section 5.

Chapter III. Action of Micro-Differential Operators on Holomorphic Functions

In Section 1 we clarify the action of $\mathfrak{E}(G; D)$ on holomorphic functions. Here D is a G-round open set and $\mathfrak{E}(G; D)$ is the space of operators with finite propagation speed. (See [19] § 3 for the definition of $\mathfrak{E}(G; D)$ etc.) The action of $\mathfrak{E}(G; D)$ is defined in [19] in a purely cohomological way, especially by the aid of residue maps. So we first chase the residue map concretely by making use of the Čech cohomology (§ 1.2). Next we consider a subclass of $\mathfrak{E}(G; D)$ which is easy to manipulate and, at the same time, ample enough for later applications. (§ 1.3.) For an element in such a subclass we can concretely find its representative as a cohomology class of a cohomology group of a Stein covering. Such a representation enables us to write down, as an integral operator, the action of the element on a suitable relative cohomology group with the sheaf of holomorphic functions as coefficients (Proposition 3.1.5).

In Section 2 we apply the results obtained in Section 1 to study the action of micro-differential operators on a space of holomorphic functions (Proposition 3.2.1). At the end of this section we exemplify our result by applying it to the case where micro-differential operators act on the sheaf of microfunctions etc.

In Section 3 we introduce a special class of micro-differential operators which we call $\tilde{\mathscr{E}}^\infty$. As we show there, $\tilde{\mathscr{E}}^\infty$ can be identified with a subsheaf of $\mathscr{E}^\infty$ and, at the same time, it is contained in $\mathfrak{E}(G; D)$ for G contained in a complex line. The sheaves $\tilde{\mathscr{E}}^\infty$ and $\tilde{\mathscr{E}}$ play important roles in Chapter IV.

In Section 4 we first review some basic notions concerning multi-valued holomorphic functions after Sém. Cartan-Serre 1951/52, and next we study concretely how an element in $\tilde{\mathscr{E}}^\infty_0$, i.e., a germ of $\tilde{\mathscr{E}}^\infty$ at 0, acts on a space of multivalued functions considered there. We also introduce the notion of the holonomic $\mathscr{D}$-Module $\mathscr{L}(\mathfrak{a})$ of D-type with singularities along a hypersurface S and with the monodromy type $\mathfrak{a}$ for an ideal $\mathfrak{a}$ of $\boldsymbol{C}[\pi_1(X-S)]$.

In Section 5 we construct a special resolution of a holonomic $\mathscr{E}$-Module whose characteristic variety is in a generic position so that we may analyze the structure of holomorphic solutions of such a system. For this purpose we introduce a subring R (resp., R^∞) of $\tilde{\mathscr{E}}_0$ (resp., $\tilde{\mathscr{E}}^\infty_0$) which is easy to manipulate algebraically. Note that the principal symbol of an element in R of order 0 belongs to $\mathscr{O}_X[\xi_1/\tau,\dots, \xi_n/\tau]$, where $(t, x; \tau, \xi)$ is the coordinate system of $T^*X(\cong T^*\boldsymbol{C}^{n+1})$. The precise conditions on the special resolution which we use in Chapter IV are stated in Theorem 3.5.8.

Chapter IV. Embedding Holonomic Systems in Holonomic Systems of D-Type

In Section 1 we give a precise statement of the embedding theorem appearing as a title of this chapter. The proof is given in the subsequent sections of this chapter. The theorem (Theorem 4.1.1) is as follows:

Let $\mathscr{M}$ be a holonomic $\mathscr{E}_X$-Module defined on a neighborhood of $p_0 \in T^*X - T^*_XX$. Assume that the characteristic variety Λ of $\mathscr{M}$ is in a generic position at p_0. Then there exist a holonomic $\mathscr{D}_X$-Module $\mathscr{N}$ defined on a neighborhood of $q_0 = \pi(p_0)$ and a $\mathscr{D}^{\infty}_{X,q_0}$-linear homomorphism ϕ from $\mathscr{M}^{\infty}_{p_0} \underset{\text{def}}{=} (\mathscr{E}^{\infty}_X \underset{\mathscr{E}_X}{\otimes} \mathscr{M})_{p_0}$ into $\mathscr{N}^{\infty}_{q_0} \underset{\text{def}}{=} (\mathscr{D}^{\infty}_X \underset{\mathscr{D}_X}{\otimes} \mathscr{N})_{q_0}$ which satisfy the following conditions:

(0.8) *There exist an integer r and a holonomic system $\mathscr{L}$ of D-type with singularities along $\pi(\Lambda)$ such that $\mathscr{N} = \mathscr{L}/\mathscr{O}^r_X$ holds.*

(0.9) *The homomorphism $\tilde{\phi}$ from $\mathscr{M}^{\infty}_{p_0}$ into $\mathscr{E}^{\infty}_{p_0} \underset{\mathscr{D}_{q_0}}{\otimes} \mathscr{N}_{q_0} = \mathscr{E}^{\infty}_{p_0} \underset{\mathscr{D}^{\infty}_{q_0}}{\otimes} \mathscr{N}^{\infty}_{q_0}$ defined by $\tilde{\phi}(s) = 1 \otimes \phi(s)$ is an injective $\mathscr{E}^{\infty}_{p_0}$-linear homomorphism.*

In Section 2 we prepare some elementary results concerning the geometry of $S = \pi(\Lambda) \subset X$ under the condition that a Lagrangian variety $\Lambda \subset T^*X - T^*_XX$ is in a generic position. Throughout this chapter we assume that $\dim X = 1 + n$ and take a suitable coordinate system $(t, x; \tau, \xi) = (t, x_1, \dots, x_n; \tau, \xi_1, \dots, \xi_n)$ of T^*X such that the fundamental 1-form ω equals $\tau dt + \sum_{j=1}^{n} \xi_j dx_j$. We denote the point $(0; dt)$ by p_0 and $\pi(p_0) (=0)$ by q_0. The projection from X to $\mathbb{C}^n$ defined by $(t, x) \mapsto x$ shall be denoted by F. We also denote by $B(\varepsilon, \delta)$ (resp., $B(\varepsilon)$) the set $\{(t, x) \in X; |t| < \delta, |x| < \varepsilon\}$ (resp., $\{x \in \mathbb{C}^n; |x| < \varepsilon\}$. If follows from the assumption that there exist positive constants δ_0 and ε_0 with $\varepsilon_0 < \delta_0$ and an analytic subset $H \subset B(\varepsilon_0)$ such that

(0.10) $\quad S \cap (B(\varepsilon_0, \delta_0) - F^{-1}(H)) \xrightarrow{F} B(\varepsilon_0) - H$ is a finite covering.

We denote by G_0 the closed convex cone $\{(t, x) \in \mathbb{C}^{1+n}; x = 0, \operatorname{Im} t = 0, \operatorname{Re} t \leqq 0\}$.

In Section 3 we construct the following resolution of $\mathscr{M}$:

(0.11)
$$0 \leftarrow \mathscr{M} \leftarrow \mathscr{E}^{N_0}_X \xleftarrow{P_0} \mathscr{E}^{N_1}_X \leftarrow \cdots \xleftarrow{P_{r-1}} \mathscr{E}^{N_r}_X \longleftarrow 0,$$

where P_j are matrices whose components belong to $\tilde{\mathscr{E}}_{p_0}$ and are of strictly negative order.

Furtheremore (0.11) is exact on $\{(t, x; \tau, \xi) \in \Lambda; |t|, |x| \ll 1\}$ and

$$0 \longleftarrow \mathscr{E}_X^{N_0} \xleftarrow{P_0} \mathscr{E}_X^{N_1} \longleftarrow \cdots \xleftarrow{P_{r-1}} \mathscr{E}_X^{N_r} \longleftarrow 0$$

is exact on $\{(t, x; \tau, \xi) \in T^*X - \Lambda; \tau \neq 0\}$. Then we can find a integro-differential operator $K_j(t_1, t_2, x, D_x)$ defined on $\{(t_1, t_2, x); |t_1|, |t_2| < \delta_0, |x| < \varepsilon_0\}$ so that P_j has the form K_j. Setting $D = B(\varepsilon_0, \delta_0)$, we obtain a complex $\mathfrak{M}_.$ of $\mathfrak{E}(G_0; D)$-module by

$$\mathfrak{M}_. : 0 \leftarrow \mathfrak{E}(G_0; D)^{N_0} \xleftarrow{K_0} \cdots \xleftarrow{K_{r-1}} \mathfrak{E}(G_0; D)^{N_r} \longleftarrow 0.$$

We will use this complex to discuss the extensibility of holomorphic solutions of $\mathscr{M}$.

In Section 4 we prove some vanishing theorems for relative cohomology groups related to $\mathscr{M}$ so that we may later (§ 6) apply the results to extend multi-valued holomorphic solutions of $\mathscr{M}$ across (a family of) non-characteristic hypersurfaces. Their proof essentially relies on Theorem 4.5.1 of [19].

In Section 5 we apply the method developed in [13] to prove that holomorphic solutions of $\mathscr{M}$ can be prolonged to a multi-valued holomorphic solutions with finite determination property. In order to clarify the meaning of "holomorphic solutions of $\mathscr{M}$", we introduce an $\mathscr{E}_{p_0}^{\mathbf{R}}$-module C. An element η in C is represented by a holomorphic function φ defined on $V - Z$ modulo holomorphic functions on V for an open neighborhood V of q_0 and a closed set $Z \subset \mathbf{C}^{1+n}$ with its tangent cone $C_{q_0}(Z)$ at q_0 being contained in $\{(t, x) \in \mathbf{C}^{1+n}; \operatorname{Re} t \geqq 0\}$. We call the holomorphic function φ a representative of η. In the sequel we denote by $\mathscr{Z}$ the set of closed subsets $Z \subset \mathbf{C}^{1+n}$ such that its normal cone at q_0 $C_{q_0}(Z)$ is contained in $\{(t, x) \in \mathbf{C}^{1+n}; \operatorname{Re} t \geqq 0\}$. Then the main result (Theorem 4.5.2) in this section is as follows:

Let ϕ be in $\operatorname{Hom}_{\mathscr{E}_{p_0}}(\mathscr{M}_{p_0}, C)$, s in $\mathscr{M}_{p_0}$ and φ a representative of $\phi(s) \in \mathscr{C}$. Then there exist an open neighborhood of q_0 and a multi-valued holomorphic function $\tilde{\varphi}$ on $V - S$ such that a branch of $\tilde{\varphi}$ coincides with φ on $V - Z$ for some $Z \in \mathscr{Z}$.

Furthermore, the monodromy property of thus obtained $\tilde{\varphi}$ is essentially invariant under the action of micro-differential operator $P \in \mathscr{E}_{p_0}$ (Theorem 4.5.3).

In Section 6 and Section 7 we give the proof of Theorem 4.1.1. We first describe the structure of $\operatorname{Hom}_{\mathscr{E}_{p_0}}(\mathscr{M}_{p_0}, C)$ by using the results proved in Section 4. For this purpose we take a point x_1 in $B(\varepsilon_0) - H$ and denote by p_j $(j = 1, \ldots, N)$ the points in $S \cap F^{-1}(x_1)$. Then we have

$$(0.12) \qquad \mathrm{Hom}_{\mathscr{E}_{p_0}}(\mathscr{M}_{p_0}, C) = \bigoplus_{j=1}^{N} \mathscr{H}om_{\mathscr{E}}(\mathscr{M}, \mathscr{C}_{S|X}^{\mathbf{R}})_{p_j}$$

In particular, (0.12) implies that $\mathrm{Hom}_{\mathscr{E}_{p_0}}(\mathscr{M}_{p_0}, C)$ is finite-dimensional.

For generators s_j $(1 \leqq j \leqq N_0)$ of $\mathscr{M}$ and $\phi \in \mathrm{Hom}_{\mathscr{E}_{p_0}}(\mathscr{M}_{p_0}, C)$, we denote by φ_j a representative of $\phi(s_j)$. Then φ_j can be extended to a multi-valued holomorphic function $\tilde{\varphi}_j$ on $B(\varepsilon_1, \delta_1) - S$. Next, for $\sigma \in \pi \underset{\mathrm{def}}{=} \pi_1(B(\varepsilon_0, \delta_0) - S)$ and $\phi \in \mathrm{Hom}(\mathscr{M}_{p_0}^{\infty}, C)$, we define ϕ^{σ} as follows:

For $s \in \mathscr{M}_{p_0}^{\infty}$, take a representative φ of $\phi(s)$ and continue φ to a multi-valued holomorphic function $\tilde{\varphi}$ on $V - S$. Then $\phi^{\sigma}(s)$ is defined by the element given by $\sigma(\tilde{\varphi}) \in C$.

Thus we obtain a finite-dimensional representation $\mathrm{Hom}(\mathscr{M}_{p_0}, C)$ of π. We define ideals $\mathfrak{c}$ and $\mathfrak{a}$ of $C[\pi]$ by the following:

(0.13) $\mathfrak{c} = \{\sigma \in C[\pi];\ \sigma(\varphi)$ is holomorphic near q_0 for any $\phi \in \mathrm{Hom}_{\mathscr{E}_{p_0}}(\mathscr{M}_{p_0}, C)$ and any representative φ of any element of $\phi(\mathscr{M}_{p_0})\} = \{\sigma \in C[\pi];\ \phi^{\sigma} = 0$ for any $\phi \in \mathrm{Hom}(\mathscr{M}_{p_0}, C)\}$.

$$(0.14) \qquad \mathfrak{a} = \sum_{\gamma \in \pi} (\gamma - 1)\mathfrak{c}.$$

We denote by $\mathscr{L}$ the holonomic system of D-type with the monodromy type $\mathfrak{a}$. Then $\mathscr{L}$ contains $\mathscr{O}$ as a $\mathscr{D}$-sub-Module and

$$(0.15) \qquad \mathscr{H}om_{\mathscr{D}}(\mathscr{O}, \mathscr{L}/\mathscr{O})_{q_0} = 0.$$

Let $\mathscr{N}$ denote $\mathscr{L}/\mathscr{O}$. After these preparations, we easily find the following $\mathscr{D}_{q_0}^{\infty}$-linear map $E(\phi)$ from $\mathscr{M}_{p_0}^{\infty}$ to $\mathscr{N}_{q_0}^{\infty}$ is well-defined for $\phi \in \mathrm{Hom}_{\mathscr{E}_{p_0}}(\mathscr{M}_{p_0}, C)$.

(0.16) For $s \in \mathscr{M}_{p_0}^{\infty}$ we choose a representative φ of $\phi(s)$. Then $E(\phi)(s)$ is, by definition, $\varphi \bmod \mathscr{O}_{q_0}$. Furthermore, if we define a C-linear homomorphism $F(\phi)$ from $\mathscr{M}_{p_0}^{\infty}$ to $\mathscr{E}_{p_0}^{\infty} \underset{\mathscr{D}_{q_0}^{\infty}}{\otimes} \mathscr{N}_{q_0}^{\infty}$ by

$$F(\phi)\colon s \longmapsto 1 \otimes E(\phi)(s),$$

we can verify that $F(\phi)$ is actually $\mathscr{E}_{p_0}^{\infty}$-linear. (Proposition 4.7.1.) Finally, we define an $\mathscr{E}_{p_0}^{\infty}$-linear map Φ from $\mathscr{M}_{p_0}^{\infty}$ into $\mathscr{E}_{p_0}^{\infty} \underset{\mathscr{D}_{q_0}}{\otimes} \mathscr{N}_{q_0}^{r}$ for a base $\{\phi_1, \dots, \phi_r\}$ of $\mathrm{Hom}_{\mathscr{E}_{p_0}}(\mathscr{M}_{p_0}, C)$ by

$$\Phi = F(\phi_1) \oplus \cdots \oplus F(\phi_r),$$

and we verify that Φ is injective. At last, this completes the proof of Theorem 4.1.1 stated in Section 1.

Chapter V. Basic Properties of Holonomic Systems with R.S.

In Section 1 we prove several basic properties of holonomic systems with R.S. which are derived from the embedding theorem proved in Chapter IV. The first one (Theorem 5.1.1) asserts that a holonomic $\mathscr{E}$-Module $\mathscr{M}$ with R.S. whose characteristic variety is in a generic position is actually a $\mathscr{D}$-Module; more precisely, we have the following result:

Let $\mathscr{M}$ *be a holonomic* $\mathscr{E}$*-Module with R.S. defined near* $p\in T^*X-T^*_XX$. *Assume that* Supp $\mathscr{M}$ *is in a generic position at p. Then* $\mathscr{M}_p$ *is a finitely generated* $\mathscr{D}_{\pi(p)}$*-module. Furthermore we have*

$$(0.17)\qquad \mathscr{E}_q \underset{\mathscr{D}_{\pi(p)}}{\otimes} \mathscr{M}_p = \begin{cases} \mathscr{M}_p, & (q=p), \\ 0, & (q\in \pi^{-1}\pi(p)-T^*_XX-\boldsymbol{C}^\times p). \end{cases}$$

The second main result (Theorem 5.1.5) in this section is as follows:

Let $\mathscr{M}$ *be a holonomic* $\mathscr{E}$*-Module with R.S. defined near* $p\in T^*X-T^*_XX$. *Assume that* Sup $\mathscr{M}$ *is in a generic position at p. Let* $\mathscr{M}_0$ *be a coherent* $\mathscr{E}(0)$*-sub-Module of* $\mathscr{M}$. *Then* $\mathscr{M}_{0,p}$ *is an* $\mathscr{O}_{\pi(p)}$*-module of finite type.*

The third main result (Theorem 5.1.6) implies that, for any holonomic $\mathscr{E}$-Module with R.S. $\mathscr{M}$, we can canonically construct a coherent $\mathscr{E}(0)$-sub-Module $\mathscr{M}_0$ by the aid of the notion of orders. It reads as follows:

Let c be a real number and $\mathscr{M}$ *a holonomic* $\mathscr{E}$*-Module with R.S. defined on an open set* $\Omega\subset T^*X-T^*_XX$. *Denote* Supp $\mathscr{M}$ *by* Λ. *Let* $\mathscr{M}_0$ *be the subsheaf of* $\mathscr{M}$ *given by* $U\mapsto\{s\in\mathscr{M}(U);\ \mathrm{ord}_p(s)\subset\{\lambda\in\boldsymbol{C};\ \mathrm{Re}\,\lambda\leqq c\}$ *for any point p of* $U\cap\Lambda_{\mathrm{reg}}\}$. *Then* $\mathscr{M}_0$ *satisfies the following conditions*:

(i) $\mathscr{M}_0$ *is a coherent* $\mathscr{E}(0)|_\Omega$*-Module.*

(ii) $\mathscr{M}=\mathscr{E}\mathscr{M}_0$ *and* $\mathscr{M}_0=\mathscr{E}_\Lambda\mathscr{M}_0$.

(iii) *For any closed analytic subset W of an open subset U of* T^*X *such that* codim $W\geqq\dim X+1$, *we have* $\mathscr{H}^0_W(\mathscr{M}/\mathscr{M}_0)=0$.

As an important corollary of this result we find the following:

Let $\mathscr{M}$ *be a holonomic* $\mathscr{E}$*-Module with R.S. Let V be an involutory analytic set containing* Supp $\mathscr{M}$. *Then* $\mathscr{M}$ *has regular singularities along* $V-T^*_XX$ *(Corollary 5.1.7).*

Theorem 5.1.6 is also used to prove the global existence of a good filtration of a holonomic $\mathscr{D}$-Module with R.S. (Corollary 5.1.11).

In Section 2 we give the proof of our main result (Theorem 5.2.1) which asserts that, for any holonomic $\mathscr{E}_X$-Module $\mathscr{M}$ defined near $p_0 \in T^*X$, $\mathscr{M}_{\text{reg}}$ is a holonomic (in particular, coherent) $\mathscr{E}_X$-Module, and $\mathscr{E}^\infty \underset{\mathscr{E}}{\otimes} \mathscr{M} = \mathscr{E}^\infty \underset{\mathscr{E}}{\otimes} \mathscr{M}_{\text{reg}}$ holds near p_0. The proof of this theorem follows from Theorem 4.1.1 on $T^*X - T^*_X X$, while near $T^*_X X$ the proof requires further considerations. As its consequence we obtain the following result:

Let $\mathscr{L}$ be a holonomic system of D-type. Then $\mathscr{L}$ is a holonomic $\mathscr{D}$-Module with R.S.

In Section 3 we prove that the restriction of a holonomic $\mathscr{E}$-Module with R.S. to a non-characteristic submanifold yields a holonomic system with R.S. and that the integration of a holonomic $\mathscr{E}_Y$-Module with R.S. $\mathscr{N}$ along fiber $\varphi: Y \to X$ yields a holonomic $\mathscr{E}_X$-Module with R.S. $\varphi_* \mathscr{N}$, if $\rho_\varphi^{-1}\text{Supp}\,\mathscr{N} \cap \varpi_\varphi^{-1}(U) \to U$ is a finite map. The proof again makes essential use of the embedding theorem. We also use the fact proved in Section 2 that a holonomic system of D-type is with R.S.

In Section 4 we discuss the restriction of a holonomic $\mathscr{D}$-Module with R.S. $\mathscr{M}$ to an arbitrary submanifold, which is not necessarily non-characteristic with respect to $\mathscr{M}$. In the course of the discussion, we obtain some results which are used in Chapter VI for the proof of several comparison theorems. The main result (Theorem 5.4.1) of this section is as follows:

Let Y be an analytic subset of X and $\mathscr{M}$ a holonomic $\mathscr{D}_X$-Module with R.S. Then we have

(i) *$\mathscr{H}^k_{[Y]}(\mathscr{M})$ and $\mathscr{H}^k_{[X|Y]}(\mathscr{M})$ have R.S. for any k.*

(ii) *$\mathscr{D}_X^\infty \underset{\mathscr{D}_X}{\otimes} (\mathscr{H}^k_{[Y]}(\mathscr{M})) = \mathscr{H}^k_Y(\mathscr{D}_X^\infty \underset{\mathscr{D}_X}{\otimes} \mathscr{M})$ holds for any k.*

(iii) *$\mathscr{D}_X^\infty \underset{\mathscr{D}_X}{\otimes} (\mathscr{H}^k_{[X|Y]}(\mathscr{M})) = \mathscr{H}^k_{X|Y}(\mathscr{D}_X^\infty \underset{\mathscr{D}_X}{\otimes} \mathscr{M})$ holds for any k.*

We prove this result first for a holonomic system of D-type. For such a system, this follows easily from the results in Chapter II, Section 3. The general case is proved by the induction on the condimension of Supp $\mathscr{M}$. We note that (ii) is obtained by Mebkhout [20] for the special case where $\mathscr{M} = \mathscr{O}_X$. As an immediate consequence of the result stated above we see that, for a submanifold Y of X, $\mathscr{T}or_k^{\mathscr{O}_X}(\mathscr{O}_Y, \mathscr{M})$ is a holonomic $\mathscr{D}_Y$-Module with R.S. for any k, if $\mathscr{M}$ is a holonomic $\mathscr{D}_X$-Module with R.S. (Corollary 5.4.6). In particular, $\mathscr{M}_Y \underset{\text{def}}{=} \mathscr{O}_Y \underset{\mathscr{O}_X}{\otimes} \mathscr{M}$ is a holonomic $\mathscr{D}_Y$-Module with R.S.

Chapter VI. Comparison Theorems

In Section 1 we prove the following comparison theorem (Theorem 6.1.3).

Let $\mathcal{M}$ and $\mathcal{N}$ be two holonomic $\mathcal{E}_X$-Modules with R.S. Then

$$\mathbf{R}\mathcal{H}om_{\mathcal{E}_X}(\mathcal{M}, \mathcal{N}) = \mathbf{R}\mathcal{H}om_{\mathcal{E}_X}(\mathcal{M}, \mathcal{E}_X^\infty \underset{\mathcal{E}_X}{\otimes} \mathcal{N}).$$

This means that the solutions are not altered for holonomic systems with R.S. whether we allow the solutions to have essential singularities or not.

We prove this result first for $\mathcal{D}_X$-Modules by using Theorem 5.4.1 in the preceding chapter, and then prove the general case by using the result obtained for $\mathcal{D}$-Modules.

In Section 2 we generalize a part of the results proved in Chapter V, Section 3 as follows:

Let $F: X \to Y$ be a projective map and $\mathcal{M}$ a holonomic $\mathcal{D}_X$-Module with R.S. Then $R^kF_*(\mathcal{D}_{Y\leftarrow X} \overset{\mathbf{L}}{\underset{\mathcal{D}_X}{\otimes}} \mathcal{M})$ is a holonomic $\mathcal{D}_Y$-Module with R.S. (Theorem 6.2.1.)

The proof of Theorem 6.2.1 is based on the comparison theorem proved in Section 1. Theorem 6.2.1 improves several results of our previous works which make use of the integration along projective fibers of a holonomic $\mathcal{D}$-Modules, in that we find the resulting holonomic $\mathcal{D}$-Module to be with R.S. As an example of such an improvement, we give Theorem 6.2.5, which asserts that the hyperfunction $\prod_{j=1}^{n} f_{j+}^{s_j}$ (Re $s_j \geqq 0$) satisfies a holonomic $\mathcal{D}$-Module with R.S. (Cf. [11])

In Section 3 we prove a comparison theorem between formal power series category and convergent power series category. The theorem (Theorem 6.3.1) reads as follows:

Let $\mathcal{M}$ be a holonomic $\mathcal{D}_X$-Module with R.S. Then for any point x in X and any j, the natural homomorphism

$$\mathcal{E}xt^j_{\mathcal{D}_X}(\mathcal{M}, \mathcal{O}_X)_x \longrightarrow \mathcal{E}xt^j_{\mathcal{D}_X}(\mathcal{M}, \hat{\mathcal{O}}_{X,x}) \tag{0.18}$$

is an isomorphism.

We prove this result by Theorem 6.1.1 by the aid of the duality argument.

In Section 4 we prove the converse of Theorem 6.3.1, namely, we prove the following:

Let $\mathcal{M}$ be a holonomic $\mathcal{D}_X$-Module. Assume that

$$\mathbf{R}\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \mathcal{O}_X)_x \cong \mathbf{R}\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \hat{\mathcal{O}}_{X,x})$$

holds for any x in X. Then $\mathcal{M}$ is with R.S.

We prove this result by the induction on the dimension of X. Note that this has been proved by Malgrange [21] for X with $\dim X=1$. We make essential use of his result. In the course of the proof we prove and use the following result (Theorem 6.4.5), which is interesting by its right.

Let $\mathscr{M}$ be a holonomic $\mathscr{E}_X$-Module with a smooth Largrangian manifold Λ as its characteristic variety defined near $p\in\Lambda$. Let $f(x,\xi)$ be a homogeneous function on T^*X of degree 0 such that $f(p)=0$. Assume that $df(p)$ and $\omega(p)$ are linearly independent and that $df|_\Lambda\neq 0$ at p. Assume furthermore that the restriction of $\mathscr{M}$ to $V_a=\{(x,\xi)\in T^*X; f(x,\xi)=a\}$ has R.S. for any a with $|a|\ll 1$. Then $\mathscr{M}$ itself has R.S. in a neighborhood of p.

At the end of this section, we discuss the relationship between the notion of holonomic $\mathscr{D}$-Modules with R.S. and the notion of Fuchsian systems introduced in an interesting paper of Ramis [23]. He defines the notion of a Fuchsian system for a complex of $\mathscr{D}$-Modules by using the validity of the comparison theorem as its characteristic property. Our results show that a complex of $\mathscr{D}$-Modules is Fuchsian if and only if any of its cohomology groups is with R.S. in our sense. We emphasize that we have derived comparison theorems from the micro-local properties of the systems in question.

Appendix

In the appendix we give proofs of the several statements which are used in this paper and whose reference are difficult to find in spite of the fact that the results themselves are well-known to specialists.

In Section A we give a detailed recipe how to derive results for $\mathscr{D}$-Modules from the corresponding results for $\mathscr{E}$-Modules outside the zero section (i.e., T_X^*X) by adding a dummy variable, namely, by considering $\Phi(\mathscr{M})\underset{\text{def}}{=}\mathscr{E}_C\delta(t)\hat{\otimes}\mathscr{M}$ on $T^*(\boldsymbol{C}\times X)$ for an $\mathscr{E}_X$-Module $\mathscr{M}$. We also discuss the monodromy structure of an $\mathscr{E}$-Module with R.S. (§ A.4) and a good filtation of a $\mathscr{D}$-Module (§ A.5).

In Section B we give a proof of a result on constructible sheaves, as we could not find a suitable reference for its proof.

In Section C we show how to deduce the results in Chapter II, Section 2 from the results proved by Deligne [3], namely, we prove Theorem 2.2.1 in Section C.1 and Theorem 2.2.3 in Section C.2.

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List of Notations

$\boldsymbol{C}^\times = \boldsymbol{C} - \{0\}$

$\boldsymbol{R}^+ = \{c \in \boldsymbol{R};\ c > 0\}$

$\boldsymbol{Z}_+ = \{0, 1, 2, \ldots\}$

X : A complex manifold.

$M \underset{L}{\times} N$: The fiber product of topological spaces M and N over a topological space L.

TX : The tangent bundle of X.

T_xX for a point $x \in X$: The tangent space of X at x.

T^*X : The cotangent bundle of X. The canonical projection from T^*X to X is denoted by π.

T^*_xX for a point $x \in X$: The cotangent space of X at x.

$\boldsymbol{C}^\times p$ for a point p in T^*X: The orbit through p of the multiplicative group $\boldsymbol{C}^\times$ by the action of $\boldsymbol{C}^\times$ on T^*X by $\boldsymbol{C}^\times \ni c$: $(x, \xi) \mapsto (x, c\xi)$ for $(x, \xi) \in T^*X$.

Y_{reg} for an analytic subset Y of X: The submanifold $\{x \in Y$; there exists a neighborhood U of x such that $Y \cap U$ is non-singular.$\}$

$Y_{\text{sing}} \underset{\text{def}}{=} Y - Y_{\text{reg}}$

T^*_YX, where Y is an analytic subset of X: The conormal bundle of Y. If Y is not regular, the conormal bundle T^*_YX means, by definition, the closure of $T^*_{Y_{\text{reg}}}X$ in $\pi^{-1}(Y)$.

P^*X : The projective cotangent bundle, i.e., $(T^*X - T^*_XX)/\boldsymbol{C}^\times$. The canonical projection from $T^*X - T^*_XX$ to P^*X is denoted by γ.

ρ_f, where f is a holomorphic map from Y to X: The canonical projection from $Y \underset{X}{\times} T^*X$ to T^*Y.

ϖ_f, where f is a map from Y to X: The canonical projection from $Y \underset{X}{\times} T^*X$

to T^*X. If there is no fear of confusions, we sometimes omit the subscript f in ρ_f and ϖ_f.

$C_x(S; V)$ for a point x in a manifold M and $S, V \subset M$: The normal cone of S and V at x, i.e., $\{v \in T_xM$; there exist sequences $\{x_n\}$ in S, $\{y_n\}$ in V and $\{a_n\}$ in $\boldsymbol{R}^+$ such that $\{x_n\}$ and $\{y_n\}$ converge to x and that $a_n(x_n - y_n)$ converges to $v\}$.

$C(S; V) = \bigcup_{x \in M} C_x(S; V)$

$C_x(S) = C_x(S; \{x\})$

ω_X : The fundamental 1-form $\sum_j \xi_j dx_j$ on T^*X.

$\{f, g\}$ for holomorphic functions f and g on T^*X: The Poisson bracket of f and g.

$\mathcal{H}om_{\mathcal{R}}(\mathcal{A}, \mathcal{B})$, where $\mathcal{A}$ and $\mathcal{B}$ are sheaves of (left) $\mathcal{R}$-Modules for a sheaf of rings $\mathcal{R}$: The sheaf of $\mathcal{R}$-homomorphisms from $\mathcal{A}$ to $\mathcal{B}$.

$\mathbf{R}\mathcal{H}om(\ ,\ )$: The right derived functor of $\mathcal{H}om(\ ,\ )$.

$\mathcal{E}xt^j_{\mathcal{R}}(\mathcal{A}, \mathcal{B})$: The j-th right derived functor of $\mathcal{H}om_{\mathcal{R}}(\mathcal{A}, \mathcal{B})$. (=The j-th extension group.)

$\mathcal{A} \underset{\mathcal{R}}{\otimes} \mathcal{B}$, where $\mathcal{A}$(resp., $\mathcal{B}$) is a sheaf of left (resp., right) $\mathcal{R}$-Modules: The tensor product of $\mathcal{A}$ and $\mathcal{B}$ over $\mathcal{R}$.

$\overset{\mathbf{L}}{\otimes}$: The left derived functor of $\otimes$.

$\mathcal{T}or^{\mathcal{R}}_j(\mathcal{A}, \mathcal{B})$: The j-th left derived functor of $\otimes$. (=The j-th torsion group.)

$\Gamma(U; \mathcal{F})$, where U is an open set of a topological space M and $\mathcal{F}$ is a sheaf on M: The section module of $\mathcal{F}$ over U.

$\Gamma_Z(U; \mathcal{F})$, where Z is a closed subset of M: The module of sections of $\mathcal{F}$ over U supported in Z.

$\Gamma_Z(\mathcal{F})$: The sheaf defined by $U \mapsto \Gamma_Z(U; \mathcal{F})$.

$\mathbf{R}\Gamma, \mathbf{R}\Gamma_Z$: The right derived functors of Γ and Γ_Z, respectively.

$\mathcal{H}^j_Z(\mathcal{F})$: The j-th right derived functor of Γ_Z.

$\mathcal{H}^j_{X|Z}(\mathcal{F})$: The j-th right derived functor of Γ_{X-Z}.

$H^j(U; \mathcal{F})$: The j-th cohomology group of $\mathcal{F}$ over U.

$H^j_Z(U; \mathcal{F})$: The j-th relative cohomology group of $\mathcal{F}$ over U with the support Z.

$\Gamma_{[Y]}(\mathcal{F})$, where Y is an analytic subset of a complex manifold X: $\varinjlim_m \mathcal{H}om_{\mathcal{O}_X}(\mathcal{O}_X/\mathcal{I}^m, \mathcal{F})$, where $\mathcal{O}_X$ is the sheaf of holomorphic functions on X and $\mathcal{I}$ is an $\mathcal{O}_X$-Ideal such that $\mathrm{Supp}\,(\mathcal{O}_X/\mathcal{I}) = Y$.

$\Gamma_{[X|Y]}(\mathcal{F})$: $\varinjlim_m \mathcal{H}om_{\mathcal{O}_X}(\mathcal{I}^m, \mathcal{F})$

$\mathbf{R}\Gamma_{[Y]}, \mathbf{R}\Gamma_{[X|Y]}$: The right derived functor of $\Gamma_{[Y]}$ and $\Gamma_{[X|Y]}$, respectively.

$\mathcal{H}^j_{[Y]}(\mathcal{F})$, $\mathcal{H}^j_{[X|Y]}(\mathcal{F})$: The j-th right derived functor of $\Gamma_{[Y]}(\mathcal{F})$ and $\Gamma_{[X|Y]}(\mathcal{F})$, respectively.

$\varphi_*\mathcal{F}$, where φ is a continuous map from a topological space M to a topological space N and $\mathcal{F}$ is a sheaf on M: The direct image of $\mathcal{F}$ by φ.

$\varphi_!\mathcal{F}$: The sheaf on N given by $U \mapsto \{s \in \Gamma(f^{-1}(U); \mathcal{F}); f|_{\text{supp}\, s}$: supp $s \to U$ is proper.$\}$

$\mathbf{R}\varphi_*$, $\mathbf{R}\varphi_!$: The right derived functor of φ_* and $\varphi_!$, respectively.

$R^j\varphi_*$, $R^j\varphi_!$: The j-th right derived functor of φ_* and $\varphi_!$, respectively.

$\mathcal{O}_X$: The sheaf of holomorphic functions on a complex manifold X. Here and in what follows, the subscript X is often omitted.

$\hat{\mathcal{O}}_{X,x}$, where x is a point in X: The ring of formal power series at x, i.e., $\hat{\mathcal{O}}_{X,x} = \varprojlim_k \mathcal{O}_{X,x}/\mathfrak{m}^k$, where $\mathfrak{m}$ is the maximal ideal of $\mathcal{O}_{X,x}$.

$\mathcal{O}_{T^*X}(m)$: The sheaf of holomorphic functions on T^*X which are homogeneous of degree m with respect to the fiber coordinate.

Ω^p_X : The sheaf of holomorphic p-forms.

$\Omega_X = \Omega^{\dim X}_X$

$\mathcal{D}_X$: The sheaf of linear differential operators of finite order on X. The subscript X is often omitted.

$\mathcal{D}_X(m)$: The sheaf of linear differential operators of order at most m.

$\mathcal{D}^\infty_X$: The sheaf of linear differential operator of infinite order.

$\mathcal{B}^\infty_{Y|X}$, where Y is a complex submanifold of X of codimension d: $\mathcal{H}^d_Y(\mathcal{O}_X)$.

$\mathcal{B}_{Y|X}$: $\mathcal{H}^d_{[Y]}(\mathcal{O}_X)$.

$\mathcal{C}^{\mathbf{R}}_{Y|X}$, where Y is a submanifold of X of codimension d: $\mathcal{H}^d_{T^*_YX}(\pi^{-1}\mathcal{O}_X)^a$, where a: $T^*X \to T^*X$ is the antipodal map, i.e., $a(x, \xi) \underset{\text{def}}{=} (x, -\xi)$ and π is the projection from the comonoidal transform $\widetilde{{}^YX^*}$ onto X (S-K-K [24] Chap. II § 1, Definition 1.1.4).

$\mathcal{C}^\infty_{Y|X}$: $\mathcal{C}^\infty_{Y|X}|_{T^*X - T^*_XX} \underset{\text{def}}{=} \gamma^{-1}\gamma_*\mathcal{C}^{\mathbf{R}}_{Y|X}$ and $\mathcal{C}^\infty_{Y|X}|_{T^*_XX} \underset{\text{def}}{=} \mathcal{H}^d_Y(\mathcal{O}_X)$.

$\mathcal{C}_{Y|X}$: The subsheaf of $\mathcal{C}^\infty_{Y|X}$ consisting of sections of finite order.

$\mathcal{E}^{\mathbf{R}}_X$: $\mathcal{C}^{\mathbf{R}}_{X|X\times X} \underset{p_2^{-1}\mathcal{O}_X}{\otimes} p_2^{-1}\Omega_X$, where p_2 is the second projection from $X \times X$ onto X.

$\mathcal{E}^\infty_X$: $\mathcal{C}^\infty_{X|X\times X} \underset{p_2^{-1}\mathcal{O}_X}{\otimes} p_2^{-1}\Omega_X$, i.e., the sheaf of micro-differential (=pseudo-differential) operators of infinite order on T^*X.(*)

(*) In S-K-K [24], $\mathcal{E}^\infty_X$ (resp., $\mathcal{E}_X$) is denoted by $\mathcal{P}_X$ (resp., $\mathcal{P}^f_X$). In addition to these changes of notations, we want to call the reader's attention to the fact that we consider $\mathcal{E}_X$ and $\mathcal{E}^\infty_X$ all over T^*X, i.e., including T^*_XX as their domain of definition. Needless to say, $\mathcal{E}_X|_{T^*_XX}$ and $\mathcal{E}^\infty_X|_{T^*_XX}$ are $\mathcal{D}_X$ and $\mathcal{D}^\infty_X$, respectively.

$\mathscr{E}_X$: $\mathscr{C}_{X|X\times X} \underset{p_2^{-1}\mathscr{O}_X}{\otimes} p_2^{-1}\Omega_X$, i.e., the sheaf of micro-differential operators of finite order.(*)

$\mathscr{F}\hat{\otimes}\mathscr{G}$ for an $\mathscr{E}_{X_1}$-Module $\mathscr{F}$ and an $\mathscr{E}_{X_2}$-Module $\mathscr{G}$: Let p_j ($j=1, 2$) denote the projection from $T^*(X_1\times X_2)$ to T^*X_j ($j=1, 2$, respectively). Then $\mathscr{F}\hat{\otimes}\mathscr{G}$ is, by definition, the $\mathscr{E}_{X_1\times X_2}$-Module

$$\mathscr{E}_{X_1\times X_2} \underset{p_1^{-1}\mathscr{E}_{X_1}\underset{C}{\otimes}p_2^{-1}\mathscr{E}_{X_2}}{\otimes} (p_1^{-1}\mathscr{F}\underset{C}{\otimes}p_2^{-1}\mathscr{G}).$$

$\mathscr{M}^R$, where $\mathscr{M}$ is an $\mathscr{E}$-Module: $\mathscr{E}^R\underset{\mathscr{E}}{\otimes}\mathscr{M}$.

$\mathscr{M}^\infty=\mathscr{E}^\infty\underset{\mathscr{E}}{\otimes}\mathscr{M}$ or $\mathscr{D}^\infty\underset{\mathscr{D}}{\otimes}\mathscr{M}$, according as $\mathscr{M}$ is an $\mathscr{E}$-Module or $\mathscr{D}$-Module.

$\mathscr{M}^*=\mathbf{R}\mathscr{H}om_{\mathscr{E}_X}(\mathscr{M}, \mathscr{E}_X)\underset{\mathscr{O}_X}{\otimes}\Omega_X^{\otimes-1}[\dim X]$ or $\mathbf{R}\,\mathscr{H}om_{\mathscr{D}_X}(\mathscr{M}, \mathscr{D}_X)\underset{\mathscr{O}_X}{\otimes}\Omega_X^{\otimes-1}[\dim X]$ according as $\mathscr{M}$ is an $\mathscr{E}_X$-Module or $\mathscr{D}_X$-Module. When $\mathscr{M}$ is holonomic, they are $\mathscr{E}xt^{\dim X}_{\mathscr{E}_X}(\mathscr{M}, \mathscr{E}_X)\underset{\mathscr{O}_X}{\otimes}\Omega_X^{\otimes-1}$ or $\mathscr{E}xt^{\dim X}_{\mathscr{D}_X}(\mathscr{M}, \mathscr{D}_X)\underset{\mathscr{O}_X}{\otimes}\Omega_X^{\otimes-1}$, respectively.

$\mathscr{M}_{\mathrm{reg}}$, where $\mathscr{M}$ is a holonomic $\mathscr{E}$-Module: The regular part of $\mathscr{M}$. See Chapter I, Section 1 for its definition (Definition 1.1.19).

$\mathscr{E}^\infty_{Y\xrightarrow{f}X}$, where f is a holomorphic map from Y to X: $\mathscr{C}^\infty_{Y|Y\times X}\underset{\mathscr{O}_X}{\otimes}\Omega_X^{\dim X}$. Here Y is identified with the graph of f in $Y\times X$ and $T^*_Y(Y\times X)$ is identified with $T^*X\underset{X}{\times}Y$. $\mathscr{E}^\infty_{Y\xrightarrow{f}X}$ is a $(\rho_f^{-1}\mathscr{E}^\infty_Y, \varpi_f^{-1}\mathscr{E}^\infty_X)$-bi-Module. In what follows, we often omit f in this symbol.

$\mathscr{E}^\infty_{X\xleftarrow{f}Y}$: $\mathscr{C}^\infty_{Y|Y\times X}\underset{\mathscr{O}_Y}{\otimes}\Omega_Y^{\dim Y}$. This is a $(\varpi_f^{-1}\mathscr{E}^\infty_X, \rho_f^{-1}\mathscr{E}^\infty_Y)$-bi-Module.

$\mathscr{E}_{Y\xrightarrow{f}X}$: $\mathscr{C}_{Y|Y\times X}\underset{\mathscr{O}_X}{\otimes}\Omega_X^{\dim X}$.

$\mathscr{E}_{X\xleftarrow{f}Y}$: $\mathscr{C}_{Y|Y\times X}\underset{\mathscr{O}_Y}{\otimes}\Omega_Y^{\dim Y}$.

$\mathscr{E}(m)$: The sheaf of micro-differential operators of order at most m.

σ_m : The symbol map from $\mathscr{E}(m)$ to $\mathscr{O}_{T^*X}(m)$, namely, the map which assigns the principal symbol to a micro-differential operator of order m.

$\mathscr{M}(m)$, where $\mathscr{M}$ is an $\mathscr{E}(0)$-Module: $\mathscr{E}(m)\underset{\mathscr{E}(0)}{\otimes}\mathscr{M}$.

SS $\mathscr{M}$ for a $\mathscr{D}_X$-Module $\mathscr{M}$: The characteristic variety of $\mathscr{M}$, i.e., Supp $(\mathscr{E}_X\underset{\pi^{-1}\mathscr{D}_X}{\otimes}\mathscr{M})$.

$\tilde{\mathscr{E}}^\infty$ and $\tilde{\mathscr{E}}$: A special subclass of micro-differential operators. See Chapter III, Section 3 for their definitions.

$\mathfrak{E}(G, D)$: $H^n_G(D\times D; \mathscr{O}^{(0,n)})$, see [19] Section 3, Definition 3.1.5.

$\mathscr{I}_V$, where V is a homogeneous involutory subvariety of $T^*X-T^*_XX$: $\{P\in\mathscr{E}_X(1);\ \sigma_1(P)$ vanishes on $V\}$.

(*) See the footnote in p. 830.

$\mathscr{E}_V$: The sub-Algebra of $\mathscr{E}_X$ generated by $\mathscr{I}_V$.

$\mathscr{E}_V(m) \underset{\text{def}}{=} \mathscr{E}_V\mathscr{E}(m) (=\mathscr{E}(m)\mathscr{E}_V)$

ord u for a section u of a holonomic $\mathscr{E}$-Module with R.S.: The order of u. See Chapter I, Section 5.3 for the definition.

$\sigma(u)$: The principal symbol of u. See Chapter I, Section 5.3 for the definition.

Chapter I. Basic Properties of Holonomic Systems

In this chapter we shall give the definition of holonomic systems of micro-differential equations with regular singularities. The notion of the systems with regular singularities was introduced in [18] in order to investigate the boundary value problems. We also study the elementary properties of holonomic systems with regular singularities.

§1.

In this section we extend the notion of regular singularities introduced in [18]. In order to perform this we start by an algebraic preparation.

1.1. Let X be an arbitrary topological space and $\mathscr{A}$ a sheaf of (not necessarily commutative) rings with the unit.

Definition 1.1.1. We say that $\mathscr{A}$ is *Noetherian* from the left if $\mathscr{A}$ satisfies the following conditions.

(a) $\mathscr{A}$ is coherent as a left $\mathscr{A}$-Module.

(b) For any point $x \in X$, the stalk $\mathscr{A}_x$ is a left Noetherian ring.

(c) For any open set U of X, a sum of left coherent $(\mathscr{A}|_U)$-Ideals is also coherent.

In the sequel, we omit the word "left" if there is no fear of confusion.

Example 1.1.2. (a) The sheaf $\mathscr{O}$ (resp., $\mathscr{D}$) of holomorphic functions (resp., linear differential operators) on a complex manifold is Noetherian.

(b) For a complex manifold X, the sheaves $\mathscr{E}_X$ and $\mathscr{E}_X(0)$ are Noetherian Rings on T^*X.

As the following propositions are easy to prove, we leave the proofs to the reader.

Proposition 1.1.3. *Let $\mathscr{A}$ be a Noetherian Ring and $\mathscr{M}$ a coherent $\mathscr{A}$-*

Module. Then a sum of coherent $\mathscr{A}$-sub-Modules of $\mathscr{M}$ is also coherent.

Proposition 1.1.4. *Let $\mathscr{A}$ be a Noetherian Ring and $\mathscr{R}$ an Algebra finitely generated over $\mathbf{Z}$. Then $\mathscr{A}\underset{\mathbf{Z}}{\otimes}\mathscr{R}$ is also Noetherian.*

Proposition 1.1.5. *Let $\mathscr{A}=\bigcup_{j\in\mathbf{Z}}\mathscr{A}_j$ be a filtered Ring (i.e., $\mathscr{A}_j\supset\mathscr{A}_{j-1}$ for any j, $\mathscr{A}_0\ni 1$ and $\mathscr{A}_j\cdot\mathscr{A}_k\subset\mathscr{A}_{j+k}$). Suppose that $\mathscr{A}_0$ and $\bigoplus_{j=0}^{\infty}(\mathscr{A}_j/\mathscr{A}_{j-1})$ are Noetherian and that $\mathscr{A}_j$ is coherent over $\mathscr{A}_0$ for any j. Then we have*

(i) *$\mathscr{A}$ is a Noetherian Ring.*

(ii) *Let $\mathscr{M}$ be an $\mathscr{A}$-sub-Module of $\mathscr{A}^N$. Then, $\mathscr{M}$ is a coherent $\mathscr{A}$-Module if and only if $\mathscr{M}\cap(\mathscr{A}_j)^N$ is coherent over $\mathscr{A}_0$ for any j.*

Definition 1.1.6. An $\mathscr{A}$-Module $\mathscr{M}$ is called pseudo-coherent if any $\mathscr{A}$-sub-Module of $\mathscr{M}$ that is locally of finite type on an open subset U of X is coherent over U.

Proposition 1.1.7. *Let $\mathscr{A}=\bigcup\mathscr{A}_j$ be as in Proposition 1.1.5. Then any coherent $\mathscr{A}$-Module is a pseudo-coherent $\mathscr{A}_0$-Module.*

Example. Let X be a complex manifold and U an open subset of $T^*X-T^*_XX$. Then any coherent $\mathscr{E}_X|_U$-Module is a pseudo-coherent $\mathscr{E}_X(0)|_U$-Module.

1.2. We shall recall the notion of regular singularities introduced in [18].

Let X be a complex manifold and we shall use the notations in the list of notations, e.g., $\mathscr{E}_X$, $\mathscr{E}_X(m)$, T^*X, $\mathscr{O}_{T^*X}(m)$, etc. Let V be a homogeneous involutory subvariety of $T^*X-T^*_XX$. The subvariety V may have singular points. Let I_V be the sheaf of holomorphic functions on $T^*X-T^*_XX$ which vanish on V, and let $I_V(m)$ denote $I_V\cap\mathscr{O}_{T^*X}(m)$.

The sheaf $\{P\in\mathscr{E}_X(1);\ \sigma_1(P)\in I_V(1)\}$ shall be denoted by $\mathscr{I}_V$. We denote by $\mathscr{E}_V$ the sub-Algebra of $\mathscr{E}_X$ generated by $\mathscr{I}_V$, and by $\mathscr{E}_V(m)$ the sheaf $\mathscr{E}_V\mathscr{E}(m)=\mathscr{E}(m)\mathscr{E}_V$. Note that $\mathscr{I}_V^k=\underbrace{\mathscr{I}_V\cdots\mathscr{I}_V}_{k}$ is a coherent (left and right) $\mathscr{E}(0)$-Module for any $k\geqq 0$.

Proposition 1.1.8. *$\mathscr{E}_V$ is a Noetherian Ring.*

Proof. Set $\mathscr{A}=\mathscr{E}_V$, $\mathscr{A}_m=\mathscr{I}_V^m\ (m\geqq 0)$, $\mathscr{A}_m=\mathscr{E}(m)$ for $m\leqq 0$. Then $\mathscr{A}=\bigcup\mathscr{A}_m$ is a filtered Ring and $\mathscr{A}_m$ is coherent over a Noetherian Ring $\mathscr{A}_0$ for any m. Hence we can apply Proposition 1.1.5 and it is sufficient to prove that $\bigoplus_{m=0}^{\infty}(\mathscr{A}_m/\mathscr{A}_{m-1})$ is Noetherian. It is easy to verify that $\bigoplus_{m=0}^{\infty}(\mathscr{A}_m/\mathscr{A}_{m-1})$ is a commutative Ring. Let $\{f_1,\dots,f_N\}$ be a system of generators of the coherent

$\mathcal{O}_{T^*X}(0)$-Module $I_V(1)$, and let P_j be a section of $\mathscr{I}_V$ such that $\sigma_1(P_j)=f_j$. We define the homomorphism Φ from the polynomial ring over $\mathcal{O}_{T^*X}$ $\mathcal{O}_{T^*X}(0)[T_1,\dots, T_N]$ into $\bigoplus_{m=0}^{\infty}(\mathscr{A}_m/\mathscr{A}_{m-1})$ by $T_i \overset{\Phi}{\longmapsto} P_i$. Then Φ is a surjective homomorphism of graded Rings. On the other hand, if we denote by Φ_m the homogeneous part of Φ of degree m, then Ker Φ_m is a coherent $\mathcal{O}_{T^*X}(0)$-Module. Hence the proposition follows from Proposition 1.1.4 and Proposition 1.1.5. Q. E. D.

By applying Proposition 1.1.5, we also obtain

Proposition 1.1.9. *Let $\mathscr{M}$ be an $\mathscr{E}_V$-sub-Module of $(\mathscr{E}_V)^N$. If $\mathscr{M}\cap(\mathscr{I}_V^k)^N$ is a coherent $\mathscr{E}(0)$-Module for any $k\geqq 0$, then $\mathscr{M}$ is a coherent $\mathscr{E}_V$-Module.*

Proposition 1.1.10. *A coherent $\mathscr{E}_X$-Module is pseudo-coherent over $\mathscr{E}_V$.*

Proof. Let $\mathscr{N}$ be an $\mathscr{E}_V$-sub-Module of a coherent $\mathscr{E}_X$-Module $\mathscr{M}$. Suppose that $\mathscr{N}$ is locally of finite type over $\mathscr{E}_V$. Let $s_1,\dots, s_N$ be a system of generators of $\mathscr{N}$. Let $\mathscr{N}'$ be the kernel of the homomorphism $\varphi:\mathscr{E}_V^N\to\mathscr{M}$ defined by $\varphi(P_1,\dots, P_N)=\sum_j P_j s_j$. Since $\mathscr{M}$ is pseudo-coherent over $\mathscr{E}(0)$, $\mathscr{N}'\cap(\mathscr{I}_V^k)^N$ is coherent over $\mathscr{E}(0)$ for any k. Therefore $\mathscr{N}'$ is coherent by Proposition 1.1.5, which implies that $\mathscr{N}$ is a coherent $\mathscr{E}_V$-Module. Q. E. D.

Definition 1.1.11. Let $\mathscr{M}$ be a coherent $\mathscr{E}$-Module defined on $\Omega\subset T^*X-T_X^*X$. We say that $\mathscr{M}$ has regular singularities along V if the following equivalent conditions are satisfied.

(i) For any point p of Ω, there are a neighborhood U of p and an $\mathscr{E}_V$-sub-Module $\mathscr{M}_0$ of $\mathscr{M}$ defined on U which is coherent over $\mathscr{E}(0)$, and which generates $\mathscr{M}$ as an $\mathscr{E}$-Module.

(ii) For any coherent $\mathscr{E}(0)$-sub-Module $\mathscr{L}$ of $\mathscr{M}$ defined on an open subset of Ω, $\mathscr{E}_V\mathscr{L}$ is coherent over $\mathscr{E}(0)$.

(iii) Any coherent $\mathscr{E}_V$-sub-Module of $\mathscr{M}$ that is defined on an open set of Ω is coherent over $\mathscr{E}(0)$.

The equivalence of these three conditions can be proved in the same way as in the proof of Theorem 1.7 of [18].

We denote by $IR(\mathscr{M}; V)$ the set of the points x such that $\mathscr{M}$ has not regular singularities along V on any neighborhood of x.

Lemma 1.1.12. *$IR(\mathscr{M}; V)$ is a closed analytic subset of Ω.*

Proof. The question being local, we may assume that $\mathscr{M}$ has an $\mathscr{E}(0)$-

sub-Module $\mathscr{M}_0$ such that $\mathscr{M}_0$ is coherent over $\mathscr{E}(0)$ and that $\mathscr{M}=\mathscr{E}\mathscr{M}_0$. By (i) and (ii) in Definition 1.1.11, $\mathscr{M}$ has regular singularities in a neighborhood of x if and only if $\mathscr{E}_V\mathscr{M}_0$ is coherent over $\mathscr{E}(0)$. Set $\mathscr{M}_k=\mathscr{J}_V^k\mathscr{M}_0$ for $k\geqq 1$. Then it is clear that $\mathscr{M}_k$ is coherent over $\mathscr{E}(0)$ and that $\mathscr{E}_V\mathscr{M}=\bigcup_{k\geqq 0}\mathscr{M}_k$. If $\mathscr{M}_k=\mathscr{M}_{k-1}$ for some $k=k_0$, then $\mathscr{M}_k=\mathscr{M}_{k-1}$ for $k\geqq k_0$. Therefore Supp $(\mathscr{M}_k/\mathscr{M}_{k-1})$ is a decreasing sequence of analytic subsets, hence locally stationary. Set $Y=\bigcap \operatorname{Supp}(\mathscr{M}_k/\mathscr{M}_{k-1})$. Then we have $Y=IR(\mathscr{M}; V)$.

Q. E. D.

Lemma 1.1.13. *If $\mathscr{M}$ has regular singularities along V, then* Supp $\mathscr{M}\subset V$.

Proof. Take $\mathscr{M}_0$ as in the condition (i) of Definition 1.1.1. Then $\mathscr{J}_V\mathscr{M}_0 \subset\mathscr{M}_0$. Hence Supp $(\mathscr{M}_0/\mathscr{E}(-1)\mathscr{M}_0)$ is contained in V. The lemma follows from this fact because the support of $\mathscr{M}$ coincides with that of $\mathscr{M}_0/\mathscr{E}(-1)\mathscr{M}_0$.

Q. E. D.

Proposition 1.1.14. *Let*

$$0\longrightarrow\mathscr{M}'\xrightarrow{\varphi}\mathscr{M}\xrightarrow{\psi}\mathscr{M}''\longrightarrow 0$$

be an exact sequence of coherent $\mathscr{E}$-Modules. Then $\mathscr{M}$ has regular singularities along V if and only if $\mathscr{M}'$ and $\mathscr{M}''$ have regular singularities along V.

Proof First we shall show that $\mathscr{M}$ has regular singularities along V if so are $\mathscr{M}'$ and $\mathscr{M}''$. Let $\mathscr{N}$ be a coherent $\mathscr{E}_V$-sub-Module of $\mathscr{M}$. We set $\mathscr{N}''=\psi(\mathscr{N})$ and $\mathscr{N}'=\varphi^{-1}(\mathscr{N})$. Since $\mathscr{M}''$ is pseudo-coherent over $\mathscr{E}_V$, $\mathscr{N}''$ is also a coherent $\mathscr{E}_V$-Module. Hence $\mathscr{N}''$ is coherent over $\mathscr{E}(0)$.

We shall show that $\mathscr{N}'$ is a coherent $\mathscr{E}_V$-Module. Let $\mathscr{L}$ (resp., $\mathscr{L}'$) be a coherent $\mathscr{E}(0)$-sub-Module of $\mathscr{N}$ (resp., $\mathscr{M}'$) which generates $\mathscr{N}$ (resp., $\mathscr{M}'$) as an $\mathscr{E}_V$-Module (resp., $\mathscr{E}$-Module). Then $\mathscr{N}'$ is a union of $\mathscr{E}_V(\mathscr{E}(m)\mathscr{L}'\cap \varphi^{-1}(\mathscr{J}_V^k\mathscr{L}))$, and hence $\mathscr{N}'$ is a union of coherent sub-Modules of $\mathscr{N}$. Hence $\mathscr{N}'$ is also a coherent $\mathscr{E}_V$-Module. Therefore, $\mathscr{N}'$ is coherent over $\mathscr{E}(0)$. Hence it follows from the exact sequence

$$0\longrightarrow\mathscr{N}'\longrightarrow\mathscr{N}\longrightarrow\mathscr{N}''\longrightarrow 0$$

that $\mathscr{N}$ is also coherent over $\mathscr{E}(0)$. Thus we have proved that $\mathscr{M}$ has regular singularities.

Conversely assume that $\mathscr{M}$ has regular singularities along V. Then, by the property (iii) of Definition 1.1.11, $\mathscr{M}'$ has regular singularities along V, and, by the property (i) of Definition 1.1.11, $\mathscr{M}''$ has regular singularities along V.

Q. E. D.

Proposition 1.1.15. *Let X and Y be two complex manifolds, V(resp., W) a homogeneous involutory subvariety in $T^*X - T^*_XX$ (resp., $T^*Y - T^*_YY$)* and let $\mathcal{M}$ (resp., $\mathcal{N}$) *be a coherent $\mathcal{E}_X$-Module (resp., coherent $\mathcal{E}_Y$-Module) with regular singularities along V (resp., W). Then $\mathcal{M}\hat{\otimes}\mathcal{N}$ is a coherent $\mathcal{E}_{X\times Y}$-Module with regular singularities along $V\times W$.*

Proof. Clearly $\mathcal{I}_{V\times W}$ is generated by $\mathcal{I}_V$ and $\mathcal{I}_W$, i.e.,

$$\mathcal{I}_{V\times W} = \mathcal{E}_{X\times Y}(0)\mathcal{I}_V + \mathcal{E}_{X\times Y}(0)\mathcal{I}_W.$$

Choose a coherent $\mathcal{E}_X(0)$-sub-Module $\mathcal{M}_0$ (resp., a coherent $\mathcal{E}_Y(0)$-sub-Module $\mathcal{N}_0$) of $\mathcal{M}$ (resp., $\mathcal{N}$) such that $\mathcal{M} = \mathcal{E}_X\mathcal{M}_0$ (resp., $\mathcal{N} = \mathcal{E}_Y\mathcal{N}_0$) and $\mathcal{I}_V\mathcal{M}_0 \subset \mathcal{M}_0$ (resp., $\mathcal{I}_W\mathcal{N}_0 \subset \mathcal{N}_0$). Then $\mathcal{L}_0 = \mathcal{M}_0\hat{\otimes}\mathcal{N}_0$ is a coherent $\mathcal{E}_{X\times Y}(0)$-sub-Module of $\mathcal{M}\hat{\otimes}\mathcal{N}$ and it satisfies the conditions $\mathcal{E}_{X\times Y}\mathcal{L}_0 = \mathcal{M}\hat{\otimes}\mathcal{N}$ and $\mathcal{I}_{V\times W}\mathcal{L}_0 \subset \mathcal{L}_0$. Hence $\mathcal{M}\hat{\otimes}\mathcal{N}$ has regular singularities along $V\times W$.

Q. E. D.

Definition 1.1.16. A holonomic $\mathcal{E}$-Module $\mathcal{M}$ is said to have R.S. on a Lagrangian variety Λ if $\Lambda \cap IR(\mathcal{M}; \Lambda - T^*_XX)$ is nowhere dense in $\Lambda - T^*_XX$. We say that $\mathcal{M}$ has R.S. if $\mathcal{M}$ has R.S. on Supp $\mathcal{M}$. A holonomic $\mathcal{D}$-Module $\mathcal{M}$ is said to have R.S. if $\mathcal{E} \underset{\mathcal{D}}{\otimes} \mathcal{M}$ has R.S.

If Supp $\mathcal{M}$ is contained in a locally finite union of Lagrangian varieties Λ_j, then $\mathcal{M}$ is said to have R.S. if and only if $\mathcal{M}$ has R.S. on any Λ_j.

Note that the notion given in Definition 1.1.16 is different from that given in Definition 1.1.11. However, we shall prove later (Corollary 5.1.7 in Chapter V) that, if a holonomic system $\mathcal{M}$ has R.S., then $\mathcal{M}$ has regular singularities along any involutory variety which contains Supp $\mathcal{M}$.

The following propositions immediately follow from Proposition 1.1.14 and Proposition 1.1.15, respectively.

Proposition 1.1.17. *Let*

$$0 \longrightarrow \mathcal{M}' \longrightarrow \mathcal{M} \longrightarrow \mathcal{M}'' \longrightarrow 0$$

be an exact sequence of holonomic systems. If $\mathcal{M}$ is with R.S., then so are $\mathcal{M}'$ and $\mathcal{M}''$. Conversely, if $\mathcal{M}'$ and $\mathcal{M}''$ are with R.S., then so is $\mathcal{M}$.

Proposition 1.1.18. *If $\mathcal{M}$ and $\mathcal{N}$ are holonomic systems with R.S., then so is $\mathcal{M}\hat{\otimes}\mathcal{N}$.*

Definition 1.1.19. Let $\mathcal{M}$ be a holonomic $\mathcal{E}_X$-Module. We define the subsheaf $\mathscr{M}_{\text{reg}}$ of $\mathscr{M}^\infty$ by assigning

(1.1.1) $\mathscr{M}_{\text{reg}}(U)=\{s\in\mathscr{M}^\infty(U)$; for any point x in U, there is a coherent Ideal $\mathscr{I}$ of $\mathscr{E}_X$ defined in a neighborhood of x such that $\mathscr{E}_X/\mathscr{I}$ has R.S. and that $\mathscr{I}s=0\}$

to each open subset U of X.

Proposition 1.1.20. *The sheaf $\mathscr{M}_{\text{reg}}$ is an $\mathscr{E}$-sub-Module of $\mathscr{M}^\infty$.*

Proof. We first show that $Pu\in\mathscr{M}_{\text{reg}}$ for $P\in\mathscr{E}$ and $u\in\mathscr{M}_{\text{reg}}$.

If we take $\mathscr{I}$ as in (1.1.1), then $\mathscr{I}'=\{Q\in\mathscr{E};\ QP\in\mathscr{I}\}$ is a coherent Ideal of $\mathscr{E}$ and $\mathscr{E}/\mathscr{I}'$ has R.S. (because $\mathscr{E}/\mathscr{I}'\subset\mathscr{E}/\mathscr{I}$). Moreover, $\mathscr{I}'Pu=0$. Hence Pu belongs to $\mathscr{M}_{\text{reg}}$. Next we show that u_1+u_2 belongs to $\mathscr{M}_{\text{reg}}$, if u_1 and u_2 are in $\mathscr{M}_{\text{reg}}$. Then for any point x we can choose coherent Ideals $\mathscr{I}_1$ and $\mathscr{I}_2$ as in (1.1.1). Let $\mathscr{I}$ be defined as the annihilator of $1\oplus 1$ in $\mathscr{E}/\mathscr{I}_1\oplus\mathscr{E}/\mathscr{I}_2$, then $\mathscr{E}/\mathscr{I}$ has R.S. and $\mathscr{I}(u_1+u_2)=0$. Hence u_1+u_2 belongs to $\mathscr{M}_{\text{reg}}$.

Q.E.D.

Proposition 1.1.21. *Let f be an $\mathscr{E}^\infty$-linear homomorphism from $\mathscr{M}^\infty$ to $\mathscr{N}^\infty$, where $\mathscr{M}$ and $\mathscr{N}$ are holonomic $\mathscr{E}$-Modules. Then $f(\mathscr{M}_{\text{reg}})\subset\mathscr{N}_{\text{reg}}$.*

This immediately follows from the definition.

§2.

In this section we will prove that $\mathscr{E}xt^j_{\mathscr{E}_X,Z}(\mathscr{M},\mathscr{N})$, $\mathscr{E}xt^j_{\mathscr{E}_X,Z}(\mathscr{M},\mathscr{N}^\infty)$ and $\mathscr{E}xt^j_{\mathscr{E}_X,Z}(\mathscr{M},\mathscr{N}^\infty/\mathscr{N})$ all vanish if $j<\operatorname{codim} Z-\operatorname{proj\,dim}\mathscr{N}$,(*) where $\mathscr{M}$ and $\mathscr{N}$ are coherent $\mathscr{E}_X$-Modules, not necessarily holonomic. This result may be regarded as a kind of Hartogs' theorem for $\mathscr{E}_X$-Modules (cf. [16], Theorem 1) and it will be used frequently in our later arguments.

Theorem 1.2.1. *Let $\mathscr{M}$ and $\mathscr{N}$ be coherent left $\mathscr{E}_X$-Modules. Let Z be a (not necessarily homogeneous) closed analytic subset of T^*X. Then*

$$\mathscr{E}xt^j_{\mathscr{E}_X,Z}(\mathscr{M},\mathscr{N}^\infty/\mathscr{N})=0 \tag{1.2.1}$$

holds for $j<\operatorname{codim}_{T^*X}Z-\operatorname{proj\,dim}\mathscr{N}$.

Proof. (I) The case where $Z\subset T^*X-T^*_XX$ and Z is homogeneous.

(*) Here proj dim $\mathscr{N}$ means the (local) projective dimension of $\mathscr{N}$, i.e., the largest integer j such that $\mathscr{E}xt^j_{\mathscr{E}}(\mathscr{N};\mathscr{E})\neq 0$.

First, by the induction on proj dim $\mathcal{N}$, we shall reduce the problem to the case where $\mathcal{N}$ is a free $\mathcal{E}_X$-Module. In fact, if proj dim $\mathcal{N}>0$, we choose a locally free $\mathcal{E}_X$-Module $\mathcal{L}$ and a surjective homomorphism

$$\psi: \mathcal{L} \longrightarrow \mathcal{N}. \tag{1.2.2}$$

Denote by $\mathcal{N}'$ the kernel of ψ. Then we have the following exact sequence:

$$\begin{aligned}\cdots \to \mathcal{E}xt^j_{\mathcal{E}_X,Z}(\mathcal{M}, \mathcal{N}^\infty/\mathcal{N}) &\longrightarrow \mathcal{E}xt^j_{\mathcal{E}_X,Z}(\mathcal{M}, \mathcal{L}^\infty/\mathcal{L}) \\ &\longrightarrow \mathcal{E}xt^{j+1}_{\mathcal{E}_X,Z}(\mathcal{M}, \mathcal{N}'^\infty/\mathcal{N}') \to \cdots,\end{aligned} \tag{1.2.3}$$

because $\mathcal{E}_X^\infty/\mathcal{E}_X$ is flat over $\mathcal{E}_X$.

On the other hand, it follows from the definition that proj dim $\mathcal{N}=$ proj dim $\mathcal{N}'+1$. Therefore it suffices to show the theorem when proj dim $\mathcal{N}=0$, and hence we may assume without loss of generality that $\mathcal{N}$ is free. Since $\mathcal{E}_X=\mathcal{C}_{X|X\times X}$ by the definition, it is then enough to show that

$$\begin{aligned}&\mathcal{E}xt^j_{\mathcal{E}_X,Z}(\mathcal{M}, \mathcal{C}^\infty_{X|X\times X}/\mathcal{C}_{X|X\times X}) \\ &\quad = \mathcal{E}xt^j_{\mathcal{E}_{X\times X},Z}(\mathcal{E}_{X\times X} \underset{\mathcal{E}_X}{\otimes} \mathcal{M}, \mathcal{C}^\infty_{X|X\times X}/\mathcal{C}_{X|X\times X})=0\end{aligned} \tag{1.2.4}$$

for a closed subset $Z\subset T^*_X(X\times X)$, if $j<\operatorname{codim}_{T^*X} Z=\operatorname{codim}_{T^*_X(X\times X)} Z$. Since $\mathcal{C}_{X|X\times X}$ defines a simple holonomic system supported by $T^*_X(X\times X)$, (1.2.4) can be reduced to the following assertion:

$$\mathcal{E}xt^j_{\mathcal{E}_X,Z}(\mathcal{M}, \mathcal{N}^\infty/\mathcal{N})=0 \tag{1.2.5}$$

for $j<\operatorname{codim}_{T^*X} Z-\dim X$, if $\mathcal{N}$ is a simple holonomic system and if Z is a closed subset of Supp $\mathcal{N}$.

Now we shall prove (1.2.5). If we choose the following exact sequence (1.2.6) with a free $\mathcal{E}_X$-Module $\mathcal{L}$,

$$0 \longrightarrow \mathcal{M}' \longrightarrow \mathcal{L} \longrightarrow \mathcal{M} \longrightarrow 0, \tag{1.2.6}$$

then we find that if suffices to show (1.2.5) only when $\mathcal{M}$ is a free $\mathcal{E}_X$-Module. In fact, we may use the induction on j in view of the following exact sequence

$$\begin{aligned}\cdots \to \mathcal{E}xt^{j-1}_{\mathcal{E}_X,Z}(\mathcal{M}', \mathcal{N}^\infty/\mathcal{N}) &\longrightarrow \mathcal{E}xt^j_{\mathcal{E}_X,Z}(\mathcal{M}, \mathcal{N}^\infty/\mathcal{N}) \\ \longrightarrow \mathcal{E}xt^j_{\mathcal{E}_X,Z}(\mathcal{L}, \mathcal{N}^\infty/\mathcal{N}) &\longrightarrow \mathcal{E}xt^j_{\mathcal{E}_X,Z}(\mathcal{M}', \mathcal{N}^\infty/\mathcal{N}) \longrightarrow \cdots.\end{aligned} \tag{1.2.7}$$

Thus in proving (1.2.5) we may assume without loss of generality that $\mathcal{M}=\mathcal{E}_X$. Therefore it suffices to show

$$\mathcal{H}^j_Z(\mathcal{N}^\infty/\mathcal{N})=0 \tag{1.2.8}$$

for $j<\operatorname{codim}_{T^*X} Z-\dim X$ on the condition that $\mathcal{N}$ is a simple holonomic system and that Z is a homogeneous closed analytic subset of $\operatorname{Supp}\mathcal{N}$. Furthermore we may assume that $\mathcal{N}=\mathcal{C}_{Y|X}$ for a non-singular hypersurface Y of X. Note that $Z=\pi^{-1}(\pi(Z))$ and that $\operatorname{codim}_{T^*X} Z-\dim X=\operatorname{codim}_Y \pi(Z)=\operatorname{codim}_Y Z$ holds. Thus we have reduced the problem to the following claim:

(1.2.9) $$\mathscr{H}_Z^j(\mathcal{C}_{Y|X}^\infty/\mathcal{C}_{Y|X})=0$$

holds for $j<\operatorname{codim}_Y Z$, if Z is a closed subset of a non-singular hypersurface Y. Here $\mathcal{C}_{Y|X}^\infty$ and $\mathcal{C}_{Y|X}$ are regarded as sheaves on Y.

Next we shall show that we have to consider only the case when Z is non-singular. In fact, by noting the fact $\operatorname{codim}_Y Z_{sing}$(*) $\leqq \operatorname{codim}_Y Z+1$ and making use of the induction on the dimension of Z, we may suppose that $\mathscr{H}_{Z_{sing}}^j(\mathcal{C}_{Y|X}^\infty/\mathcal{C}_{Y|X})=0$ for $j<\operatorname{codim}_Y Z_{sing}$. If (1.2.9) holds at non-singular points of Z, then

(1.2.10) $$\operatorname{Supp}\mathscr{H}_Z^j(\mathcal{C}_{Y|X}^\infty/\mathcal{C}_{Y|X})\subset Z_{sing}$$

holds for $j<\operatorname{codim}_Y Z$.

Then, considering the spectral sequence $E_2^{p,q}=\mathscr{H}_{Z_{sing}}^p(\mathscr{H}_Z^q(\mathcal{C}_{Y|X}^\infty/\mathcal{C}_{Y|X}))$, we find

(1.2.11) $$\begin{cases} E_2^{p,q}=0, \ p\neq 0, \ q<\operatorname{codim}_Y Z \\ E_2^{p,q}=\mathscr{H}_Z^q(\mathcal{C}_{Y|X}^\infty/\mathcal{C}_{Y|X}), \ p=0, \ q<\operatorname{codim}_Y Z. \end{cases}$$

Therefore we can conclude that

(1.2.12) $$\mathscr{H}_Z^j(\mathcal{C}_{Y|X}^\infty/\mathcal{C}_{Y|X})=\mathscr{H}_{Z_{sing}}^j(\mathcal{C}_{Y|X}^\infty/\mathcal{C}_{Y|X})$$

if $j<\operatorname{codim}_Y Z$. Furthermore the right-hand side of (1.2.12) is zero by the hypothesis of the induction.

Now we embark on the proof of (1.2.9) under the additional assumption that Z is non-singular. First we recall the following commutative diagram:

(1.2.13) $$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathscr{B}_{Y|X}^\infty & \longrightarrow & \mathcal{C}_{Y|X}^\infty & \longrightarrow & \mathcal{O}_X|_Y & \longrightarrow & 0 \\ & & \uparrow & & \uparrow & & \wr\uparrow & & \\ 0 & \longrightarrow & \mathscr{B}_{Y|X} & \longrightarrow & \mathcal{C}_{Y|X} & \longrightarrow & \mathcal{O}_{X|Y} & \longrightarrow & 0. \end{array}$$

This diagram shows that $\mathscr{B}_{Y|X}^\infty/\mathscr{B}_{Y|X}$ is isomorphic to $\mathcal{C}_{Y|X}^\infty/\mathcal{C}_{Y|X}$. Hence it suffices to show that

(1.2.14) $$\mathscr{H}_Z^j(\mathscr{B}_{Y|X}^\infty/\mathscr{B}_{Y|X})=0$$

(*) Z_{sing} denotes the set of the singular points of Z.

for $j<\operatorname{codim}_Y Z$. As we have assumed that Z is non-singular, we may assume

(1.2.15) $$X=\boldsymbol{C}\times\boldsymbol{C}^l\times\boldsymbol{C}^n\supset Y=\{0\}\times\boldsymbol{C}^l\times\boldsymbol{C}^n\supset Z=\{0\}\times\{0\}^l\times\boldsymbol{C}^n.$$

We shall denote a point in X by $(t, x, y)\in\boldsymbol{C}\times\boldsymbol{C}^l\times\boldsymbol{C}^n$. Since there is nothing to prove if $l=0$, we assume $l\geqq 1$. In order to compute $\mathscr{H}^j_Z(\mathscr{B}^\infty_{Y|X}/\mathscr{B}_{Y|X})$, we introduce three families of sets defined as follows:

(1.2.16) $$K_{\delta,\varepsilon}=\{(0, x, y)\in Y;\ \delta\leqq|x|\leqq\varepsilon,\ |y|\leqq\varepsilon\}$$

(1.2.17) $$K_\varepsilon=\{(0, x, y)\in Y;\ |x|\leqq\varepsilon,\ |y|\leqq\varepsilon\}$$

(1.2.18) $$U_{\delta,\varepsilon,\rho}=\{(0, x, y)\in Y;\ \delta-\rho<|x|\leqq\varepsilon,\ |y|\leqq\varepsilon\}.$$

In order to prove (1.2.14), we want to prove

(1.2.19) $$\varinjlim_{\varepsilon>0} H^j_{Z\cap K_\varepsilon}(K_\varepsilon, \mathscr{B}^\infty_{Y|X}/\mathscr{B}_{Y|X})=0.$$

Let $\mathscr{B}^{(m)}_{Y|X}$ be the subsheaf $\mathscr{D}_X(m)\delta(t)$ of $\mathscr{B}_{Y|X}$. Then $\mathscr{B}_{Y|X}=\varinjlim_m \mathscr{B}^{(m)}_{Y|X}$ holds. Since K_ε and $K_{\delta,\varepsilon}$ are compact, we have

(1.2.20) $$H^j_{K_\varepsilon-K_{\delta,\varepsilon}}(K_\varepsilon, \mathscr{B}^\infty_{Y|X}/\mathscr{B}_{Y|X})=\varinjlim_m H^j_{K_\varepsilon-K_{\delta,\varepsilon}}(K_\varepsilon\mathscr{B}^\infty_{Y|X}/\mathscr{B}^{(m)}_{Y|X})$$

by virtue of the long exact sequence of cohomology groups and the fact that the inductive limit operation is commutative with the (absolute) cohomology operation on compact sets. Here $H^j_{K_\varepsilon-K_{\delta,\varepsilon}}(K_\varepsilon, *)$ means $\varinjlim_{\rho>0} H^j_{K_\varepsilon-U_{\delta,\varepsilon,\rho}}(K_\varepsilon, *)$.

Since $\mathscr{B}^\infty_{Y|X}/\mathscr{B}^{(m)}_{Y|X}$ is isomorphic to $\mathscr{B}^\infty_{Y|X}$ as sheaves, $H^j_{K_\varepsilon-K_{\delta,\varepsilon}}(K_\varepsilon, \mathscr{B}^\infty_{Y|X}/\mathscr{B}^{(m)}_{Y|X})$ is isomorphic to $H^j_{K_\varepsilon-K_{\delta,\varepsilon}}(K_\varepsilon, \mathscr{B}^\infty_{Y|X})$. We shall now show that

(1.2.21) $$H^j_{K_\varepsilon-K_{\delta,\varepsilon}}(K_\varepsilon, \mathscr{B}^\infty_{Y|X})=0$$

for $j<l$. If we prove (1.2.21), then by the Mittag-Leffler theorem ([4]) we find

(1.2.22) $$\varprojlim_{\delta>0} H^j_{K_\varepsilon-K_{\delta,\varepsilon}}(K_\varepsilon, \mathscr{B}^\infty_{Y|X})=H^j_{Z\cap K_\varepsilon}(K_\varepsilon, \mathscr{B}^\infty_{Y|X}).$$

Here we have used the fact that

$$Z\cap K_\varepsilon=\bigcap_{\delta<0}(K_\varepsilon-K_{\delta,\varepsilon}).$$

In order to prove (1.2.21), it suffices to show that

(1.2.23) $$H^j_{S_{\delta,\varepsilon'}}(V_{\varepsilon',\varepsilon''}, \mathscr{B}^\infty_{Y|X})=0 \qquad \text{for } j<l,$$

where $V_{\varepsilon',\varepsilon''}=\{(t, x, y)\in X;\ |t|<\varepsilon'',\ |x|<\varepsilon',\ |y|<\varepsilon'\}$ and $S_{\delta,\varepsilon'}=\{(t, x, y)\in X;\ t=0,\ |x|\leqq\delta,\ |y|<\varepsilon'\}$. However, it is known (e.g. [20]) that

(1.2.24) $$H^j_{S_{\delta,\varepsilon'}}(U_{\varepsilon'\varepsilon''}, \mathscr{B}^\infty_{Y|X})=H^{j+1}_{S_{\delta,\varepsilon'}\cap Y}(U_{\varepsilon',\varepsilon''}, \mathscr{O}_X)=0$$

holds for $j+1\neq l+1$. This proves (1.2.23), and hence it finally completes the proof of Theorem 1.2.1 under the assumption that $Z\subset T^*X-T^*_XX$ and Z is homogeneous.

(II) General case.

In the sequel, $(t, x; \tau, \xi)$ shall denote a point in $T^*(\mathbb{C}\times X)$. Define $\mathscr{E}_{\mathbb{C}\times X}$-Modules $\tilde{\mathscr{M}}$ (resp., $\tilde{\mathscr{N}}$) by $\mathscr{E}_{\mathbb{C}}\delta(t)\hat{\otimes}\mathscr{M}$ (resp., $\mathscr{E}_{\mathbb{C}}\delta(t)\hat{\otimes}\mathscr{N}$). Define $\tilde{Z}$ by $\{(t, x; \tau, \xi)\in T^*(\mathbb{C}\times X); (x, \tau^{-1}\xi)\in Z, t=0, \tau\neq 0\}$. Then we have $\operatorname{proj\,dim}\tilde{\mathscr{N}}=\operatorname{proj\,dim}\mathscr{N}+1$ and $\operatorname{codim}_{T^*(\mathbb{C}\times X)}\tilde{Z}=\operatorname{codim}_{T^*X}Z+1$. In what follows we regard T^*X as a closed subset of $T^*(\mathbb{C}\times X)$ by $(x, \xi)\mapsto(0, x; 1, \xi)$. Then the result obtained in (I) proves that

$$\mathscr{E}xt^j_{\mathscr{E}_{\mathbb{C}\times X},Z}(\tilde{\mathscr{M}}, \tilde{\mathscr{N}}^\infty/\tilde{\mathscr{N}})|_{T^*X}=0$$

holds if $j<\operatorname{codim}_{T^*X}Z-\operatorname{proj\,dim}\mathscr{N}$.

Hence it suffices to prove

$$\mathscr{E}xt^j_{\mathscr{E}_{\mathbb{C}\times X},\tilde{Z}}(\tilde{\mathscr{M}}, \tilde{\mathscr{N}}^\infty/\tilde{\mathscr{N}})|_{T^*X}\cong\mathscr{E}xt^j_{\mathscr{E}_X,Z}(\mathscr{M}, \mathscr{N}^\infty/\mathscr{N}).$$

For this purpose, it is enough to prove that

$$\mathscr{E}xt^j_{\mathscr{E}_{\mathbb{C}\times X},\tilde{Z}}(\tilde{\mathscr{M}}, \tilde{\mathscr{N}})|_{T^*X}\cong\mathscr{E}xt^j_{\mathscr{E}_X,Z}(\mathscr{M}, \mathscr{N}) \tag{1.2.25}$$

and

$$\mathscr{E}xt^j_{\mathscr{E}_{\mathbb{C}\times X},\tilde{Z}}(\tilde{\mathscr{M}}, \tilde{\mathscr{N}}^\infty)|_{T^*X}\cong\mathscr{E}xt^j_{\mathscr{E}_X,Z}(\mathscr{M}, \mathscr{N}^\infty) \tag{1.2.26}$$

hold for all j. In proving this, we may assume without loss of generality that $\mathscr{M}=\mathscr{N}=\mathscr{E}_X$, because $\mathscr{M}$ and $\mathscr{N}$ admit free resolutions.

Let us define the projection F from $\{(t, x; \tau, \xi)\in T^*(\mathbb{C}\times X); t=0, \tau\neq 0\}$ to T^*X by $F(0, x; \tau, \xi)=(x, \tau^{-1}\xi)$. Then we have

$$\mathbb{R}\mathscr{H}om_{\mathbb{C}\times X}(\mathscr{E}_{\mathbb{C}}\delta(t)\hat{\otimes}\mathscr{E}_X, \mathscr{E}_{\mathbb{C}}\delta(t)\hat{\otimes}\mathscr{E}_X)\cong F^{-1}\mathscr{E}_X.$$

On the other hand, $\tilde{Z}=F^{-1}Z$ holds in $\{\tau\neq 0\}$. Hence we have

$$\mathbb{R}\Gamma_{\tilde{Z}}(F^{-1}\mathscr{E}_X)|_{T^*X}\cong F^{-1}\mathbb{R}\Gamma_Z\mathscr{E}_X|_{T^*X}\cong\mathbb{R}\Gamma_Z(\mathscr{E}_X).$$

This proves (1.2.25). The proof of (1.2.26) is the same as this. Thus we have completed the proof for the general case. Q.E.D.

The proof given above also proves $\mathscr{H}^j_Z(\mathscr{B}_{Y|X})=\mathscr{H}^j_Z(\mathscr{B}^\infty_{Y|X})=0$ holds for $j<l$. Therefore we have the following

Theorem 1.2.2. *Let $\mathscr{M}$ and $\mathscr{N}$ be coherent left $\mathscr{E}_X$-Modules. Let Z be a closed subset of T^*X. Then*

(1.2.27) $$\mathscr{E}xt^j_{\mathscr{E}_X,Z}(\mathscr{M},\mathscr{N})=\mathscr{E}xt^j_{\mathscr{E}_X,Z}(\mathscr{M},\mathscr{N}^\infty)=0$$

holds for $j<\operatorname{codim}_{T^*X}Z-\operatorname{proj}\dim\mathscr{N}$.

In this article we often use these theorems in the following form:

Corollary 1.2.3. *Let Ω be an open subset of T^*X and Z a closed analytic subset of Ω. Let $\mathscr{M}$ be a coherent $\mathscr{E}_X|_\Omega$-Module and $\mathscr{N}$ a coherent $\mathscr{E}_X|_\Omega$-sub-Module of $\mathscr{M}$.*

(I) *Suppose that* $\operatorname{codim}_{T^*X}Z\geqq\dim X+1$.

(i) *If $s\in\Gamma(\Omega;\mathscr{M}^\infty)$ is contained in $\mathscr{M}$ outside Z, then $s\in\Gamma(\Omega;\mathscr{M})$.*

(ii) *If $s\in\Gamma(\Omega;\mathscr{M}^\infty)$ is contained in $\mathscr{N}^\infty$ outside Z, then s is contained in $\Gamma(\Omega;\mathscr{N}^\infty)$.*

(II) *Suppose that* $\operatorname{codim}_{T^*X}Z\geqq\dim X+2$. *Then any section of $\mathscr{M}^\infty$ defined on $\Omega-Z$ is uniquely extended to a section of $\mathscr{M}^\infty$ defined on Ω.*

§ 3.

In this section we investigate some basic properties of holonomic systems with regular singularities. The main tool used here is the classification of holonomic systems having a non-singular Lagrangian manifold as characteristic variety.

First we recall the following classification theorem ([9]).

Theorem 1.3.1. *Let Λ be a non-singular homogeneous Lagrangian subvariety of $T^*X-T^*_XX$ and $\mathscr{M}_0$ a holonomic system with support in Λ and of multiplicity 1 along Λ. Let γ be the projection from T^*X-X onto P^*X.*

(i) *For any holonomic system $\mathscr{M}$ with support in Λ, $\mathscr{E}xt^j_{\mathscr{E}}(\mathscr{M}_0,\mathscr{M}^{\mathbf{R}})=0$ $(j\neq0)$ and $\mathscr{H}om_{\mathscr{E}}(\mathscr{M}_0,\mathscr{M}^{\mathbf{R}})$ is a locally constant sheaf on Λ. The rank of $\mathscr{H}om_{\mathscr{E}}(\mathscr{M}_0,\mathscr{M}^{\mathbf{R}})$ coincides with the multiplicity m of $\mathscr{M}$ along Λ and the canonical homomorphism*

$$\mathscr{M}_0^{\mathbf{R}}\underset{\mathbf{C}}{\otimes}\mathscr{H}om_{\mathscr{E}}(\mathscr{M}_0,\mathscr{M}^{\mathbf{R}})\longrightarrow\mathscr{M}^{\mathbf{R}}$$

is an isomorphism.

(ii) *Suppose that $\Lambda=\gamma^{-1}\gamma\Lambda$, $\mathscr{M}_0=\gamma^{-1}\gamma_*\mathscr{M}_0$ and $\mathscr{M}=\gamma^{-1}\gamma_*\mathscr{M}$. Then*

$$\mathscr{M}^\infty=\gamma^{-1}\gamma_*(\mathscr{M}_0^{\mathbf{R}}\underset{\mathbf{C}}{\otimes}\mathscr{H}om_{\mathscr{E}}(\mathscr{M}_0,\mathscr{M}^{\mathbf{R}})).$$

This theorem says that the structure of $\mathscr{M}^\infty$ is determined by $\mathscr{H}om_{\mathscr{E}}(\mathscr{M}_0,\mathscr{M}^{\mathbf{R}})$. We fix one preferred reference system $\mathscr{M}_0$ in our argument.

Since $\mathcal{H}om_{\mathcal{E}}(\mathcal{M}_0, \mathcal{M}^R)$ is a locally constant sheaf on Λ and the fiber of γ is $\boldsymbol{C}-\{0\}$, we can associate an $m\times m$ constant matrix T_l (up to inner automorphism) with $l\in\pi_1(\gamma^{-1}(\gamma(p)), p)\cong\boldsymbol{Z}$ for $p\in\Lambda$. Let e be a generator of $\pi_1(\gamma^{-1}(\gamma(p)), p)$. We call $T\equiv T_e$ to be the monodromy of $\mathcal{M}$ (with respect to the reference system $\mathcal{M}_0$). Using this terminology, we can restate (ii) symbolically as follows:

The structure of $\mathcal{M}^\infty$ is determined by the monodromy of $\mathcal{M}$.

In what follows we shall explicitly show how $\mathcal{M}^\infty$ is determined by the monodromy of $\mathcal{M}$, assuming Λ has a simple form. The simplifying assumption on Λ given there is not restrictive, because any non-singular Lagrangian variety can be transformed into that form by a homogeneous canonical transformation.

Take $X=\boldsymbol{C}^n$ and let Λ be given by

$$\{(x, \xi)\in T^*X;\ x_1=\xi_2=\cdots=\xi_n=0,\ |x|<\varepsilon,\ \xi_1\neq 0\}. \tag{1.3.1}$$

We set $\mathcal{M}_0=\mathcal{E}/(\mathcal{E}x_1+\mathcal{E}D_2+\cdots+\mathcal{E}D_n)$ and

$$\mathcal{M}_{\lambda,m}=\mathcal{E}/(\mathcal{E}(x_1D_1-\lambda)^m+\mathcal{E}D_2+\cdots+\mathcal{E}D_n).$$

Then we have the following

Lemma 1.3.2. $\mathcal{E}xt^j_{\mathcal{E}}(\mathcal{M}_{\lambda,m}, \mathcal{M}_{\lambda',m'})\cong\mathcal{E}xt^j_{\mathcal{E}}(\mathcal{M}_{\lambda,m}, \mathcal{M}^\infty_{\lambda',m'})$.

Proof. If $m\geqq 2$, there is an exact sequence

$$0\longrightarrow\mathcal{M}_{\lambda,m-1}\longrightarrow\mathcal{M}_{\lambda,m}\longrightarrow\mathcal{M}_{\lambda,1}\longrightarrow 0.$$

Hence, by using the induction on m and m', we can reduce the problem to the case where $m=m'=1$. Set $\mathcal{N}_\lambda=\mathcal{E}_C/\mathcal{E}_C(tD_t-\lambda)$. Then we have

$$\mathcal{E}xt^j_{\mathcal{E}_X}(\mathcal{M}_\lambda, \mathcal{M}_{\lambda'})\xrightarrow{\sim}\mathcal{E}xt^j_{\mathcal{E}_C}(\mathcal{N}_\lambda, \mathcal{N}_{\lambda'})$$
$$\mathcal{E}xt^j_{\mathcal{E}_X}(\mathcal{M}_\lambda, \mathcal{M}^\infty_{\lambda'})\xrightarrow{\sim}\mathcal{E}xt^j_{\mathcal{E}_C}(\mathcal{N}_\lambda, \mathcal{N}^\infty_{\lambda'}).$$

(Cf. S–K–K [24], Chapter II, Theorem 5.3.1. Even though $\mathcal{E}^\infty_X$ is needed there, $\mathcal{E}_X$ suffices in our case.) Hence it suffices to show

$$\mathcal{E}xt^j_{\mathcal{E}_C}(\mathcal{N}_\lambda, \mathcal{N}_{\lambda'})\xrightarrow{\sim}\mathcal{E}xt^j_{\mathcal{E}_C}(\mathcal{N}_\lambda, \mathcal{N}^\infty_{\lambda'}). \tag{1.3.2}$$

Let v be the generator $1 \bmod \mathcal{E}(x_1D_1-\lambda')$ of $\mathcal{N}_{\lambda'}$. Then by an immediate calculation we find

$$\mathcal{E}xt^0_{\mathcal{E}_C}(\mathcal{N}_\lambda, \mathcal{N}_{\lambda'})$$
$$=\{A(D_1)v;\ (x_1D_1-\lambda)(A(D_1)v)=0,\ A(D_1)\in\mathcal{E}_C\}\cong\begin{cases}0 & \text{if } \lambda-\lambda'\notin\boldsymbol{Z}\\ \boldsymbol{C} & \text{if } \lambda-\lambda'\in\boldsymbol{Z}\end{cases}$$

and

$$\mathscr{E}xt^0_{\mathscr{E}_C}(\mathscr{N}_\lambda, \mathscr{N}^\infty_{\lambda'})$$
$$=\{\tilde{A}(D_1)v;\ (x_1D_1-\lambda)(\tilde{A}(D_1)v)=0,\ \tilde{A}(D_1)\in\mathscr{E}^\infty_C\}\cong\begin{cases}0 & \text{if } \lambda-\lambda'\notin \boldsymbol{Z}\\ \boldsymbol{C} & \text{if } \lambda-\lambda'\in\boldsymbol{Z}.\end{cases}$$

Similarly, we can verify

$$\mathscr{E}xt^1_{\mathscr{E}_C}(\mathscr{N}_\lambda, \mathscr{N}_{\lambda'})=\begin{cases}0 & \text{if } \lambda-\lambda'\notin \boldsymbol{Z}\\ \boldsymbol{C} & \text{if } \lambda-\lambda'\in\boldsymbol{Z}\end{cases}$$

and

$$\mathscr{E}xt^1_{\mathscr{E}_C}(\mathscr{N}_\lambda, \mathscr{N}^\infty_{\lambda'})=\begin{cases}0 & \text{if } \lambda-\lambda'\notin \boldsymbol{Z}\\ \boldsymbol{C} & \text{if } \lambda-\lambda'\in\boldsymbol{Z}.\end{cases}$$

Thus we have verified (1.3.2). Q. E. D.

Under the assumption (1.3.1) the monodromy matrix T is associated with the loop $e=\{(x;\xi)=(0;e^{i\theta},0,\dots,0)\}_{0\leqq\theta\leqq 2\pi}$. Hence we have the following

Lemma 1.3.3. *The monodromy of $\mathscr{M}_{\lambda,m}$ is equivalent to the following $m\times m$ matrix*:

$$\begin{bmatrix} e^{2\pi i\lambda} & 1 & & \\ & e^{2\pi i\lambda} & 1 & \\ & & \ddots & \ddots\ \ 1 \\ & & & e^{2\pi i\lambda}\end{bmatrix}. \tag{1.3.3}$$

Proof. We first note

$$\mathscr{H}om_{\mathscr{E}_X}(\mathscr{M}_0, \mathscr{M}^\infty_{\lambda,m})=\mathscr{H}om_{\mathscr{E}_X}(\mathscr{M}^*_{\lambda,m}, \mathscr{M}^{*\boldsymbol{R}}_0).$$

Since $\mathscr{M}^*_0=\mathscr{M}_0$ and $\mathscr{M}^*_{\lambda,m}=\mathscr{M}_{-\lambda-1,m}$, it is sufficient to study the monodromy of $\mathscr{H}om_{\mathscr{E}_X}(\mathscr{M}_{-\lambda-1,m}, \mathscr{M}^{\boldsymbol{R}}_0)$.

Set $Y=\{x_1=0\}$ and $\mathscr{M}^{\boldsymbol{R}}_0=\mathscr{C}^{\boldsymbol{R}}_{Y|X}$. Then $\mathscr{H}om(\mathscr{M}_{-\lambda-1,m}, \mathscr{M}^{\boldsymbol{R}}_0)=\{u\in\mathscr{C}^{\boldsymbol{R}}_{Y|X}, (x_1D_1+\lambda+1)^m u=D_2u=\cdots=D_nu=0\}$. Clearly the elements in this space are given by

$$x_1^{-\lambda-1}(\log x_1)^j\ (0\leqq j\leqq m-1),\ \text{when } -\lambda-1\neq 0,1,2,\dots$$

and

$$x_1^{-\lambda-1}(\log x_1)^j\ (1\leqq j\leqq m),\ \text{when } -\lambda-1=0,1,2,\dots\ .$$

It is then easy to see that this gives the monodromy (1.3.3). Q. E. D.

If we transform the monodromy matrix T of $\mathcal{M}$ in the Jordan form, we see that there exist λ_j and m_j such that T is equivalent to the monodromy of $\oplus\mathcal{M}_{\lambda_j,m_j}$. Since the monodromy determines the structure of the holonomic system over $\mathscr{E}^\infty$ (Theorem 1.3.1 (ii)), we obtain the following lemma.

Lemma 1.3.4. *$\mathcal{M}^\infty$ is isomorphic to a finite direct sum $\oplus\mathcal{M}^\infty_{\lambda_j,m_j}$.*

This lemma proves the following proposition.

Proposition 1.3.5. *Let Λ be a non-singular homogeneous Lagrangian variety and let $\mathcal{M}$ be a holonomic system whose support is contained in Λ. Then*

(i) *$\mathcal{M}_{\mathrm{reg}}$ is a coherent $\mathscr{E}_X$-Module and it has regular singularities along Λ.*

(ii) *$\mathscr{E}^\infty \underset{\mathscr{E}}{\otimes} \mathcal{M}_{\mathrm{reg}} \to \mathcal{M}^\infty$ is an isomorphism.*

(iii) *If $\mathcal{M}$ has regular singularities along Λ on a non-void open set of Λ and if Λ is connected, then $\mathcal{M}$ has regular singularities along Λ on the whole Λ.*

Proof. Let us first prove (i) and (ii). The question being local, we may suppose that Λ has the form (1.3.1). Therefore, it follows from Lemma 1.3.4 that $\mathcal{M}^\infty$ is isomorphic to the finite direct sum of $\mathcal{M}^\infty_{\lambda,m}$'s. Therefore, in proving (i) and (ii), we may suppose that $\mathcal{M}=\mathcal{M}_{\lambda,m}$. We shall show that $\mathcal{M}_{\mathrm{reg}} =\mathcal{M}$ holds in this case. This immediately implies (i) and (ii). Since $\mathcal{M}$ has regular singularities along Λ, $\mathcal{M}_{\mathrm{reg}}$ contains $\mathcal{M}$. In order to show that $\mathcal{M}_{\mathrm{reg}} =\mathcal{M}$, it is enough to show that we have $f(\mathcal{N})\subset\mathcal{M}$ for any holonomic system $\mathcal{N}$ with R.S. and an $\mathscr{E}$-linear homomorphism $f:\mathcal{N}\to\mathcal{M}^\infty$. First suppose that $\mathcal{N}$ has regular singularities along Λ. Then, by Lemma 3.7 of [18], $\mathcal{N}$ is a quotient of the system defined by

$$\begin{cases} D_2u=0 \\ \vdots \\ D_nu=0 \\ (x_1D_1-A)u=0, \end{cases}$$

where A is an $N\times N$ constant matrix and u is a column vector of size N. By transforming the matrix A into the Jordan form, we find that this system is isomorphic to a direct sum of $\mathcal{M}_{\lambda,m}$'s. Therefore $\mathcal{N}$ is a quotient of $\oplus\mathcal{M}_{\lambda_j,m_j}$. Then, by Lemma 1.3.2, we find $f(\mathcal{N})\subset\mathcal{M}$.

Now suppose that $\mathcal{N}$ has R.S. Then $Z=IR(\mathcal{N};\Lambda)\cap\Lambda$ is a nowhere dense analytic subset of Λ. We have already verified that $f(\mathcal{N})\subset\mathcal{M}$ on $\Lambda-Z$. Hence it follows from Corollary 1.2.3 in Section 2 that $f(\mathcal{N})\subset\mathcal{M}$.

Thus (i) and (ii) are proved.

Lastly we prove (iii). Set $IR(\mathcal{M};\Lambda)=Z$. Then Z is a nowhere dense analytic set of Λ. We shall show that $\mathcal{M}_{\text{reg}}\supset\mathcal{M}$. Let u be a section of $\mathcal{M}$. Then u is contained in $\mathcal{M}_{\text{reg}}$ outside Z. Hence Corollary 1.2.3 in Section 2 implies $u\in\mathcal{M}_{\text{reg}}$. Since it follows from (i) that $\mathcal{M}_{\text{reg}}$ has regular singularities along Λ, $\mathcal{M}$ has regular singularities along Λ. Q.E.D.

Proposition 1.3.6. *Let $\mathcal{M}$ be a holonomic system with R.S. (We do not assume that its characteristic variety is non-singular). Then $\mathcal{M}_{\text{reg}}=\mathcal{M}$.*

Proof. It is trivial that $\mathcal{M}_{\text{reg}}$ contains $\mathcal{M}$. We shall show that $\mathcal{M}$ contains $\mathcal{M}_{\text{reg}}$. In order to show this, it is enough to show that, for any holonomic system $\mathcal{N}$ with R.S. and an $\mathcal{E}$-linear homomorphism $f:\mathcal{N}\to\mathcal{M}^{\infty}$, $f(\mathcal{N})$ is contained in $\mathcal{M}$. However, $\mathcal{M}=\mathcal{M}_{\text{reg}}$ holds in the non-singular locus Λ' of Λ. Hence, for any section u of $\mathcal{N}$, $f(u)$ is contained in $\mathcal{M}$ on $\Lambda-\Lambda'$. Then Corollary 1.2.3 in Section 2 entails that $f(u)$ is contained in $\mathcal{M}$. This completes the proof of the proposition. Q.D.E.

Proposition 1.3.7. *Let Λ be a connected non-singular homogeneous Lagrangian variety and $\mathcal{M}$ and $\mathcal{N}$ two coherent $\mathcal{E}_X$-Modules supported by Λ. Let $\varphi:\mathcal{M}^{\infty}\to\mathcal{N}^{\infty}$ be an $\mathcal{E}^{\infty}$-linear homomorphism. Then we have the following:*

(i) *The support of the cokernel of φ is Λ or an empty set.*

(ii) *The support of the kernel of φ is Λ or an empty set.*

(iii) *If φ is surjective (resp., injective), then the homomorphism $\varphi^*:\mathcal{N}^{*\infty}\to\mathcal{M}^{*\infty}$ is injective (resp., surjective).*

Proof. By considering $\mathcal{M}_{\text{reg}}$ and $\mathcal{N}_{\text{reg}}$, we may assume that $\mathcal{M}$ and $\mathcal{N}$ have R.S. Therefore there is a homomorphism $\psi:\mathcal{M}\to\mathcal{N}$ such that $\varphi=\mathcal{E}^{\infty}\otimes\psi$. Since the cokernel of φ is equal to the tensor product of $\mathcal{E}^{\infty}$ and the cokernel of ψ, we obtain (i). The assertion (ii) is verified in the same way. Now note that φ is surjective (resp., injective) if and only if ψ is so. Hence we obtain (iii).

As an application of the structure theorem (Theorem 1.3.1) of holonomic systems and Theorem 1.2.2, we can prove that the support of a section of a holonomic system is an analytic subset. More precisely we have the following proposition.

Proposition 1.3.8. *Let $\mathcal{M}$ be a coherent $\mathcal{E}_X$-Module with the characteristic variety V. Suppose that there is an integer r such that $\mathcal{E}xt^j_{\mathcal{E}}(\mathcal{M}, \mathcal{E})=0$ for $j \neq r$. Let s be a section of $\mathcal{M}^\infty$ defined on an open set Ω. Then* supp *s is a union of irreducible components of $\Omega \cap V$.*

Proof. We shall prove this proposition in several steps.

(a) In the case where V is smooth and $\omega|_V \neq 0$, where ω is the fundamental 1-form. In this case, we can transform V to $\{(x, \xi); \xi_1 = \cdots = \xi_r = 0\}$. By the condition of the vanishing of cohomology groups, there is a coherent $\mathcal{E}_X$-Module $\mathcal{M}'$ such that $\mathcal{M}^\infty \oplus \mathcal{M}'^\infty \cong \mathcal{L}^{\infty N}$ where $\mathcal{L} = \mathcal{E}/(\mathcal{E}D_1 + \cdots + \mathcal{E}D_r) \underset{\text{def}}{=} \mathcal{E}u_0$ (S-K-K [24] Chapter II, Theorem 5.3.7). Hence we may assume from the beginning that $\mathcal{M} = \mathcal{L}$. Any element s of $\mathcal{L}^\infty$ is written in the unique form $s = P(x, D'')u_0$, where $P(x, D'') = \sum P_j(x, D'')$ is a micro-differential operator which commutes with $x_1, \ldots, x_r$. If s vanishes on a neighborhood of a point $p \in \Omega \cap V$, then P vanishes on a neighborhood of p and hence every $p_j(x, \xi'')$ vanishes. By analytic continuation each p_j vanishes on a connected component of $\Omega \cap V$ containing p, and hence $s=0$ on this connected component.

(b) In the case where V is smooth and $r < \dim X$. Set $Z = \{p \in V; (\omega|_V)(p) = 0\}$. Then $\operatorname{codim} Z \geqq r+1$. By the result for the preceding case (a), $\operatorname{supp} s \cap (V-Z)$ is a union of connected components of $V-Z$. Hence the closure S of $\operatorname{supp} s \cap (V-Z)$ is a union of connected components of V. We have $S \subset \operatorname{supp} s \subset S \cup Z$. Therefore, $s|_{V-S}$ gives a section of $\mathcal{H}^0_Z(\mathcal{M}^\infty)|_{V-S}$, which vanishes by Theorem 1.2.2. Thus we have $S = \operatorname{supp} s$.

(c) In the case where V is smooth Lagrangian. By a quantized contact transformation, we may assume that $V = \{x_1 = 0, \xi_2 = \cdots = \xi_n = 0\}$. Then, by Lemma 1.3.4, $\mathcal{M}^\infty$ is isomorphic to $(\mathcal{E}^N/\mathcal{E}^N(x_1 D_1 - A) + \mathcal{E}^N D_2 + \cdots + \mathcal{E}^N D_n)^\infty$ for an $N \times N$ constant matrix, and hence any element of $\mathcal{M}^\infty$ is written in the unique form $P(x_1, D_2, \ldots, D_n)$, where P is a vector of micro-differential operators of length N. Hence we can apply the same argument as in (a).

(d) The general case. In (a)$\sim$(c), we proved the proposition when V is smooth. Hence $\operatorname{supp} s \cap V_{\text{reg}}$ is closed and open in V_{reg}. Therefore $\operatorname{supp} s \subset \overline{\operatorname{supp} s \cap V_{\text{reg}}} \cup V_{\text{sing}}$. Note that $V' = \overline{\operatorname{supp} s \cap V_{\text{reg}}}$ is a union of irreducible components of V. If $p \in V_{\text{sing}} - V'$, then s belongs to $\mathcal{H}^0_{V_{\text{sing}}}(\mathcal{M}^\infty)_{p_0}$. Since $\mathcal{H}^0_{V_{\text{sing}}}(\mathcal{M}^\infty)_{p_0} = 0$ by Theorem 1.2.2, $s=0$ at p. Therefore we obtain $\operatorname{supp} s \subset V'$.

Evidently supp $s \supset V'$. Thus we obtain the desired result. Q. E. D.

We conjecture that the following general statement be true.

Conjecture: *For any coherent $\mathscr{E}_X$-Module $\mathscr{M}$ and $s \in \mathscr{M}^\infty$,* supp s *is an analytic set.*

§4.

In [6] it is shown that, for any holonomic $\mathscr{D}_X$-Module $\mathscr{M}$, $\mathscr{E}xt^j_{\mathscr{D}_X}(\mathscr{M}, \mathscr{O}_X)$ is a constructible sheaf. We shall prove here that these sheaves of the solutions determine $\mathscr{M}$.

Definition 1.4.1. We call a sheaf $\mathscr{F}$ of $\boldsymbol{C}$-vector spaces on X constructible,(*) if there is a decreasing sequence $X = X_0 \supset X_1 \supset X_2 \supset \cdots$ of closed analytic subset of X such that $\bigcap_j X_j = \phi$ and that $\mathscr{F}|_{X_j - X_{j+1}}$ is a locally constant sheaf of finite rank.

We first recall the following propositions of constructible sheaves. (See [25] Exposé 7 and Appendix § C of this article.)

Proposition 1.4.2. (i) *If $\mathscr{F}^\cdot$ and $\mathscr{G}^\cdot$ are bounded complex of sheaves with constructible sheaves as their cohomologies, then $\mathbf{R}\mathscr{H}om_{\boldsymbol{C}}(\mathscr{F}^\cdot, \mathscr{G}^\cdot)$ has constructible sheaves as their cohomology.*

(ii) *Under the same conditions as in (i), we have*

$$\mathbf{R}\mathscr{H}om_{\boldsymbol{C}}(\mathscr{F}^\cdot, \mathscr{G}^\cdot) = \mathbf{R}\mathscr{H}om_{\boldsymbol{C}}(\mathbf{R}\mathscr{H}om(\mathscr{G}^\cdot, \boldsymbol{C}_X) \underset{\boldsymbol{C}}{\otimes} \mathscr{F}^\cdot, \boldsymbol{C}_X)$$
$$= \mathbf{R}\mathscr{H}om_{\boldsymbol{C}}(\mathbf{R}\mathscr{H}om_{\boldsymbol{C}}(\mathscr{G}^\cdot, \boldsymbol{C}_X), \mathbf{R}\mathscr{H}om_{\boldsymbol{C}}(\mathscr{F}^\cdot, \boldsymbol{C}_X)).$$

In particular, $\mathbf{R}\mathscr{H}om_{\boldsymbol{C}}(\mathbf{R}\mathscr{H}om_{\boldsymbol{C}}(\mathscr{F}^\cdot, \boldsymbol{C}_X), \boldsymbol{C}_X) = \mathscr{F}^\cdot$. Here and in the sequel $\boldsymbol{C}_X$ denotes the constant sheaf on X with $\boldsymbol{C}$ as its stalk.

(iii) *Let $\mathscr{F}^\cdot$ be a bounded complex with constructible sheaves as its cohomologies, and let $\mathscr{L}^\cdot$ be any bounded complex. Then*

$$\mathbf{R}\Gamma_\Delta(p_1^{-1}\mathscr{F}^\cdot \underset{\boldsymbol{C}}{\otimes} p_2^{-1}\mathscr{L}^\cdot) = \mathbf{R}\mathscr{H}om_{\boldsymbol{C}}(\mathbf{R}\mathscr{H}om_{\boldsymbol{C}}(\mathscr{F}^\cdot, \boldsymbol{C}_X), \mathscr{L}^\cdot)[-2n],$$

where $n = \dim X$, Δ is the diagonal set of $X \times X$ and p_1 (resp., p_2) denotes the projection from $X \times X$ to the first (resp., second) component X.

Proposition 1.4.3. *Let $\mathscr{M}$ be a holonomic $\mathscr{D}_X$-Module and Y a complex*

(*) Another terminology "finitistic" was used in [6].

manifold. Let p_1 and p_2 denote the projection from $X \times Y$ onto X and Y, respectively. Then

$$\mathbf{R}\mathcal{H}om_{p_1^{-1}\mathcal{D}_X}(p_1^{-1}(\mathcal{M}), \mathcal{O}_{X\times Y}) \xleftarrow{\sim} p_1^{-1}\mathbf{R}\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \mathcal{O}_X) \underset{\mathbf{C}}{\otimes} p_2^{-1}\mathcal{O}_Y$$

holds.

Proof. The question being local, it is enough to show that for $x \in X$ and $y \in Y$,

$$\mathcal{E}xt^j_{p_1^{-1}\mathcal{D}_X}(p_1^{-1}\mathcal{M}, \mathcal{O}_{X\times Y})_{(x,y)} = \mathcal{E}xt^j_{\mathcal{D}_X}(\mathcal{M}, \mathcal{O}_X)_x \underset{\mathbf{C}}{\otimes} \mathcal{O}_{Y,y}.$$

Let us take a resolution of $\mathcal{M}$ in a neighborhood of x:

$$0 \longleftarrow \mathcal{M} \longleftarrow \mathcal{E}_X^{N_0} \xleftarrow{P_0} \mathcal{E}_X^{N_1} \xleftarrow{P_1} \cdots \xleftarrow{P_{N_{r-1}}} \mathcal{E}_X^{N_r} \longleftarrow 0.$$

Then $\mathcal{E}xt^j_{p_1^{-1}\mathcal{D}_X}(p_1^{-1}\mathcal{M}, \mathcal{O}_X)_x$ is the j-th cohomology group of the complex

$$\mathcal{O}^{\cdot}_{X,x}: \mathcal{O}^{N_0}_{X,x} \xrightarrow{P_0} \mathcal{O}^{N_1}_{X,x} \xrightarrow{P_1} \cdots \xrightarrow{P_{N_{r-1}}} \mathcal{O}^{N_r}_{X,x}$$

and

$$\mathcal{E}xt^j_{\mathcal{D}_X}(\mathcal{M}, \mathcal{O}_{X\times Y})_{(x,y)}$$

is the j-th cohomology group of the complex

$$\mathcal{O}^{\cdot}_{X\times Y,(x,y)}: \mathcal{O}^{N_0}_{X\times Y,(x,y)} \xrightarrow{P_0} \mathcal{O}^{N_1}_{X\times Y,(x,y)} \xrightarrow{P_1} \cdots \xrightarrow{P_{N_{r-1}}} \mathcal{O}^{N_r}_{X\times Y,(x,y)}.$$

Since $\mathcal{O}_{X,x}$ is a nuclear DFS-space and $\mathcal{O}_{X\times Y,(x,y)} = \mathcal{O}_{X,x}\hat{\hat{\otimes}}\mathcal{O}_{Y,y}$ (the completion of the tensor product of topological linear spaces) and since the cohomology of $\mathcal{O}^{\cdot}_{X,x}$ has finite dimension, we have

$$H^j(\mathcal{O}^{\cdot}_{X\times Y,(x,y)}) = H^j(\mathcal{O}^{\cdot}_{X,x})\hat{\hat{\otimes}}\mathcal{O}_{Y,y}. \qquad \text{Q.E.D.}$$

This proposition immediately implies the following corollary.

Corollary 1.4.4. *If $\mathcal{M}^{\cdot}$ is a bounded complex of $\mathcal{D}_X$-Modules whose cohomologies are holonomic, then*

$$\mathbf{R}\mathcal{H}om_{p_1^{-1}\mathcal{D}_X}(p_1^{-1}\mathcal{M}^{\cdot}, \mathcal{O}_{X\times Y}) \xleftarrow{\sim} \mathbf{R}\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}^{\cdot}, \mathcal{O}_X) \underset{\mathbf{C}}{\otimes} p_2^{-1}\mathcal{O}_Y.$$

Corollary 1.4.5. *Let $\mathcal{M}^{\cdot}$ be a bounded complex of $\mathcal{D}_X$-Modules whose cohomologies are holonomic and let $\mathcal{N}^{\cdot}$ be a bounded complex of $\mathcal{D}_Y$-Modules whose cohomologies are coherent. Then*

$$\mathbf{R}\mathcal{H}om_{\mathcal{D}_{X\times Y}}(\mathcal{M}^{\cdot}\hat{\otimes}\mathcal{N}^{\cdot}, \mathcal{O}_{X\times Y}) = p_1^{-1}\mathbf{R}\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}^{\cdot}, \mathcal{O}_X) \underset{\mathbf{C}}{\otimes} p_2^{-1}\mathbf{R}\mathcal{H}om_{\mathcal{D}_Y}(\mathcal{N}^{\cdot}, \mathcal{O}_Y)$$

holds.

Proof. We may assume that $\mathscr{M}^{\cdot}$ and $\mathscr{N}^{\cdot}$ are simple complexes (i.e., complexes consisting of only one non-trivial component). The question being local, we may assume further that $\mathscr{N}$ admits a free resolution. Thus we can assume that $\mathscr{N}=\mathscr{D}_Y$. Then

$$\mathbf{R}\mathscr{H}om_{\mathscr{D}_{X\times Y}}(\mathscr{M}\hat{\otimes}\mathscr{N},\ \mathscr{O}_{X\times Y})=\mathbf{R}\mathscr{H}om_{\mathscr{D}_X}(\mathscr{M},\ \mathscr{O}_{X\times Y})$$

and

$$\mathbf{R}\mathscr{H}om_{\mathscr{D}_Y}(\mathscr{N},\ \mathscr{O}_Y)=\mathscr{O}_Y.$$

The corollary follows then from Proposition 1.4.3. Q.E.D.

Proposition 1.4.6. *Let $\mathscr{M}^{\cdot}$ be a bounded complex of $\mathscr{D}_X$-Modules with holonomic systems as cohomologies. Then*

$$\mathbf{R}\mathscr{H}om_{\mathscr{D}_X}(\mathscr{O}_X,\ \mathscr{M}^{\cdot})\xrightarrow{\sim}\mathbf{R}\mathscr{H}om_{\mathbf{C}}(\mathbf{R}\mathscr{H}om_{\mathscr{D}_X}(\mathscr{M}^{\cdot},\ \mathscr{O}_X),\ \mathbf{C}_X)$$

holds.

Proof. Let $I^{\cdot}\leftarrow\mathscr{M}^{\cdot}$ be a complex of injective $\mathscr{D}_X$-Modules quasi-isomorphic to $\mathscr{M}^{\cdot}$ and let $J^{\cdot}$ be an injective resolution of $\mathscr{O}_X$ as $\mathscr{D}_X$-Modules. Then $\mathscr{H}om_{\mathscr{D}}(\mathscr{O}_X,\ J^{\cdot})$ is an injective resolution of $\mathbf{C}_X$, because $\mathscr{E}xt^j_{\mathscr{D}}(\mathscr{O}_X,\ \mathscr{O}_X)=H^j(\mathscr{H}om_{\mathscr{D}}(\mathscr{O}_X,\ J^{\cdot}))$. Hence we have

$$\mathbf{R}\mathscr{H}om_{\mathscr{D}}(\mathscr{O},\ \mathscr{M}^{\cdot})\cong\mathscr{H}om_{\mathscr{D}}(\mathscr{O},\ I^{\cdot}),$$
$$\mathbf{R}\mathscr{H}om_{\mathscr{D}}(\mathscr{M}^{\cdot},\ \mathscr{O})\cong\mathscr{H}om_{\mathscr{D}}(I^{\cdot},\ J^{\cdot}),$$

and

$$\mathbf{R}\mathscr{H}om_{\mathbf{C}}(\mathbf{R}\mathscr{H}om_{\mathscr{D}}(\mathscr{M}^{\cdot},\ \mathscr{O}),\ \mathbf{C}_X)\cong\mathscr{H}om_{\mathbf{C}}(\mathscr{H}om_{\mathscr{D}}(I^{\cdot},\ J^{\cdot}),\mathscr{H}om_{\mathscr{D}}(\mathscr{O},\ J^{\cdot})).$$

We have the homomorphism

$$\mathscr{H}om_{\mathscr{D}}(\mathscr{O},\ I^{\cdot})\longrightarrow\mathscr{H}om_{\mathbf{C}}(\mathscr{H}om_{\mathscr{D}}(I^{\cdot},\ J^{\cdot}),\ \mathscr{H}om_{\mathscr{D}}(\mathscr{O},\ J^{\cdot})),$$

and hence we can define the homomorphism

$$\mathbf{R}\mathscr{H}om(\mathscr{O},\ \mathscr{M}^{\cdot})\longrightarrow\mathbf{R}\mathscr{H}om_{\mathbf{C}}(\mathbf{R}\mathscr{H}om_{\mathscr{D}}(\mathscr{M}^{\cdot},\ \mathscr{O}),\ \mathbf{C}_X).$$

We shall show that this is an isomorphism. We may assume without loss of generality that $\mathscr{M}^{\cdot}$ is a single $\mathscr{D}_X$-Module $\mathscr{M}$. It is enough to show that the homomorphism

$$\mathbf{R}\mathscr{H}om(\mathscr{O},\ \mathscr{M})_x\longrightarrow\mathbf{R}\mathscr{H}om_{\mathbf{C}}(\mathbf{R}\mathscr{H}om_{\mathscr{D}}(\mathscr{M},\ \mathscr{O}),\ \mathbf{C}_X)_x \tag{1.4.1}$$

is quasi-isomorphism for any $x\in X$. For this purpose we recall the following

Lemma 1.4.7. *For any bounded complex $F^{\cdot}$ with constructible sheaves* *as its cohomologies, we have*

$$\mathbf{R}\mathcal{H}om_{\mathbf{C}}(F^{\cdot}, \mathbf{C}_X)_x \cong Hom_{\mathbf{C}}(\mathbf{R}\Gamma_{\{x\}}(F^{\cdot})_x, \mathbf{C})[-2n],$$

where n denotes dim X.

Proof. Let $\mathbf{C}_x$ be the sheaf with the support at $\{x\}$ whose stalk at x is $\mathbf{C}$. Then

$$\begin{aligned}(\mathbf{R}\mathcal{H}om_{\mathbf{C}}(F^{\cdot}, \mathbf{C}_X)_x)^* &= \mathbf{R}\mathcal{H}om_{\mathbf{C}}(\mathbf{R}Hom_{\mathbf{C}}(F^{\cdot}, \mathbf{C}_X), \mathbf{C}_x)_x \\ &= \mathbf{R}\mathcal{H}om_{\mathbf{C}}(\mathbf{R}\mathcal{H}om_{\mathbf{C}}(\mathbf{C}_x, \mathbf{C}), F^{\cdot})_x \\ &= \mathbf{R}\mathcal{H}om_{\mathbf{C}}(\mathbf{C}_x[-2n], F^{\cdot})_x \\ &= \mathbf{R}\Gamma_{\{x\}}(F^{\cdot})_x[2n].\end{aligned}$$

Here * means the dual vector space. Q.E.D.

We return to the proof of Proposition 1.4.6. We shall show that (1.4.1) is a quasi-isomorphism. By [6], we have

$$(\mathbf{R}\mathcal{H}om(\mathcal{O}, \mathcal{M})_x)^* = \mathbf{R}\mathcal{H}om_{\mathcal{D}}(\mathcal{M}, \mathcal{B}^{\infty}_{\{x\}|X})[n] = \mathbf{R}\Gamma_{\{x\}}\mathbf{R}\mathcal{H}om_{\mathcal{D}}(\mathcal{M}, \mathcal{O})[2n].$$

Thus, together with Lemma 1.4.7, we can conclude that (1.4.1) is a quasi-isomorphism. This completes the proof of Proposition 1.4.6.

Proposition 1.4.8. *Let $\mathcal{M}^{\cdot}$ and $\mathcal{N}^{\cdot}$ be bounded complexes of $\mathcal{D}_X$-Modules whose cohomologies are holonomic. Then*

$$\mathbf{R}\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}^{\cdot}, \mathcal{N}^{\cdot\infty}) = \mathbf{R}\mathcal{H}om_{\mathbf{C}}(\mathbf{R}\mathcal{H}om_{\mathcal{D}_X}(\mathcal{N}^{\cdot}, \mathcal{M}^{\cdot\infty}), \mathbf{C})$$

holds.

Proof. By [8], we have

$$\mathbf{R}\mathcal{H}om_{\mathcal{D}_X}(\mathcal{N}^{\cdot}, \mathcal{M}^{\cdot\infty}) = \mathbf{R}\mathcal{H}om_{\mathcal{D}_{X\times X}}(\mathcal{N}^{\cdot}\hat{\otimes}\mathcal{M}^{\cdot *}, \mathcal{B}^{\infty}_{\Delta|X\times X})[n],$$

where Δ is the diagonal set of $X\times X$ and $\mathcal{M}^{\cdot *} = \mathbf{R}\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}^{\cdot}, \mathcal{D}_X) \underset{\mathcal{O}_X}{\otimes} \Omega_X^{\otimes -1}[n]$. Hence by Proposition 1.4.2 we have

$$\begin{aligned}\mathbf{R}\mathcal{H}om_{\mathcal{D}_X}(\mathcal{N}^{\cdot}, \mathcal{M}^{\cdot\infty}) &= \mathbf{R}\mathcal{H}om_{\mathcal{D}_{X\times X}}(\mathcal{N}^{\cdot}\hat{\otimes}\mathcal{M}^{\cdot *}, \mathbf{R}\Gamma_{\Delta}(\mathcal{O}_{X\times X})[n])[n] \\ &= \mathbf{R}\Gamma_{\Delta}(\mathbf{R}\mathcal{H}om_{\mathcal{D}_{X\times X}}(\mathcal{N}^{\cdot}\hat{\otimes}\mathcal{M}^{\cdot *}, \mathcal{O}_{X\times X}))[2n] \\ &= \mathbf{R}\Gamma_{\Delta}(\mathbf{R}\mathcal{H}om(\mathcal{N}^{\cdot}, \mathcal{O}_X)\hat{\otimes}\mathbf{R}\mathcal{H}om(\mathcal{M}^{\cdot *}, \mathcal{O}_X))[2n].\end{aligned}$$

On the other hand, it follows from Proposition 1.4.6 that

$$\begin{aligned}&\mathbf{R}\mathcal{H}om_{\mathbf{C}}(\mathbf{R}\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}^{\cdot *}, \mathcal{O}_X), \mathbf{C}_X)) \\ &\qquad = \mathbf{R}\mathcal{H}om_{\mathcal{D}_X}(\mathcal{O}_X, \mathcal{M}^{\cdot *}) = \mathbf{R}\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}^{\cdot}, \mathcal{O}_X).\end{aligned}$$

Hence by Proposition 1.4.2 (iii) we obtain

$$\mathbf{R}\mathcal{H}om_{\mathcal{D}_X}(\mathcal{N}^{\cdot}, \mathcal{M}^{\cdot\infty}) = \mathbf{R}\mathcal{H}om_{\mathbf{C}}(\mathbf{R}\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}^{\cdot}, \mathcal{O}_X), \mathbf{R}\mathcal{H}om_{\mathcal{D}_X}(\mathcal{N}^{\cdot}, \mathcal{O}_X)).$$

In the same way we have

$$\mathbf{R}\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}^{\cdot}, \mathcal{N}^{\cdot\infty}) = \mathbf{R}\mathcal{H}om_{\mathbf{C}}(\mathbf{R}\mathcal{H}om_{\mathcal{D}}(\mathcal{N}^{\cdot}, \mathcal{O}_X), \mathbf{R}\mathcal{H}om_{\mathcal{D}}(\mathcal{M}^{\cdot}, \mathcal{O}_X)).$$

Thus the required result follows from Proposition 1.4.2 (ii). Q.E.D.

Now we shall show the following theorem which tells us how to reconstruct a holonomic system by its solutions.

Theorem 1.4.9. *Let $\mathcal{M}^{\cdot}$ be a bounded complex of $\mathcal{D}_X$-Modules whose cohomologies are holonomic. Then*

$$\mathbf{R}\mathcal{H}om_{\mathbf{C}}(\mathbf{R}\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}^{\cdot}, \mathcal{O}_X), \mathcal{O}_X) = \mathcal{M}^{\cdot\infty}.$$

Proof. By the definition of $\mathcal{B}^{\infty}_{\Delta|X\times X}$ and Proposition 1.4.2 (iii), we find the following equalities.

$$\begin{aligned}
\mathcal{M}^{\cdot\infty} &= \mathcal{D}^{\infty}_X \overset{\mathbf{L}}{\underset{\mathcal{D}_X}{\otimes}} \mathcal{M}^{\cdot} = \mathcal{B}^{(0,n)\infty}_{\Delta|X\times X} \overset{\mathbf{L}}{\underset{\mathcal{D}_X}{\otimes}} \mathcal{M}^{\cdot} \\
&= \mathbf{R}\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}^{\cdot *}, \mathcal{B}^{\infty}_{\Delta|X\times X})[n] \\
&= \mathbf{R}\Gamma_{\Delta}(\mathbf{R}\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}^{\cdot *}, \mathcal{O}_{X\times X}))[2n] \\
&= \mathbf{R}\Gamma_{\Delta}(\mathcal{O}_X \hat{\otimes} \mathbf{R}\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}^{\cdot *}, \mathcal{O}_X))[2n] \\
&= \mathbf{R}\mathcal{H}om_{\mathbf{C}}(\mathbf{R}\mathcal{H}om_{\mathbf{C}}(\mathbf{R}\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}^{\cdot *}, \mathcal{O}_X), \mathbf{C}), \mathcal{O}_X) \\
&= \mathbf{R}\mathcal{H}om_{\mathbf{C}}(\mathbf{R}\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}^{\cdot}, \mathcal{O}_X), \mathcal{O}_X).
\end{aligned}$$

Q.E.D.

§ 5.

In [18] we developed the notion of principal symbols for a system of micro-differential equations with regular singularities. Here we apply it to holonomic systems whose characteristic variety is non-singular.

5.1. Let X be a complex manifold and Λ a Lagrangian submanifold of $T^*X - T^*_XX$.

Let $\mathcal{Q}$ be an invertible $\mathcal{O}_\Lambda$-Module such that $\mathcal{Q}^{\otimes 2} \underset{\text{def}}{=} \mathcal{Q} \underset{\mathcal{O}_\Lambda}{\otimes} \mathcal{Q} \cong \Omega_\Lambda \otimes \Omega_X^{\otimes -1}$. In general, such a $\mathcal{Q}$ does not exist globally on Λ; however, $\mathcal{Q}$ exists locally on Λ, and a local existence of $\mathcal{Q}$ is sufficient for our subsequent discussion.

Let $\mathcal{Q}'$ be another invertible $\mathcal{O}_\Lambda$-Module such that $\mathcal{Q}'^{\otimes 2} \cong \Omega_\Lambda \otimes \Omega_X^{\otimes -1}$, then there is locally an isomorphism $\varphi: \mathcal{Q} \to \mathcal{Q}'$ such that $\varphi(s) \otimes \varphi(s) = s \otimes s$ as a section of $\Omega_\Lambda \otimes \Omega_X^{\otimes -1}$ for any $s \in \mathcal{Q}$. If φ is such an isomorphism, then any isomorphism φ' satisfying the same condition as φ must be either φ or $-\varphi$; in fact, we have $\varphi(s)^{\otimes 2}=\varphi'(s)^{\otimes 2}$ for any $s\in\mathscr{Q}$. Therefore $\mathscr{Q}$ is uniquely determined up to sign.

Let $\mathscr{X}$ be the vector field defined by $\sum_{j=1}^{n}\xi_j\frac{\partial}{\partial\xi_j}$. Then $\mathscr{X}$ does not depend on the choice of local coordinate systems. Since Λ is homogeneous, $\mathscr{X}$ acts on $\Omega_\Lambda\otimes\Omega_X^{\otimes -1}$ as a derivation, and hence $\mathscr{X}$ acts also on $\mathscr{Q}$ as a derivation by the formula: $2s\otimes\mathscr{X}(s)=\mathscr{X}(s^{\otimes 2})$ for $s\in\mathscr{Q}$.

Let $\mathscr{A}$ be the sheaf of linear differential operators of finite order from $\mathscr{Q}$ into $\mathscr{Q}$. Although $\mathscr{Q}$ does not exist globally, $\mathscr{A}$ is canonically defined and exists globally because $\mathscr{Q}$ is uniquely determined up to sign.

For an integer m, we shall denote by $\mathscr{A}(m)$ the subsheaf of $\mathscr{A}$ consisting of all homogeneous differential operators from $\mathscr{Q}$ into $\mathscr{Q}$ homogeneous of order m; in other words, $\mathscr{A}(m)=\{P\in\mathscr{A};\ [\mathscr{X}, P]=mP\}$. We shall denote by $\mathscr{O}_\Lambda(m)$ the sheaf of homogeneous holomorphic functions of degree m defined on Λ. The Algebra $\mathscr{A}(0)$ contains $\mathscr{O}_\Lambda(0)$ as a sub-Algebra and we have the following relations:

$$\mathscr{O}_\Lambda\underset{\mathscr{O}_\Lambda(0)}{\otimes}\mathscr{A}(0)\cong\mathscr{A}(0)\underset{\mathscr{O}_\Lambda(0)}{\otimes}\mathscr{O}_\Lambda\cong\mathscr{A},$$
$$\mathscr{O}_\Lambda(m)\underset{\mathscr{O}_\Lambda(0)}{\otimes}\mathscr{A}(0)\cong\mathscr{A}(0)\underset{\mathscr{O}_\Lambda(0)}{\otimes}\mathscr{O}_\Lambda(m)\cong\mathscr{A}(m).$$

In [18] we defined the homomorphism $L^{(m)(*)}$ from $\mathscr{I}_\Lambda(m)=\mathscr{O}_\Lambda(m)\mathscr{I}_\Lambda$ into $\mathscr{A}(m)$ as follows

$$(1.5.1)\quad L^{(m)}(P)(s)=\left\{\left(H_{P_{m+1}}+P_m-\frac{1}{2}\sum_{i=1}^{n}\frac{\partial^2 P_{m+1}}{\partial x_i\partial\xi_i}\right)(s\otimes\sqrt{dx})\right\}\otimes(\sqrt{dx})^{\otimes -1}$$

for $P=P_{m+1}(x, D)+P_m(x, D)+\cdots\in\mathscr{I}_\Lambda$. Here $dx=dx_1\wedge\cdots\wedge dx_n$. This homomorphism is uniquely extended to a homomorphism from $\mathscr{E}_\Lambda(m)$ into $\mathscr{A}(m)$ under the condition

$$(1.5.2)\qquad L^{(m_1)}(P_1)L^{(m_2)}(P_2)=L^{(m_1+m_2)}(P_1P_2)$$

for $P_1\in\mathscr{E}_\Lambda(m_1)$ and $P_2\in\mathscr{E}_\Lambda(m_2)$. One can check easily $L^{(m)}(\mathscr{E}_\Lambda(m-1))=0$. In fact, we have a more precise statement.

Lemma 1.5.1. *The sequence*

$$0\longrightarrow\mathscr{E}_\Lambda(m-1)\longrightarrow\mathscr{E}_\Lambda(m)\xrightarrow{L^{(m)}}\mathscr{A}(m)\longrightarrow 0$$

is exact.

Proof. By multiplying an invertible micro-differential operator of order $(-m)$, we can reduce the statement to the case where $m=0$.

(*) In [18] this homomorphism is denoted by $L^{(m+1)}$ instead of $L^{(m)}$.

Let us denote by $J_\Lambda(k)$ the intersection $J_\Lambda \cap \mathcal{O}_{T^*X}(k)$. Then, for any k, we have

$$\mathcal{I}_\Lambda^k/(\mathcal{I}_\Lambda^{k-1}+\mathcal{I}_\Lambda^{k+1}(-1)) \cong J_\Lambda(1)^k/(J_\Lambda^{k+1}\cap \mathcal{O}_{T^*X}(k)) \cong S^k(J_\Lambda(1)/(J_\Lambda^2 \cap \mathcal{O}_{T^*X}(1))),$$

where S^k signifies the k-th symmetric product. On the other hand, if we denote by $\Theta_\Lambda(1)$ the sheaf of homogeneous vector fields on Λ, and if $\mathcal{A}_k(0)$ denotes the sheaf of homogeneous differential operators from $\mathcal{Q}$ into $\mathcal{Q}$ of order $\leqq k$, then we have $\mathcal{A}_k(0)/\mathcal{A}_{k-1}(0) \cong S^k(\Theta_\Lambda(1))$. Since the Hamiltonian map H induces an isomorphism from $J_\Lambda(1)/(J_\Lambda^2 \cap \mathcal{O}_{T^*X}(1))$ onto $\Theta_\Lambda(1)$, $L^{(0)}$ also induces an isomorphism

$$\mathcal{I}_\Lambda^k/(\mathcal{I}_\Lambda^{k-1}\cap \mathcal{I}_\Lambda^{k+1}(-1)) \xrightarrow{\sim} \mathcal{A}_k(0)/\mathcal{A}_{k-1}(0).$$

In particular, we have

$\mathcal{I}_\Lambda^k \cap \operatorname{Ker} L^{(0)} \subset (\mathcal{I}_\Lambda^{k-1} \cap \operatorname{Ker} L^{(0)}) + \mathcal{E}_\Lambda(-1)$ for any k,

which implies $\operatorname{Ker} L^{(0)} \subset \mathcal{E}_\Lambda(-1)$. Q.E.D.

Let θ be a section of $\mathcal{I}_\Lambda$ such that $L^{(0)}(\theta)=\mathcal{X}$. As is easily seen, θ is characterised by the following conditions (1.5.3)$\sim$(1.5.6):

(1.5.3) $\theta=\theta_1(x, D)+\theta_0(x, D)+\cdots$ is a micro-differential operator of order 1.

(1.5.4) $$\theta_1|_\Lambda=0.$$

(1.5.5) $$d\theta_1 \equiv -\omega \bmod J_\Lambda \Omega^1_{T^*X}.$$

(1.5.6) $$\theta_0=\frac{1}{2}\sum_{i=1}^n \frac{\partial^2\theta_1}{\partial x_i \partial \xi_i} \quad \text{on } \Lambda.$$

We then find the following properties of θ.

(1.5.7) For any $P \in \mathcal{E}_\Lambda(m)$, $[\theta, P] \equiv mP \bmod \mathcal{E}_\Lambda(m-1)$.

(1.5.8) Let P be an invertible micro-differential operator of order m. Then, for any polynomial $g(\theta)$ of one variable, we have

$$Pg(\theta)P^{-1} \equiv g(\theta+m) \bmod \mathcal{E}_\Lambda(-1).$$

The property (1.5.7) immediately follows from Lemma 1.5.1 and (1.5.8) is an easy consequence of (1.5.7).

5.2. Let $\mathcal{M}$ be a coherent $\mathcal{E}_X$-Module with regular singularities along a Lagrangian submanifold Λ. In particular, the support of $\mathcal{M}$ is contained in Λ and hence $\mathcal{M}$ is a holonomic $\mathcal{E}_X$-Module.

Let $\mathcal{M}_0$ be a coherent $\mathcal{E}_\Lambda$-sub-Module of $\mathcal{M}$. Since $\mathcal{M}$ has regular singularities, $\mathcal{M}_0$ is coherent over $\mathcal{E}(0)$. We denote by $\overline{\mathcal{M}_0}$ the quotient sheaf

$\mathscr{M}_0/\mathscr{M}_0(-1)$. Then $\overline{\mathscr{M}_0}$ is endowed with a canonical structure of $\mathscr{A}(0)$-Module, and $\overline{\mathscr{M}_0}$ is a coherent $\mathcal{O}_\Lambda(0)$-Module. Since $\mathscr{A} \underset{\mathscr{A}(0)}{\otimes} \overline{\mathscr{M}_0} = \mathcal{O}_\Lambda \underset{\mathcal{O}_\Lambda(0)}{\otimes} \overline{\mathscr{M}_0}$ is an $\mathscr{A}$-Module coherent over $\mathcal{O}_\Lambda$, we can apply the theory of systems of linear differential equations and $\mathcal{O}_\Lambda \underset{\mathcal{O}_\Lambda(0)}{\otimes} \overline{\mathscr{M}_0}$ is locally isomorphic to a finite direct sum of copies of $\mathscr{Q}$ as an $\mathscr{A}$-Module. In particular, $\mathcal{O}_\Lambda \underset{\mathcal{O}_\Lambda(0)}{\otimes} \overline{\mathscr{M}_0}$ is a locally free $\mathcal{O}_\Lambda$-Module of finite rank, and hence $\overline{\mathscr{M}_0}$ is a locally free $\mathcal{O}_\Lambda(0)$-Module of finite rank.

Let $\mathscr{F}$ be the C_Λ-Module $\mathscr{H}om_{\mathscr{A}}(\mathcal{O}_\Lambda \underset{\mathcal{O}_\Lambda(0)}{\otimes} \overline{\mathscr{M}_0}, \mathscr{Q}) = \mathscr{H}om_{\mathscr{A}(0)}(\overline{\mathscr{M}_0}, \mathscr{Q})$. Then $\mathscr{F}$ is a locally free C_Λ-Module of finite rank and we have an isomorphism $\mathcal{O}_\Lambda \underset{\mathcal{O}_\Lambda(0)}{\otimes} \overline{\mathscr{M}_0} \xrightarrow{\sim} \mathscr{H}om_{C_\Lambda}(\mathscr{F}, \mathscr{Q})$ of $\mathscr{A}$-Modules. Since $C[\theta]$ is the center of $\mathscr{A}(0)$, $\mathscr{F}$ is endowed with a structure of $C[\theta]$-Module. One can check easily that

$$\overline{\mathscr{M}_0} \xrightarrow{\sim} \mathscr{H}om_{C[\theta]}(\mathscr{F}, \mathscr{Q}). \tag{1.5.9}$$

Since $\mathscr{F}$ is a locally free C_Λ-Module of finite rank, there exists a non-zero polynomial $b(\theta)$ such that $b(\theta)\mathscr{F}=0$. This condition is equivalent to the condition $b(\theta)\overline{\mathscr{M}_0}=0$ by the isomorphism (1.5.9). We shall denote by $b(\theta: \mathscr{M}_0)$ the monic polynomial $b(\theta)$ with the smallest degree such that $b(\theta)\overline{\mathscr{M}_0}=0$. The above investigation assures the existence of $b(\theta; \mathscr{M}_0)$.

For $\lambda \in C$, we define $\mathscr{F}\langle\lambda\rangle$ and $\overline{\mathscr{M}_0}\langle\lambda\rangle$ by

$$\mathscr{F}\langle\lambda\rangle = \{s \in \mathscr{F};\ (\theta-\lambda)^N s = 0 \quad \text{for} \quad N \gg 0\} \tag{1.5.10}$$

and

$$\overline{\mathscr{M}_0}\langle\lambda\rangle = \{s \in \overline{\mathscr{M}_0};\ (\theta-\lambda)^N s = 0 \quad \text{for} \quad N \gg 0\}. \tag{1.5.11}$$

Then, three conditions $\mathscr{F}\langle\lambda\rangle=0$, $\overline{\mathscr{M}_0}\langle\lambda\rangle=0$ and $b(\lambda; \mathscr{M}_0)\neq 0$ are obviously equivalent. We have also

$$\mathscr{F} = \bigoplus_\lambda \mathscr{F}\langle\lambda\rangle \tag{1.5.12}$$

$$\overline{\mathscr{M}_0} = \bigoplus_\lambda \overline{\mathscr{M}_0}\langle\lambda\rangle. \tag{1.5.13}$$

$$\overline{\mathscr{M}_0}\langle\lambda\rangle \cong \mathscr{H}om_{C[\theta]}(\mathscr{F}\langle\lambda\rangle, \mathscr{Q}) \tag{1.5.14}$$

$$\mathscr{F}\langle\lambda\rangle \cong \mathscr{H}om_{\mathscr{A}(0)}(\overline{\mathscr{M}_0}\langle\lambda\rangle, \mathscr{Q}). \tag{1.5.15}$$

Lemma 1.5.2. $b(\theta; \mathscr{M}_0(k)) = b(\theta-k; \mathscr{M}_0)$.

Proof. Let P be an invertible micro-differential operator of order k. Then, for a polynomial $b(\theta)$, by (1.5.8), we have the following chain of equivalent statements:

$$b(\theta)\overline{\mathscr{M}_0}=0 \Longleftrightarrow b(\theta)\mathscr{M}_0 \subset \mathscr{M}_0(-1) \Longleftrightarrow b(\theta)P^{-1}\mathscr{M}_0(k) \subset P^{-1}\mathscr{M}_0(k-1)$$
$$\Longleftrightarrow Pb(\theta)P^{-1}\mathscr{M}_0(k) \subset \mathscr{M}_0(k-1) \Longleftrightarrow b(\theta+k)\mathscr{M}_0(k) \subset \mathscr{M}_0(k-1).$$

Thus we have the desired result. Q.E.D.

Lemma 1.5.3. *If $\mathscr{M}'_0$ is a coherent $\mathscr{E}_\Lambda$-sub-Module of a coherent $\mathscr{E}_\Lambda$-sub-Module $\mathscr{M}_0$ of $\mathscr{M}$. Then $b(\theta; \mathscr{M}'_0)$ is a divisor of $b(\theta; \mathscr{M}_0)b(\theta+1; \mathscr{M}_0)\cdots b(\theta+N; \mathscr{M}_0)$ for some N.*

Proof. There is N such that $\mathscr{M}'_0 \cap \mathscr{M}_0(-N-1) \subset \mathscr{M}'_0(-1)$. We have

$$\begin{aligned} &b(\theta; \mathscr{M}_0)b(\theta+1; \mathscr{M}_0)\cdots b(\theta+N; \mathscr{M}_0)\mathscr{M}'_0 \\ &\subset b(\theta; \mathscr{M}_0(-N))b(\theta; \mathscr{M}_0(1-N))\cdots b(\theta; \mathscr{M}_0)\mathscr{M}_0 \\ &\subset b(\theta; \mathscr{M}_0(-N))\cdots b(\theta; \mathscr{M}_0(-1))\mathscr{M}_0(-1) \subset \cdots \subset \mathscr{M}_0(-N-1). \end{aligned}$$

and hence we obtain

$$\begin{aligned} &b(\theta;\mathscr{M}_0)b(\theta+1; \mathscr{M}_0)\cdots b(\theta+N; \mathscr{M}_0)\mathscr{M}'_0 \\ &\subset \mathscr{M}'_0 \cap \mathscr{M}_0(-N-1) \subset \mathscr{M}'_0(-1). \end{aligned}$$

This gives the desired result. Q.E.D.

Lemma 1.5.4. *Let $\mathscr{M}_0$ be a coherent $\mathscr{E}_\Lambda$-sub-Module of $\mathscr{M}$ which generates $\mathscr{M}$ as an $\mathscr{E}_X$-Module. Let $\varphi(\theta)$ be a polynomial prime to $b(\theta-k; \mathscr{M}_0)$ for any $k=1, 2,\dots$. Then we have*

$$\mathscr{M}_0=\{s \in \mathscr{M};\ \varphi(\theta)s \in \mathscr{M}_0\}.$$

Proof. Let s be a section of $\mathscr{M}$ such that $\varphi(\theta)s \in \mathscr{M}_0$. There is N such that $s \in \mathscr{M}_0(N)$. By the induction on N, we shall prove that s belongs to $\mathscr{M}_0$. If $N \leqq 0$, then there is nothing to prove. Suppose that $N>0$. Then $b(\theta; \mathscr{M}_0(N))=b(\theta-N; \mathscr{M}_0)$ and $\varphi(\theta)$ are prime to each other. Therefore $b(\theta; \mathscr{M}_0(N))s \in \mathscr{M}_0(N-1)$ and $\varphi(\theta)s \in \mathscr{M}_0(N-1)$ imply $s \in \mathscr{M}_0(N-1)$. By the hypothesis of induction, we obtain $s \in \mathscr{M}_0$. Q.E.D.

Lemma 1.5.5. *Let $\mathscr{M}'_0$ be a coherent $\mathscr{E}_\Lambda$-sub-Module of a coherent $\mathscr{E}_\Lambda$-sub-Module $\mathscr{M}_0$ of $\mathscr{M}$. Then, for any $\lambda \in \mathbf{C}$, $\overline{\mathscr{M}'_0}\langle\lambda\rangle \to \overline{\mathscr{M}_0}\langle\lambda\rangle$ is injective if $b(\lambda+k; \mathscr{M}_0) \neq 0$ for $k=1, 2, 3,\dots$.*

Proof. In order to prove this lemma, it is sufficient to show that for any $s \in \mathscr{M}'_0 \cap \mathscr{M}_0(-1)$ and an integer $m (\theta-\lambda)^m s \in \mathscr{M}'_0(-1)$ implies $s \in \mathscr{M}'_0(-1)$. Let us take N such that $\mathscr{M}'_0 \cap \mathscr{M}_0(-N-1) \subset \mathscr{M}'_0(-1)$ and set $b(\theta)=b(\theta; \mathscr{M}_0(-1))\cdots b(\theta; \mathscr{M}_0(-N))$. Then we have $b(\theta)\mathscr{M}_0(-1) \subset \mathscr{M}_0(-N-1)$ and hence $b(\theta)s \in \mathscr{M}_0(-N-1) \cap \mathscr{M}'_0 \subset \mathscr{M}'_0(-1)$. Since $b(\theta)$ and $(\theta-\lambda)^m$ are prime to each other, we obtain $s \in \mathcal{M}'_0(-1)$. Q.E.D.

Lemma 1.5.6. *Let $\mathcal{M}_0$ be a coherent $\mathcal{E}_\Lambda$-sub-Module of $\mathcal{M}$ and let λ be a complex number. Choose a polynomial $b(\theta)$ and a non-negative integer N so that $b(\theta; \mathcal{M}_0)=b(\theta)(\theta-\lambda)^N$ with $b(\lambda)\neq 0$. Let $\mathcal{M}'_0$ be the kernel of $\mathcal{M}_0 \to \mathcal{M}_0\langle\lambda\rangle$. Then $b(\theta; \mathcal{M}'_0)$ is a divisor of $b(\theta)(\theta-\lambda+1)^N$.*

Proof. If $N=0$, then $\mathcal{M}_0\langle\lambda\rangle=0$ and hence $\mathcal{M}_0=\mathcal{M}'_0$. Therefore the lemma is obvious.

Suppose $N\geqq 1$. We have $\overline{\mathcal{M}_0}=\overline{\mathcal{M}_0}\langle\lambda\rangle\oplus\{s\in\overline{\mathcal{M}_0};\, b(\theta)s=0\}$. Hence $b(\theta)(\mathcal{M}'_0/\mathcal{M}_0(-1))=0$, or equivalently, $b(\theta)\mathcal{M}'_0\subset\mathcal{M}_0(-1)$. Now, we shall show that $(\theta-\lambda+1)^N b(\theta)\mathcal{M}'_0\subset\mathcal{M}'_0(-1)$. We have $(\theta-\lambda+1)^N b(\theta)\mathcal{M}'_0\subset(\theta-\lambda+1)^N\mathcal{M}_0(-1)$. Since $0\to\mathcal{M}'_0(-1)\to\mathcal{M}_0(-1)\to(\mathcal{M}_0(-1))\langle\lambda-1\rangle$ is exact and $(\theta-\lambda+1)^N(\mathcal{M}_0(-1))\langle\lambda\rangle=0$, $(\theta-\lambda+1)^N\mathcal{M}_0(-1)$ is contained in $\mathcal{M}'_0(-1)$. Thus we obtain the desired result. Q.E.D.

5.3. Let u be a section of $\mathcal{M}$. A root of $b(\theta; \mathcal{E}_\Lambda u)=0$ is called an order of u and the set of roots of u will be denoted by $\operatorname{ord} u$. It is easy to see that $\operatorname{ord}(Pu)=\operatorname{ord} u+\operatorname{ord} P$, if a micro-differential operator P satisfies $\sigma(P)|_\Lambda\not\equiv 0$.

A solution of the system of linear differential equations for $\varphi\in\mathcal{Q}$

(1.5.16) $$L_P^{(0)}\varphi=0 \text{ for any } P\in\mathcal{E}_\Lambda \text{ satisfying } Pu=0$$

is called a *principal symbol* of u and the linear hull of principal symbols of u is denoted by $\sigma(u)$. The space of principal symbol is nothing but $\mathcal{H}om_{\mathcal{A}(0)}(\mathcal{E}_\Lambda u/\mathcal{E}_\Lambda(-1)u, \mathcal{Q})$. If m is the multiplicity of $\mathcal{E}_X u$, then $\mathcal{E}_\Lambda u/\mathcal{E}_\Lambda(-1)u$ is locally isomorphic to $\mathcal{Q}^m$ and hence $\sigma(P)$ has dimension m. If P is a micro-differential operator such that $\sigma(P)|_\Lambda\not\equiv 0$, then $\sigma(Pu)=\sigma(P)\sigma(u)$.

Let $\mathcal{M}_0$ be a coherent $\mathcal{E}_\Lambda$-sub-Module of $\mathcal{M}$ which generates $\mathcal{M}$ as an $\mathcal{E}_X$-Module and let $\lambda_1,\ldots,\lambda_N$ be the roots of $b(\theta; \mathcal{M}_0)=0$. Then, by Lemma 1.5.2. and Lemma 1.5.3, any order of any section has the form $\lambda_j-\nu$ for some j and $\nu\in\boldsymbol{Z}$. By Lemma 1.5.3, any order of any section of $\mathcal{M}_0$ has the form $\lambda_j-\nu$ for some j and $\nu\in\boldsymbol{Z}_+$. More precisely, we have the following lemma, which is an immediate consequence of Lemma 1.5.5.

Lemma 1.5.7. *Let $\mathcal{M}_0$ be a coherent $\mathcal{E}_\Lambda$-sub-Module in $\mathcal{M}$ and let $\{\lambda_1,\ldots,\lambda_N\}$ be the set of roots of $b(\lambda; \mathcal{M}_0)=0$. Then for any section u of the kernel of $\mathcal{M}_0\to\overline{\mathcal{M}_0}\langle\lambda\rangle$,* ord u *has either the form $\lambda_1-\nu\,(\nu=1, 2,\ldots)$ or the form $\lambda_j-\nu\,(j=2,\ldots, N,\ \nu=0, 1, 2,\ldots)$.*

Proposition 1.5.8. *Let Z be a subset of $\boldsymbol{C}$ satisfying the following three conditions*:

(1.5.17) If $z \in Z$ and $k \in \boldsymbol{Z}_+$ then $z-k \in Z$.

(1.5.18) For any $z \in \boldsymbol{C}$, there is $k \in \boldsymbol{Z}$ such that $z+k \in Z$.

(1.5.19) For any $z \in \boldsymbol{C}$, there is $k \in \boldsymbol{Z}$ such that $z+k \notin Z$.

Then $\mathcal{N}=\{u \in \mathcal{M};\ \operatorname{ord} u \subset Z\}$ *is a coherent* $\mathcal{E}_A$*-Module.*

Proof. Let $\mathcal{M}_0$ be a coherent $\mathcal{E}_A$-sub-Module of $\mathcal{M}$ such that $\mathcal{M}=\mathcal{E}\mathcal{M}_0$. Let $\{\lambda_1,\dots,\lambda_N\}$ be the set of roots of $b(\lambda;\mathcal{M}_0)=0$. Let m_j be the maximum of $\{m \in \boldsymbol{Z};\ \lambda_j - m \in Z\}$. By replacing $\mathcal{M}_0$ with $\mathcal{M}_0(N)$ for $N \gg 0$, we may assume that $m_j \geqq 0$. Hence there is an integer $N \geqq 0$ such that the following condition $(1.5.20)_N$ holds:

$(1.5.20)_N$ There is a coherent $\mathcal{E}_A$-sub-Module $\mathcal{M}_0$ of $\mathcal{M}$ such that $\mathcal{E}\mathcal{M}_0=\mathcal{M}$ and that, for any root λ of $b(\lambda;\mathcal{M}_0)=0$, $\lambda+1 \notin Z$ and $\lambda-N \in Z$.

We shall prove $(1.5.20)_N$ implies $(1.5.20)_{N-1}$ for $N \geqq 1$. Let $\lambda_1,\dots,\lambda_l$ be sets of roots of $b(\lambda;\mathcal{M}_0)=0$ such that $\lambda \notin Z$. We define $\mathcal{M}_j$ as the kernel of $\mathcal{M}_{j-1} \to \mathcal{M}_{j-1}\langle\lambda_j\rangle$ $(j=1,\dots,l)$. Then, by the repeated application of Lemma 1.5.6, we can easily see that $b(\lambda_j;\mathcal{M}_l) \neq 0$ $(1 \leqq j \leqq l)$ and $b(\lambda;\mathcal{M}_l)=0$ implies either $\lambda=\lambda_j-1$ $(1 \leqq j \leqq l)$ or $b(\lambda;\mathcal{M}_0)=0$. Hence $b(\lambda;\mathcal{M}_l)=0$ implies $\lambda+1 \notin Z$ and $\lambda-(N-1) \in Z$. Thus $(1.5.20)_{N-1}$ holds. Therefore, by the induction, $(1.5.20)_N$ holds for $N=0$, i.e., there is a coherent $\mathcal{E}_A$-sub-Module $\mathcal{M}_0$ of $\mathcal{M}$ such that $\mathcal{E}\mathcal{M}_0=\mathcal{M}$ and that $b(\lambda;\mathcal{M}_0)=0$ implies $\lambda \in Z$ and $\lambda+1 \notin Z$.

Now we shall show that $\mathcal{N}=\mathcal{M}_0$. Lemma 1.5.3 implies $\mathcal{N} \supset \mathcal{M}_0$. We shall show the converse inclusion relation. Let u be a section of $\mathcal{N}$ and let $b(\theta)=b(\theta;\mathcal{E}_A u)$. Further let $\{\lambda_1,\dots,\lambda_N\}$ be the set of roots of $b(\lambda;\mathcal{M}_0)=0$. Then λ_j is not a root of $b(\lambda-\nu)=0$ for $\nu=1,2,\dots$. On the other hand

$$b(\theta-k)\cdots b(\theta)u \in \mathcal{E}_A(-k-1)u \subset \mathcal{M}_0 \qquad \text{for} \quad k \gg 0.$$

Hence Lemma 1.5.4 implies that $u \in \mathcal{M}_0$. Q.E.D.

In the course of the proof of Proposition 1.5.8, we also obtained the following proposition.

Proposition 1.5.9. *Let Z and $\mathcal{N}$ be as in Proposition* 1.5.8, *and let $\mathcal{M}_0$ be a coherent $\mathcal{E}(0)$-sub-Module of $\mathcal{M}$ which generates $\mathcal{M}$ as an $\mathcal{E}$-Module. Then the following two conditions are equivalent*

(i) $\mathcal{N}=\mathcal{M}_0$.

(ii) *Any root* λ *of* $b(\lambda;\ \mathcal{M}_0)=0$ *satisfies* $\lambda\in Z$ *and* $\lambda+1\notin Z$.

Before ending this section, we shall remark the following. If $\mathcal{M}_0$ is a coherent $\mathscr{E}_\Lambda$-sub-Module of $\mathcal{M}$ defined on Ω, then $\mathscr{H}om_{\mathscr{A}(0)}(\overline{\mathcal{M}}_0,\ \mathscr{Q})$ is a locally constant sheaf. Hence the monic polynomial $b(\theta)$ with the smallest degree such that $b(\theta)(\overline{\mathcal{M}}_0)_p=0\,(p\in\Lambda\cap\Omega)$ does not depend on p when $\Lambda\cap\Omega$ is connected. In particular, for a section u of $\mathcal{M}$ defined on an open set Ω, ord u is well-defined on each connected component Λ_i of $\Lambda\cap\Omega$, which we shall denote by $\mathrm{ord}_{\Lambda_i}(u)$ (or $\mathrm{ord}_x(u)$ for $x\in\Lambda_i$).

§6.

In this section we prepare some geometric results in symplectic geometry. Even though they might be well-known, we include their proofs for completeness. These results will be frequently used in our later discussions.

Proposition 1.6.1. *Let* (V, E) *be a* $(2n)$*-dimensional symplectic vector space*(*) *and let* Λ *be an isotropic homogeneous analytic subset of* V. *Then there is a Lagrangian (linear) subspace* λ *such that* $\Lambda\cap\lambda\subset\{0\}$.

Proof. We shall prove by the induction on n. If $n=1$ then all lines are Lagrangian and $\dim\Lambda\leqq 1$. Therefore the proposition is evident. Suppose $n>1$. Then, there is a line μ such that $\mu\cap\Lambda\subset\{0\}$. Then $\mu^\perp\cap\Lambda\to\mu^\perp/\mu=V'$ is a finite map. Let Λ' be its image. Then Λ' is isotropic in V' (see Proposition 4.9 in [7]). By the hypothesis of the induction on n, there is a Lagrangian subspace λ' of V' such that $\Lambda'\cap\lambda'\subset\{0\}$. Then the inverse image λ of λ' by the map $\mu^\perp\to V'$ satisfies the required condition. Q.E.D.

The following corollary is an immediate consequence of Proposition 1.6.1.

Corollary 1.6.2. *Let* (V, E) *be a symplectic vector space,* Λ *a homogeneous isotropic analytic subset of* V, *and* $X(V)$ *the space of all Lagrangian subspaces. Then* $Y=\{\lambda\in X(V);\ \lambda\cap\Lambda\not\subset\{0\}\}$ *is a proper closed analytic subset of* $X(V)$.

In order to state another corollary of the proposition (Corollary 1.6.4), we introduce the following notion.

(*) This means, by definition, that V is a 2n-dimensional vector space and E is a non-degenerate skew-symmetric form on V.

Definition 1.6.3. Let Λ be a Lagrangian variety of $T^*X - T^*_XX$. We say Λ is in a generic position at a point p in $\Lambda - T^*_XX$ if and only if $\Lambda \cap \pi^{-1}(\pi(p)) = \boldsymbol{C}^\times p$ in a neighborhood of p.

Corollary 1.6.4. *Let X be a complex manifold, p a point in $T^*X - T^*_XX$, Λ_j $(j=1,\dots, N)$ Lagrangian manifolds of T^*X and Λ a Lagrangian variety of T^*X. Then we can find a homogeneous canonical transformation Φ: $(T^*X, p) \to (T^*X, p)$ such that $\Phi(\Lambda)$ is in a generic position at p and that $\Phi(\Lambda_j)$ is the conormal bundle of a smooth hypersurface S_j of X $(j=1,\dots, n)$.*

Proof. Set $\mu = T_p(\boldsymbol{C}^\times p)$ and $V = \mu^\perp/\mu \subset T_p(T^*X)/\mu$. Then V is a symplectic vector space. By [19] Proposition 10.4.1, $C_p(\Lambda \cup \bigcup_j \Lambda_j)/\mu$ is a Lagrangian variety of V. Therefore we can find a Lagrangian subspace λ of $T_p(T^*X)$ such that $\lambda \supset \mu$ and that $(\lambda/\mu) \cap C_p(\Lambda \cup \bigcup_j \Lambda_j)/\mu = 0$. If there is a homogeneous canonical transformation Φ such that $\Phi(\lambda) = T_p\pi^{-1}\pi(p)/\mu$, then, by replacing Λ and Λ_j with $\Phi\Lambda$ and $\Phi\Lambda_j$ respectively, we may assume that

$$T_p\pi^{-1}\pi(p) \cap C_p(\Lambda) \subset \mu$$

and

$$T_p\pi^{-1}\pi(p) \cap T_p\Lambda_j \subset \mu \qquad (j=1,\dots, n)$$

hold. Then we immediately find the desired results. Therefore it suffices to show the existence of such Φ. This is an immediate consequence of the following lemma.

Lemma 1.6.5. *Let X be a complex manifold, p a point of $T^*X - T^*_XX$, $\mu = T_p(\boldsymbol{C}^\times p)$ and g a symplectic transformation of $T_p(T^*X)$ such that $g|_\mu = \mathrm{id}_\mu$. Then there is a homogeneous canonical transformation Φ: $(T^*X, p) \to (T^*X, p)$ such that $T_p\Phi = g$.*

Since the proof of this lemma is easy, we omit it.

Chapter II. Holonomic Systems of $\boldsymbol{D}$-type

§1.

Let X be a complex manifold, Y a hypersurface (possibly with singularities) of X and j the inclusion map from $X - Y$ into X. In [3] Deligne proved the correspondance of locally constant sheaves on $X - Y$ of finite rank with integrable connections with regular singularities along Y. We shall re-interpret his result by the terminology of $\mathscr{D}_X$-Modules.

Let X be a complex manifold, Θ_X the sheaf of the vector fields and Ω_X^k the sheaf of the k-forms. Note that $\mathscr{D}_X$ contains Θ_X. We write Ω_X for $\Omega_X^{\dim X}$. In what follows n denotes $\dim X$.

Let $\mathscr{F}$ be an $\mathscr{O}_X$-Module. Remember that an integrable connection on $\mathscr{F}$ is a map $\Theta_X \underset{\mathbf{C}}{\otimes} \mathscr{F} \ni v\otimes s \mapsto \nabla_v s \in \mathscr{F}$ satisfying the properties

$$\begin{aligned} &\nabla_{v_1+v_2}s = \nabla_{v_1}s + \nabla_{v_2}s \\ &\nabla_v(s_1+s_2) = \nabla_v s_1 + \nabla_v s_2 \\ &\nabla_{av}(s) = a\nabla_v s \\ &\nabla_v(as) = a\nabla_v s + (v(a))s \\ &\nabla_{[v_1,v_2]}(s) = \nabla_{v_1}\nabla_{v_2}s - \nabla_{v_2}\nabla_{v_1}s \end{aligned}$$

for $\quad v, v_1, v_2 \in \Theta_X,\ s, s_1, s_2 \in \mathscr{F}$ and $a \in \mathscr{O}_X$.

Then it is easy to see that we can endow $\mathscr{F}$ with the structure of $\mathscr{D}_X$-Module so that $\nabla_v(s) = vs$ and that the structure of $\mathscr{O}_X$-Module on $\mathscr{F}$ coincides with that induced from the structure of $\mathscr{D}_X$-Module, namely, an integrable connection is nothing but a $\mathscr{D}_X$-Module. Let ∇ be an integrable connection on $\mathscr{F}$ and consider the associated de Rham complex

$$(2.1.1) \qquad \mathscr{F} \underset{\mathscr{O}_X}{\otimes} \Omega_X^{\cdot}: \mathscr{F} \xrightarrow{d} \mathscr{F} \underset{\mathscr{O}_X}{\otimes} \Omega_X^1 \xrightarrow{d} \mathscr{F} \underset{\mathscr{O}_X}{\otimes} \Omega_X^2 \xrightarrow{d} \cdots \xrightarrow{d} \mathscr{F} \underset{\mathscr{O}_X}{\otimes} \Omega_X^n \longrightarrow 0.$$

Here the operator is defined so that the following relations (2.1.2) are satisfied by the aid of a local coordinate system $(x_1,\dots, x_n)$.

$$(2.1.2) \qquad \begin{cases} ds = \sum_{j=1}^{n} \nabla_{\frac{\partial}{\partial x_j}} s\otimes dx_j & \text{for} \quad s \in \mathscr{F} \\ d(s\wedge\omega) = (-1)^p s\wedge d\omega + ds\wedge\omega & \text{for} \quad s \in \mathscr{F}\underset{\mathscr{O}_X}{\otimes}\Omega_X^p \quad \text{and} \quad \omega\in\Omega_X^q. \end{cases}$$

Here $s\wedge\omega$, for example, is considered as a section of $\mathscr{F} \underset{\mathscr{O}_X}{\otimes} \Omega_X^{p+q}$.

On the other hand, we can consider the following resolution of $\mathscr{O}_X$ as a $\mathscr{D}_X$-Module:

$$\begin{aligned} (2.1.3) \qquad 0 \longleftarrow \mathscr{O}_X \longleftarrow \mathscr{D}_X \xleftarrow{\delta} \mathscr{D}_X \underset{\mathscr{O}_X}{\otimes} \Theta_X \xleftarrow{\delta} \mathscr{D}_X \underset{\mathscr{O}_X}{\otimes} \overset{2}{\wedge}\Theta_X \xleftarrow{\delta} \\ \cdots \xleftarrow{\delta} \mathscr{D}_X \underset{\mathscr{O}_X}{\otimes} \overset{n}{\wedge}\Theta_X \longleftarrow 0. \end{aligned}$$

Here the homomorphism $\mathscr{D}_X \to \mathscr{O}_X$ is given by $P \mapsto P1$ and δ is defined by

(2.1.4) $\delta(P\otimes v_1\wedge\cdots\wedge v_p)$

$$= \sum_{i=1}^{p}(-1)^{i-1}Pv_i\otimes(v_1\wedge\cdots\wedge v_{i-1}\wedge v_{i+1}\wedge\cdots\wedge v_p)$$
$$+\sum_{1\leqq i<j\leqq p}(-1)^{i+j}P\otimes([v_i, v_j]\wedge v_1\wedge\cdots\wedge v_{i-1}\wedge v_{i+1}\wedge \cdots\wedge v_{j-1}\wedge v_{j+1}\wedge\cdots\wedge v_p)$$

Then we find that $\mathscr{F}\underset{\mathcal{O}_X}{\otimes}\Omega_X^{\cdot}$ is nothing but $\mathscr{H}om_{\mathscr{D}_X}(\mathscr{D}_X\underset{\mathcal{O}_X}{\otimes}\overset{\cdot}{\wedge}\Theta_X, \mathscr{F})$. Considering their cohomologies we also find

(2.1.5) $$\mathscr{H}^k(\mathscr{F}\underset{\mathcal{O}_X}{\otimes}\Omega_X^{\cdot})=\mathscr{E}xt^k_{\mathscr{D}_X}(\mathcal{O}_X, \mathscr{F}).$$

In other words, we obtain

(2.1.6) $$\mathbf{R}\mathscr{H}om_{\mathscr{D}_X}(\mathcal{O}_X, \mathscr{F})=\mathscr{F}\underset{\mathcal{O}_X}{\otimes}\Omega_X^{\cdot}.$$

§ 2.

Let Y be a hypersurface of a complex manifold. Let L be a locally constant sheaf of finite rank on $X-Y$ (i.e., locally free $\boldsymbol{C}_{X-Y}$-Module of finite rank). Let j be the inclusion map from $X-Y$ into X. Then $j_*(L\underset{\boldsymbol{C}}{\otimes}\mathcal{O}_{X-Y})$ has canonically a structure of $\mathscr{D}_X$-Module. Fix a non-singular point y_0 of Y and choose a local coordinate system $(x_1,\dots, x_n)$ in a neighborhood U of y_0 so that $y_0=0$ and that $U\cap Y$ is defined by $x_1=0$ in U and $U=\{x; |x|<1\}$. Set $U^{\pm}=\{x\in U; \pm\operatorname{Re}x_1>-|\operatorname{Im}x_1|\}$. Then L is a constant sheaf on $U^{\pm}$. Let $f_{\pm}$ be the isomorphism $L|_{U^{\pm}}\to\boldsymbol{C}^m_{U^{\pm}}$ and $f_{\pm,i}$ $(i=1,\dots, m)$ be the composition of $f_{\pm}$ and the i-th projection: $\boldsymbol{C}^m_{U^{\pm}}\to\boldsymbol{C}_{U^{\pm}}$. We say a section s of $j_*(L\underset{\boldsymbol{C}}{\otimes}\mathcal{O}_{X-Y})$ at y_0 is in a Nilsson class (resp., in a strict Nilsson class) at y_0 if there are $C>0$, $N>0$ and $\varepsilon>0$ such that

(2.2.1) $$|f_{\pm,i}(s)(x)|\leqq C|x_1|^{-N}$$

(2.2.2) $$(\text{resp., } |f_{\pm,i}(s)(x)|\leqq C(\log(1/|x_1|))^N)$$

for $x\in U^{\pm}$ with $|x|<\varepsilon$ and for any $i=1,\dots, m$. This notion does not depend on the choice of local coordinate systems.

For any section $s\in\Gamma(U; j_*(L\otimes\mathcal{O}_{X-Y}))$, $f_{\pm,i}(s)$ can be prolonged to a multi-valued function on $U-Y$ with finite determination. Therefore we can write $f_{\pm,i}(s)$ in the form $\sum_{\lambda\in I}\sum_{j=1}^{N}a_{\lambda,j}(x)x_1^{\lambda}(\log x_1)^j$, where I is a finite subset of $\boldsymbol{C}$ such that $\lambda-\mu\notin\boldsymbol{Z}$ for $\lambda\neq\mu\in I$ and $a_{\lambda,j}(x)\in\Gamma(U-Y;\mathcal{O}_X)$. Then the condition (2.2.1) is equivalent to saying that $a_{\lambda,j}(x)$ are meromorphic functions with pole along Y. Therefore $f_{\pm,i}(s)$ satisfies a holonomic system of linear differential equations whose characteristic variety is contained in $T^*_Y X \cup T^*_X X$. Furthermore the holonomic system is with R.S. Hence s itself satisfies a holonomic system of linear differential equations with R.S. Conversely, if s satisfies a system of linear differential equations with R.S., then so does $f_{\pm,i}(s)$. Therefore the condition (2.2.1) is satisfied.

More generally, we say that a section of $j_*(L\otimes\mathcal{O}_{X-Y})$ is in the (strict) Nilsson class if so is it at any non-singular point of Y. We shall denote by $\mathscr{L}(L)$ (resp., $\mathscr{L}_0(L)$) the subsheaf of $j_*(L\otimes\mathcal{O}_{X-Y})$ consisting of the sections in the (resp., strict) Nilsson class.

The following theorem is proved in [3] Chapter II (Proposition 5.7, Théorème 6.2 and Théorème 4.1. See also Appendix § C of this article). For brevity, we shall write $\mathscr{L}$ and $\mathscr{L}_0$ for $\mathscr{L}(L)$ and $\mathscr{L}_0(L)$.

Theorem 2.2.1. (i) *$\mathscr{L}_0$ is a coherent $\mathcal{O}_X$-Module.*

(ii) $\mathscr{L}=\mathscr{H}^0_{[X|Y]}(\mathscr{L}_0)$ *and* $\mathscr{H}^k_{[X|Y]}(\mathscr{L})=0$ for $k\neq 0$; $\mathscr{H}^k_{[Y]}(\mathscr{L})=0$ for any k.

(iii) Let s be a section of $j_*(L\otimes\mathcal{O}_{X-Y})$ on an open set U of X. Suppose that s belongs to $\mathscr{L}$ (resp., $\mathscr{L}_0$) outside a closed analytic subset of U of codimension at least 2. Then s belongs to $\mathscr{L}$ (resp., $\mathscr{L}_0$).

This theorem immediately implies the following theorem.

Theorem 2.2.2. (i) *$\mathscr{L}$ is a coherent $\mathscr{D}_X$-Module, and, moreover, $\mathscr{L}$ is holonomic.*

(ii) *$\mathscr{L}$ is a system with regular singularities along $T^*_{Y_{\mathrm{reg}}}X$.*

Proof. It is clear that $\mathscr{D}_X\mathscr{L}_0$ is a coherent $\mathscr{D}_X$-Module and holonomic on $X-Y$. Therefore $\mathscr{L}=\mathscr{H}^0_{[X|Y]}(\mathscr{D}_X\mathscr{L}_0)$ is also holonomic ([8]). The assertion (ii) is clear.

In [3] the following theorem is also proved. (See also Appendix § C of this article.)

Theorem 2.2.3. $\mathbf{R}\mathscr{H}om_{\mathscr{D}_X}(\mathcal{O}_X, \mathscr{L})=\mathbf{R}j_*(L)$.

This theorem combined with Theorem 1.4.9 of Chapter I, Section 4 implies the following theorem.

Theorem 2.2.4. $\mathscr{L}^{\infty}=j_*(L\underset{C}{\otimes}\mathscr{O}_{X-Y})$.

Proof. It follows from Theorem 1.4.9 that

$$\mathscr{L}^{\infty}=\mathbf{R}\mathscr{H}om_{C}(\mathbf{R}\mathscr{H}om_{\mathscr{D}_X}(\mathscr{L},\mathscr{O}_X),\mathscr{O}_X).$$

On the other hand we have

$$\begin{aligned}\mathbf{R}\mathscr{H}om_{\mathscr{D}_X}(\mathscr{L},\mathscr{O}_X)&=\mathbf{R}\mathscr{H}om_{\mathscr{D}_X}(\mathbf{R}\mathscr{H}om(\mathscr{O}_X,\mathscr{L}),\boldsymbol{C}_X)\\&=\mathbf{R}\mathscr{H}om_{C}(\mathbf{R}j_*(L),\boldsymbol{C}_X)=\mathbf{R}j_!(L^*),\end{aligned}$$

where L^* is the sheaf on $X-Y$ defined by $\mathscr{H}om_{C}(L,\boldsymbol{C}_{X-Y})$, because $\mathbf{R}\mathscr{H}om_{C}(\mathbf{R}j_!L^*,\boldsymbol{C}_X)=\mathbf{R}j_*L$. Thus we obtain $\mathscr{L}^{\infty}=\mathbf{R}\mathscr{H}om_{C}(\mathbf{R}j_!(L^*),\mathscr{O}_X)$ $=\mathbf{R}j_*(\mathbf{R}\mathscr{H}om_{C}(L^*,\mathscr{O}_{X-Y}))=\mathbf{R}j_*(L\underset{C}{\otimes}\mathscr{O}_{X-Y})$. Q.E.D.

§3.

In this section we first introduce the notion of a holonomic system of D-type, and, then we investigate some of its basic properties.

Definition 2.3.1. Let Y be a hypersurface of X. A holonomic system $\mathscr{L}$ of linear differential equations(*) on X is said to be of D-type with singularities(*) along Y if it satisfies the following conditions:

(2.3.1) $$SS(\mathscr{L})\subset\pi^{-1}(Y)\cup T^*_X X,$$

(2.3.2) $\mathscr{L}$ has R.S. on $T^*_Y X$.

(2.3.3) $$\mathscr{H}^k_{[Y]}(\mathscr{L})=0\quad\text{for any}\quad k.$$

We saw in the last section that, for a locally constant sheaf L of finite rank defined on $X-Y$, $\mathscr{L}(L)$ is a holonomic system of D-type.

Conversely, suppose that $\mathscr{L}$ is of D-type along Y. Then $L=\mathscr{H}om_{\mathscr{D}_X}(\mathscr{O}_X,\mathscr{L})|_{X-Y}$ is a locally constant sheaf of finite rank on $X-Y$. Consider the $\mathscr{D}_X$-linear homomorphism $\psi\colon\mathscr{L}\to j_*(L\otimes\mathscr{O}_{X-Y})$. The homomorphism ψ is an isomorphism on $X-Y$. Since the kernel of ψ is contained in $\mathscr{H}^0_Y(\mathscr{L})$ $=\mathscr{H}^0_{[Y]}(\mathscr{L})$, ψ is injective. By condition (2.3.2), $\psi(\mathscr{L})$ is contained in $\mathscr{L}(L)$. Since $\psi(\mathscr{L})$ and $\mathscr{L}(L)$ coincide outside Y, the condition (2.3.3) implies that $\psi(\mathscr{L})=\mathscr{L}(L)$. Thus we have obtained the following theorem.

(*) In the sequel we shall simply say "holonomic system $\mathscr{L}$ of D-type", namely, omit "linear differential equations". We also often omit "with singularities".

Theorem 2.3.2. (i) *The category of holonomic systems of D-type along Y and that of locally constant sheaves of finite rank on $X-Y$ are isomorphic under the correspondences* $\mathscr{L}\mapsto\mathscr{H}om_{\mathscr{D}_X}(\mathscr{O}_X, \mathscr{L})|_{X-Y}$ *and* $L\mapsto\mathscr{L}(L)$.

(ii) *If a holonomic system $\mathscr{L}$ is of D-type along Y, then we have*

(iia) $\mathscr{L}^\infty \xrightarrow{\sim} j_* j^{-1}\mathscr{L}$

and

$\mathscr{H}^k_Y(\mathscr{L}^\infty)=0 \qquad$ *for any* k.

(iib) *If the support of a section s of $\mathscr{L}^\infty/\mathscr{L}$ is contained in an analytic set of codimension$\geqq 2$, then s is zero.*

(iii) *For any coherent $\mathscr{O}_X$-sub-Module $\mathscr{G}$ of $\mathscr{L}$ and an analytic subset Z of codimension>2, $\mathscr{H}^0_{[X|Z]}(\mathscr{G})$ is a coheretn $\mathscr{O}_X$-Module.*

Proof. All assertions have already been proved except (iii). Let us prove (iii). Suppose that Y is defined by $f=0$. Then, by Hilbert's Nullstellensatz, $\mathscr{G}$ is contained in $f^{-N}\mathscr{L}_0$ for $N\gg 0$. Since $\mathscr{H}^0_{[X|Z]}(\mathscr{L}_0)=\mathscr{L}_0$ (Theorem 2.2.1 (iii)), $\mathscr{H}^0_{[X|Z]}(\mathscr{G})$ is contained in $f^{-N}\mathscr{L}_0$, and thus coherent. Q. E. D.

In the rest of this section we show several basic properties of a holonomic system of D-type.

Proposition 2.3.3. *Let $\mathscr{L}$ be a holonomic system of D-type along a hypersurface Y. Let S be a hypersurface of X. Then*

(i) $\mathscr{H}^k_{[X|S]}(\mathscr{L})=0 \qquad$ for $\quad k\neq 0$

and

$\mathscr{H}^0_{[X|S]}(\mathscr{L})$ *is of D-type along* $S\cup Y$.

(ii) $(\mathscr{H}^k_{[S]}(\mathscr{L}))^\infty=\mathscr{H}^k_S(\mathscr{L}^\infty)$.

Proof. The assertion that $\mathscr{H}^k_{[X|S]}(\mathscr{L})=0$ for $k\neq 0$ is obvious, because S is a hypersurface. Let us prove that $\mathscr{L}'=\mathscr{H}^0_{[X|S]}(\mathscr{L})$ is of D-type along $S\cup Y$. The condition (2.3.1) in Definition 2.3.1 is clearly satisfied. We show that the condition (2.3.3) in Definition 2.3.1 is satisfied. By (1.2.7) of [8], we have

$$\mathbf{R}\Gamma_{[S\cup Y]}\mathbf{R}\Gamma_{[X|S]}(\mathscr{L})=\mathbf{R}\Gamma_{[X|S]}\mathbf{R}\Gamma_{[S\cup Y]}(\mathscr{L}). \tag{2.3.4}$$

On the other hand, we have the following commutative diagram:

$$\begin{array}{ccc} & \mathbf{R}\Gamma_{[S\cap Y]}(\mathscr{L}) & \\ \swarrow & & \nwarrow {\scriptstyle +1} \\ \mathbf{R}\Gamma_{[S]}(\mathscr{L})\oplus\mathbf{R}\Gamma_{[Y]}(\mathscr{L}) & \longrightarrow & \mathbf{R}\Gamma_{[S\cup Y]}(\mathscr{L}). \end{array} \tag{2.3.5}$$

Since $\mathbf{R}\Gamma_{[Y]}(\mathscr{L})=0$, we have $\mathbf{R}\Gamma_{[S\cup Y]}(\mathscr{L})=\mathbf{R}\Gamma_{[S]}\mathbf{R}\Gamma_{[Y]}(\mathscr{L})=0$. Hence (2.3.5) implies

$$\mathbf{R}\Gamma_{[S]}(\mathscr{L})=\mathbf{R}\Gamma_{[S\cup Y]}(\mathscr{L}). \tag{2.3.6}$$

Therefore we have by (2.3.4) and (2.3.6)

$$\mathbf{R}\Gamma_{[S\cup Y]}\mathbf{R}\Gamma_{[X|S]}(\mathscr{L})=\mathbf{R}\Gamma_{[X|S]}\mathbf{R}\Gamma_{[S]}(\mathscr{L}).$$

Since $\mathbf{R}\Gamma_{[S]}(\mathscr{L})$ is supported by S, this clearly vanishes. Thus we have verified the condition (2.3.3).

Next let us prove that $\mathscr{L}'$ has R.S. on $T^*_{(S\cup Y)}X$. We write $S=S_0\cup S_1$ so that S_0 and S_1 are hypersurfaces and that $S_0\subset Y$ and $S_1\cap Y$ is codimension greater than 1. Since $\mathscr{L}'=\mathscr{L}$ on $X-S_1$, $\mathscr{L}'$ has R.S. on T^*_YX. Let us prove that $\mathscr{L}'$ has R.S. on $T^*_{S_1}X$. Let x be a non-singular point of S_1-Y. Then, in a neighborhood of x, $\mathscr{L}$ is isomorphic to a direct sum of finite copies of $\mathcal{O}_X$. On the other hand, $\mathscr{H}^0_{[X|S_1]}(\mathcal{O}_X)$ has regular singularities along $T^*_{S_1}X$ in a neighborhood of x. This implies that $\mathscr{L}'$ has R.S. on $T^*_{S_1}X$.

Now, we shall prove

$$\mathscr{H}^k_{[S]}(\mathscr{L})^\infty=\mathscr{H}^k_S(\mathscr{L}^\infty).$$

First note that Theorem 2.3.3 (iia) entails that

$$\mathbf{R}\Gamma_{S\cup Y}(\mathbf{R}\Gamma_{[X|S]}(\mathscr{L})^\infty)=0$$

holds, because $\mathscr{L}'=\mathbf{R}\Gamma_{[X|S]}(\mathscr{L})$ is of D-type along $S\cup Y$. Therefore we have

$$\mathbf{R}\Gamma_{S\cup Y}(\mathscr{L}^\infty)=\mathbf{R}\Gamma_{S\cup Y}(\mathbf{R}\Gamma_{[S]}(\mathscr{L})^\infty).$$

Thus we have

$$\mathbf{R}\Gamma_S(\mathscr{L}^\infty)=\mathbf{R}\Gamma_S\mathbf{R}\Gamma_{S\cup Y}(\mathscr{L}^\infty)=\mathbf{R}\Gamma_S\mathbf{R}\Gamma_{S\cup Y}(\mathbf{R}\Gamma_{[S]}(\mathscr{L})^\infty)=\mathbf{R}\Gamma_{[S]}(\mathscr{L})^\infty.$$

Q.E.D.

Proposition 2.3.4. *Let $\mathscr{M}$ be a holonomic $\mathscr{D}_X$-Module, Z an analytic subset of X, and Z_0 the union of irreducible components of Z of codimension 1. Assume that $\mathscr{M}$ has R.S. on $T^*_{Z_0}X$ and that $\mathrm{SS}(\mathscr{M})\subset\pi^{-1}(Z)\cup T^*_XX$. Then $\mathscr{L}=\mathscr{H}^0_{[X|Z]}(\mathscr{M})$ is of D-type.*

Proof. We may assume that Z is a proper analytic subset. Let Z_1 be the union of the irreducible components of codimension greater than 1. Let i be the imbedding from $X-Z$ into $X-Z_0$. Let L be the locally constant sheaf $\mathscr{H}om_{\mathbf{C}}(\mathcal{O}_X, \mathscr{M})|_{X-Z}$. Then $\mathscr{M}$ is isomorphic to $L\underset{\mathbf{C}}{\otimes}\mathcal{O}_X$ on $X-Z$. Since Z_1

has codimension greater than one, $\tilde{L}=i_*L$ is also a locally constant sheaf on $X-Z_0$. Let $\mathscr{L}'$ be the holonomic system of D-type with singularities along Z_0 corresponding to $\tilde{L}$. Then we can construct the homomorphism $\psi: \mathscr{M}\to\mathscr{L}'^\infty$, because $\mathscr{L}'^\infty=j_*(L\otimes\mathscr{O}_X|_{X-Z})=j_*(\mathscr{M}|_{X-Z})$ holds. Here j denotes the imbedding of $X-Z$ into X. Since $\mathscr{M}$ has R.S. on $T^*_{Z_0}X$, the image of ψ is contained in $\mathscr{L}'$. Thus we obtain the homomorphism $\mathscr{M}\to\mathscr{L}'$, which is an isomorphism on $X-Z$. Therefore $\mathscr{L}=\mathscr{H}^0_{[X|Z]}(\mathscr{M})$ is isomorphic to $\mathscr{L}'=\mathscr{H}^0_{[X|Z]}(\mathscr{L}')$. Q.E.D.

Proposition 2.3.5. *Let $\mathscr{L}$ be a holonomic $\mathscr{D}_X$-Module of D-type along a hypersurface Y, and let s be a section of $\mathscr{L}^\infty$. If s satisfies a holonomic system of micro-differential equations with R.S. at each point on $T^*_{Y_{\mathrm{reg}}}X-T^*_XX$, then s belongs to $\mathscr{L}$.*

Proof. Let $\tilde{s}$ denote the section $1\otimes s$ of $\mathscr{E}^\infty\underset{\pi^{-1}\mathscr{D}^\infty}{\otimes}\pi^{-1}\mathscr{L}^\infty$. Then $\tilde{s}$ belongs to $(\mathscr{E}\underset{\mathscr{D}}{\otimes}\mathscr{L})_{\mathrm{reg}}$ on $T^*_{Y_{\mathrm{reg}}}X-T^*_XX$. Since $\mathscr{L}$ has R.S. on $T^*X-\pi^{-1}(Y_{\mathrm{sing}})$, we have $(\mathscr{E}\underset{\mathscr{D}}{\otimes}\mathscr{L})_{\mathrm{reg}}|_{T^*X-\pi^{-1}(Y_{\mathrm{sing}})}=(\mathscr{E}\underset{\mathscr{D}}{\otimes}\mathscr{L})|_{T^*X-\pi^{-1}(Y_{\mathrm{sing}})}$. Therefore $\tilde{s}$ belongs to $\mathscr{E}\underset{\mathscr{D}}{\otimes}\mathscr{L}$ on $T^*X-\pi^{-1}(Y_{\mathrm{sing}})-T^*_XX$. Then we can apply Theorem 1.2.1 to conclude that $\tilde{s}$ belongs to $\mathscr{E}\underset{\mathscr{D}}{\otimes}\mathscr{L}$ on $T^*X-\pi^{-1}(Y_{\mathrm{sing}})$. Therefore s belongs to $\mathscr{L}$ outside Y_{sing}. The desired result then follows from Theorem 2.3.2 (iib). Q.E.D.

Proposition 2.3.6. *Let $\mathscr{L}$ be a holonomic $\mathscr{D}$-Module of D-type with singularities along a hypersurface Y. Let s be a section of $\mathscr{L}^\infty$ and $\tilde{s}=1\otimes s$ the corresponding section of $\mathscr{E}^\infty\underset{\mathscr{D}^\infty}{\otimes}\mathscr{L}^\infty$. Suppose that* supp $(\tilde{s})\cap(T^*_YX-T^*_XX)$ *is a nowhere dense subset of $T^*_YX-T^*_XX$. Then $\mathscr{D}s$ is a holonomic $\mathscr{D}$-Module locally isomorphic to a direct sum of finite copies of $\mathscr{O}_X$.*

Proof. By the preceding proposition, s belongs to $\mathscr{L}$. On Y_{reg}, this proposition is evident. Hence $\mathscr{D}s$ is a holonomic $\mathscr{D}$-Module locally isomorphic to a direct sum of finite copies of $\mathscr{O}$ on $X-Y_{\mathrm{sing}}$. Set $L=\mathscr{H}om_{\mathscr{D}}(\mathscr{O},\mathscr{D}s)|_{X-Y_{\mathrm{sing}}}$. Then L is a locally constant sheaf on $X-Y_{\mathrm{sing}}$. Since $\pi_1(X)=\pi_1(X-Y_{\mathrm{sing}})$, L can be extended to a locally constant sheaf $\tilde{L}$ on X. Set $\mathscr{N}=\mathscr{O}_X\underset{\boldsymbol{C}}{\otimes}\tilde{L}$. Then $\mathscr{N}$ is isomorphic to $\mathscr{D}s$ outside Y_{sing}. Hence an injection $j: \mathscr{N}|_{X-Y_{\mathrm{sing}}}\to\mathscr{L}^\infty|_{X-Y_{\mathrm{sing}}}$ can be prolonged to $\tilde{j}: \mathscr{N}\to\mathscr{L}^\infty$. Since $\mathscr{H}^1_{Y_{\mathrm{sing}}}(\mathscr{N})=\mathscr{N}\underset{\mathscr{O}}{\otimes}\mathscr{H}^1_{Y_{\mathrm{sing}}}(\mathscr{O})=0$, there is a section $\tilde{s}$ of $\mathscr{N}$ such that $\tilde{s}=j^{-1}(s)$ on $X-Y_{\mathrm{sing}}$. Hence $\tilde{j}(\tilde{s})=s$ on $X-Y_{\mathrm{sing}}$, which implies $\tilde{s}=s$. This shows that $\mathscr{D}s$ is a quotient of $\mathscr{N}$ and we obtain the desired result. Q.E.D.

Let $\mathscr{L}$ be a holonomic $\mathscr{D}_X$-Module of D-type with singularities along Y. Then $\mathscr{L}=\mathscr{L}(L)$ for a locally constant sheaf L. We call a section s of $\mathscr{L}^\infty$ is in the strict Nilsson class if s belongs to $\mathscr{L}_0(L)$. Now fix a non-singular point y_0 of Y and define $f_{\pm,i}$ as in Section 2. Then $f_{\pm,i}(s)=\sum_{\lambda\in I}\sum_{j=0}^{N} a_{\lambda,j}(x)x_1^{\lambda}(\log x_1)^j$, where I is a finite subset of $\boldsymbol{C}$ such that $0\leqq \operatorname{Re}\lambda<1$ and $a_{\lambda,j}(x)\in\mathscr{O}(U-Y)$. Then s is in the strict Nilsson class if and only if $a_{\lambda,j}(x)\in\mathscr{O}(U)$. Thus we have

Proposition 2.3.7. *A section s of $\mathscr{L}$ is in the strict Nilsson class if and only if* $\operatorname{ord}_{T^*_{Y_{\mathrm{reg}}}X}(s)\subset\{\lambda\in\boldsymbol{C};\ \operatorname{Re}\lambda\leqq -1/2\}$.

Proposition 2.3.8. *Let $\mathscr{L}$ be a holonomic $\mathscr{D}$-Module of D-type with singularities along $\{x_1=0\}$ and $\mathscr{L}_0$ the subsheaf of $\mathscr{L}$ consisting of the sections in the strict Nilsson class. Then the following properties hold.*

(i) $(x_1D_1)\mathscr{L}_0\subset\mathscr{L}_0$, $D_j\mathscr{L}_0\subset\mathscr{L}_0$ $\quad(j=2,\dots,n)$.

(ii) *There is a polynomial $b(\lambda)$ such that $D_1b(x_1D_1)\mathscr{L}_0\subset\mathscr{L}_0$ and any root λ of $b(\lambda)=0$ satisfies* $0\leqq\operatorname{Re}\lambda<1$.

Proof. (i) is obvious. Let us prove (ii). There is a finite set I of $\{\lambda\in\boldsymbol{C};\ 0\leqq\operatorname{Re}\lambda<1\}$ and an integer N such that for any section s of $\mathscr{L}_0$

$$f_{\pm,i}(s)=\sum_{\lambda\in I}\sum_{j=0}^{N} a_{\lambda,j}(x)x_1^{\lambda}(\log x_1)^j$$

for some $a_{\lambda,j}\in\mathscr{O}_{X|Y}$.

Set $b(s)=\prod_{\lambda\in I}(s-\lambda)^N$. Then it is easy to check that $x_1^{-1}b(x_1D_1)s$ is in the strict Nilsson class. Hence we have

$$D_1b(x_1D_1)s\in\mathscr{L}_0. \qquad \text{Q.E.D.}$$

Chapter III. Action of Micro-differential Operators on Holomorphic Functions

The main purpose of this chapter is to clarify the action of $\mathfrak{E}(G;D)$ on holomorphic functions. In the course of the discussion we introduce a special class $\tilde{\mathscr{E}}$ of micro-differential operators and prove several basic properties of $\tilde{\mathscr{E}}$ (§ 3 and § 5). These materials will be effectively used in Chaper IV. We also recall some basic facts concerning multi-valued holomorphic functions for our arguments in later chapters. (§ 4)

§1.

As shown in [19], operators in $\mathfrak{E}(G;\,D)$ act on the relative cohomology groups with the sheaf of holomorphic functions as coefficients. On the other hand, [10] and [2] show explicitly how micro-differential operators act on a space of holomorphic functions. The purpose of this section is to discuss their relationship.

1.1. The action of $\mathfrak{E}(G;\,D)$ is defined in [19] in a purely cohomological way, especially by the aid of residue maps. Hence we begin our investigation by the study of residue maps. Let X and Y be two complex manifolds of dimension n and m, respecitively, and let f be a smooth holomorphic map from X to Y. One can define the residue map

$$R^l f_!(\Omega_X) \longrightarrow \Omega_Y \qquad (l=n-m)\ ([5])$$

Here Ω_X (resp., Ω_Y) denotes the sheaf of holomorphic n (resp., m)-forms on X (resp., Y). Let us recall how this homomorphism is constructed. Let $\mathscr{B}_X^{(p,q)}$ (resp., $\mathscr{B}_Y^{(p,q)}$) denote the sheaf of $(p,\,q)$-forms having hyperfunctions as their coefficients (the reader can replace here hyperfunctions with infinitely differentiable functions). Then we have the flabby resolutions of Ω_X and Ω_Y:

$$0 \longrightarrow \Omega_X \longrightarrow \mathscr{B}_X^{(n,0)} \xrightarrow{\bar\partial} \mathscr{B}_X^{(n,1)} \longrightarrow \cdots \xrightarrow{\bar\partial} \mathscr{B}_X^{(n,n)} \longrightarrow 0$$

and

$$0 \longrightarrow \Omega_Y \longrightarrow \mathscr{B}_Y^{(m,0)} \xrightarrow{\bar\partial} \mathscr{B}_Y^{(m,1)} \xrightarrow{\bar\partial} \cdots \xrightarrow{\bar\partial} \mathscr{B}_Y^{(m,m)} \longrightarrow 0.$$

On the other hand we have

$$f_!(\mathscr{B}_X^{(n,k)}) \longrightarrow \mathscr{B}_Y^{(m,k-l)} \qquad \text{for} \quad k \geqq l$$

by integration along fiber. It is easy to see that this gives the homomorphism of the complexes

$$\begin{array}{ccccccccccccc}
0 \longrightarrow & f_!\mathscr{B}_X^{(n,0)} & \xrightarrow{\bar\partial} \cdots \xrightarrow{\bar\partial} & f_!\mathscr{B}_X^{(n,l-1)} & \xrightarrow{\bar\partial} & f_!\mathscr{B}_X^{(n,l)} & \xrightarrow{\bar\partial} \cdots \xrightarrow{\bar\partial} & f_!\mathscr{B}_X^{(n,n)} & \longrightarrow 0 \\
 & \downarrow & & \downarrow & & \downarrow & & \downarrow & \\
 & 0 & \longrightarrow \cdots \longrightarrow & 0 & \longrightarrow & \mathscr{B}_Y^{(m,0)} & \xrightarrow{\bar\partial} \cdots \xrightarrow{\bar\partial} & \mathscr{B}_Y^{(m,m)} & \longrightarrow 0
\end{array}$$

and hence we obtain the homomorphism

$$\mathbf{R}f_!\Omega_X[l] = f_!\mathscr{B}_X^{(n,\cdot)}[l] \longrightarrow \mathscr{B}_Y^{(m,\cdot)} = \Omega_Y\,.$$

If $g\colon Y \to Z$ is another smooth map of fiber dimension r, then the homomorphism $\mathbf{R}(g\circ f)_!\Omega_X[l+r]\to\Omega_Z$ coincides with the composition $\mathbf{R}g_!\Omega_Y[r]\to\Omega_Z$ and $\mathbf{R}g_!\mathbf{R}f_!\Omega_X[l+r]\to\mathbf{R}g_!\Omega_Y[r]$ obtained from $\mathbf{R}f_!\Omega_X[l]\to\Omega_Y$ by applying the functor $\mathbf{R}g_![r]$.

1.2. We shall calculate the residue map when the cohomology group $R^lf_!(\Omega_X)$ is given by the Čech cohomology.

Let us consider the special case where the fiber of f is of dimension 1.

Let us suppose that there is a closed subset Z of X proper on Y and suppose that X is Stein. Then

$$H^1_Z(X;\Omega_X)=\Gamma(X-Z;\Omega_X)/\Gamma(X;\Omega_X).$$

Suppose that the section of $R^1f_!(\Omega_X)$ is given by a section $\varphi\in\Gamma(X-Z;\Omega_X)$ through the homomorphisms

$$\Gamma(X-Z;\Omega_X)\longrightarrow H^1_Z(X;\Omega_X)\longrightarrow H^0(Y;\mathbf{R}f_*\mathbf{R}\Gamma_Z(\Omega_X)[1])\longrightarrow H^0(Y;R^1f_!\Omega_X).$$

Let us calculate the corresponding holomorphic function on Y. Consider the exact sequence of the complexes

$$0\longrightarrow\Gamma_Z(X;\mathscr{B}_X^{(n,\cdot)})\longrightarrow\Gamma(X;\mathscr{B}_X^{(n,\cdot)})\longrightarrow\Gamma(X-Z;\mathscr{B}_X^{(n,\cdot)})\longrightarrow 0.$$

The homomorphism $\Gamma(X-Z;\Omega_X)\to H^1_Z(X;\Omega_X)$ is derived from the homomorphism $H^0(\Gamma(X-Z;\mathscr{B}_X^{(n,\cdot)}))\to H^1(\Gamma_Z(X;\mathscr{B}_X^{(n,\cdot)}))$. If we choose an element $\tilde\varphi$ of $\Gamma(X;\mathscr{B}_X^{(n,0)})$ such that $\tilde\varphi$ coincides with φ on $X-Z$, then $\bar\partial\tilde\varphi\in\Gamma_Z(X;\mathscr{B}_X^{(n,1)})$ gives the element of $H^1(\Gamma_Z(X;\mathscr{B}_X^{(n,\cdot)})=H^1_Z(X;\Omega_X)$. Hence the corresponding holomorphic function is given by the integral

$$\int\bar\partial\tilde\varphi$$

along the fiber.

Take an open subset D containing Z with a smooth boundary. Then, by Stoke's theorem,

$$\int\bar\partial\tilde\varphi=\int_D\bar\partial\tilde\varphi=\int_{\partial D}\tilde\varphi\,.$$

Thus we obtain

Proposition 3.1.1. *Let the fiber dimension be* 1 *and Z a closed subset of X proper on Y. Let φ be a section of Ω_X on $X-Z$, and $[\varphi]$ the element of $H^1_Z(X;\Omega_X)$ corresponding to φ. Then the corresponding element of $H^0(Y;\Omega_Y)$ is given*

$$\psi(y)=\int_{\gamma}\varphi,$$

where γ is a cycle in the fiber $f^{-1}(y)$ around $Z\cap f^{-1}(y)$.

In what follows we shall use relative cohomology groups of covering (Čech cohomology groups). See e.g. [20] for the definition of the relative Čech cohomology groups.

The following Corollary 3.1.2 is an immediate consequence of Proposition 3.1.1.

Corollary 3.1.2. *Let X, Y and Z be as in the preceding proposition, and let $\mathscr{U}=\{U_i\}_{i\in I}$ be an open covering of Y, $\mathscr{U}'=\{U_i\}_{i\in I_0}$ a subcovering of $\mathscr{U}$ and $T=Y-\bigcup_{i\in I_0}U_i$. Let $\tilde{\mathscr{U}}$ be the open covering $(f^{-1}(U_i)-Z)_{i\in I}$ of $X-Y$ and $\tilde{\mathscr{U}}'=\{f^{-1}(U_i)-Z\}_{i\in I_0}$ the subcovering of $\tilde{\mathscr{U}}$. Let $(\varphi_{i_0,\dots,i_r})$ be a relative cocycle $Z^r(\tilde{\mathscr{U}}\bmod\tilde{\mathscr{U}}';\Omega_X)$. Then the image of*

$$Z^r(\tilde{\mathscr{U}}\bmod\tilde{\mathscr{U}}';\Omega_X)\longrightarrow H^r_{f^{-1}(T)}(X-Z;\Omega_X)\longrightarrow H^{r+1}_{f^{-1}(T)\cap Z}(X;\Omega_X)\longrightarrow H^r_T(Y;\Omega_Y)$$

is the image of the cocycle $\{\psi_{i_0,\dots,i_r}\}\in Z^r(\mathscr{U}\bmod\mathscr{U}';\Omega_Y)$ by the homomorphism

$$Z^r(\mathscr{U}\bmod\mathscr{U}';\Omega_Y)\longrightarrow H^r_T(Y;\Omega_Y),$$

where

$$\psi_{i_0,\dots,i_r}=\int_{\gamma}\varphi_{i_0,\dots,i_r}$$

for a cycle γ in the fiber of f around Z.

Corollary 3.1.3. *Let $X, Y, Z, \mathscr{U}, \mathscr{U}', \tilde{\mathscr{U}}$ and $\tilde{\mathscr{U}}'$ be as in the preceding corollary. Let $\tilde{\tilde{\mathscr{U}}}$ be the open covering of X given by $\{\tilde{\tilde{U}}_{a(i)}, \tilde{\tilde{U}}_{b(i)}\}_{i\in I}$, where $\tilde{\tilde{U}}_{a(i)}=f^{-1}(U_i)$ and $\tilde{\tilde{U}}_{b(i)}=f^{-1}(U_i)-Z$, and $\tilde{\tilde{\mathscr{U}}}'$ the subcovering*

$$\{\tilde{\tilde{U}}_{a(i)}\}_{i\in I_0}\cup\{\tilde{\tilde{U}}_{b(i)}\}_{i\in I}.$$

Let φ be a cocycle in $Z^{r+1}(\tilde{\tilde{\mathscr{U}}}\bmod\tilde{\tilde{\mathscr{U}}}';\mathcal{O}_X)$. Then the image of φ by the map

$$Z^{r+1}(\tilde{\tilde{\mathscr{U}}}\bmod\tilde{\tilde{\mathscr{U}}}';\Omega_X)\to H^{r+1}_{f^{-1}(T)\cap Z}(X;\Omega_X)\to H^r_T(Y;\Omega_Y)$$

is given by the cocycle $\psi\in Z^r(\mathscr{U}\bmod\mathscr{U}';\Omega_Y)$, where

$$\psi_{i_0,\dots,i_r}=\sum_{\nu=0}^{r}(-1)^{\nu+1}\int_{\gamma}\varphi_{a(i_0),\dots,a(i_\nu),b(i_\nu),\dots,b(i_{r+1})},$$

where γ is a path in the fiber of f around Z.

Proof. By taking a refinement of $\tilde{\tilde{\mathscr{U}}}$, we may assume that $U_{a(i)}$ and $U_{b(i)}$ are Stein open subsets. If φ is the image of an element ϕ in $Z^r(\tilde{\mathscr{U}}\bmod\tilde{\mathscr{U}}';\Omega_X)$

by the homomorphism $Z^r(\tilde{\mathscr{U}} \bmod \tilde{\mathscr{U}}'; \Omega_X) \to Z^{r+1}(\tilde{\tilde{\mathscr{U}}} \bmod \tilde{\tilde{\mathscr{U}}}'; \Omega_X)$, then we have

$$\varphi_{a(i_0),b(i_0),\dots,b(i_r)} = -\phi_{i_0,\dots,i_r}$$

and other φ's are zero.

It is easy to see that if φ is a coboundary then so is ψ. On the other hand, $H^r(\tilde{\mathscr{U}} \bmod \tilde{\mathscr{U}}'; \Omega_X) \cong H^r(\tilde{\tilde{\mathscr{U}}} \bmod \tilde{\tilde{\mathscr{U}}}'; \Omega_X) = H^{r+1}_{Z \cap f^{-1}(T)}(X; \Omega_X)$ and hence

$$Z^{r+1}(\tilde{\tilde{\mathscr{U}}} \bmod \tilde{\tilde{\mathscr{U}}}'; \Omega_X) = B^{r+1}(\tilde{\tilde{\mathscr{U}}} \bmod \tilde{\tilde{\mathscr{U}}}'; \Omega_X) + Im\,(Z^r(\tilde{\mathscr{U}} \bmod \tilde{\mathscr{U}}'; \Omega_X) \to Z^{r+1}(\tilde{\tilde{\mathscr{U}}} \bmod \tilde{\tilde{\mathscr{U}}}'; \Omega_X)).$$

This proves the corollary.

By the repeated application of the preceding corollary, we obtain the following corollary.

Corollary 3.1.4. *Let Y be a complex manifold, $\mathscr{U} = \{U_i\}_{i \in I}$ an open covering of Y and $\mathscr{U}' = \{U_i\}_{i \in I_0}$ an open subcovering of $\mathscr{U}$ and $T = Y - \bigcup_{i \in I_0} U_i$. Let X be an open subset of $Y \times \boldsymbol{C}^l$, f the projection from X to Y and Z_j a closed subset of $Y \times \boldsymbol{C}(j=1,\dots, l)$. Suppose that $Z = Z_1 \underset{Y}{\times} \cdots \underset{Y}{\times} Z_l \subset Y \times \boldsymbol{C}^l$ is a subset of X. Set $V_j = X - \boldsymbol{C} \times \cdots \times \boldsymbol{C} \times Z_j \times \boldsymbol{C} \times \cdots \times \boldsymbol{C}$. For $p, q \in \boldsymbol{Z}$ with $p \leqq q$, let $[p, q]$ denote the set of integers in such that $p \leqq i \leqq q$. Let $\tilde{\mathscr{W}}$ be the open covering $\{W_{(j,i)}\}_{(j,i) \in [1,l] \times I}$, where $W_{(j,i)} = V_j \cap U_i$. Let $\tilde{\mathscr{W}}'$ be the subcovering of $\mathscr{W}$ consisting of $W_{(j,i)}$ with $i \in I_0$. Let φ be a cocycle in $Z^{r+l-1}(\tilde{\mathscr{W}} \bmod \tilde{\mathscr{W}}'; \mathcal{O}_X)$. Then the image of φ by the map*

$$Z^{r+l-1}(\tilde{\mathscr{W}} \bmod \tilde{\mathscr{W}}'; \mathcal{O}_X) \to H^{r+l-1}_{f^{-1}(T)}(X - Z; \mathcal{O}_X) \to H^{r+l}_{Z \cap f^{-1}(T)}(X; \mathcal{O}_X) \to H^r_T(Y; \mathcal{O}_Y)$$

is given by the image of the cocycle $\phi \in Z^r(\mathscr{U} \bmod \mathscr{U}'; \mathcal{O}_Y)$ by the map $Z^r(\mathscr{U} \bmod \mathscr{U}'; \mathcal{O}_Y) \to H^r_T(Y; \mathcal{O}_Y)$. Here ϕ is given by (up to sign)

$$\phi_{i_0,\dots,i_r}(y) = \sum (-1)^k \int_{\gamma_1 \times \cdots \times \gamma_l} \varphi_{(\alpha(1), i_{\beta(1)}),\dots,(\alpha(r+l), i_{\beta(r+l)})}(y, t) dt_1 \cdots dt_l$$

where $k = \sum_{\nu=2}^{r+l} \beta(\nu)(\alpha(\nu) - \alpha(\nu-1))$ and γ_j is a path in $(Y \times \boldsymbol{C}) \underset{Y}{\times} \{y\}$ around Z_j $(j=1,\dots, l)$. Here the summation is over the set of (α, β) such that α is a nondecreasing surjective map from $[1, r+l]$ to $[1, l]$ and β is a nondecreasing surjective map from $[1, r+l]$ to $[0, r]$. (Note that we may assume that $(\alpha(\nu), \beta(\nu)) - (\alpha(\nu-1), \beta(\nu-1))$ is $(0, 1)$ or $(1, 0)$; otherwise $\varphi = 0$.)

Proof. We shall prove this corollary by the induction on l.

We may assume that $X = Y \times \boldsymbol{C}^l$. Let $\tilde{\mathscr{W}}$ be the open covering $\{W_{(j,i)}\}_{j \in [0,l] \times I}$ and $\tilde{\mathscr{W}}'$ the open covering $\{W_{(j,i)}\}_{j \in [0,l] \times I_0 \cup [l,l] \times I}$, where $W_{(0,i)} = f^{-1} U_i$. Consider the image of φ by the map $H^{r+l-1}(\mathscr{W} \bmod \mathscr{W}'; \mathcal{O}_X)$

$\to H^{r+l}(\tilde{\mathscr{W}} \bmod \tilde{\mathscr{W}}'; \mathcal{O}_X)$. Then the image is given by $\psi \in Z^{r+l}(\tilde{\mathscr{W}} \bmod \tilde{\mathscr{W}}'; \mathcal{O}_X)$, where

$$\psi_{(0,i_1),(j_1,i_1),\dots,(j_{r+l},i_{r+l})} = -\varphi_{(j_1,i_1),\dots,(j_{r+l},i_{r+l})}, \quad \text{if} \quad 0 < j_1 \leqq j_2 \leqq, \dots, \leqq j_{r+l}$$

and ψ takes the value zero otherwise. Let X_1 be defined by $Y \times \mathbb{C}^{l-1}$, g the projection from X onto X_1 defined by $(y, t_1,\dots, t_l) \to (y, t_1,\dots, t_{l-1})$, f_1 the projection from X_1 onto Y, and $Z_1 = g(Z)$. Let $\tilde{\mathscr{W}}_1 = \{W_{1(j,i)}\}_{(j,i)\in[0,l-1]\times I}$ and $\tilde{\mathscr{W}}'_1 = \{W_{1(j,i)}\}_{(j,i)\in[0,l-1]\times I_0 \cup [1,l]\times I}$ be the open covering of X_1 and its subcovering defined in the same way as $\tilde{\mathscr{W}}$ and $\tilde{\mathscr{W}}'$. Let $\tilde{\tilde{\mathscr{W}}}_1$ be the covering of X given by $\tilde{\tilde{W}}_{a(j,i)} = g^{-1}W_{1(j,i)} = W_{j,i}$ (where $W_{1(j,i)} = U_i \times \mathbb{C}^{l-1} - \mathbb{C} \times \cdots \times \mathbb{C} \times Z_j \times \mathbb{C} \times \cdots \times \mathbb{C}$) and $\tilde{\tilde{W}}_{b(j,i)} = g^{-1}W_{1(j,i)} \cap V_l = W_{(j,i)} \cap V_l$ for $j \in [0, l-1]$ and $i \in I$, and $\tilde{\tilde{\mathscr{W}}}_1$ the subcovering $\{\tilde{\tilde{W}}_{a(j,i)}\}_{(j,i)\in[0,l-1]\times I_0} \cup \{\tilde{\tilde{W}}_{b(j,i)}\}_{(j,i)\in[0,l-1]\times I}$. Then $W_{a(j,i)} \subset W_{j,i}$ and $\tilde{\tilde{W}}_{b(j,i)} \subset W_{l,i}$. Hence, by the preceding corollary, the image of ψ by the map

$$Z^{r+l}(\tilde{\mathscr{W}} \bmod \tilde{\mathscr{W}}'; \mathcal{O}_X) \longrightarrow Z^{r+l}(\tilde{\tilde{\mathscr{W}}} \bmod \mathscr{W}'; \mathcal{O}_X) \longrightarrow H^{r+l}_{Z\cap f^{-1}(T)}(X; \mathcal{O}_X) \longrightarrow H^{r+l-1}_{Z_1\cap f_1^{-1}(T)}(X_1; \mathcal{O}_{X_1})$$

is given by the cocycle $\tilde{\phi} \in Z^{r+l-1}(\tilde{\tilde{\mathscr{W}}}_1 \bmod \tilde{\tilde{\mathscr{W}}}'_1; \mathcal{O}_{X_1})$, where

$$\tilde{\phi}_{(j_0,i_0),\dots,(j_{r+l-1},i_{r+l-1})} = \sum_{\nu} (-1)^{\nu} \int_{\gamma_l} \psi_{(j_0,i_0),\dots,(j_\nu,i_\nu)(l,i_\nu),\dots,(l,i_{r+l-1})}.$$

Therefore $\tilde{\phi} = 0$ unless $j_u \neq 0$ except one μ, and

$$\tilde{\phi}_{(0,i),(j_1,i_1),\dots,(j_{r+l},i_{r+l})} = \sum_{\nu} (-1)^{\nu+1} \int_{\gamma_l} \varphi_{(j_1,i_1),\dots,(j_\nu,i_\nu)(l,i_\nu),\dots,(l,i_{r+l})}.$$

Therefore $\tilde{\phi}$ is the image of $\tilde{\varphi} \in Z^{r+l-2}(\mathscr{W}_1 \bmod \mathscr{W}'_1; \mathcal{O}_X)$ given by

$$\tilde{\varphi}_{(j_1,i_1)\dots(j_{r+l},i_{r+l})} = \sum_{\nu} (-1)^{\nu} \int_{\gamma_l} \varphi_{(j_1,i_1)\dots(j_\nu,i_\nu)(l,i_\nu)\dots(l,i_{r+l})}.$$

Therefore, by the induction of l, ϕ is given by

$$\begin{aligned}\phi_{i_0,\dots,i_r} &= \sum (-1)^k \int_{\gamma_1\times\cdots\times\gamma_{l-1}} \tilde{\varphi}_{(\alpha(1),i_{\beta(1)}),\dots,(\alpha(r+l-1),i_{\beta(r+l-1)})} \\ &= \sum (-1)^{k+\nu} \int_{\gamma_1\times\cdots\times\gamma_l} \varphi_{(\alpha(1),i_{\beta(1)})\dots(\alpha(\nu),i_{\beta(\nu)})(l,i_{\beta(\nu+1)})\dots(l,i_{\beta(r+l)})}\end{aligned}$$

where $k=\sum_{\mu=2}^{r+l-1}\beta(\mu)(\alpha(\mu)-\alpha(\mu-1))$. Set $\tilde{\alpha}(\mu)=\alpha(\mu)(\mu\leqq\nu)$ and $\tilde{\alpha}(\mu)=l$ for $\mu>\nu$. Since $\alpha(\nu)=l-1$, and $\beta(\mu)=\mu-l$ for $\mu>\nu$ (other terms do not give any contribution),

$$\sum_{\mu=2}^{r+l}(\tilde{\alpha}(\mu)-\tilde{\alpha}(\mu-1))\beta(\mu)=\sum_{\mu=2}^{\nu}(\tilde{\alpha}(\mu)-\tilde{\alpha}(\mu-1))\beta(\mu)+\beta(\nu+1)\equiv k+(\nu-l+1),$$

we have the desired result. Q.E.D.

1.3. We shall consider the action of $\mathfrak{E}(G; D)$ in the following special case. Set $X=\boldsymbol{C}^n$ and let c and $a_j(j=2,\dots,n)$ be positive numbers.(*) Set

$$Z=Z(a_2,\dots,a_n;\,c)=\{z\in\boldsymbol{C}^n;\ a_j|z_1|\geqq|z_j| \text{ for } j=2,\dots,n,\ c\operatorname{Re}z_1\geqq|\operatorname{Im}z_1|\} \tag{3.1.1}$$

and let $G=G(a_2,\dots,a_n;\,c)$ be the convex hull of Z, i.e.,

$$G(a_2,\dots,a_n;\,c)=\{z\in\boldsymbol{C}^n;\ a_j(1+c^2)^{1/2}\operatorname{Re}z_1\geqq|z_j| \text{ for } j=2,\dots,n \text{ and } c\operatorname{Re}z_1\geqq|\operatorname{Im}z_1|\}. \tag{3.1.2}$$

The dual cone of G is given by

$$\Xi=\{\zeta\in\boldsymbol{C}^n;\ c|\operatorname{Im}\zeta_1|+(1+c^2)^{1/2}\sum_{j=2}^{n}a_j|\zeta_j|<\operatorname{Re}\zeta_1\}.$$

Set

$$\tilde{Z}=\{(z,w)\in\boldsymbol{C}^n\times\boldsymbol{C}^n;\ w-z\in Z\}$$

and

$$\tilde{G}=\{(z,w)\in\boldsymbol{C}^n\times\boldsymbol{C}^n;\ w-z\in G\}.$$

For a G-round(**) open set D, $\mathfrak{E}(G; D)$ is defined by

$$H^n_{\tilde{G}}(D\times D;\,\mathcal{O}_{\boldsymbol{C}^{2n}})$$

([19], § 3). The elements of $H^n_{\tilde{G}}(D\times D;\,\mathcal{O}_{\boldsymbol{C}^{2n}})$ being difficult to express explicitly, we shall consider

$$H^n_{\tilde{Z}}(D\times D;\,\mathcal{O}_{\boldsymbol{C}^{2n}})$$

instead of the relative cohomology group with support in $\tilde{G}$.

Clearly $\boldsymbol{C}^{2n}-\tilde{Z}$ has an open Stein covering $\bigcup_{j=1}^{n}V_j$, where $V_j=\{(z,w);\ c\operatorname{Re}(w_1-z_1)<|\operatorname{Im}(w_1-z_1)|\}$ for $j=1$ and $V_j=\{(z,w);\ a_j|z_1-w_1|<|z_j-w_j|\}$ for $j=2,\dots,n$. Hence, if D is G-round and holomorphically convex, then

(*) Actually c and a_j may be zero.

(**) An open set D is called G-round if $(D+G)\cap(D+G^a)=D$, where $G^a=\{z;\ -z\in G\}$.

$$H^n_Z(D\times D;\,\mathcal{O})=\mathcal{O}(\bigcap_{j=1}^{n} V_j\cap D\times D)/(\sum_{k=1}^{n}\mathcal{O}(\bigcap_{j\neq k} V_j\cap D\times D)).$$

Set $V=\bigcap_{j=1}^{n} V_j=\{(z, w);\ c\operatorname{Re}(w_1-z_1)<\operatorname{Im}(w_1-z_1),\ a_j|z_1-w_1|<|z_j-w_j|$ for $j=2,\dots, n\}$.

Any holomorphic function $f(z, w)$ defined on $V\cap D\times D$ determines an element in $H^n_Z(D\times D;\,\mathcal{O})$ and hence an element in $\mathfrak{E}(G;\,D)$, which we shall denote by $[f(z, w)]$.

Let Ω_1 and Ω_2 be two G-open and holomorphically convex sets such that $\Omega_1\supset\Omega_2$ and $\Omega_1-\Omega_2\subset\subset D$. Then $\mathfrak{E}(G;\,D)$ operates on $H^1_{\Omega_1-\Omega_2}(\Omega_1;\,\mathcal{O})=\mathcal{O}(\Omega_2)/\mathcal{O}(\Omega_1)=\mathcal{O}(\Omega_2\cap D)/\mathcal{O}(\Omega_1\cap D)$. In the next subsection we shall write down this operation explicitly.

1.4. Let c' be a positive number greater than c, and let l be a line in $\boldsymbol{C}$ such that $\{w_1\in l;\ c'|\operatorname{Re} w_1|\geqq|\operatorname{Im} w_1|\}$ is compact. Let α and β be two points on l such that $\operatorname{Im}\alpha>\operatorname{Im}\beta$. We define $L=\{w\in\boldsymbol{C}^n;\ w_1\in l\}$, $L_\pm=\{w\in\boldsymbol{C}^n;\ w_1\in l+\mathbf{R}^\pm\}$ and $L_0=\{w\in\boldsymbol{C}^n;\ w_1\in[\alpha, \beta]\}$. Here $[\alpha, \beta]$ means the segment joining α and β. We shall denote by G' the closed cone $G(a_2,\dots, a_n;\, c')$ defined by the formula (3.1.2). The cone G' contains G. We shall assume further that D is holomorphically convex and G'-round. Let $\tilde{D}$ be the open subset $\{z\in D;\ (z+G)\cap L\subset L_0\cap D, (z+G')\cap L\subset D, c'\operatorname{Re}(\alpha-z_1)>\operatorname{Im}(\alpha-z_1)>c\operatorname{Re}(\alpha-z_1)$ and $c'\operatorname{Re}(\beta-z_1)>-\operatorname{Im}(\beta-z_1)>c\operatorname{Re}(\beta-z_1)\}$. One can easily verify that $\tilde{D}$ is also holomorphically convex.

Let $f(z, w)$ be a holomorphic function defined on $V\cap D\times D$ and Ω a G'-open subset of $\boldsymbol{C}^n$. We shall define a homomorphism

$$K^\beta_\alpha(f):\ \mathcal{O}(\Omega\cap D)\longrightarrow\mathcal{O}(\Omega\cap\tilde{D})$$

by the formula

$$K^\beta_\alpha(f)(u)(z)=\int_\alpha^\beta dw_1\oint dw_2\cdots\oint dw_n f(z, w)u(w).$$

Here $\oint dw_j$ means the contour integral along the cycle $\{w_j;\ |z_j-w_j|=a_j|z_1-w_1|+\varepsilon\}$ with $0<\varepsilon\ll 1$.

Let us verify that $K^\beta_\alpha(f)$ is well-defined. For a holomorphic function $u(w)$ defined on $\Omega\cap D$, we put

$$g(z, w_1)=\oint dw_2\cdots\oint dw_n f(z, w)u(w).$$

For $(z, w_1)\in\boldsymbol{C}^n\times\boldsymbol{C}$, we define

$$T(z, w_1)=\{(w_1, w')\in \boldsymbol{C}^n;\ |w_j-z_j|\leqq a_j|w_1-z_1| \quad \text{for} \quad j=2,\dots, n\}$$

Then it is obvious that $g(z, w_1)$ is defined at (z, w_1) if $z\in D$, $c\,\mathrm{Re}\,(w_1-z_1)<|\mathrm{Im}\,(w_1-z_1)|$ and $T(z, w_1)\subset \Omega\cap D$.

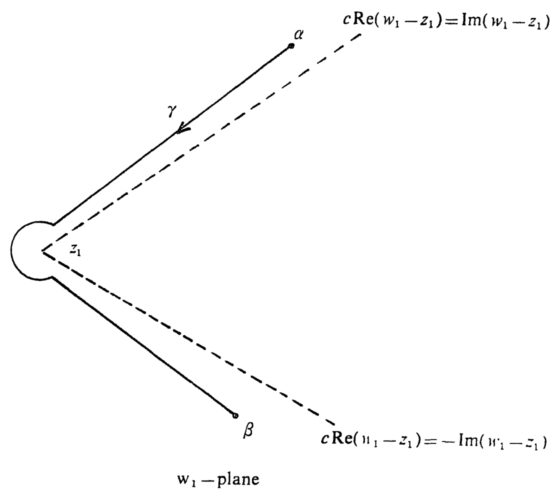

Fig. 3.1.1

Consider a path γ in the w_1-plane as figured in Fig. 3.1.1, that is, γ is a path starting from α and ending at β and γ is contained in

$$\begin{aligned}&\{w_1;\ |w_1-z_1|=\varepsilon,\ c\,\mathrm{Re}\,(w_1-z_1)<|\mathrm{Im}\,(w_1-z_1)|\}\\ &\cup\{w_1;\ w_1=z_1+t(\alpha-z_1),\ 0<t\leqq 1\}\\ &\cup\{w_1;\ w_1=z_1+t(\beta-z_1),\ 0<t\leqq 1\}\end{aligned}$$

for $0<\varepsilon\ll 1$. Such a path γ can be described in the w_1-plane because $z\in\tilde{D}$. We define

$$K_\alpha^\beta(f)(u)(z)=\int_\gamma g(z, w_1)dw_1.$$

Hence, in order to see that $K_\alpha^\beta(f)(u)$ is defined on $\Omega\cap\tilde{D}$, it is enough to verify that $T(z, w_1)\subset\Omega\cap D$ and $c\,\mathrm{Re}\,(w_1-z_1)<|\mathrm{Im}\,(w_1-z_1)|$ if $z\in\Omega\cap\tilde{D}$ and $w_1\in\gamma$. The second condition is satisfied because $w_1\in\gamma$. $T(z, w_1)$ is contained in any neighborhood of z if w_1 is sufficiently close to z. Hence we may assume that $w_1=z_1+t(\alpha-z_1)$ or $w_1=z_1+t(\beta-z_1)$ with $0<t\leqq 1$. Then $T(z, w_1)$ is contained in $(z+G')\cap\overline{L_-}$. Hence $T(z, w_1)\subset z+G'\subset\Omega\cap(D+G')$. Moreover we have $T(z, w_1)\subset((z+G')\cap L)+G'^a\subset D+G'^a$. Hence we obtain $T(z, w_1)\subset(D+G'^a)\cap(D+G')=D$. Thus we have shown that $K_\alpha^\beta(f)$ is a well-defined

homomorphism from $\mathcal{O}(\Omega \cap D)$ into $\mathcal{O}(\Omega \cap \tilde{D})$. Q.E.D.

Proposition 3.1.5. *Let $\Omega_1 \supset \Omega_2$ be two G'-open subsets such that $\Omega_1 - \Omega_2 \Subset \tilde{D}$ and that $\Omega_1 \cap D$ and $\Omega_2 \cap D$ are holomorphically convex. Then for $f \in \mathcal{O}(V \cap D \times D)$, the action of the corresponding element $[f]$ in $\mathfrak{E}(G; D)$ on $H^1_{\Omega_1-\Omega_2}(\Omega_1; \mathcal{O}_X)$ coincides with*

$$\begin{aligned} &H^1_{\Omega_1-\Omega_2}(\Omega_1; \mathcal{O}_X) \cong \mathcal{O}(\Omega_2 \cap D)/\mathcal{O}(\Omega_1 \cap D) \\ &\xrightarrow{K^\beta_\alpha(f)} \mathcal{O}(\Omega_2 \cap \tilde{D})/\mathcal{O}(\Omega_1 \cap \tilde{D}) \\ &\xrightarrow{\sim} H^1_{\Omega_1-\Omega_2}(\Omega_1; \mathcal{O}_X)\,. \end{aligned}$$

Proof. Let $(\Omega_1 - \Omega_2)_{G'}$ be the topological space $\Omega_1 - \Omega_2$ endowed with the G'-topology. Then the action of $[f]$ on $H^1_{\Omega-\Omega_2}(\Omega; \mathcal{O}_X)$ is a sheaf endomorphism of the sheaf $\Omega - \Omega_2 \mapsto H^1_{\Omega-\Omega_2}(\Omega; \mathcal{O}_X)$ on $(\Omega_1 - \Omega_2)_{G'}$. The endomorphism induced from $K^\beta_\alpha(f)$ gives also a sheaf endomorphism of the same sheaf on $(\Omega_1 - \Omega_2)_{G'}$. If they coincide on an open basis of $(\Omega_1 - \Omega_2)_{G'}$, then they are equal. Thus it is enough to show that, for $\Omega = z_0 + (\mathrm{Int}\, G')$, the action on $H^1_{\Omega-\Omega_2}(\Omega\colon \mathcal{O}_X)$ of $[f]$ and that of $K^\beta_\alpha(f)$ coincide. Then $\Omega \cap \Omega_2$ and Ω are holomorphically convex. Hence we may assume from the first that Ω_1 and Ω_2 are holomorphically convex.

The action of $H^n_{\tilde{Z}}(D \times D; \mathcal{O}_{X \times X})$ on $H^1_{\Omega_1-\Omega_2}(\Omega_1; \mathcal{O}_X)$ is given by

$$\begin{aligned} &H^n_{\tilde{Z}}(D \times D; \mathcal{O}_{X\times X}) \otimes H^1_{\Omega_1-\Omega_2}(\Omega_1; \mathcal{O}_X) \\ &\xrightarrow{\alpha} H^{n+1}_{\tilde{Z}\cap(\Omega_1\cap D)\times(\Omega_1-\Omega_2)}((\Omega_1 \cap D) \times (\Omega_1 \cap D); \mathcal{O}_{X\times X}) \\ &\xrightarrow{\beta} H^1_{\Omega_1-\Omega_2}(\Omega_1 \cap \tilde{D}; \mathcal{O}_X)\,. \end{aligned}$$

Here α is the cup-product and β is the residue map.

Let u be a holomorphic function defined on $\Omega_2 \cap D$ and $[u]$ the corresponding element of $H^1_{\Omega_1-\Omega_2}(\Omega_1; \mathcal{O}_X) = \mathcal{O}_X(\Omega_2 \cap D)/\mathcal{O}_X(\Omega_1 \cap D)$. We shall express the image of $[f] \otimes [u]$ by α using Čech cohomology.

Take a Stein open covering $\mathscr{W} = \{W_j\}_{j=0,\dots,n}$ of $((\Omega_1 \cap D) \times (\Omega_1 \cap D) - \tilde{Z}) \cup (\Omega_1 \cap D) \times (\Omega_2 \cap D)$, where $W_0 = (\Omega_1 \cap D) \times (\Omega_2 \cap D)$ and $W_j = V_j \cap (\Omega_1 \cap D) \times (\Omega_1 \cap D)$ for $j = 1, 2, \dots, n$. Then $f(z, w)u(w) \in \mathcal{O}(\bigcap_{j=1}^n W_j)$ determines a cohomology class in $H^n(\mathscr{W}; \mathcal{O}_{X\times X}) = \mathcal{O}(\bigcap_{j=0}^n W_j)/(\sum_{k=0}^n \mathcal{O}(\bigcap_{j\neq k} W_j))$, which we shall denote by $[f(z, w)u(w)]$. Then $\alpha([f] \otimes [u]) = [f(z, w)u(w)]$ holds if we identify

$$H^n(\mathscr{W}; \mathcal{O}_{X\times X}) = H^n((\Omega_1 \cap D) \times (\Omega_1 \cap D) - \tilde{Z}) \cup ((\Omega_1 \cap D) \times (\Omega_2 \cap D)); \mathcal{O}_{X\times X})$$

with

$$H^{n+1}_{\tilde{Z}\cap(\Omega_1\cap D)\times(\Omega_1-\Omega_2)}((\Omega_1 \cap D) \times (\Omega_1 \cap D); \mathcal{O}_{X\times X})\,.$$

For any holomorphic function $\varphi(z, w)$ defined on $\bigcap_{j=0}^{n} W_j$, let us denote by $[\varphi]$ the corresponding element of $H^{n+1}_{\tilde{Z}_0 \cap (\Omega_1 \cap D)\times(\Omega_1-\Omega_2)}((\Omega_1 \cap D)\times(\Omega_1 \cap D); \mathcal{O}_{X\times X}) \cong H^n(\mathscr{W}; \mathcal{O}_{X\times X}) \cong \mathcal{O}(\bigcap_{j=0}^{n} W_j)/(\sum_{k=0}^{n} \mathcal{O}(\bigcap_{j\neq k} W_j))$.

Thus, in order to prove the proposition, it is sufficient to show (3.1.3) for any $\varphi \in \mathcal{O}(\bigcap_{j=0}^{n} W_j)$,

$$\int_\gamma dw_1 \oint dw_2 \cdots \oint dw_n \varphi(z, w) \text{ modulo } \mathcal{O}(\Omega_1 \cap D)$$

coincides with the image of $[\varphi]$ by β. Here γ is the path from α to β given in Fig. 3.1.1.

Since $\tilde{Z} \cap (\Omega_1 \cap D)\times(\Omega_1-\Omega_2) = \tilde{Z} \cap (\Omega_1 \cap D)\times(\boldsymbol{C}^n-\Omega_2)$, we can apply the excision theorem for relative cohomology groups and we obtain the isomorphism

$$\begin{aligned} &H^{n+1}_{\tilde{Z} \cap (\Omega_1 \cap D)\times(\boldsymbol{C}^n-\Omega_2)}((\Omega_1 \cap D)\times \boldsymbol{C}^n; \mathcal{O}_{X\times X}) \\ &\quad \xrightarrow{\sim} H^{n+1}_{\tilde{Z} \cap (\Omega_1 \cap D)\times(\Omega_1-\Omega_2)}((\Omega_1 \cap D)\times(\Omega_1 \cap D); \mathcal{O}_{X\times X}). \end{aligned}$$

Let $\mathscr{W}' = \{W'_j\}_{j=0,\dots,n}$ be the Stein open covering of $((\Omega_1 \cap D)\times \boldsymbol{C}^n - \tilde{Z}) \cup (\Omega_1 \cap D)\times\Omega_2$ given by

$$W'_0 = (\Omega_1 \cap D)\times\Omega_2, \; W_j = V_j \cap (\Omega_1 \cap D)\times \boldsymbol{C}^n \qquad \text{for} \quad j=1, \dots, n.$$

Then we have the isomorphism

$$H^n(\mathscr{W}'; \mathcal{O}_{X\times X}) \xrightarrow{\sim} H^n(\mathscr{W}; \mathcal{O}_{X\times X}).$$

Hence we have

$$\varphi \in \mathcal{O}(\bigcap_{j=0}^{n} W'_j) + \sum_{k=0}^{n} \mathcal{O}(\bigcap_{j\neq k} W_j).$$

We shall show that, if $\varphi \in \mathcal{O}(\bigcap_{j\neq k} W_j)$ for some k, then $\int_\gamma dw_1 \oint dw_2 \cdots \oint dw_n \varphi(z, w)$ is holomorphic on $\Omega_1 \cap \tilde{D}$. If $k=2, \dots, n$, then Cauchy's integral formula implies

$$\oint dw_k \varphi(z, w) = 0$$

and hence $\int_\gamma dw_1 \oint dw_2 \cdots \oint dw_n \varphi(z, w) = 0$. If $k=0$, $\int_\gamma dw_1 \oint dw_2 \cdots \oint dw_n \varphi(z, w)$ is holomorphic on $\Omega_1 \cap \tilde{D}$ by the argument employed in order to prove the holomorphic character of $K^\beta_\alpha(f)(u)$.

Now, let us suppose that φ is holomorphic on $\bigcap_{j\neq 1} W_j = (\Omega_1 \cap D)\times(\Omega_2 \cap D) \cap \bigcap_{j=2}^{n} V_j$. As was shown earlier, the holomorphic function

$$g(z, w_1)=\oint dw_2\cdots\oint dw_n\varphi(z, w)$$

is holomorphic if $z\in\Omega_1\cap D$ and $T(z, w_1)\underset{\text{def}}{=}\{(w_1, w')\in\mathbf{C}^n;\ |w_j-z_j|\leqq a_j|w_1-z_1|$ for $j=2,\dots, n\}\subset\Omega_2\cap D$. Let γ_1' be the straight path joining α and β. Then for $z\in\Omega_2\cap\tilde{D}$, we have

$$\int_\gamma g(z, w_1)dw_1=\int_{\gamma_1'} g(z, w_1)dw_1.$$

We shall prove that the second term is holomorphic on $z\in\Omega_1\cap\tilde{D}$. In order to prove this, it is sufficient to show that, if $z\in\Omega_1\cap\tilde{D}$ and $w_1\in\gamma_1'$, then $T(z, w_1)\subset\Omega_2\cap D$. Since $z\in\tilde{D}$ and $w_1\in\gamma_1'$, we have

$$c'\operatorname{Re}(w_1-z_1)\geqq|\operatorname{Im}(w_1-z_1)|$$

and hence $T(z, w_1)\subset(z+G')\cap L$. Therefore we have $T(z, w_1)\subset\Omega_1\cap D$. On the other hand $\tilde{D}\cap L=\phi$. Hence we obtain $T(z, w_1)\subset\Omega_1\cap(D-\tilde{D})\subset\Omega_2\cap\tilde{D}$.

Therefore, in proving (3.1.3), we may assume from the first that φ is holomorphic on $\bigcap_{j=0}^{n} W_j'$.

Let us take a sufficient large positive number R such that $|\operatorname{Re}(w_1-z_1)|\leqq R$ for $w, z\in\Omega_1-\Omega_2$. Set $Z_0=\{(z, w)\in\tilde{Z};\ \operatorname{Re}(w_1-z_1)\leqq R\}$. Then $\tilde{Z}\cap(\Omega_1\cap D)\times(\mathbf{C}^n-\Omega_2)=\tilde{Z}\cap(\Omega_1-\Omega_2)\times(\Omega_1-\Omega_2)$ is a closed subset of $Z_0\cap(\Omega_1-\Omega_2)\times\mathbf{C}^n$. Hence the residue map

$$H^{n+1}_{Z(\Omega_1\cap D)\times(\mathbf{C}^n-\Omega_2)}((\Omega_1\cap D)\times\mathbf{C}^n;\ \mathcal{O}_{X\times X})\longrightarrow H^1_{\Omega_1-\Omega_2}(\Omega_1\cap\tilde{D};\ \mathcal{O}_X)$$

decomposes into

$$\begin{aligned}&H^{n+1}_{\tilde{Z}\cap(\Omega_1\cap D)\times(\mathbf{C}^n-\Omega_2)}((\Omega_1\cap D)\times\mathbf{C}^n;\ \mathcal{O}_{X\times X})\\ &\xrightarrow{\ \gamma\ } H^{n+1}_{Z_0\cap(\Omega_1-\Omega_2)\times\mathbf{C}^n}((\Omega_1\cap D)\times\mathbf{C}^n;\ \mathcal{O}_{X\times X})\\ &\xrightarrow{\ \beta'\ } H^1_{\Omega_1-\Omega_2}(\Omega_1\cap\tilde{D};\ \mathcal{O}_{X\times X})\end{aligned}$$

We put $V_1'=\{(z, w)\in\mathbf{C}^n\times\mathbf{C}^n;\ \operatorname{Re}(w_1-z_1)>R \text{ or } c^1\operatorname{Re}(w_1-z_1)<|\operatorname{Im}(w_1-z_1)|\}$ and $V_j'=V_j$ $(j=2,\dots, n)$. Then $\{V_j'\}_{j=0,\dots,n}$ is a Stein open covering of $\mathbf{C}^n\times\mathbf{C}^n-Z_0$. Let $\mathscr{W}''=\{W_j''\}_{j=0,\dots,n}$ be the Stein covering of $((\Omega_1\cap D)\times\mathbf{C}^n-Z_0)\cup(\Omega_2\cap D)\times\mathbf{C}^n$ given by $W_0''=(\Omega_2\cap D)\times\mathbf{C}^n$ and $W_j''=V_j'\cap(\Omega_1\cap D)\times\mathbf{C}^n$ for $j=1,\dots, n$. Then we have

$$\begin{aligned}H^n(\mathscr{W}'';\ \mathcal{O}_{X\times X})&\cong H^n(((\Omega_1\cap D)\times\mathbf{C}^n-Z_0)\cup(\Omega_2\cap D)\times\mathbf{C}^n;\ \mathcal{O}_{X\times X})\\ &=H^{n+1}_{Z_0\cap(\Omega_1-\Omega_2)\times\mathbf{C}^n}((\Omega_1\cap D)\times\mathbf{C}^n;\ \mathcal{O}_{X\times X})\end{aligned}$$

and γ induces the homomorphism

$$H^n(\mathscr{W}'; \mathscr{O}_{X\times X}) \longrightarrow H^n(\mathscr{W}''; \mathscr{O}_{X\times X}) = \mathscr{O}_{X\times X}(\bigcap_{j=0}^{n} W''_j)/(\sum_{k=0}^{n} \mathscr{O}_{X\times X}(\bigcap_{j\neq k} W''_j)).$$

Let $\psi(z, w)$ be a holomorphic function defined on $\bigcap_{j=0}^{n} W''_j$ which represents $\gamma([\varphi(z, w)])$. Then it follows from Corollary 3.1.4 that the image $\beta'\circ\gamma([\varphi(z, w)]) = \beta'([\psi(z, w)]$ is given by

$$\int_{\gamma_1\times\cdots\times\gamma_n} \psi(z, w)dw \qquad \text{modulo} \quad \mathscr{O}(\Omega_1 \cap \tilde{D}).$$

Here $\gamma_j (j=2,\ldots, n)$ is a cycle in the w_j-plane around $\{w_j; |w_j - z_j| \leqq a_j|w_1 - z_1|\}$ and γ_1 is a cycle in the w_1-plane around $(w_1; \mathrm{Re}(w_1 - z_1) \leqq R, c\,\mathrm{Re}(w_1 - z_1) \geqq |\mathrm{Im}(w_1 - z_1)|\}$. Now, we shall investigate the relation between $\varphi(z, w)$ and $\psi(z, w)$ in order to show that

$$\int_{\alpha}^{\beta} dw_1 \oint dw_2 \cdots \oint dw_n \varphi(z, w) \equiv \int_{\gamma_1\times\cdots\times\gamma_n} \psi(z, w)dw \qquad \text{modulo} \quad \mathscr{O}(\Omega_1 \cap \tilde{D}).$$

Let $\mathscr{U} = \{U_j\}_{j=0,\ldots,n+1}$ be the Stein covering given by $U_0 = V'_1 \cap (\Omega_1 \cap D) \times \Omega_2$, $U_j = V_j \cap (\Omega_1 \cap D) \times \mathbf{C}^n$ $(j=1,\ldots, n)$, and $U_{n+1} = (\Omega_2 \cap D) \times \Omega_2$. Then $\mathscr{U}$ is also a Stein covering of $((\Omega_1 \cap D) \times \mathbf{C}^n - Z_0) \cup (\Omega_2 \cap D) \times \mathbf{C}^n$, and $\mathscr{U}$ is a refinement of $\mathscr{W}'$ and $\mathscr{W}''$ at once.

Since $U_0 \subset W'_0$, $U_j \subset W'_j (j=1,\ldots, n)$ and $U_{n+1} \subset W'_0$, the image of $[\varphi(z, w)]$ in $H^{n+1}_{Z_0\cap(\Omega_1-\Omega_2)\times\mathbf{C}^n}((\Omega_1 \cap D) \times \mathbf{C}^n; \mathscr{O}_{X\times X})$ is expressed by $\{\varphi_{i_0,\ldots,i_n}\} \in Z^n(\mathscr{U}; \mathscr{O}_{X\times X})$, where $\varphi_{i_0,\ldots,i_n} (i_0 < \cdots < i_n)$ are given by

$$\begin{cases} \varphi_{0,2,\ldots,n+1} = \varphi(z, w), \\ \varphi_{1,2,\ldots,n+1} = (-1)^n \varphi(z, w), \\ \varphi_{i_0,\ldots,i_n} = 0 \qquad \text{otherwise.} \end{cases}$$

In the same way, we have $U_0 \subset W''_1$, $U_j \subset W''_j$ $(j=1,\ldots, n)$ and $U_{n+1} \subset W''_0$. Hence $[\psi(z, w)]$ is expressed by $\{\psi_{i_0,\ldots,i_n}\} \in Z^n(\mathscr{U}; \mathscr{O}_{X\times X})$, where $\psi_{i_0,\ldots,i_n}$ $(i_0 < \cdots < i_n)$ are given by

$$\begin{cases} \psi_{0,2,\ldots,n+1} = (-1)^n \psi(z, w), \\ \psi_{1,2,\ldots,n+1} = (-1)^n \psi(z, w), \\ \psi_{i_0,\ldots,i_n} = 0 \qquad \text{otherwise.} \end{cases}$$

As $\{\varphi_{i_0,\ldots,i_n}\}$ and $\{\psi_{i_0,\ldots,i_n}\}$ give the same cohomology class in $H^n(\mathscr{U}; \mathscr{O}_{X\times X})$, $\{\varphi_{i_0,\ldots,i_n} - \psi_{i_0,\ldots,i_n}\}$ is a coboundary. Hence there are holomorphic functions $\phi_{j,k} (\phi_{j,k} = -\phi_{k,j})$ defined on $\bigcap_{i\neq j,k} U_i$ $(j, k=0, 1,\ldots, n+1)$ such that

$$\varphi(z, w) = \sum_i \phi_{i,n+1}(z, w) \qquad \text{on} \quad \bigcap_{i\neq n+1} U_i = (\Omega_1 \cap D) \times \Omega_2 \cap \bigcap_{j=1}^{n} V_j,$$

and

$$\psi(z, w) = -\sum_i \phi_{i,1}(z, w) \qquad \text{on} \quad \bigcap_{i \neq 1} U_i = (\Omega_2 \cap D) \times \Omega_2 \cap \bigcap_{j=1}^{n} V'_j .$$

For $z \in \Omega_2 \cap \tilde{D}$ and w_1 such that $T(z, w_1) \subset \Omega_2$,

$$\int_{\gamma_2 \times \cdots \times \gamma_n} \varphi(z, w) dw' = \sum_i \int_{\gamma_2 \times \cdots \times \gamma_n} \phi_{i,n+1}(z, w) dw' .$$

By Cauchy's integral formula, we find

$$\int_{\gamma_i} dw_i \phi_{i,n+1}(z, w) = 0 \qquad \text{for} \quad i = 2, \ldots, n$$

and hence we obtain

$$\int_{\gamma_2 \times \cdots \times \gamma_n} \varphi(z, w) dw' = \int_{\gamma_2 \times \cdots \times \gamma_n} \phi_{0,n+1}(z, w) dw' + \int_{\gamma_2 \times \cdots \times \gamma_n} \phi_{1,n+1}(z, w) dw' .$$

Since $\phi_{0,n+1}(z, w)$ is holomorphic on $V \cap (\Omega_1 \cap D) \times \mathbb{C}^n$, $\int_{\gamma \times \gamma_2 \times \cdots \times \gamma_n} \phi_{0,n+1}(z, w) dw$ is holomorphic on $\Omega_1 \cap \tilde{D}$. Here γ is the path from α to β given in Fig. 3.1.1. Hence we obtain

(3.1.4)

$$\int_{\gamma \times \gamma_2 \times \cdots \times \gamma_n} \varphi(z, w) dw \equiv \int_{\gamma \times \gamma_2 \times \cdots \times \gamma_n} \phi_{1,n+1}(z, w) dw \qquad \text{modulo} \quad \mathcal{O}(\Omega_1 \cap \tilde{D}) .$$

Similarly, by using Cauchy's integral formula, we find

$$\begin{aligned} \int_{\gamma_2 \times \cdots \times \gamma_n} \psi(z, w) dw' &= \sum_i \int_{\gamma_2 \times \cdots \times \gamma_n} \phi_{1,i}(z, w) dw' \\ &= \int_{\gamma_2 \times \cdots \times \gamma_n} (\phi_{1,0}(z, w) + \phi_{1,n+1}(z, w)) dw' . \end{aligned}$$

However, again by Cauchy's integral formula,

$$\int_{\gamma_1 \times \cdots \times \gamma_n} \phi_{1,0}(z, w) dw = 0 \qquad \text{for} \quad z \in \Omega_2 \cap \tilde{D} ,$$

because $\phi_{1,0}(z, w)$ is defined on $(\Omega_2 \cap D) \times \Omega_2 \cap \bigcap_{j=2} V_j$. Thus we obtain

$$\int_{\gamma_1 \times \cdots \times \gamma_n} \psi(z, w) dw = \int_{\gamma_1 \times \cdots \times \gamma_n} \phi_{1,n+1}(z, w) dw .$$

Together with (3.1.4) it suffices to prove

$$\int_{\gamma_1 \times \cdots \times \gamma_n} \phi_{1,n+1}(z, w) dw \equiv \int_{\gamma \times \gamma_2 \times \cdots \times \gamma_n} \phi_{1,n+1}(z, w) dw \qquad \text{modulo} \quad \mathcal{O}(\Omega_1 \cap \tilde{D}) .$$

The function $\phi_{1,n+1}(z, w)$ is holomorphic on $\bigcap_{i \neq 1, n+1} U_i = (\Omega_1 \cap D) \times \Omega_2 \cap$

$\bigcap_{j=1}^{n} V'_j$. It is easy to see that $h(z, w_1) \underset{\text{def}}{=} \int_{\gamma_2 \times \cdots \times \gamma_n} \phi_{1,n+1}(z, w) dw'$ is holomorphic if $z \in \Omega_1 \cap D$, $T(z, w_1) \subset \Omega_2$ and $w_1 - z_1 \notin K \underset{\text{def}}{=} \{t \in \boldsymbol{C};\ c \operatorname{Re} t \geqq |\operatorname{Im} t|, \operatorname{Re} t \leqq R\}$.

Now let γ' be the path from β to α described in Fig. 3.1.2.

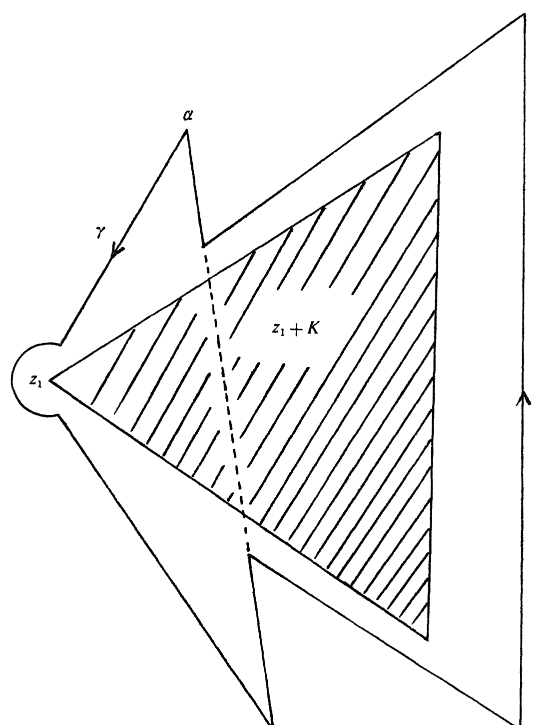

Fig. 3.1.2

If $z \in \Omega_2 \cap \tilde{D}$ and $w_1 \in \gamma \cup \gamma'$, then $T(z, w_1) \subset \Omega_2$ and hence

$$\int_{\gamma_1} h(z, w_1) dw_1 - \int_{\gamma} h(z, w_1) dw_1 = \int_{\gamma'} h(z, w_1) dw_1 \qquad \text{for} \quad z \in \Omega_2 \cap \tilde{D}.$$

Next suppose that $z \in \Omega_1 \cap \tilde{D}$ and $w_1 \in \gamma'$. Then $c' \operatorname{Re}(w_1 - z_1) \geqq |\operatorname{Im}(w_1 - z_1)|$, and hence $T(z, w_1) \in z + G' \subset \Omega_1$. Since $\tilde{D} \cap \overline{L_+} = \phi$, $T(z, w_1) \subset \Omega_1 \cap \overline{L_+} \subset \Omega_1 - \tilde{D} \subset \Omega_2$. Thus $\int_{\gamma'} h(z, w_1) dw_1$ is holomorphic on $\Omega_1 \cap \tilde{D}$. This completes the proof of Proposition 3.1.5.

§ 2.

The results in the preceding section yield the explicit formula for the action of micro-differential operators on the sheaf of microfunctions etc., as described in [10] and [2].

Let us first recall what is the kernel function of a micro-differential operator (see S-K-K [24], Chapter II, § 1.4). Let $\Phi_\lambda(\tau)$ be a holomorphic function defined on $\tau \in \boldsymbol{C} - \overline{\mathbf{R}^+}$ given by

$$\Phi_\lambda(\tau) = \frac{\Gamma(\lambda)}{(-\tau)^\lambda},$$

where we choose a branch $\Phi_\lambda(-1) = \Gamma(\lambda)$. It follows from the definition that Φ_λ has a pole at $\lambda = 0, -1, \ldots$. We define by convention

$$\Phi_{-n}(\tau) = -\frac{\tau^n}{n!}\{\log(-\tau) + \gamma - (1 + \cdots + 1/n)\},$$

for $n = 0, 1, 2, \ldots$, where $\gamma = 0.57721\ldots$ is the Euler constant. Then

$$\Phi_\lambda(\tau) = \frac{\tau^n}{(\lambda+n)n!} + \Phi_{-n}(\tau) + (\lambda+n)F(\lambda, \tau)$$

in a neighborhood of $\lambda = -n$, where $F(\lambda, \tau)$ is holomorphic if λ is in a neighborhood of $-n$ and $\tau \in \boldsymbol{C} - \overline{\mathbf{R}^+}$. Thus we have

$$\frac{\partial}{\partial \tau}\Phi_\lambda(\tau) = \Phi_{\lambda+1}(\tau).$$

For $\alpha = (\alpha_1, \ldots, \alpha_n)$, and $z = (z_1, \ldots, z_n)$, we set

$$\Phi_\alpha(z) = \Phi_{\alpha_1}(z_1) \cdots \Phi_{\alpha_n}(z_n).$$

Let $P(z, D_z)$ be a micro-differential operator defined in a neighborhood of (z^0, dz_1). Then, we can expand P as a power series in $D_1, \ldots, D_n$;

$$\sum_{\alpha \in \boldsymbol{Z}^n} a_\alpha(z) D_Z^\alpha .$$
$$a_j \geqq 0 \quad (j = 2, \ldots, n)$$

We set

$$K(z, w) = \frac{1}{(2\pi i)^{n_\alpha}} \sum a_\alpha(z) \Phi_{\alpha+\delta}(z-w),$$

where $\delta = (\overbrace{1, \ldots, 1}^{n})$.

Then $K(z, w)$ has the form

$$K(z, w)=K_0(z, w)+\frac{1}{2\pi i}K_1(z, w)\log(w_1-z_1)+K_2(z, w),$$

where

$$K_0(z, w)=\frac{1}{(2\pi i)^n}\sum_{\alpha_1\geqq 0} a_\alpha(z)\Phi_{\alpha+\delta}(z-w)$$

$$K_1(z, w)=\frac{1}{(2\pi i)^{n-1}}\Bigg(\sum_{\substack{\alpha_1<0\\ \alpha=(\alpha_1,\alpha')}}\frac{a_\alpha(z)}{(-\alpha_1-1)!}(z_1-w_1)^{-\alpha_1-1}\Phi_{\alpha'+\delta'}(z'-w')\Bigg).$$

Here $\delta'=(\overbrace{1,\dots, 1}^{n-1})$, $z'=(z_2,\dots, z_n)$ and $w'=(w_2,\dots, w_n)$. Then there are positive numbers a_j $(j=2,\dots, n)$ and a neighborhood D of z^0 such that

(3.2.1) $K_0(z, w)$ is holomorphic on $\{(z, w)\in D\times D;\ z_j\neq w_j\ (j=1,\dots, n)\}$ and $K_1(z, w)$ and $K_2(z, w)$ are holomorphic on $\{(z, w)\in D\times D;\ |z_j-w_j|>a_j|z_1-w_1|$ for $j=2,\dots, n\}$.

We shall define Z, G and V as in Section 1.3 and use the same notations as those used there with $c=0$. This means, in particular, $Z=G=\{z\in\mathbb{C}^n;\ a_j|z_1|\geqq|z_j|$ for $j=2,\dots, n$ and $\operatorname{Im} z_1=0$, $\operatorname{Re} z_1\geqq 0\}$. Then K is holomorphic on $V\cap D\times D$. Therefore K determines an element of $H^n_Z(D\times D;\ \mathcal{O}_{X\times X})$ and hence that of $\mathfrak{E}(G; D)$.

Set

$$P_0(z, D_z)=\sum_{\alpha_1\geqq 0} a_\alpha(z)D_z^\alpha.$$

Then $P_0(z, D_z)$ is a differential operator of infinite order.

Set

$$A(z, w_1, D_{z'})=\sum_{\substack{\alpha_1<0\\ \alpha=(\alpha_1,\alpha')}}\frac{a_\alpha(z)}{(-\alpha_1-1)!}(z_1-w_1)^{-\alpha_1-1}D_{z'}^{\alpha'}.$$

For any $c'>0$, we take α, β, G' and $\tilde{D}$ as in Section 1.4.

Proposition 3.2.1. (i) *If $v(w')$ is holomorphic in a neighborhood of $\{w'\in\mathbb{C}^{n-1};\ |w_j-z_j|\leqq a_j|z_1-w_1|,\ j=2,\dots, n\}$, then $A(z, w_1, D_{z'})v(z')$ is well-defined in a neighborhood of (z, w_1).*

(ii) *Let Ω_1 and Ω_2 be G'-open subsets such that $\Omega_1-\Omega_2\subset\tilde{D}$ and $u(z)$ a holomorphic function defined on $\Omega_2\cap D$. Then the function $K^\beta_\alpha(K)(u)$ is equal to*

(3.2.2) $$(P)_\alpha(u)\underset{\text{def}}{=}P_0(z, D_z)u(z)+\int_\alpha^{z_1}A(z, w_1, D_{z'})u(w_1, z')dw_1$$

modulo $\mathcal{O}(\Omega_1\cap\tilde{D})$.

Proof. The first assertion is obvious, because

$$A(z, w_1, D_{z'})v(z')=-\int_{\gamma_2\times\cdots\times\gamma_n} K_1(z, w)v(w')dw',$$

where $\gamma_j=\{w_j\in\boldsymbol{C};\ |w_j-z_j|=a_j|z_1-w_1|+\varepsilon\}$ for $0<\varepsilon\ll 1$.

Let us prove the second assertion. Clearly $K_\alpha^\beta(K_2)$ is holomorphic on $\Omega_1\cap\tilde{D}$.

On the other hand, it follows from the definition that

$$(K_\alpha^\beta(K_0-K_2)u)(z)=\int_{\gamma\times\gamma_2\times\cdots\times\gamma_n}(K_0(z, w)+\frac{1}{2\pi i}K_1(z, w)\log(z_1-w_1))u(w)dw,$$

where γ is the path from α to β described in Fig. 3.2.1.

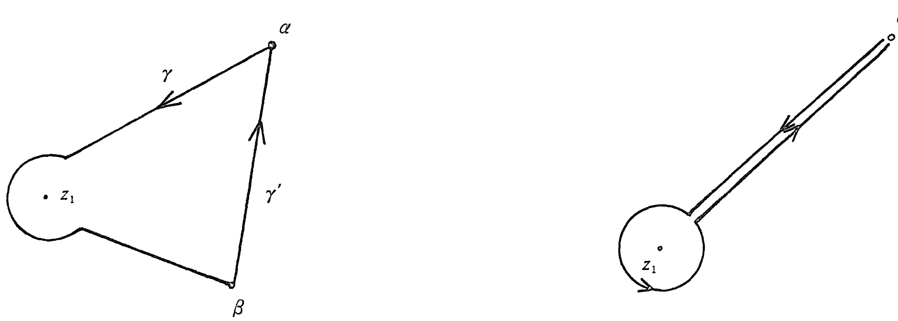

Fig. 3.2.1

Let γ' be the straight path $[\beta, \alpha]$ from β to α. Then we can easily verify

$$\int_{\gamma'\times\gamma_2\times\cdots\times\gamma_n}(K_0(z, w)+\frac{1}{2\pi i}K_1(z, w)\log(z_1-w_1))u(w)dw$$

is holomorphic on $\Omega_1\cap\tilde{D}$. Thus

$$\int_{\gamma\times\gamma_2\times\cdots\times\gamma_n}K_0(z, w)u(w)dw\equiv\int_{(\gamma+\gamma')\times\gamma_2\times\cdots\times\gamma_n}K_0(z, w)u(w)dw$$

modulo $\mathcal{O}(\Omega_1\cap\tilde{D})$ and the last term is equal to $P_0(z, D_z)u(z)$ by Cauchy's integral formula.

On the other hand, we have

$$\frac{1}{2\pi i}\int_{(\gamma+\gamma')\times\gamma_2\times\cdots\times\gamma_n}K_1(z, w)\log(w_2-z_2)u(w)dw$$
$$=-\frac{1}{2\pi i}\int_{\gamma+\gamma'}A(z, w_1, D_{z'})u(w_1, z')\log(w_1-z_1)dw_1.$$

Taking the difference of the branch of $\log(w_1-z_1)$, we find that this equals

$$\int_{\alpha}^{z_1} A(z, w_1, D_{z'})u(w_1, z')dw_1. \qquad \text{Q.E.D.}$$

This proposition can be effectively used to clarify the action of micro-differential operators on various sheaves, as we see below.

Case 1. Let φ be a real valued real analytic (or C^1) function on an open set W of $\boldsymbol{C}^n$ and $S=\varphi^{-1}(0)$. Suppose that $\Omega=\{z\in W; \varphi(z)<0\}$ is pseudo-convex and $d\varphi$ does not vanish on S. Set $\mathscr{C}_{S|W}=\mathscr{H}^1_{W-\Omega}(\mathscr{O}_{C^n})|_S$ and $(T_S^*W)^+=\{(z, k\partial_z\varphi)\in T^*\boldsymbol{C}^n; k>0, z\in S\}$ and let π_+ be the projection from $(T_S^*W)^+$ onto S. Then $\mathscr{E}^\infty|_{(T_S^*W)^+}$ acts on $\pi_+^{-1}\mathscr{C}_{S|W}$. Let p be a point in $(T_S^*W)^+$. By an affine transformation, let us assume that $p=(0, dz_1)$. Let P be a micro-differential operator defined at p. Then we can take $\{a_j\}$ and an open neighborhood D of 0 such that the condition (3.1.1) is satisfied. Defining G as in Section 1.3, and shrinking D so that D is G-round, and taking l, α and β sufficiently near to 0 and c' sufficiently large, we may assume that $\tilde{D}$ is a neighborhood of 0. Hence, by Proposition 3.2.1, we find that P operates on $\mathscr{O}(\Omega\cap\tilde{D})/\mathscr{O}(\Omega)$ by the action $(P)_\alpha$ defined by the formula (3.2.2). Since $\mathscr{C}_{S|W,p}$ is an inductive limit of $\mathscr{O}(U\cap\Omega)/\mathscr{O}(U)$, where U runs over the set of neighborhood of 0, P operates on $\mathscr{C}_{S|W,p}$ by the action $(P)_\alpha$ with $0<-\alpha\ll 1$.

Case 2. Let Y be a non-singular complex hypersurface of a complex manifold X. We defined in S-K-K [24], Chapter II, Section 1.1, the sheaf $\mathscr{C}_{Y|X}^{\boldsymbol{R}}$ on S_Y^*X (or $T_Y^*X-T_X^*X$). This is an $\mathscr{E}^\infty$-Module. Let us choose a coordinate system $z=(z_1,\dots, z_n)$ of X such that Y is given by $z_1=0$. At $q=(0, dz_1)$ the action of a micro-differential operator P is given as follows: Set $\Omega_\delta=\{z\in\boldsymbol{C}^n; \operatorname{Re} z_1<\delta|\operatorname{Im} z_1|\}$. Then $\mathscr{C}_{Y|X,q}^{\boldsymbol{R}}=\varinjlim_{U\ni 0,\delta>0}\mathscr{O}(U\cap\Omega_\delta)/\mathscr{O}(U)$. Thus, in the same way as in Case 1, P acts on $\mathscr{C}_{Y|X,q}^{\boldsymbol{R}}$ as $(P)_\alpha$ with $0<-\alpha\ll 1$.

Case 3. Let M be a real analytic manifold and let X be its complexification. Let $\mathscr{C}_M$ be the sheaf of microfunctions. The sheaf $\mathscr{C}_M$ is defined on T_M^*X and $\mathscr{E}_X^\infty$ acts on $\mathscr{C}_M$. Let us choose a coordinate system $z=(z_1,\dots, z_n)$ such that $M=\{z\in\boldsymbol{C}; z_j\in\boldsymbol{R}, j=1,\dots, n\}$. Consider a point $q=(0, \sqrt{-1}\,dz_1)$. Then $\mathscr{C}_{M,q}$ is an inductive limit of $H^n_{T_\delta}(U; \mathscr{O})$ with $T_\delta=\{z\in\boldsymbol{C}^n, -\delta\operatorname{Im} z_1\geqq\operatorname{Im} z_j$ for $j=2,\dots, n$ and $-\delta\operatorname{Im} z_1\geqq-(\operatorname{Im} z_2+\cdots+\operatorname{Im} z_n)\}$ and U is a holomorphically convex open neighborhood of 0. Then we have $U-T_\delta=\bigcup_{j=1}^n U\cap V_j(\delta)$, where $V_j(\delta)=\{-\delta\operatorname{Im} z_1<\operatorname{Im} z_j\}$, $j=2,\dots, n$, and $V_1(\delta)=\{-\delta\operatorname{Im} z_1<-\operatorname{Im}(z_2+\cdots+z_n)\}$. Hence

$$H^n_{T_\delta}(U;\mathcal{O})=\mathcal{O}(U\cap\bigcap_{j=1}^{n}V_j(\delta))/(\sum_{k=1}^{n}\mathcal{O}(U\cap\bigcap_{j\neq k}V_j(\delta))).$$

Let P be a micro-differential operator defined on a neighborhood of q. Choose corresponding G and D as before. Since the action of P as an element of $\mathfrak{E}(G;D)$ is local with respect to G-topology, P acts on $\mathscr{C}_{M,q}$ by the action of $(P)_\alpha$ on $\mathcal{O}(U\cap\bigcap_{j=1}^{n}V_j(\delta))$ and $\mathcal{O}(U\cap\bigcap_{j\neq k}V_j(\delta))$ $(k=1,\dots,n)$.

§3.

As seen in the preceding sections, operators in $\mathfrak{E}(G;D)$ has a local property with respect to G-topology, or, roughly speaking, operators in $\mathfrak{E}(G;D)$ have influence domain to the direction G.

We shall here introduce the class of operators where G is contained in a complex line.

Let X and Y be complex manifolds, and let F be a smooth holomorphic map from X to Y. Assume the fiber dimension of F is one. Set $n=\dim Y$ and $\dim X=1+n$. The $2(1+n)$-dimensional manifold $X\times X$ contains $X\underset{Y}{\times}X$ and the diagonal set X. If we employ a coordinate system $(t,y)=(t,y_1,\dots,y_n)$ of X and $y=(y_1,\dots,y_n)$ of Y such that F is given by $(t,y)\mapsto y$, then $X\times X=\{(t_1,y,t_2,y')\}$, $X\underset{Y}{\times}X=\{(t_1,y,t_2,y');\ y=y'\}$, and $X=\{(t_1,y,t_2,y');t_1=t_2,y=y'\}$. Note that $X\underset{Y}{\times}X$ is of codimension n in $X\times X$ and that X is a hypersurface of $X\underset{Y}{\times}X$. Let p_1 and p_2 be the first and the second projections from $X\times X$ onto X, respectively. Consider

$$\mathscr{K}=\mathscr{B}^\infty_{X\underset{Y}{\times}X|X\times X}\underset{p_2^{-1}\mathcal{O}_X}{\otimes}p_2^{-1}\Omega_X=\mathscr{H}^n_{X\underset{Y}{\times}X}(\mathcal{O}_{X\times X}\underset{p_2^{-1}\mathcal{O}_X}{\otimes}p_2^{-1}\Omega_X).$$

Here Ω_X is the sheaf of $(1+n)$-forms on X. If we fix coordinate systems

$$(t,y),\ y=(y_1,\dots,y_n)\quad\text{and}\quad(t_1,t_2,y)$$

of X, Y and $X\underset{Y}{\times}X$, respectively, we can identify $\mathscr{K}$ with the sheaf of all linear differential operators (of infinite order) defined on $X\underset{Y}{\times}X$ that contains neither D_{t_1} nor D_{t_2}.

Now we employ the same procedure as that used in constructing the sheaf $\mathscr{C}_{Y|X}$ in S-K-K [24], Chapter II, Section 1.1, replacing the sheaf $\mathcal{O}_X$ used there with the sheaf $\mathscr{K}$ introduced above. Let $S^*_X(X\underset{Y}{\times}X)$ be the conormal sphere bundle, $\widetilde{X\underset{Y}{\times}X}^*$ the comonoidal transform of $X\underset{Y}{\times}X$ with center X and π the projection from $(\widetilde{X \underset{Y}{\times} X})^*$ onto $X \underset{Y}{\times} X$. Set

$$\tilde{\mathscr{E}}^{\infty} = \pi_* \mathscr{H}^1_{S^*_X(X \underset{Y}{\times} X)}(\pi^{-1}\mathscr{K}) .$$

Then, in the same way as in S-K-K [24], Chapter II, Section 1.1, we obtain exact sequence

$$0 \longrightarrow \mathscr{H}^1_X(\mathscr{K}) \longrightarrow \tilde{\mathscr{E}}^{\infty} \longrightarrow \mathscr{K}|_X \longrightarrow 0 .$$

On the other hand,

$$\begin{aligned} \mathscr{H}^1_X(\mathscr{K}) &= \mathscr{H}^1_X \mathscr{H}^n_{X \underset{Y}{\times} X}(\mathcal{O}_{X \times X} \underset{p_2^{-1}\mathcal{O}_X}{\otimes} p_2^{-1}\Omega_X) \\ &= \mathscr{H}^{1+n}_X(\mathcal{O}_{X \times X} \underset{p_2^{-1}\mathcal{O}_X}{\otimes} p_2^{-1}\Omega_X) \\ &= \mathscr{D}^{\infty}_X . \end{aligned}$$

Hence we obtain

$$0 \longrightarrow \mathscr{D}^{\infty}_X \longrightarrow \tilde{\mathscr{E}}^{\infty} \longrightarrow \mathscr{K}|_X \longrightarrow 0 .$$

Next we shall identify $\tilde{\mathscr{E}}^{\infty}$ with a subsheaf of $\mathscr{E}^{\infty}_X$. Let γ be the projection from $T^*X - X \underset{Y}{\times} T^*Y$ onto X. Then $\tilde{\mathscr{E}}^{\infty}$ can be considered as a subsheaf of $\gamma_*(\mathscr{E}^{\infty}_X|_{T^*X - X \underset{Y}{\times} T^*Y})$ as follows:

The section s of $\tilde{\mathscr{E}}^{\infty}$ is given by

$$s = \frac{1}{2\pi i}\left\{P_0(t_1, y, D_{t_1}, D_y)\frac{1}{t_2 - t_1} + K(t_1, t_2, y, D_y)\log(t_2 - t_1)\right\} . \tag{3.3.1}$$

where K belongs to $\mathscr{K}$ and $P \in \mathscr{D}^{\infty}_X$. Therefore, by expanding $K(t_1, t_2, y, D_y)$ into the form $\sum_{j=0}^{\infty} \frac{1}{j!} K_j(t_1, y, D_y)(t_1 - t_2)^j$, we assign the micro-differential operator

$$P_0(t, y, D_t, D_y) - \sum_{j=0}^{\infty} K_j(t, y, D_y) D_t^{-j}$$

to s. In S-K-K [24], Chapter II, Section 1.4, for any micro-differential operator $P(t, y, D_t, D_y) = \sum_{j \in \mathbf{Z}, \alpha \in \mathbf{Z}^n_+} a_{j,\alpha}(t, y) D_t^j D_y^{\alpha}$ defined in a neighborhood of $(0, 0; dt)$, we associated to it the kernel function

$$K(t_1, t_2, y, y') = \frac{1}{(2\pi\sqrt{-1})^{n+1}} \sum a_{j,\alpha}(t, y) \Phi_{j+1}(t_1 - t_2) \Phi_{\alpha+\delta}(y - y') , \tag{3.3.2}$$

where $\delta = (\overbrace{1, \dots, 1}^{n})$ and $\Phi_{\alpha}(z) = \Phi_{\alpha_1}(z_1) \cdots \Phi_{\alpha_n}(z_n)$.

Then P gives a section of $\mathscr{E}^{\infty}$ on $\{t, y; \tau, \zeta); t = y = 0, |\tau| > \sum_{j=1}^{n} a_j |\zeta_j|\}$, if $K(t_1, t_2, y, y')$ converges for $0<|y_j-y'_j|<a_j|t_1-t_2|$ $(j=1,\dots, n)$.

We set

$$P_0(t, y, D_t, D_y)=\sum_{(j,\alpha)\in Z_+^{n+1}} a_{j,\alpha}(t, y)D_t^j D_y^\alpha \tag{3.3.3}$$

and

$$K(t_1, t_2, y, D_y)=\sum_{j<0,\alpha\in Z_+^n} \frac{1}{(-j-1)!} a_{j,\alpha}(t, y)(t_1-t_2)^{-j-1}D_t^j D_y^\alpha \tag{3.3.4}$$

Then P is a section of $\tilde{\mathscr{E}}^\infty$ if and only if $K(t_1, t_2, y, y')$ defined by (3.3.4) converges for $t_1\neq t_2$. This is equivalent to saying that $K(t_1, t_2, y, y')$ given by (3.3.2) converges for $y_j\neq y'_j$ and $t_1\neq t_2$. Hence, by this correspondence, $\tilde{\mathscr{E}}^\infty$ can be identified with a subsheaf of $\gamma_*(\mathscr{E}_X^\infty|_{T^*X-X\times_Y T^*Y})$. Therefore it also implies that, for any $G_\theta=\{(t, x)\in \mathbf{C}^{1+n};\ x=0,\ t\in e^{2\pi i\theta}\overline{\mathbf{R}^+}\}$, an element of $\tilde{\mathscr{E}}^\infty$ defines an element of $\mathfrak{E}(G_\theta, D)$ for some D.

The advantage in introducing the sheaf $\tilde{\mathscr{E}}^\infty$ lies in the fact that the action of $\tilde{\mathscr{E}}^\infty$ on holomorphic functions is local with respect to y-coordinates (but not local with respect to t-coordinate).

This enables us to define the action of operators in $\tilde{\mathscr{E}}^\infty$ on a special kind of multi-valued function. (See § 4.)

We shall call the pair $(P_0(t, y, D_t, D_y), K(t_1, t_2, y, D_y))$ given by (3.3.1) the kernel of $s\in\tilde{\mathscr{E}}^\infty$. We denote by $\tilde{\mathscr{E}}$ the subsheaf $\tilde{\mathscr{E}}^\infty\cap\gamma_*(\mathscr{E}_X|_{(T^*X-X)\times_Y T^*Y})$ of $\tilde{\mathscr{E}}^\infty$.

§4.

In this section we first review some basic notions concerning multi-valued holomorphic function after Séminaire Cartan-Serre 1951/52 (Seminaire sur les fonctions de plusieurs variables). Then we prepare some results needed in Chapter IV.

4.1. A pair (X', ι) is called a manifold étalé over X if X' is a complex manifold and ι is a local isomorphism from X' to X. A manifold (X', ι) étalé over X is called a covering space of X if for any point x in X, there is a neighborhood U of x such that $X'\cap\iota^{-1}(U)$ is isomorphic to a disjoint sum of the copies of U. Hereafter we assume that X is a connected complex manifold.

Let F be a connected subset of X and φ a germ of a holomorphic function on F (i.e., φ is a holomorphic function defined on a neighborhood of F). We say that φ is continued to a multi-valued holomorphic function defined on X if there are a holomorphic function $\tilde{\varphi}$ on the universal covering $\iota_0: \tilde{X} \to X$ and a connected set $\tilde{F}$ in $\tilde{X}$ such that $\iota_0(\tilde{F}) = F$ and the germ of $\tilde{\varphi}$ at $\tilde{F}$ equals the pull back of φ.

For the germ of any holomorphic function φ, there is a maximal continuation $((X', \iota), F', \varphi')$ in the following sense: there is a "largest" connected manifold (X', ι) étalé over X, a closed subset F' of X' and a holomorphic function φ' on X' such that F' is homeomorphic to F by ι and that the germ of φ' at F' equals φ. This means: if (X'', ι'), F'' and φ'' satisfy the same conditions as (X', ι), F' and φ', then there exists a unique morphism ι'' from X'' to X' such that $\iota' = \iota \circ \iota''$, $\iota''(F'') = F'$ and $\varphi'' = \varphi' \circ \iota''$. Hence φ is continued to a multi-valued holomorphic function on X if the maximal continuation in this sense is a covering space over X.

Proposition 3.4.1. *Let Z_1 and Z_2 be two disjoint closed sets in X. Suppose that $X - Z_1$, $X - Z_2$ and $X - (Z_1 \cup Z_2)$ are connected and that $\pi_1(X - (Z_1 \cup Z_2)) \to \pi_1(X - Z_j)$ is surjective $(j = 1, 2)$. If a germ of holomorphic function φ at a point $x_0 \in X - (Z_1 \cup Z_2)$ is continued to a multi-valued holomorphic function on $X - Z_1$ and to a multi-valued holomorphic function on $X - Z_2$, then φ is continued to a multi-valued holomorphic function on X.*

Proof. Let $((X', \iota), x'_0, \varphi')$ be the maximal continuation of φ (in the sense defined above). Let X'_j be the connected component of $X' \cap \iota^{-1}(X - Z_j)$ containing x'_0 $(j = 1, 2)$. Then it follows from the assumption that X'_j is a covering space of $X - Z_j$. By the condition on the fundamental groups, $X'_j \cap \iota^{-1}(X - (Z_1 \cup Z_2))$ is connected.

Since $X'_1 \cap X'_2$ is a covering space of $X - (Z_1 \cup Z_2)$, we have

$$X'_j \cap \iota^{-1}(X - (Z_1 \cup Z_2)) = X'_1 \cap X'_2 \qquad (j = 1, 2).$$

Thus we obtain

$$X'_1 \cap \iota^{-1}(X - (Z_1 \cup Z_2)) = X'_2 \cap \iota^{-1}(X - (Z_1 \cup Z_2)).$$

This implies that $X'_1 \cup X'_2$ is a covering space of X. Hence φ is continued to a multi-valued holomorphic function on X. Q.E.D.

Proposition 3.4.2. *Let Y_1 and Y_2 be closed analytic susbets of X such that the codimension of $Y_1 \cap Y_2$ is strictly greater than 1. Let φ be a germ of a holomorphic function at $x_0 \in X - (Y_1 \cup Y_2)$. If φ is continued to a multi-valued function on $X - Y_j$ for $j = 1, 2$, then φ is continued to a multi-valued holomorphic function on X.*

Proof. By the preceding proposition, φ is continued to a multi-valued holomorphic function on $X-Y$, where $Y=Y_1\cap Y_2$. Since $\pi_1(X-Y)=\pi_1(X)$, if $\tilde{X}\xrightarrow{\iota_0}X$ is the universal covering of X, then $\iota_0^{-1}(X-Y)$ is a universal covering of $X-Y$. Any holomorphic function on $\tilde{X}-\iota_0^{-1}(Y)$ is continued to a holomorphic function on $\tilde{X}$, because codim $\iota_0^{-1}(Y)\geqq 2$. This implies the desired result. Q.E.D.

Let X be a connected complex manifold, π the fundamental group of X and $\tilde{X}$ the universal covering of X. Then π acts on $\tilde{X}$ and $X=\tilde{X}/\pi$. A holomorphic function on $\tilde{X}$ is called multi-valued holomorphic function on X.

Since the fundamental group π of X acts on the space $\mathcal{O}(\tilde{X})$ of holomorphic functions on the universal covering $\tilde{X}$ of X, $\mathcal{O}(\tilde{X})$ is a module over the group ring $\boldsymbol{C}[\pi]$ of π. Let $\mathfrak{a}$ be a left ideal of $\boldsymbol{C}[\pi]$. We say that a holomorphic function f on $\tilde{X}$ has monodromy type $\mathfrak{a}$ if $\mathfrak{a}f=0$. We say that f has finite determination if f is of monodromy type $\mathfrak{a}$ for a left ideal $\mathfrak{a}$ such that $\dim_{\boldsymbol{C}}(\boldsymbol{C}[\pi]/\mathfrak{a})<\infty$.

4.2. Let X, Y and F be as in Section 2. That is, X is a complex manifold of dimension $(1+n)$, Y a complex manifold of dimension n and F a smooth holomorphic map from X to Y.

Assume furthermore the following conditions:

(3.4.1) F is topologically trivial locally on Y, and Y is connected.

(3.4.2) F admits a section, i.e., there is a submanifold Σ of X such that Σ is isomorphic to Y through F.

(3.4.3) All fibers of F are connected.

Let $\tilde{X}$ and $\tilde{Y}$ be a universal covering of X and Y, respectively. By Hurewicz-Steenrod isomorphism, we have an exact sequence.

$$\pi_2(X)\longrightarrow\pi_2(Y)\longrightarrow\pi_1(F^{-1}(y))\longrightarrow\pi_1(X)\longrightarrow\pi_1(Y)$$

for $y\in Y$. Since F admits a section Σ, $\pi_2(X)\to\pi_2(Y)$ and $\pi_1(X)\to\pi_1(Y)$ are surjective and hence the sequence

$$1\longrightarrow\pi_1(F^{-1}(y))\longrightarrow\pi_1(X)\longrightarrow\pi_1(Y)\longrightarrow 1$$

is exact. Set $X'=\tilde{X}/\pi_1(\Sigma)$ and let ι' be the natural projection from X' onto X. The map $\iota'^{-1}(\Sigma)\to\Sigma$ admits a section, which we shall denote by the same letter Σ. By applying the same argument to $F'\underset{\text{def}}{=}F\cdot\iota'\colon X'\to Y$, we have an exact sequence

$$1 \longrightarrow \pi_1(F'^{-1}(y)) \longrightarrow \pi_1(X') \longrightarrow \pi_1(Y) \longrightarrow 1.$$

On the other hand, $\pi_1(X')=\pi_1(\Sigma)=\pi_1(Y)$ and hence $\pi_1(F'^{-1}(y))=1$. This means that any fiber of F' is simply connected. Also, one can easily show that $\tilde{X} \cong X' \underset{Y}{\times} \tilde{Y}$.

We shall assume further

(3.4.4) F decomposes into $X \xrightarrow{i} \boldsymbol{C} \times Y \xrightarrow{p} Y$, where p is the projection from $\boldsymbol{C} \times Y$ onto Y and i is a local isomorphism.

Hence there exists a canonical map from $X' \underset{Y}{\times} X'$ into $\boldsymbol{C} \times \boldsymbol{C}$ so that $X' \underset{Y}{\times} X'$ is étalé over $\boldsymbol{C} \times \boldsymbol{C} \times Y$. Let η denote the projection from $X' \underset{Y}{\times} X'$ onto Y and $\mathscr{R}$ the sheaf $\eta_* \mathscr{D}^{\infty}_{X' \underset{Y}{\times} X' \to Y}$. Then $\mathscr{R}$ has a structure of sheaf of rings (without the unit) in the following way: Let us denote by $(t_1, t_2, y) \in \boldsymbol{C} \times \boldsymbol{C} \times Y$ a local coordinate system of $\boldsymbol{C} \times \boldsymbol{C} \times Y$ or $X' \underset{Y}{\times} X'$. For $P=P(t_1, t_2, y, D_y)$ and $Q=Q(t_1, t_2, y, D_y)$, the product $R(t_1, t_2, y, D_y)$ of P and Q is

$$R(t_1, t_2, y, D_y)=\int_{t_1}^{t_2} dt_3 P(t_1, t_3, y, D_y) Q(t_3, t_2, y, D_y),$$

Then this is well-defined because any fiber of $X' \to Y$ is simply connected. The Ring $\mathscr{R}$ operates on $F'_* \mathscr{O}_{X'}$ by

$$\mathscr{R} \ni P(t_1, t_2, y, D_y): \varphi(t, y) \longmapsto \psi(t, y),$$

where

$$\psi(t, y)=\int_{\Sigma \cap F'^{-1}(y)}^{t} P(t, s, y, D_y)\varphi(s, y) ds.$$

This integral is well-defined, but this action does depend on the choice of Σ. We shall denote by P_Σ this action of $P \in \mathscr{R}$.

Let Σ' be another section of F'. Then we have

$$P_\Sigma \varphi(t, y) - P_{\Sigma'} \varphi(t, y)=\int_{\Sigma \cap F'^{-1}(y)}^{\Sigma' \cap F'^{-1}(y)} P(t, s, y, D_y)\varphi(s, y) ds$$

and hence this is a function defined on Y. Therefore $P_\Sigma \varphi$ does not depend on Σ, as a section of $F'_* \mathscr{O}_{X'}/\mathscr{O}_Y$.

4.3. Let F be the projection from $\boldsymbol{C}^{1+n}$ onto $\boldsymbol{C}^n$ defined by $(t, x)=(t, x_1,\dots, x_n) \mapsto x=(x_1,\dots, x_n)$. Let S be a hypersurface of $\boldsymbol{C}^{1+n}$ containing the origin. Suppose that the origin is an isolated point of $S \cap F^{-1}(0)$. Let $B(\varepsilon, \delta)$ and $B(\varepsilon)$ denote the open sets $\{(t, x) \in \boldsymbol{C}^{1+n}; |t|<\delta, |x|<\varepsilon\}$ and $\{x \in \boldsymbol{C}^n; |x|<\varepsilon\}$, respectively. Then there are positive ε_0 and δ_0 such that the following hold:

(3.4.5) $S \cap B(\varepsilon_0, \delta_0)$ is a closed hypersurface of $B(\varepsilon_0, \delta_0)$,

(3.4.6) $\bar{S} \cap \{(t, x); |t| = \delta_0, |x| \leqq \varepsilon_0\} = \emptyset$

(3.4.7) $S \cap F^{-1}(0) \cap B(\varepsilon_0, \delta_0) = \{0\}$.

For $\delta > 0$, let $\varepsilon(\delta)$ be $\inf\{|x|$; there exists t such that (t, x) belongs to $S \cap B(\varepsilon_0, \delta_0) - B(\varepsilon_0, \delta)\}$. Then $\lim_{\delta \downarrow 0} \varepsilon(\delta) = 0$. By (3.4.6), $S \cap B(\varepsilon_0, \delta_0) \xrightarrow{F} B(\varepsilon_0)$ is a finite map. Let H be the subset of $B(\varepsilon_0)$ consisting of the points over which $S \cap B(\varepsilon_0, \delta_0) \to B(\varepsilon_0)$ is not a local isomorphism. Then H is a closed analytic subset of $B(\varepsilon_0)$ with codimension greater than zero. We shall set $X = B(\varepsilon_0, \delta_0)$, $Y = B(\delta_0)$ and $S_0 = (S \cup F^{-1}(H)) \cap X$.

By replacing ε_0 with a smaller one, we can assume further

(3.4.8) $B(\varepsilon) - H \hookrightarrow B(\varepsilon_0) - H$, $B(\varepsilon, \delta_0) - S \hookrightarrow B(\varepsilon_0, \delta_0) - S$, $B(\varepsilon, \delta_0) - S_0 \hookrightarrow B(\varepsilon_0, \delta_0) - S_0$ are isomorphisms up to homotopy.

Let $\widetilde{X - S_0}$ and $\widetilde{Y - H}$ be a universal covering space of $X - S_0$ and $Y - H$, respectively. Let π (resp., π_0) denote the fundamental group of $X - S_0$ (resp., $Y - H$). By using the section $\{(t, x); t = \lambda, x \in Y - H\}$ of F, where $\lambda \in \mathbb{C}$ such that $0 < \delta_0 - |\lambda| \ll 1$, π_0 can be regarded as a subgroup of π. Let π_1 be the fundamental group of a fiber of $X - S_0 \to Y - H$. Then π_1 is a normal subgroup of π and π is a semi-direct product of π_1 and π_0.

Set $X' = (\widetilde{X - S_0})/\pi_0$ and we define ι, ι', α, α_0, α_1, α_2 and $\tilde{F}$ as in Fig. 3.4.1. The map $\tilde{F}$ commutes with the action of π.

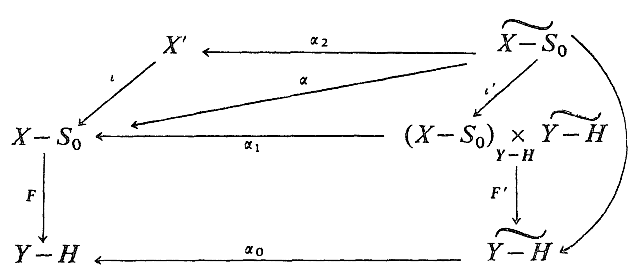

Fig. 3.4.1

We denote by X'' the fiber product $X \underset{Y}{\times} \widetilde{Y - H}$ and by β the projection from X'' onto $X - F^{-1}(H)$ (hence α is the restriction of β). Set

(3.4.9) $$F = \varinjlim_{U} \mathcal{O}_{\widetilde{X - S_0}}(\alpha^{-1}(U - S_0))$$

and

(3.4.10)
$$G=\varinjlim_{U} \mathcal{O}_{X''}(\beta^{-1}(U-F^{-1}(H))),$$

where U ranges over a fundamental neighborhood system of the origin of C^{1+n}. By definition, G is identified with a subspace of F. Clearly π acts on F, G and F/G.

Lemma 3.4.3. *Let $u(t, x)$ be a holomorphic function on $\widetilde{X-S_0}$ and x_0 a point of $\widetilde{Y-H}$. Let γ be a closed cycle in $\tilde{F}^{-1}(x_0)$. Then we have $\int_{\gamma} u(t, x_0)dt=0$.*

Proof. Since the fiber of $\tilde{F}$ is simply connected, this lemma is obvious by Cauchy's integral formula. Q.E.D.

Let φ be a holomorphic function defined on $\alpha^{-1}(B(\varepsilon, \delta)-S_0)$ for ε and δ such that $0<\delta\leqq\delta_0$ and $0<\varepsilon<\varepsilon_0$, $\varepsilon(\delta)$. Let $P(t, x, D_t, D_x)$ be an element of $\mathscr{D}_X^{\infty}(B(\varepsilon, \delta))$ and let $K(t_1, t_2, x, D_x)$ be an element of $\mathscr{D}^{\infty}_{C\times X\to Y}(\{(t_1, t_2, x); |t_1|, |t_2|<\varepsilon, |x|<\delta\})$. Then $\mu=(P, K)$ determines an element of $\tilde{\mathscr{E}}_0^{\infty}$.

We choose a section Σ of

$$\iota^{-1}(B(\varepsilon, \delta)-S_0) \longrightarrow B(\varepsilon)-H.$$

We define the action of μ on φ by

(3.4.11)
$$\mu_{\Sigma}\varphi(p)=P(t, x, D_t, D_x)\varphi(p)+\int_{\alpha_2^{-1}(\Sigma)\cap\tilde{F}^{-1}(\tilde{F}(p))}^{p} K(t, s, x, D_x)\varphi(q)ds,$$

where $\alpha(q)=(s, x)$ and the integral is calculated along the path in $\tilde{F}^{-1}(\tilde{F}(p))$ which starts from $\alpha_2^{-1}(\Sigma)\cap\tilde{F}^{-1}(\tilde{F}(p))$ and ends at p. The preceding lemma guarantees the right-hand side of (3.4.11) is well-defined on $\alpha^{-1}(B(\varepsilon, \delta)-S_0)$.

By defining the action of $\mu\in\tilde{\mathscr{E}}_0^{\infty}$ in this way, we obtain the following

Proposition 3.4.4. *F/G is an $\tilde{\mathscr{E}}_0^{\infty}$-module.*

Proof. It is easy to see that the definition (3.4.11) does not depend on the choice of Σ modulo G, and that, for $\varphi\in\mathcal{O}_{X''}(\beta^{-1}(B(\varepsilon, \delta)-F^{-1}(H)))$, $\mu_{\Sigma}\varphi$ belongs to $\mathcal{O}_{X''}(\beta^{-1}(B(\varepsilon, \delta)-F^{-1}(H)))$. (See the preceding subsection 4.2.)

Set $G_0=\{(t, x); x=0, t\geqq 0\}$. Then $\tilde{\mathscr{E}}_0^{\infty}$ is a subring of $\mathfrak{E}\underset{\text{def}}{=}\varinjlim_{D}\mathfrak{E}(G_0; D)$, where D ranges over the set of G_0-round neighborhoods of the origin.

Now, F/G is identified with a subspace of

$$\begin{aligned}K&=\varinjlim_{U}\mathcal{O}_{X''}(\beta^{-1}(U-(S_0+G_0^a)))/\mathcal{O}_{X''}(\beta^{-1}(U-F^{-1}(H)))\\&=\varinjlim_{V,U}\Gamma(\alpha_0^{-1}(V-H); \alpha_0^{-1}\mathscr{H}_U),\end{aligned}$$

where U (resp., V) ranges over a neighborhood system of the origin of $\mathbf{C}^{1+n}$ (resp., $\mathbf{C}^n$), and $\mathscr{H}_U$ is the sheaf on $Y-H$ associated with the presheaf

$$W \longmapsto H^1_{U\cap F^{-1}(W)\cap(S_0+G^a)}(U\cap F^{-1}(W);\ \mathcal{O}_X)\,.$$

Since we can take a neighborhood system of the origin of $\mathbf{C}^{1+n}$ formed by U such that $U\cap(S_0+G_0^a)$ is a locally closed open susset of D with respect to the G-topology, $\mathfrak{E}(G_0;\ D)$ acts on $\mathscr{H}_U$ and hence on K. Thus K is an $\mathfrak{E}$-module, in particular, K is an $\tilde{\mathscr{E}}_0^\infty$-module. By Propositions 3.1.5 and 3.2.1, the action of $\tilde{\mathscr{E}}_0^\infty$ on K coincides with (3.4.11). Thus we obtain the desired result. Q.E.D.

Remark that the action of $\tilde{\mathscr{E}}_0^\infty$ on F/G commutes with that of π.

Let $\mathfrak{a}$ be an ideal of $\mathbf{C}[\pi]$. Set

(3.4.12) $$F(\mathfrak{a})=\{\varphi\in F;\ \sigma(\varphi)=0 \quad \text{for any} \quad \sigma\in\mathfrak{a}\}$$
and
$$G(\mathfrak{a})=G\cap F(\mathfrak{a})\,.$$

Suppose further that $\dim_{\mathbf{C}}\mathbf{C}[\pi]/\mathfrak{a}<\infty$. Let $L(\mathfrak{a})$ be the locally constant $\mathbf{C}_{X-S_0}$-module defined by $(\mathbf{C}[\pi]/\mathfrak{a})^*$; namely, for any open set U of $X-S_0$, $\Gamma(U;\ L(\mathfrak{a}))=\{\psi;\ \psi$ is a $\mathbf{C}$-linear map from $\mathbf{C}[\pi]/\mathfrak{a}$ into the space of locally constant function on $\alpha^{-1}(U)$ such that $\psi(\gamma\sigma)(x)=\psi(\sigma)(\gamma^{-1}x)$ for $\sigma\in\mathbf{C}[\pi]/\mathfrak{a}$, $\gamma\in\pi$ and $x\in\alpha^{-1}(U)\}$. Let $\mathscr{L}(\mathfrak{a})$ be the associated holonomic $\mathscr{D}_X$-Module of D-type with singularities along S_0. Hence, for any $U\subset X$, $\Gamma(U;\ \mathscr{L}(\mathfrak{a})^\infty)$ is the space of holomorphic functions φ defined on $\alpha^{-1}(U-S_0)$ such that $\sigma(\varphi)=0$ for $\sigma\in\mathfrak{a}$. Therefore we have

(3.4.13) $$F(\mathfrak{a})=\mathscr{L}(\mathfrak{a})_0^\infty\,.$$

We call $\mathscr{L}(\mathfrak{a})$ the holonomic $\mathscr{D}$-Module of D-type with singularities along S_0 and with the monodromy type $\mathfrak{a}$.

Let $\mathfrak{a}'$ be the image of $\mathfrak{a}$ by the map $\mathbf{C}[\pi]\to\mathbf{C}[\pi_1(X-F^{-1}(H))]=\mathbf{C}[\pi_1]$ and let $\mathscr{G}(\mathfrak{a})$ be the holonomic $\mathscr{D}_X$-Module of D-type with singularities along $F^{-1}(H)$ and with the monodromy type $\mathfrak{a}'$. Then we have $\mathscr{G}(\mathfrak{a})\subset\mathscr{F}(\mathfrak{a})$ and

(3.4.14) $$G(\mathfrak{a})=\mathscr{G}(\mathfrak{a})_0^\infty\,.$$

Now, we shall show that $\tilde{\mathscr{E}}_0^\infty(F(\mathfrak{a})/G(\mathfrak{a}))$ is contained in $F(\mathfrak{b})/G(\mathfrak{b})$ for some $\mathfrak{b}$ such that $\dim\mathbf{C}[\pi]/\mathfrak{b}<\infty$.

In order to show this, we shall prepare several lemmas.

Lemma 3.4.5. *Let π be a group generated by finite elements and $\mathfrak{a}$ a* *left ideal of* $\boldsymbol{C}[\pi]$. *If* $\boldsymbol{C}[\pi]/\mathfrak{a}$ *is finite-dimensional, then* $\mathfrak{a}$ *is a finitely generated ideal of* $\boldsymbol{C}[\pi]$.

Proof. Suppose that π is generated by $\sigma_0=1, \sigma_1,\dots, \sigma_N$ as a semi-group. Let A_j ($j=0, 1,\dots$) be the subset of $\boldsymbol{C}[\pi]$ defined by $A_0=\{1\}$, $A_j=\sum_{i=0}^{N}\sigma_i A_{j-1}$ ($j\geqq 1$). Then $\boldsymbol{C}[\pi]=\bigcup_j A_j$. Since $\boldsymbol{C}[\pi]/\mathfrak{a}$ is finite-dimensional by the assumption, there is an integer m such that $\boldsymbol{C}[\pi]=\mathfrak{a}+A_m$. Let $\gamma_1,\dots, \gamma_r$ be a base of the $\boldsymbol{C}$-vector space A_{m+1}. Then there is γ'_ν in A_m such that $\gamma_\nu-\gamma'_\nu$ is contained in $\mathfrak{a}$ ($\nu=1,\dots, r$). We shall show that

$$\mathfrak{a}=(\mathfrak{a}\cap A_m)+\sum_{\nu=1}^{r}\boldsymbol{C}[\pi](\gamma_\nu-\gamma'_\nu).$$

Let $\mathfrak{a}'$ be the right-hand side. We shall show that $\mathfrak{a}\cap A_k\subset\mathfrak{a}'$ by the induction on k. If $k\leqq m$, this is evident. Suppose $k>m$. Let s be an element of $\mathfrak{a}\cap A_k$. Then we can write $s=\sum_{\nu=1}^{r}s_\nu\gamma_\nu$ with $s_\nu\in A_{k-m-1}$. Hence

$$s=\sum_{\nu=1}^{r}s_\nu\gamma'_\nu+\sum_{\nu=1}^{r}s_\nu(\gamma_\nu-\gamma'_\nu)\equiv\sum s_\nu\gamma'_\nu \bmod \mathfrak{a}'.$$

Since $\sum s_\nu\gamma'_\nu$ is contained in $A_{k-1}\cap\mathfrak{a}$, this is contained in $\mathfrak{a}'$ by the hypothesis of induction. Q. E. D.

Lemma 3.4.6. *Let* π *be a group generated by finite elements and let* $\mathfrak{a}_1$ *and* $\mathfrak{a}_2$ *be two left ideals such that* $\boldsymbol{C}[\pi]/\mathfrak{a}_\nu$ *is finite-dimensional* ($\nu=1, 2$). *Then* $\boldsymbol{C}[\pi]/\mathfrak{a}_1\mathfrak{a}_2$ *is also finite-dimensional.*

Proof. By the preceding lemma, $\mathfrak{a}_2$ is finitely generated. Let $\gamma_1,\dots, \gamma_N$ be a system of generators of $\mathfrak{a}_2$. We have an exact sequence

$$0\longrightarrow\mathfrak{a}_2/\mathfrak{a}_1\mathfrak{a}_2\longrightarrow\boldsymbol{C}[\pi]/\mathfrak{a}_1\mathfrak{a}_2\longrightarrow\boldsymbol{C}[\pi]/\mathfrak{a}_2\longrightarrow 0.$$

Since $\mathfrak{a}_2/\mathfrak{a}_1\mathfrak{a}_2$ is a quotient of $(\boldsymbol{C}[\pi]/\mathfrak{a}_1)^N$ by the homomorphism $(s_1,\dots, s_N)\mapsto\sum_{j=1}^{N}s_j\gamma_j$, $\mathfrak{a}_2/\mathfrak{a}_1\mathfrak{a}_2$ is finite-dimensional. Since $\boldsymbol{C}[\pi]/\mathfrak{a}_2$ is finite-dimensional, $\boldsymbol{C}[\pi]/\mathfrak{a}_1\mathfrak{a}_2$ is also finite-dimensional. Q. E. D.

Lemma 3.4.7. *For any left ideal* $\mathfrak{a}$ *of* $\boldsymbol{C}[\pi]$ *such that* $\boldsymbol{C}[\pi]/\mathfrak{a}$ *is a finite-dimensional* $\boldsymbol{C}$*-vector space, there is a left ideal* $\mathfrak{b}$ *of* $\boldsymbol{C}[\pi_0]$ *satisfying the following properties*:

(3.4.15) $$\dim_{\boldsymbol{C}}\boldsymbol{C}[\pi_0]/\mathfrak{b}<\infty.$$

(3.4.16) *Let* ε *and* δ *be positive numbers satisfying* $0<\delta\leqq\delta_0$ *and* $0<\varepsilon<\varepsilon_0$, $\varepsilon(\delta)$, *and let* Σ *be a section of* $\iota^{-1}(B(\varepsilon, \delta)-S_0)\to B(\varepsilon)-H$. *Then, for any* *multi-valued holomorphic function* u *on* $B(\varepsilon, \delta)-S_0$ *with the monodromy type* $\mathfrak{a}$ *and for* $\gamma_1, \gamma_2 \in \pi$,

$$v(q)=\int_{\gamma_1\alpha_2^{-1}(\Sigma)\cap\tilde{F}^{-1}(q)}^{\gamma_2\alpha_2^{-1}(\Sigma)\cap\tilde{F}^{-1}(q)} u\,dt \qquad (q\in\widetilde{B(\varepsilon)-H})$$

is a multi-valued holomorphic function on $B(\varepsilon)-H$ *with the monodromy type* $\mathfrak{b}$.

Proof. Let $\mathbf{C}[\pi/\pi_0]$ be the vector space generated by π/π_0. We denote by V the $\mathbf{C}$-vector space $\mathbf{C}[\pi/\pi_0]\underset{\mathbf{C}}{\otimes}\mathbf{C}[\pi/\pi_0]\underset{\mathbf{C}}{\otimes}(\mathbf{C}[\pi]/\mathfrak{a})$, and we endow V with a structure of π-module by

$$\gamma(\gamma_1\otimes\gamma_2\otimes w)=\gamma\gamma_1\otimes\gamma\gamma_2\otimes\gamma w$$

for $\gamma_1, \gamma_2\in\pi/\pi_0$, $\gamma\in\pi$ and $w\in\mathbf{C}[\pi]/\mathfrak{a}$. We denote by W the $\mathbf{C}$-vector subspace of V generated by $\gamma_1\otimes\gamma_3\otimes w-\gamma_1\otimes\gamma_2\otimes w-\gamma_2\otimes\gamma_3\otimes w$, $\gamma_1\otimes\gamma_2\otimes w+\gamma_2\otimes\gamma_1\otimes w$ and $\sigma(\gamma_1\otimes\gamma_2\otimes w)-\gamma_1\otimes\gamma_2\otimes w$ for $\gamma_1, \gamma_2, \gamma_3\in\pi/\pi_0$, $w\in\mathbf{C}[\pi]/\mathfrak{a}$ and $\sigma\in\pi_1$. The vector space W becomes a π-submodule of V.

We now claim that V/W is finite-dimensional. In fact, since π is a semi-direct product of π_1 and π_0, V/W is generated by $\gamma_1\otimes\gamma_2\otimes w$ ($\gamma_1, \gamma_2\in\pi_1$, $w\in\mathbf{C}[\pi]/\mathfrak{a}$) as a vector space. On the other hand, we have

$$\gamma_1\otimes\gamma_2\otimes w\equiv\gamma_1\otimes 1\otimes w-\gamma_2\otimes 1\otimes w \mod W$$

and

$$\gamma_1\gamma_2\otimes 1\otimes w=\gamma_2\otimes 1\otimes\gamma_1^{-1}w+\gamma_1\otimes 1\otimes w \mod W.$$

Hence, if $\{\gamma_1,\dots,\gamma_N\}$ is a system of generators of π_1, then $\gamma_j\otimes 1\otimes(\mathbf{C}[\pi]/\mathfrak{a})$ generate V/W.

Moreover π_1 acts trivially on V/W. We shall define a homomorphism Φ: $V/W\to\mathcal{O}(\widetilde{B(\varepsilon)-H})$ by

$$\gamma_1\otimes\gamma_2\otimes\sigma\longmapsto v(q)=\int_{\gamma_2\alpha_2^{-1}(\Sigma)\cap\tilde{F}^{-1}(q)}^{\gamma_1\alpha_2^{-1}(\Sigma)\cap\tilde{F}^{-1}(q)}(\sigma u)dt \quad (q\in\widetilde{B(\varepsilon)-H})$$

for $\gamma_1, \gamma_2\in\pi/\pi_0$, $\sigma\in\pi$.

Then it is easy to see that Φ is a well-defined π_0-linear homomorphism. Then $\mathfrak{b}=\{\sigma\in\mathbf{C}[\pi_0];\ \sigma(V/W)=0\}$ satisfies the required property. Q.E.D.

Lemma 3.4.8. *Let* $\mathfrak{a}$ *be a left ideal of* $\mathbf{C}[\pi]$ *such that* $\dim\mathbf{C}[\pi]/\mathfrak{a}<\infty$. *Then there is a left ideal* $\mathfrak{a}'$ *of* $\mathbf{C}[\pi]$ *such that* $\dim\mathbf{C}[\pi]/\mathfrak{a}'<\infty$ *and that* $F(\mathfrak{a})\subset D_tF(\mathfrak{a}')$.

Proof. Let $u(t, x)$ be a multi-valued holomorphic function defined on $B(\varepsilon, \delta)-S$ for some ε, δ such that $0<\delta\leqq\delta_0$, $0<\varepsilon<\varepsilon_0, \varepsilon(\delta)$. Suppose that u is with the monodromy type $\mathfrak{a}$. We choose a section Σ of $\iota^{-1}(B(\varepsilon, \delta)-S_0)\to B(\varepsilon)-H$, and define a multi-valued holomorphic function $v(p)$ defined on $B(\varepsilon, \delta)-S_0$ by

$$v(p)=\int_{\alpha_2^{-1}(\Sigma)\cap\tilde{F}^{-1}(\tilde{F}(p))}^{p} u\,dt \qquad (p\in\alpha^{-1}(B(\varepsilon, \delta)-S_0))$$

Then, for $\gamma\in\pi$, we have

$$(\gamma v)(p)=\int_{\alpha_2^{-1}(\Sigma)\cap\tilde{F}^{-1}\tilde{F}(\gamma^{-1}p)}^{\gamma^{-1}p} u\,dt=\int_{\gamma\alpha_2^{-1}(\Sigma)\cap\tilde{F}^{-1}(\tilde{F}(p))}^{p}(\gamma u)\,dt$$
$$=\int_{\alpha_2^{-1}(\Sigma)\cap\tilde{F}^{-1}(\tilde{F}(p))}^{p}(\gamma u)\,dt-\int_{\alpha_2^{-1}(\Sigma)\cap\tilde{F}^{-1}(\tilde{F}(p))}^{\gamma\alpha_2^{-1}(\Sigma)\cap\tilde{F}^{-1}(\tilde{F}(p))}(\gamma u)\,dt\,.$$

Hence, if $\mathfrak{b}$ is a left ideal of $\boldsymbol{C}[\pi_0]$ satisfying the conditions in the preceding lemma and if $\mathfrak{a}'$ denote the inverse image of $\mathfrak{b}$ by the map $\boldsymbol{C}[\pi]\to\boldsymbol{C}[\pi_0]$, then $(\gamma v)(p)$ equals $\int_{\alpha_2^{-1}(\Sigma)\cap\tilde{F}^{-1}(\tilde{F}(p))}^{p}\gamma u\,dt$ modulo $G(\mathfrak{a}')$. Therefore we have $\sigma(v)\in G(\mathfrak{a}')$ for $\sigma\in\mathfrak{a}$. Hence we obtain $v\in F(\mathfrak{a}'\mathfrak{a})$. Thus $\mathfrak{a}'\mathfrak{a}$ satisfies the required condition.

Q.E.D.

Proposition 3.4.9. *Let* $\mathfrak{a}$ *be a left ideal of* $\boldsymbol{C}[\pi]$ *such that* $\dim\boldsymbol{C}[\pi]/\mathfrak{a}<\infty$. *Then there is a left ideal* $\mathfrak{b}$ *of* $\boldsymbol{C}[\pi]$ *satisfying the following properties*:

(3.4.17) $\dim\boldsymbol{C}[\pi]/\mathfrak{b}<\infty$ *and* $\mathfrak{b}\subset\mathfrak{a}$.

(3.4.18) *Let* α *be the homomorphism from* $\tilde{\mathscr{E}}_0^\infty\underset{\mathscr{D}_{X,0}^\infty}{\otimes}(\mathscr{L}(\mathfrak{a}))_0^\infty$ *into* $\tilde{\mathscr{E}}_0^\infty\underset{\mathscr{D}_{X,0}^\infty}{\otimes}(\mathscr{L}(\mathfrak{b}))_0^\infty$ *and* β *the homomorphism from* $(\mathscr{L}(\mathfrak{b}))_0^\infty$ *into* $\tilde{\mathscr{E}}_0^\infty\underset{\mathscr{D}_{X,0}^\infty}{\otimes}(\mathscr{L}(\mathfrak{b}))_0^\infty$. *Then the image of* α *is contained in that of* β.

Proof. Let $\mathscr{M}$ be an arbitrary holonomic $\mathscr{D}$-Module. As shown in Section 3, we have the exact sequence

$$0\longrightarrow\mathscr{D}_X^\infty\longrightarrow\tilde{\mathscr{E}}_X^\infty\longrightarrow\mathscr{K}|_X\longrightarrow 0$$

Here $\mathscr{K}$ is a sheaf isomorphic to $\mathscr{D}_{X\underset{Y}{\times}X}^\infty/(\mathscr{D}_{X\underset{Y}{\times}X}^\infty D_{t_1}+D_{t_2}\mathscr{D}_{X\underset{Y}{\times}X}^\infty)$ and $(t_1, t_2, x)=(t_1, t_2, x_1,\dots, x_n)$ is the coordinate system of $X\underset{\boldsymbol{C}^n}{\times}X$ such that $(t_1, t_2, x)\mapsto(t_1, x)$ (resp., (t_2, x)) is the first (resp., the second) projection from $X\underset{\boldsymbol{C}^n}{\times}X$ onto X. Hence we have the exact sequence

$$\mathscr{D}_X^\infty\underset{\mathscr{D}_X}{\otimes}\mathscr{M}\longrightarrow\tilde{\mathscr{E}}^\infty\underset{\mathscr{D}_X}{\otimes}\mathscr{M}\longrightarrow\mathscr{K}|_X\underset{\mathscr{D}_X}{\otimes}\mathscr{M}\longrightarrow 0\,.$$

We shall identify $X \underset{C^n}{\times} X$ with $C \times X$ by $(t_1, t_2, x) \mapsto (t_1, (t_2, x)) \in C \times X$. Then $(\mathscr{D}^{\infty}_{X \underset{Y}{\times} X} / \mathscr{D}^{\infty}_{X \underset{Y}{\times} X} D_t) \underset{\mathscr{D}_X}{\otimes} \mathscr{M} \cong (\mathscr{O}_C \hat{\otimes} \mathscr{M})^{\infty}$. Hence we have

$$\mathscr{K}|_X \underset{\mathscr{D}_X}{\otimes} \mathscr{M} = (\mathscr{O}_C \hat{\otimes} \mathscr{M})^{\infty} / D_{t_2} (\mathscr{O}_C \hat{\otimes} \mathscr{M})^{\infty}|_X .$$

Thus we obtain the following diagram for any left ideal $\mathfrak{b}$ of $C[\pi]$ such that $\dim C[\pi]/\mathfrak{b} < \infty$.

$$\begin{array}{ccccccc} \mathscr{L}(\mathfrak{a})^{\infty} & \longrightarrow & \tilde{\mathscr{E}}^{\infty} \underset{\mathscr{D}}{\otimes} \mathscr{L}(\mathfrak{a}) & \longrightarrow & \mathscr{L}_1^{\infty} / D_{t_2} \mathscr{L}_1^{\infty}|_X & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \alpha} & & \downarrow{\scriptstyle \gamma} & & \\ \mathscr{L}(\mathfrak{b})^{\infty} & \longrightarrow & \tilde{\mathscr{E}}^{\infty} \underset{\mathscr{D}}{\otimes} \mathscr{L}(\mathfrak{b}) & \longrightarrow & \mathscr{L}_2^{\infty} / D_{t_2} \mathscr{L}_2^{\infty}|_X & \longrightarrow & 0 , \end{array}$$

where $\mathscr{L}_1 = \mathscr{O}_C \hat{\otimes} \mathscr{L}(\mathfrak{a})$ and $\mathscr{L}_2 = \mathscr{O}_C \hat{\otimes} \mathscr{L}(\mathfrak{b})$.

In order to show Proposition 3.4.9, it is sufficient to prove that $\gamma: (\mathscr{L}_1^{\infty} / D_{t_2} \mathscr{L}_1^{\infty})|_X \to (\mathscr{L}_2^{\infty} / D_{t_2} \mathscr{L}_2^{\infty})|_X$ is the zero map at 0, i.e., $(\mathscr{L}_1^{\infty}) \subset D_{t_2} (\mathscr{L}_2^{\infty})_0$. By the definition, $(\mathscr{L}_1^{\infty})_0$ (resp., $(\mathscr{L}_2^{\infty})_0$) is the set of multi-valued holomorphic functions defined on $V - C \times S_0$ with the monodromy type $\mathfrak{a}$ (resp., $\mathfrak{b}$) for some neighborhood V of $(0, 0) \in C \times X$. Hence, by the preceding lemma, we have $(\mathscr{L}_1^{\infty})_0 \subset D_{t_2} (\mathscr{L}_2^{\infty})_0$ for some $\mathfrak{b}$. Q. E. D.

Corollary 3.4.10. *For a left ideal* $\mathfrak{a}$ *of* $C[\pi]$ *such that* $\dim C[\pi]/\mathfrak{a} < \infty$, *there exists a left ideal* $\mathfrak{b}$ *of* $C[\pi]$ *such that* $\dim C[\pi]/\mathfrak{a} < \infty$ *and* $\tilde{\mathscr{E}}_0^{\infty}(F(\mathfrak{a})/G(\mathfrak{a})) \subset F(\mathfrak{b})/G(\mathfrak{b})$.

Proof. Let $\mathfrak{b}$ be an ideal of $C[\pi]$ satisfying the conditions in the preceding proposition. Consider the diagram

$$\begin{array}{ccccc} & & F(\mathfrak{b}) & & \\ & & \downarrow{\scriptstyle \beta} & & \\ \tilde{\mathscr{E}}_0^{\infty} \underset{\mathscr{D}_0^{\infty}}{\otimes} F(\mathfrak{a}) & \xrightarrow{\alpha} & \tilde{\mathscr{E}}_0^{\infty} \underset{\mathscr{D}_0^{\infty}}{\otimes} F(\mathfrak{b}) & \longrightarrow & F/G . \end{array}$$

Then the corollary immediately follows from the fact that $\alpha(\tilde{\mathscr{E}}_0^{\infty} \underset{\mathscr{D}_0^{\infty}}{\otimes} F(\mathfrak{a}))$ is contained in $\beta(F(\mathfrak{b}))$. Q. E. D.

§5.

The purpose of this section is to construct a special resolution of a holonomic $\mathscr{E}$-Module whose characteristic variety is in a generic position. This result will be effectively used in Chapter IV.

5.1. Let X be $\boldsymbol{C}^{1+n}$. We denote by $(t, x)=(t, x_1,\dots, x_n)$ a point of X. Let $(t, x; \tau, \xi)=(t, x_1,\dots, x_n; \tau, \xi_1,\dots, \xi_n)$ be a coordinate system of T^*X such that the canonical 1-form is $\tau dt+\sum_{j=1}^{n}\xi_j dx_j$. Let E be the subset $\{(t, x; \tau, \xi); \tau\neq 0\}$ of T^*X. Then, as shown in Section 3 of this chapter, $\tilde{\mathscr{E}}$ is a subsheaf of $\pi_*(\mathscr{E}|_E)$.

Let us denote by R (resp., R^∞) the ring of micro-differential operators P in $\mathscr{E}_{p_0}$ (resp., $\mathscr{E}^\infty_{p_0}$) satisfying the following condition:

(3.5.1) P is a polynomial in D_x, i.e., there exists an integer N such that $(\mathrm{ad}\,x_j)^N P=0$ for $j=1,\dots, n$.

We denote $R\cap\mathscr{E}_{p_0}(m)$ by $R(m)$. Then $R(0)/R(-1)=\mathscr{O}_{X,q_0}[\xi_1/\tau,\dots, \xi_n/\tau]$. We denote by $\tilde{\mathscr{O}}$ the sheaf $\mathscr{O}_X[\xi_1/\tau,\dots, \xi_n/\tau]$ on X. Hence $R(0)/R(-1)=\tilde{\mathscr{O}}_{q_0}$. The ring R (resp., R^∞) is a subring of $\tilde{\mathscr{E}}_{q_0}$ (resp., $\tilde{\mathscr{E}}^\infty_{q_0}$).

We set

$A^\infty=\{P\in R^\infty; [x_j, P]=0 \text{ for } j=1,\dots, n\}$, $A=A^\infty\cap R$ and $A(m)=A\cap R(m)$. Then A and $A(0)$ is a Noetherian ring (Chapter II, § 3 of S-K-K [24]) and $A(0)/A(-1)=\mathscr{O}_{X,q_0}$. The ring R (resp., $R(0)$) is generated by $D_{x_j}D_t^{-1}$ $(j=1,\dots, n)$ over A (resp., $A(0)$).

Hence, by using Propositions 1.1.4 and 1.1.5 of Chapter I, Section 1, we obtain the following

Lemma 3.5.1. *A, R, $A(0)$ and $R(0)$ are Noetherian rings.*

Moreover, R (resp., $R(0)$) is a free A (resp., $A(0)$) module with a base $(D_x D_t^{-1})^\alpha$ $(\alpha\in\boldsymbol{Z}^n_+)$, and hence we have

Lemma 3.5.2. *R (resp., $R(0)$) is faithfully flat over A (resp., $A(0)$).*

Using these results, we prove the following

Lemma 3.5.3. *Let d and r_0 be an integer and M a left submodule of R^{r_0}. Then there exist integers $r_1,\dots, r_{d+1}$ and homomorphisms f_j: $R(0)^{r_{j+1}}\to R(0)^{r_j}$ $(j=0,\dots, d)$ such that the following sequences are all exact.*

(3.5.2) $0\longleftarrow R(0)^{r_0}/(R(0)^{r_0}\cap M)\longleftarrow R(0)^{r_0}\xleftarrow{f_0} R(0)^{r_1}\longleftarrow\cdots\xleftarrow{f_d} R(0)^{r_{d+1}}$

(3.5.3) $0\longleftarrow R^{r_0}/M\longleftarrow R^{r_0}\xleftarrow{R\otimes f_0} R^{r_1}\longleftarrow\cdots\xleftarrow{R\otimes f_d} R^{r_{d+1}}$

(3.5.4) $0\longleftarrow \mathscr{E}_p^{r_0}/(\mathscr{E}_p^{r_0}M)\longleftarrow \mathscr{E}_p^{r_0}\xleftarrow{\mathscr{E}_p\otimes f_0}\mathscr{E}_p^{r_1}\longleftarrow\cdots\xleftarrow{\mathscr{E}_p\otimes f_d}\mathscr{E}_p^{r_{d+1}}$

for $p\in E\cap\pi^{-1}(q_0)$.

Proof. Since $R(0)$ is Noetherian, we can find an exact sequence (3.5.2).

Since $R(m)$ is isomorphic to $R(0)$ for every m, R is flat over $R(0)$. Hence (3.5.3) is exact. In order to prove the exactness of (3.5.4), let us consider the following diagram:

(3.5.5)

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & \cdots & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 \leftarrow & R(-1)^{r_0}/(R(-1)^{r_0}\cap M) & \leftarrow & R(-1)^{r_0} & \leftarrow & R(-1)^{r_1} & \leftarrow \cdots \leftarrow & & R(-1)^{r_{d+1}} \\
 & \downarrow & & \downarrow & & \downarrow & & & \downarrow \\
0 \leftarrow & R(0)^{r_0}/(R(0)^{r_0}\cap M) & \leftarrow & R(0)^{r_0} & \leftarrow & R(0)^{r_1} & \leftarrow \cdots \leftarrow & & R(0)^{r_{d+1}} \\
 & & & \downarrow & & \downarrow & & & \downarrow \\
 & & & \tilde{\mathcal{O}}_{q_0}^{r_0} & \leftarrow & \tilde{\mathcal{O}}_{q_0}^{r_1} & \leftarrow \cdots \leftarrow & & \tilde{\mathcal{O}}_{q_0}^{r_{d+1}} \\
 & & & \downarrow & & \downarrow & & & \downarrow \\
 & & & 0 & & 0 & & & 0
\end{array}$$

Since all columns and the first two rows are exact, the row in the bottom is also exact. On the other hand, $\mathcal{O}_{T^*C^{1+n},p}$ is flat over $\tilde{\mathcal{O}}_{q_0}$. Hence the sequence

$$\mathcal{O}_{T^*C^{1+n},p}^{r_0} \longleftarrow \mathcal{O}_{T^*C^{1+n},p}^{r_1} \longleftarrow \cdots \longleftarrow \mathcal{O}_{T^*C^{1+n},p}^{r_{d+1}} \tag{3.5.6}$$

is exact. This is the symbol sequence of (3.5.4). Therefore, by combining Theorem 3.4.1 (a) and Proposition 3.2.7 of S-K-K [24] Chapter II (cf. p. 405 of S-K-K [24]), we find (3.5.4) is exact. Q.E.D.

Corollary 3.5.4. *$\mathcal{E}_p$ is flat over R for any $p\in E\cap\pi^{-1}(q_0)$.*

Proposition 3.5.5. *Let $\mathcal{M}$ be a holonomic $\mathcal{E}_X$-Module and $\mathcal{M}_0$ a coherent $\mathcal{E}(0)$-sub-Module of $\mathcal{M}$. Suppose that the support of $\mathcal{M}$ is in a generic position at p_0. Then the stalk $\mathcal{M}_{p_0}$ (resp., $\mathcal{M}_{0,p_0}$) of $\mathcal{M}$ (resp., $\mathcal{M}_0$) at p_0 is a finitely generated left A-module (resp., $A(0)$-module), and satisfies the following:*

$$\mathcal{E}_p \underset{R}{\otimes} \mathcal{M}_{p_0} = \mathcal{E}_p \underset{\tilde{\mathcal{E}}_{q_0}}{\otimes} \mathcal{M}_{p_0} = \begin{cases} \mathcal{M}_{p_0} & \text{for } p\in C^\times p_0 \\ 0 & \text{for } p\in E\cap\pi^{-1}(q_0)-C^\times p_0 . \end{cases} \tag{3.5.7}$$

$$\mathcal{E}_p^\infty \underset{R^\infty}{\otimes} \mathcal{M}_{p_0}^\infty = \mathcal{E}_p^\infty \underset{\tilde{\mathcal{E}}_{q_0}^\infty}{\otimes} \mathcal{M}_{p_0}^\infty = \begin{cases} \mathcal{M}_{p_0}^\infty & \text{for } p\in C^\times p_0 \\ 0 & \text{for } p\in E\cap\pi^{-1}(q_0)-C^\times p_0 . \end{cases} \tag{3.5.8}$$

Proof. We may assume $\mathcal{M}=\mathcal{E}\mathcal{M}_0$. The $\mathcal{O}(0)$-Module $\mathcal{M}_0/\mathcal{M}_0(-1)$ is a coherent $\mathcal{O}(0)$-Module whose support is contained in Supp $\mathcal{M}$. Hence $(\mathcal{M}_0/\mathcal{M}_0(-1))_{p_0}$ is a finitely generated $\mathcal{O}_{X,q_0}$-module. Since p_0 is an isolated point of $\mathrm{Supp}\,\mathscr{M} \cap \pi^{-1}(q_0) \cap \{\tau=1\}$, ξ_ν/τ operates on $(\mathscr{M}_0/\mathscr{E}(-1)\mathscr{M}_0)_{p_0} \underset{\mathcal{O}_{X,q_0}}{\otimes} \boldsymbol{C}$ as a nilpotent operator for $\nu=1,\dots, n$. We choose a set of elements $\{u_j\}_{j=1,\dots,d}$ of $(\mathscr{M}_0)_{p_0}$ such that the module class $[u_j]$ of u_j in $(\mathscr{M}_0/\mathscr{E}(-1)\mathscr{M}_0)_{p_0} \underset{\mathcal{O}_{X,q_0}}{\otimes} \boldsymbol{C}$ forms a base of the finite-dimensional vector space

$$(\mathscr{M}_0/\mathscr{E}(-1)\mathscr{M}_0)_{p_0} \underset{\mathcal{O}_{X,q_0}}{\otimes} \boldsymbol{C} = \mathscr{M}_{0,p_0}/(\mathscr{E}(-1)+\sum_{\nu=1}^{n} x_\nu \mathscr{E}(0))_{p_0}\mathscr{M}_{0,p_0}.$$

Since $(\mathscr{E}(-1)+\sum_{\nu=1}^{n} x_\nu\mathscr{E}(0))_{p_0}$ is contained in the maximal ideal of the local ring $\mathscr{E}(0)_{p_0}$, Nakayama's lemma entails

$$\mathscr{M}_0 = \mathscr{E}(0)u_1 + \cdots + \mathscr{E}(0)u_d. \tag{3.5.9}$$

Let U be the column vector with $u_1,\dots, u_d$ as components. Since ξ_ν/τ acts on $(\mathscr{M}_0/\mathscr{E}(-1)\mathscr{M}_0)_{p_0} \underset{\mathcal{O}_{X,q_0}}{\otimes} \boldsymbol{C}$ as a nilpotent operator, there is a $d\times d$ matrix $A_\nu(t, x)$ of holomorphic functions on X such that $(D_\nu D_t^{-1}-A_\nu)U \in \mathscr{M}_0(-1)^d_{p_0}$ and $A_\nu(0, 0)$ is a nilpotent matrix. Hence we obtain $(D_\nu - A_\nu D_t)U \in (\mathscr{M}_0)^d_{p_0}$. By (3.5.9), there is $P_\nu \in M_d(\mathscr{E}(0)_{p_0})$(*) $(\nu=1,\dots, n)$ such that

$$(D_\nu - A_\nu D_t - P_\nu)U = 0 \qquad \text{for} \quad \nu=1,\dots, n. \tag{3.5.10}$$

Next we shall show that there exist $B_1,\dots, B_n \in M_d(\mathscr{E}(1)_{p_0})$ such that

(3.5.11) B_ν is free from $D_1,\dots, D_n$, i.e., $[D_\mu, B_\nu]=0$ for $\nu, \mu=1,\dots, n$

(3.5.12) $(D_\nu - B_\nu)U = 0$ for $\nu=1,\dots, n$

(3.5.13) $\sigma_1(B_\nu) = A_\nu(t, x)\tau$ for $\nu=1,\dots, n$.

For this purpose, we consider the following condition:

$(3.5.11)_\mu$ B_ν is free from $D_1,\dots, D_\mu$ for $\nu=1,\dots, n$.

We shall prove the existence of $B_1,\dots, B_n \in M_d(\mathscr{E}(1)_{p_0})$ satisfying $(3.5.11)_\mu$, (3.5.12) and (3.5.13) for $\mu=0,\dots, n$ by the induction on μ. This is true for $\mu=0$. For $0\leqq\mu<n$, let us suppose the existence of such B_ν satisfying $(3.5.11)_\mu$, (3.5.12) and (3.5.13). By Späth-type division theorem for micro-differential operators (S-K-K [24] Chapter II, § 2.2, Theorem 2.2.1), we divide B_ν by $D_{\mu+1}-B_{\mu+1}$; $B_\nu = S_\nu(D_{\mu+1}-B_{\mu+1}) + \tilde{B}_\nu$ where $S_\nu \in M_d(\mathscr{E}(0)_{p_0})$ and $\tilde{B}_\nu \in M_d(\mathscr{E}(1)_{p_0})$ such that $\tilde{B}_\nu$ is free from $D_{\mu+1}$. The uniqueness of the division implies that $\tilde{B}_\nu$ is also free from $D_1,\dots, D_\mu$. We have also $\sigma_1(\tilde{B}_\nu) = \sigma_1(B_\nu) = A_\nu(t, x)\tau$. Thus we

(*) Here and in what follows, $M_d(*)$ denotes the set of $d\times d$ matrices whose components belong to $*$.

see that $\tilde{B}_\nu$ satisfies $(3.5.11)_{\mu+1}$, (3.5.12) and (3.5.13), and the induction proceeds.

This completes the proof of the existence of B_ν satisfying the required conditions (3.5.11), (3.5.12) and (3.5.13).

Hence the Späth-type division theorem entails

$$\mathscr{E}(0)^d_{p_0} = \sum_{\nu=1}^n (\mathscr{E}_{p_0}(0))^d D_t^{-1}(D_\nu - B_\nu) + A(0)^d \tag{3.5.14}$$

$$\mathscr{E}_{p_0} = \sum_{\nu=1}^n (\mathscr{E}_{p_0})^d (D_\nu - B_\nu) + A^d \tag{3.5.15}$$

and

$$\mathscr{E}^\infty_{p_0} = \sum_{\nu=1}^n (\mathscr{E}^\infty_{p_0})^d (D_\nu - B_\nu) + (A^\infty)^d\,. \tag{3.5.16}$$

This implies $\mathscr{M}_{0,p_0} = \sum_{j=1}^n \mathscr{E}(0)_{p_0} u_j = \sum_{j=1}^n A(0) u_j$ and $\mathscr{M}_{p_0} = \sum_{j=1}^d \mathscr{E}_{p_0} u_j = \sum_{j=1}^d A u_j$, and hence $\mathscr{M}_{p_0}$ (resp., $\mathscr{M}_{0,p_0}$) is finitely generated over A (resp., $A(0)$).

Next we shall show (3.5.7). The relation $\mathscr{E}_p \otimes_R \mathscr{M}_{p_0} = \mathscr{E}_p \otimes_{\tilde{\mathscr{E}}_{q_0}} \mathscr{M}_{p_0} = 0$ for $p \in E \cap \pi^{-1}(q_0) - \boldsymbol{C}^\times p_0$ immediately follows from (3.5.12) and (3.5.13), because $A_\nu(0, 0)$ is a nilpotent matrix and $D_\nu - B_\nu$ is invertible at p.

Let us consider the following homomorphisms:

$$\mathscr{M}_{p_0} \xrightarrow{\ g\ } \mathscr{E}_{p_0} \otimes_R \mathscr{M}_{p_0} \xrightarrow{\ f\ } \mathscr{M}_{p_0},$$

where f and g are defined by $g(s) = 1 \otimes s$ and $f(P \otimes s) = Ps$. It is clear that $f \circ g = \mathrm{id}$. Hence the surjectivity of g will imply that both f and g are isomorphisms. Again by (3.5.15) we find $\mathscr{E}_{p_0} \otimes_R \mathscr{M}_{p_0} = \mathscr{E}_p \otimes_R (\sum R u_j) = \sum \mathscr{E}_p \otimes_R u_j = \sum 1 \otimes \sum R u_j = g(\mathscr{M}_{p_0})$ for $p \in \boldsymbol{C}^\times p_0$. In the same way, we have $\mathscr{E}_{p_0} \otimes_{\tilde{\mathscr{E}}_{q_0}} \mathscr{M}_{p_0} = \mathscr{M}_{p_0}$. Hence we obtain (3.5.7). The property (3.5.8) is also obtained in the same way, by using (3.5.16) instead of (3.5.15). We leave the detailed arguments to the reader. Q.E.D.

Corollary 3.5.6. $\mathscr{M}^\infty_{p_0} = \tilde{\mathscr{E}}^\infty_{q_0} \mathscr{M}_{p_0}$.

In fact, $\mathscr{M}^\infty_{p_0} = \sum_{j=1}^d \mathscr{E}^\infty_{p_0} u_j$, and (3.5.16) shows the desired result.

Corollary 3.5.7. *Any R-submodule of $\mathscr{M}_{p_0}$ is an $\mathscr{E}_{p_0}$-module.*

Proof. Since $\mathscr{M}_{p_0}$ is finitely generated over R, any R-submodule N of $\mathscr{M}_{p_0}$ is finitely generated. Let $s_1, \dots, s_r$ be a system of generators of N. Then $N_0 = \sum_{j=1}^r R(0) s_j$ is contained in $\sum_{j=1}^r \mathscr{E}(0) s_j$, and hence N_0 is a finitely generated $A(0)$-module. Let $u_1, \dots, u_d$ be a system of generators of N_0 as an $A(0)$-module.

Then there is $B_\nu \in A(1)$ $(\nu=1,\dots,n)$ such that $(D_\nu - B_\nu)U=0$, where U is the column vector with $u_1,\dots,u_d$ as components. Since $\mathscr{E}_{p_0}^d = \sum_{\nu=1}^{n} \mathscr{E}_{p_0}^d(D_\nu - B_\nu) + A^d$, the division theorem entails $\sum_\nu \mathscr{E}_{p_0} u_\nu = \sum_\nu A u_\nu = N$. Hence N is an $\mathscr{E}_{p_0}$-module. Q.E.D.

5.2. Let $\mathscr{M}$ be a holonomic $\mathscr{E}_X$-Module defined on a neighborhood U of p_0. Suppose that the characteristic variety Λ of $\mathscr{M}$ is in a generic position at p_0. Hence p_0 is an isolated point of $\Lambda \cap \pi^{-1}(q_0) \cap \{\tau=1\}$. Therefore, by shrinking U if necessary, we may assume that Λ is a closed analytic subset of $\pi^{-1}(V) - T_X^*X$ for a neighborhood V of q_0, and $\Lambda \cap \pi^{-1}(q_0) = \mathbf{C}^\times p_0$. Moreover $\mathscr{M}$ is defined over $\pi^{-1}(V) - T_X^*X$ with $\operatorname{Supp}\mathscr{M} = \Lambda$.

Theorem 3.5.8. *There exist integers* $r_0,\dots,r_{2n+1}$, $P_j \in M(r_{j+1}, r_j; R)$(*) $(j=0,\dots,2n)$ *and a homomorphism* $F: \mathscr{E}_{p_0}^{r_0} \to \mathscr{M}_{p_0}$ *such that the sequence*

$$(3.5.17)\qquad 0 \leftarrow \mathscr{M} \xleftarrow{F} \mathscr{E}_X^{r_0} \xleftarrow{P_0} \mathscr{E}_X^{r_1} \xleftarrow{P_1} \cdots \xleftarrow{P_{2n-1}} \mathscr{E}_X^{r_{2n}} \xleftarrow{P_{2n}} \mathscr{E}_X^{r_{2n+1}} \leftarrow 0$$

is exact on $E \cap \pi^{-1}(V)$ *for some neighborhood* V *of* q_0, *where* $E = \{(t, x; \tau, \xi) \in T^*X;\ \tau \neq 0\}$.

Proof. Let $u_1,\dots,u_d$ be a system of generators of $\mathscr{M}_{p_0}$ as an R-module. Setting $d=r_0$, we have an exact sequence by $u_1,\dots,u_{r_0}$: $0 \leftarrow \mathscr{M}_{q_0} \leftarrow R^{r_0}$. Let M be the kernel of $R^{r_0} \leftarrow \mathscr{M}_{p_0}$. Then, by Lemma 3.5.3, there is an exact sequence

$$0 \leftarrow R(0)^{r_0}/(R(0)^{r_0} \cap M) \leftarrow R(0)^{r_0} \xleftarrow{P_0} R(0)^{r_1} \leftarrow \cdots \xleftarrow{P_{2n-1}} R(0)^{r_{2n}}.$$

Let N be the kernel of $R(0)^{r_{2n}} \xrightarrow{P_{2n-1}} R(0)^{r_{2n-1}}$. Set $\bar{P}_j = (R(0)/R(-1)) \otimes P_j$ and $\bar{N} = N/R(-1)N$. Then the sequence

$$(3.5.18)\qquad \tilde{\mathscr{O}}_{q_0}^{r_0} \xleftarrow{\bar{P}_0} \tilde{\mathscr{O}}_{q_0}^{r_1} \xleftarrow{\bar{P}_1} \cdots \longleftarrow \tilde{\mathscr{O}}_{q_0}^{r_{2n}} \longleftarrow \bar{N} \longleftarrow 0$$

is exact. Since the global cohomological dimension of $\tilde{\mathscr{O}}_{q_0}$ is $2n+1$, $\bar{N}$ is a projective $\tilde{\mathscr{O}}_{q_0}$-module. On the other hand, by Grothendieck's theorem in K-theory (e.g. [27] Chapter XII. § 3), we can find an integer r such that $\bar{N} \oplus \tilde{\mathscr{O}}_{q_0}^r$ is a free $\tilde{\mathscr{O}}_{q_0}$-module. Then by replacing r_{2n} with $r_{2n}+r$ and P_{2n-1} with $P_{2n-1} \oplus 0$, respectively, we may assume from the first that $\bar{N}$ is a free $\tilde{\mathscr{O}}_{q_0}$-module. Let $v_1,\dots,v_{r_{2n+1}}$ be a set of elements of N such that the modulo classes $[v_1]$, $\dots$, $[v_{r_{2n+1}}]$ form a base of $\bar{N}$. Let P_{2n} be the $r_{2n+1} \times r_{2n}$ matrix determined by $v_1,\dots,v_{r_{2n}}$, and set $\bar{P}_{2n} = (R(0)/R(-1)) \otimes P_{2n-1}$

(*) Here and in what follows, $M(r, r'; *)$ denotes the set of $r \times r'$ matrices whose components belong to $*$.

(3.5.19) $$\tilde{\mathcal{O}}_{q_0}^{r_0} \xleftarrow{\bar{P}_0} \tilde{\mathcal{O}}_{q_0}^{r_1} \longleftarrow \cdots \longleftarrow \tilde{\mathcal{O}}_{q_0}^{r_{2n}} \xleftarrow{\bar{P}_{2n}} \tilde{\mathcal{O}}_{q_0}^{r_{2n+1}} \longleftarrow 0$$

is an exact sequence. Since $\tilde{\mathcal{O}}$ is a coherent sheaf on X,

(3.5.20) $$\tilde{\mathcal{O}}^{r_0} \xleftarrow{\bar{P}_0} \tilde{\mathcal{O}}^{r_1} \xleftarrow{\bar{P}_1} \cdots \xleftarrow{\bar{P}_{2n}} \tilde{\mathcal{O}}^{r_{2n+1}} \longleftarrow 0$$

is exact on a neighborhood of q_0. Since $\mathcal{O}_{T^*C^{1+n}}|_E$ is flat over $\tilde{\mathcal{O}}$, the sequence

(3.5.21) $$\mathcal{O}_{T^*C^{1+n}}^{r_0} \xleftarrow{\bar{P}_0} \mathcal{O}_{T^*C^{1+n}}^{r_1} \longleftarrow \cdots \xleftarrow{\bar{P}_{2n}} \mathcal{O}_{T^*C^{1+n}}^{r_{2n+1}} \longleftarrow 0$$

is exact on $E \cap \pi^{-1}(V)$ for a neighborhood V of q_0. Since this is the symbol sequence of the sequence.

(3.5.22) $$\mathscr{E}^{r_0} \xleftarrow{P_0} \mathscr{E}^{r_1} \longleftarrow \cdots \xleftarrow{P_{2n}} \mathscr{E}^{r_{2n+1}},$$

the sequence (3.5.22) is exact on $E \cap \pi^{-1}(V)$. Let $\mathcal{M}'$ be the cokernel of $\mathscr{E}^{r_0} \xrightarrow{P_0} \mathscr{E}^{r_1}$. Since $\mathcal{M}_{p_0} \leftarrow R^{r_0} \xleftarrow{P_0} R^{r_1}$ is exact, we have $\mathcal{M}'_{p_0} = \mathscr{E}_{p_0} \underset{R}{\otimes} \mathcal{M}_{p_0} = \mathcal{M}_{p_0}$. Hence $\mathcal{M}'$ is a coherent $\mathscr{E}_X$-Module defined on $E \cap \pi^{-1}(V)$ such that $\mathcal{M}_{p_0} = \mathcal{M}'_{p_0}$.

Let U be the column vector with $u_1,\dots, u_{r_0}$ as components. Then it follows from the proof of Lemma 3.5.1 that there exists $B_\nu \in M_{r_0}(R(1))$ $(\nu = 1,\dots, n)$ such that

(3.5.23) B_ν is free from D_{x_j} and its principal symbol $\sigma_1(B_\nu)$ is nilpotent at $(t, x) = (0, 0)$

(3.5.24) $$(D_\nu - B_\nu)U = 0.$$

The property (3.5.24) asserts that we can find a homomorphism Q_ν: $R^{r_0} \to R^{r_1}$ such that $D_\nu - B_\nu = P_0 \circ Q_\nu$. Hence the support of $\mathcal{M}'$ is contained in $\{(t, x; \tau, \xi); \det(\xi_\nu - \sigma_1(B_\nu)(t, x, \tau)) = 0$ for $\nu = 1,\dots, n\}$. The property (3.5.23) implies that $\operatorname{Supp} \mathcal{M}'$ is a closed analytic subset of $\pi^{-1}(V') \cap T^*_X X$ for a neighborhood V' of q_0 and $\operatorname{Supp} \mathcal{M}' \cap \pi^{-1}(q_0) = C^\times p_0$. Since $\mathcal{M} = \mathcal{M}'$ on a neighborhood of p_0, $\mathcal{M} = \mathcal{M}'$ on $\pi^{-1}(V'') \cap E$ for a neighborhood V'' of q_0. Q.E.D.

Chapter IV. Embedding Holonomic Systems into Holonomic Systems of $\mathcal{D}$-type

The purpose of this chapter is to prove a theorem which asserts that any holonomic system whose characteristic variety is in a generic position can be embedded into a holonomic system of D-type. Since holonomic systems of D-type have a rather simple structure, we can study properties of an arbitrary holonomic system by this embedding.

§1. Statement of the Results and Outline of the Proof

Our main result of this chapter is the following theorem.

Theorem 4.1.1. *Let $\mathscr{M}$ be a holonomic $\mathscr{E}_X$-Module defined on a neighborhood of $p_0 \in T^*X - T^*_XX$. Assume that the characteristic variety Λ of $\mathscr{M}$ is in a generic position. Then there exists a holonomic $\mathscr{D}_X$-Module $\mathscr{N}$ defined on a neighborhood of $q_0 = \pi(p_0)$ and a $\mathscr{D}^\infty_{X,q_0}$-linear homomorphism ϕ from $\mathscr{M}^\infty_{p_0} \underset{\mathrm{def}}{=} (\mathscr{E}^\infty_X \underset{\mathscr{E}_X}{\otimes} \mathscr{M})_{p_0}$ into $\mathscr{N}^\infty_{q_0} \underset{\mathrm{def}}{=} (\mathscr{D}^\infty_X \underset{\mathscr{D}_X}{\otimes} \mathscr{N})_{q_0}$ satisfying the following conditions:*
(4.1.1) *There exist an integer r and a holonomic system $\mathscr{L}$ of D-type with singularities along $\pi(\Lambda)$ such that $\mathscr{N}$ is isomorphic to the quotient $\mathscr{L}/\mathscr{O}^r_X$.*
(4.1.2) *The homomorphism from $\mathscr{M}^\infty_{p_0}$ into $\mathscr{E}^\infty_{p_0} \underset{\mathscr{D}_{q_0}}{\otimes} \mathscr{N}_{q_0} = \mathscr{E}^\infty_{p_0} \underset{\mathscr{D}^\infty_{q_0}}{\otimes} \mathscr{N}^\infty_{q_0}$ defined by $s \mapsto 1 \otimes \phi(s)$ is an injective $\mathscr{E}^\infty_{p_0}$-linear homomorphism.*

The idea of the proof of this theorem is as follows: Let C be the space of holomorphic functions defined on some cone whose apex is the origin modulo holomorphic functions defined on a neighborhood of the origin (see Section 5 for the exact definition of C). Then $V = \mathrm{Hom}_{\mathscr{E}^\infty_{p_0}}(\mathscr{M}^\infty_{p_0}, C)$ is a finite dimentional vector space (Proposition 4.6.1), and this space is sufficiently ample, in the sense that the homomorphism $\mathscr{M}^\infty_{p_0} \to \mathrm{Hom}_{\boldsymbol{C}}(V, C)$ is an injective map (Proposition 4.6.2). Therefore $\mathscr{M}^\infty_{p_0}$ is embedded into C^l for some $l > 0$. Let $s_1, \dots, s_N$ be a system of generators of $\mathscr{M}$ and $\{\phi_\nu\}$ a base of V. Let $\varphi_{j,\nu}$ be a holomorphic function whose modulo class is $\phi_\nu(s_j)$. Then we can prove that $\varphi_{j,\nu}$ is a multi-valued holomorphic functions defined on $X - \pi(\Lambda)$ with finite determination (§5, §6). Hence all $\varphi_{j,\nu}$ can be considered as a section of $\mathscr{L}^\infty$ for a holonomic $\mathscr{D}$-Module $\mathscr{L}$ of D-type. In Section 7 we prove that the map from $\mathscr{M}^\infty_{p_0}$ to $\mathscr{L}^{\infty l}_{q_0}$ defined by $s_j \mapsto (\varphi_{j,\nu})_\nu$ is the desired embedding of $\mathscr{M}^\infty$.

Sections 3 and 4 are devoted to the preparation for making V explicit, and Section 5 is to prove that $\varphi_{j,\nu}$ are multi-valued holomorphic functions defined on $X - \pi(\Lambda)$.

§2. Geometric Preparations

Let X be a complex manifold of dimension $1 + n$ ($n \geqq 1$), Λ a closed homogeneous Lagrangian variety in $T^*X - T^*_XX$, p_0 a point in Λ and $q_0 = \pi(p_0)$. Suppose that

(4.2.1) $$\Lambda \cap \pi^{-1}(q_0) = \mathbf{C}^{\times} p_0 .$$

Then $S=\pi(\Lambda)$ is a closed hypersurface of X on a neighborhood of q_0 and $\Lambda = T_S^*X$ on a neighborhood of q_0. Let F be a holomorphic map from (X, q_0) into $(\mathbf{C}^n, 0)$ (i.e., F is a holomorphic map from a neighborhood of q_0 into $\mathbf{C}^n$ such that $F(q_0)=0$). Suppose that F is of maximal rank and the restriction of the covector p_0 on the fiber $F^{-1}(0)$ does not vanish. Then there is a local coordinate system $(t, x)=(t, x_1,\dots, x_n)$ of X at q_0 such that $p_0=(0; dt)$ and F: $(t, x) \mapsto x$. We shall take the local coordinate system $(t, x; \tau, \xi)=(t, x_1,\dots, x_n; \tau, \xi_1,\dots, \xi_n)$ of T^*X such that the fundamental 1-form ω equals $\tau dt + \sum_{j=1}^{n} \xi_j dx_j$. For ε, $\delta>0$ we shall denote

(4.2.2) $$B(\varepsilon, \delta)=\{(t, x);\ |t|<\delta,\ |x|<\varepsilon\},$$

and

(4.2.3) $$B(\varepsilon)=\{x \in \mathbf{C}^n;\ |x|<\varepsilon\}.$$

Lemma 4.2.1. *For any $\rho>0$, there is a neighborhood U of q_0 such that*

$$S \cap U \subset \{(t, x);\ |t| \leqq \rho |x|\}.$$

Proof. S is the projection of $\{(t, x; \tau, \xi) \in \Lambda;\ \tau=1\}$ in a neighborhood of q_0. Hence, if the lemma were not true, there would be a path $p(\lambda)=(t(\lambda), x(\lambda); 1, \xi(\lambda))$ in Λ such that $p(0)=p_0$ and $|t(\lambda)|>\rho|x(\lambda)|$ for $0<\lambda \ll 1$. Since $\omega=0$ on Λ, we have

$$dt(\lambda)+\langle \xi(\lambda),\ dx(\lambda)\rangle=0, \quad \text{i.e.,} \quad \frac{dt(\lambda)}{d\lambda}=-\left\langle \xi(\lambda),\ \frac{dx(\lambda)}{d\lambda}\right\rangle.$$

Set

$$(t(\lambda),\ x(\lambda)) \equiv (t_0,\ x_0)\lambda^m \mod \lambda^{m+1}$$

with $(t_0, x_0) \neq 0$. Then $|t_0| \geqq \rho |x_0|$. Since $\xi(0)=0$, we have

$$\left\langle \xi(\lambda),\ \frac{dx(\lambda)}{d\lambda}\right\rangle \equiv 0 \mod \lambda^m$$

and hence we obtain $t_0=0$. This is a contradiction. Q.E.D.

By this lemma, there are positive numbers ε_0 and δ_0 satisfying the following conditions.

(4.2.4) $$0<\varepsilon_0<\delta_0, \quad B(\varepsilon_0, \delta_0) \subset X,$$

(4.2.5) $S \cap B(\varepsilon_0, \delta_0)$ is a closed hypersurface of $B(\varepsilon_0, \delta_0)$,

and

(4.2.6) $$S \cap B(\varepsilon_0, \delta_0) \subset \{(t, x) \in B(\varepsilon_0, \delta_0);\ |t| \geqq |x|\}.$$

The property (4.2.6) implies that $F|_{S \cap B(\varepsilon_0, \delta_0)}: S \cap B(\varepsilon_0, \delta_0) \to B(\varepsilon_0)$ is a finite map. Hence there is a closed analytic subset H of $B(\varepsilon_0)$ satisfying

(4.2.7) $$S \cap (B(\varepsilon_0, \delta_0) - F^{-1}(H)) \xrightarrow{F} B(\varepsilon_0) - H \text{ is a finite covering}.$$

Since $\Lambda \cap \pi^{-1}(q_0) = \boldsymbol{C}^\times p_0$, we may assume also

(4.2.8) $$\{(t, x; \tau, \xi) \in \Lambda;\ (t, x) \in B(\varepsilon_0, \delta_0)\} \subset \{(t, x; \tau, \xi);\ |\xi| \leqq |\tau|\}.$$

Throughout this chapter G_0 always denotes the closed convex cone $\{(t, x) \in \boldsymbol{C}^{1+n};\ x=0, \operatorname{Im} t=0, \operatorname{Re} t \leqq 0\}$ of $\boldsymbol{C}^{1+n}$. We also denote by G_0^a the antipodal set of G_0, i.e. $\{x=0, \operatorname{Im} t=0, \operatorname{Re} t \leqq 0\}$.

Lemma 4.2.2. *For any* $\rho > 0$, *there exists a point* x_1 *of* $B(\varepsilon_0) - H$ *which satisfies the following conditions*:

(4.2.9) $$|x_1| < \rho,$$

(4.2.10) *For any point* p *of* $F^{-1}(x_1) \cap S \cap B(\varepsilon_0, \delta_0)$,
$$B(\varepsilon_0, \delta_0) \cap S \cap (\{p\} + G_0) = \{p\}.$$

Proof. Let x_2 be a point of $B(\varepsilon_0) \cap B(\rho) - H$. Then, by (4.2.7), there are a neighborhood W of x_2 and holomorphic functions $\varphi_j(x)$ $(1 \leqq j \leqq m)$ defined on W such that

$$B(\varepsilon_0, \delta_0) \cap S \cap F^{-1}(W) = \bigsqcup_{j=1}^{m} \{(t, x);\ \varphi_j(x) = t,\ x \in W\}$$

and $\varphi_j(x) \neq \varphi_k(x)$ for $j \neq k$, $x \in W$. Note that $\varphi_j(x) - \varphi_k(x)$ is not a constant function for $j \neq k$ (otherwise, $S \underset{\boldsymbol{C}^n}{\times} S$ should contain $\{(t_1, t_2, x);\ t_1 - t_2 = c\}$ for $c \neq 0$ and this would contradict (4.2.6)). Therefore there is a point x_1 of W such that $\varphi_j(x_1) - \varphi_k(x_1) \notin \mathbf{R}$ for $j \neq k$ and that $|x_1 - x_2| < \rho - |x_2|$. This x_1 satisfies the required conditions. Q. E. D.

§3. Resolution of $\mathscr{M}$

We shall keep the notations of the preceding section. Let $\mathscr{M}$ be a coherent $\mathscr{E}_X$-Module defined on a neighborhood of p_0, whose support is contained in Λ. Hence we can extend $\mathscr{M}$ into a holonomic $\mathscr{E}$-Module defined on a neighborhood of $\pi^{-1}(q_0) - T_X^*X$, whose support is contained in Λ. Therefore, in the sequel, we assume that $\mathscr{M}$ is a holonomic $\mathscr{E}_X$-Module defined on a neighborhood of $\pi^{-1}(q_0)-T_X^*X$, and $\mathrm{Supp}(\mathscr{M})\subset\Lambda$.

In Chapter III, Section 5, we proved that $\mathscr{M}$ has the following resolution (4.3.1) on a neighborhood of p_0:

$$(4.3.1)\qquad 0\longleftarrow\mathscr{M}\longleftarrow\mathscr{E}_X^{N_0}\xleftarrow{P_0}\mathscr{E}_X^{N_1}\longleftarrow\cdots\xleftarrow{P_{r-1}}\mathscr{E}_X^{N_r}\longleftarrow 0,$$

where P_j are matrices whose components belong to $\tilde{\mathscr{E}}$ and are of order <0. Moreover (4.3.1) is exact on $\{(t, x; \tau, \xi)\in T^*X; |t|, |x|\ll 1, \tau\neq 0\}$. Each P_j can be expressed by a matrix of integro-differential operators $K_j(t_1, t_2, x, D_x)$ (Chapter III, § 3). Hence, for some ε_1 and δ_1 such that $0<\varepsilon_1\leqq\delta_1$, $\varepsilon_1\leqq\varepsilon_0$ and $\delta_1\leqq\delta_0$, all K_j are defined on $\{(t_1, t_2, x); |t_1|, |t_2|<\delta_1, |x|<\varepsilon_1\}$ and the sequence (4.3.1) is exact on $\{(t, x; \tau, \xi)\in T^*X; (t, x)\in B(\varepsilon_1, \delta_1), \tau\neq 0\}$. Hence, by replacing ε_0, δ_0 with ε_1, δ_1, we may assume from the beginning

(4.3.2) All $K_j(t_1, t_2, x, D_x)$ are defined on $\{(t_1, t_2, x); |t_1|, |t_2|<\delta_0, |x|<\varepsilon_0\}$.

Hence K_j gives an element of $\mathfrak{E}(G_0; D)$ with $D=B(\varepsilon_0, \delta_0)$ and $G_0=\{(t, x)\in\boldsymbol{C}^{1+n}; x=0, t\leqq 0\}$. Note that D is G_0-round. Thus we may assume the following:

(4.3.3) P_j is given by K_j.

(4.3.4) Let $\mathfrak{M}_\cdot$ be the following complex:

$$0\longleftarrow\mathfrak{E}(G_0, D)^{N_0}\xleftarrow{K_0}\cdots\xleftarrow{K_{r-1}}\mathfrak{E}(G_0, D)^{N_r}\longleftarrow 0.$$

Then $\mathscr{E}_p^{\boldsymbol{R}}\underset{\mathfrak{E}(G_0;D)}{\otimes}\mathfrak{M}_\cdot$ is quasi-isomorphic to $\mathscr{M}_p^{\boldsymbol{R}}$ for $p\in\Lambda\cap F^{-1}(D)$ and $\mathscr{E}_p^{\boldsymbol{R}}\underset{\mathfrak{E}(G_0;D)}{\otimes}\mathfrak{M}_\cdot$ is exact for $p\in\{(t, x; \tau, \xi)\notin\Lambda; \tau\neq 0, (t, x)\in D\}$.

§ 4. Vanishing Theorems

In the preceding section, we constructed a resolution $\mathfrak{M}_\cdot$ of $\mathscr{M}_p$, where $\mathfrak{M}_\cdot$ is a complex of $\mathfrak{E}(G_0; D)$-module. Hence we can apply Theorem 4.5.1 of [19]. We note that we can replace the condition (c_3) in Theorem 4.5.1 of [19] with the following weaker condition:

$\mathscr{E}_p^{\boldsymbol{R}}\otimes\mathfrak{M}_\cdot$ is exact for $p=(x, \xi)$ such that $x\in\overline{\Omega_1-\Omega_0}$ and $\langle\xi, Q(x)\rangle<0$.

The same proof in [19] can be also applicable under this weaker condition. Thus we have the following proposition in our case.

Proposition 4.4.1. *There is a G_0-open neighborhood U of q_0 contained in $D=B(\varepsilon_0, \delta_0)$ which satisfies the following condition*:

Let Ω_1 and Ω_0 be two open subsets of $\boldsymbol{C}^{1+n}$ satisfying the following conditions:

(a) $\Omega_1 \supset \Omega_0$ *and* $\Omega_1 - \Omega_0 \Subset U$.

(b) *There is an open convex cone R in TU such that $R \supset U \times (G_0 - \{0\})$ and that Ω_1 and Ω_0 are R-flat on U.*

(c) *There are an open convex cone Q of TU and a C^1-function g defined on a neighborhood of $\overline{\Omega_1 - \Omega_0}$ satisfying the following conditions*:

(c_1) $\{(p, v) \in TU;\ \langle v, dg(p)\rangle > 0,\ p \in \overline{\Omega_1 - \Omega_0}\} \supset Q \cap \tau^{-1}(\overline{\Omega_1 - \Omega_0}),\ Q \supset R$.

(c_2) $\Omega_1 - \Omega_0$ *is Q-flat in a neighborhood of* $\overline{\Omega_1 - \Omega_0}$.

(c_3) *For any point $p = (t, x; \tau, \xi)$ in T^*U, if $(t, x) \in \overline{\Omega_1 - \Omega_0}$ and if $\mathrm{Re}\,(\tau w + \langle \xi, v\rangle) < 0$ for any $(t, x; w, v) \in Q$, then p does not belong to Λ.*

Then $\Omega_1 - \Omega_0$ is locally closed in D with respect to G_0-topology and

$$\mathbf{R}\,\mathrm{Hom}_{\mathfrak{E}(G_0;D)}(\mathfrak{M}_{\bullet}, \mathbf{R}\Gamma_{\Omega_1 - \Omega_0}(\Omega_1; \varphi_* \mathcal{O}_X)) = 0 .$$

In this proposition and in the sequel, we denote by φ the continuous map from $B(\varepsilon_0, \delta_0)$ into $B(\varepsilon_0, \delta_0)_{G_0}$, where $B(\varepsilon_0, \delta_0)_{G_0}$ is the topological space $B(\varepsilon_0, \delta_0)$ endowed with the G_0-topology.

By using this proposition, we shall prove various vanishing theorems for the cohomology groups $\mathbf{R}\,\mathrm{Hom}(\mathfrak{M}_{\bullet}, \varphi_* \mathcal{O})$.

In the sequel, for $a > 0$, let $G(a)$ denote the following convex cone:

$$G(a) = \{(t, x);\ \mathrm{Re}\, t \leqq -a(|\mathrm{Im}\, t| + |x|)\} . \tag{4.4.1}$$

Then $\{G(a)\}_{a>0}$ is a decreasing family and we have $G_0 = \bigcap_{a>0} G(a)$.

Proposition 4.4.2. *Let Ω_1 and Ω_0 be two G_0-open subsets of $\boldsymbol{C}^{1+n}$ satisfying the following conditions*:

(i) $$\overline{\Omega_1 - \Omega_0} \subset U .$$

(ii) $$(\Omega_1 - \Omega_0) \cap S = \emptyset .$$

Then we have

$$\mathbf{R}\,\mathrm{Hom}_{\mathfrak{E}(G_0;D)}(\mathfrak{M}_{\bullet}, \mathbf{R}\Gamma_{\Omega_1 - \Omega_0}(\Omega_1; \varphi_* \mathcal{O}_X)) = 0 .$$

Proof. The G_0-open subset $E = \{(t, x);\ \mathrm{Re}\, t < -\varepsilon_0\}$ does not intersect U. Replacing Ω_1 and Ω_0 with $\Omega_1 \cup E$ and $\Omega_0 \cup E$, respectively, we may assume from the first that

(iii) $$\Omega_0 \cap \Omega_1 \supset E .$$

In order to prove the proposition, it is sufficient to show that

$\mathbf{R}\operatorname{Hom}_{\mathfrak{E}(G_0;D)}(\mathfrak{M}_., \mathbf{R}\Gamma_{\Omega_1-\Omega_0}(\varphi_*\mathcal{O}_X))$ is quasi-isomorphic to the zero complex of the sheaves on $(\Omega_1-\Omega_0)_{G_0}$. On the other hand, open subsets Ω of Ω_1 satisfying the three conditions:

$$\Omega\supset E,\ \overline{\Omega-\Omega_0}\subset\Omega_1-\Omega_0 \text{ and } \Omega \text{ is } G(a)\text{-open for some } a>0,$$

form a base of open subsets of $(\Omega_1-\Omega_0)_{G_0}$. Hence it is sufficient to show

$$\mathbf{R}\Gamma(\Omega; \mathbf{R}\operatorname{Hom}_{\mathfrak{E}(G_0;D)}(\mathfrak{M}_., \mathbf{R}\Gamma_{\Omega_1-\Omega_0}(\varphi_*\mathcal{O}_X)))=0$$

for such an Ω. By replacing Ω_1 with Ω, we may assume from the beginning

(0′) Ω_1 is $G(a)$-open for some $a>0$,

(i) $\overline{\Omega_1-\Omega_0}\subset U$,

(ii′) $\overline{(\Omega_1-\Omega_0)}\cap S=\emptyset$,

(iii) $\Omega_0\cap\Omega_1\supset E$.

For $b>0$, we set $\Omega_0(b)=\{x;\ x+G(b)\subset\Omega_0\}$. Then $\Omega_0(b)$ is a $G(b)$-open set and contains E. We shall prove that

$$\Omega_1\cap\Omega_0=\bigcup_{b>0}(\Omega_1\cap\Omega_0(b)), \tag{4.4.2}$$

and

$$\bigcap_{b>0}\overline{\Omega_1-\Omega_0(b)}=\overline{\Omega_1-\Omega_0}. \tag{4.4.3}$$

Let us prove first that

$$\Omega_1\cap\Omega_0\subset\bigcup_{b>0}(\Omega_1\cap\Omega_0(b)).$$

Let p be a point in $\Omega_1\cap\Omega_0$. Then $(p+G_0)-E$ is contained in $\Omega_1\cap\Omega_0$. Hence $(p+G(b))-E$ is contained in $\Omega_1\cap\Omega_0$ for some $b>0$, because $\{(p+G(b))-E\}_{b>0}$ is a decreasing family of compact sets whose intersection is $(p+G_0)-E$. This shows $p\in\Omega_0(b)$ for some $b>0$. Thus we obtain (4.4.2). Let us prove that (4.4.3). It is obvious that $\bigcap_{b>0}\overline{\Omega_1-\Omega_0(b)}\supset\overline{\Omega_1-\Omega_0}$. We shall prove the converse inclusion relation. Let p be a point outside $\overline{\Omega_1-\Omega_0}$, and we shall show $p\notin\bigcap_{b>0}\overline{\Omega_1-\Omega_0(b)}$. Since $p\notin\overline{\Omega_1-\Omega_0}$, there is an open neighborhood V of p such that $V\cap\Omega_1\subset\Omega_0$. Therefore $\Omega_0\supset((V\cap\Omega_1)+G_0)\cup E$. There are also a neighborhood V' of p_0 and $b>0$ such that, for any p' in V', we have

$$(((p'+G(a))\cap V)+G_0)\cup E\supset p'+G(b).$$

Thus we have $\Omega_1\cap V'\subset\Omega_0(b)$. This implies that $p\notin\overline{\Omega_1-\Omega_0(b)}$. Thus (4.4.3) has been proved.

Since $\{\overline{\Omega_1-\Omega_0(b)}\}_{b>0}$ is a decreasing family of compact subsets, we have

$\overline{\Omega_1-\Omega_0(b)}\subset U$ and $\overline{\Omega_1-\Omega_0(b)}\cap S=\emptyset$ for $b\gg 1$, a. Hence, if we prove the proposition for Ω_1 and $\Omega_0(b)$, then

$$\begin{aligned}&\mathrm{Ext}^j(\mathfrak{M}_., \mathbf{R}\Gamma_{\Omega_1-\Omega_0}(\Omega_1; \varphi_*\mathcal{O}_X))\\&=\varinjlim_b \mathrm{Ext}^j(\mathfrak{M}_., \mathbf{R}\Gamma_{\Omega_1-\Omega_0(b)}(\Omega_1; \varphi_*\mathcal{O}_X))=0.\end{aligned}$$

Thus we may assume from the first

(0″) Ω_1 and Ω_0 are $G(a)$-open for some a,

(i″) $\overline{\Omega_1-\Omega_0}\subset U$,

(ii″) $\overline{\Omega_1-\Omega_0}\cap S=\emptyset$.

Now we can apply Proposition 4.4.1. Set $Q=R=U\times \mathrm{Int}\, G(a)$ and $g=-\mathrm{Re}\, t$. Then all conditions in Proposition 4.4.1 are satisfied and we obtain $\mathrm{Ext}^j(\mathfrak{M}_., \mathbf{R}\Gamma_{\Omega_1-\Omega_0}(\Omega_1; \varphi_*\mathcal{O}_X)=0$ for any j. Q.E.D.

Proposition 4.4.3. *Let Ω_1 and Ω_0 be two G_0-open subsets of $\boldsymbol{C}^{1+n}$ and W an open subset of $B(\varepsilon_0)$. Assume the following conditions:*

(i) $\overline{\Omega_1-\Omega_0}\subset U$.

(ii) $S\cap F^{-1}(W)\subset\Omega_1-\Omega_0$.

Then we have

(a) *Any cohomology group of* $\mathbf{R}(F|_{\Omega_1})_*\mathbf{R}\,\mathrm{Hom}(\mathfrak{M}_., \mathbf{R}\Gamma_{\Omega_1-\Omega_0}(\varphi_*\mathcal{O}_X))$ *is a locally constant sheaf on W. Here $F|_{\Omega_1}$ is the restriction of F to Ω_1 (and hence $F|_{\Omega_1}$ is a map from Ω_1 to $\boldsymbol{C}^n$).*

(b) $\mathbf{R}(F|_{\Omega_1})_*\mathbf{R}\,\mathrm{Hom}(\mathfrak{M}_., \mathbf{R}\Gamma_{\Omega_1-\Omega_0}(\varphi_*\mathcal{O}_X))|_W$ *does not depend on the choice of Ω_1 and Ω_0. More precisely, if Ω_1' and Ω_0' are two G_0-open subsets satisfying the conditions* (i) *and* (ii), *then*

$$\mathbf{R}(F|_{\Omega_1})_*\mathbf{R}\,\mathrm{Hom}(\mathfrak{M}_., \mathbf{R}\Gamma_{\Omega_1-\Omega_0}(\varphi_*\mathcal{O}_X))|_W$$

and

$$\mathbf{R}(F|_{\Omega_1'})_*\mathbf{R}\,\mathrm{Hom}(\mathfrak{M}_., \mathbf{R}\Gamma_{\Omega_1'-\Omega_0'}(\varphi_*\mathcal{O}_X))|_W$$

are canonically isomorphic.

Remark. $F|_{\Omega_1}$ gives a continuous map from Ω_{1G_0} to $\boldsymbol{C}^n$, where Ω_{1G_0} is endowed with the G_0-topology and $\boldsymbol{C}^n$ is endowed with the usual topology. Hence the above statements make sense.

Proof. First we shall show (b).

We shall consider the case where $\Omega_0=\Omega_0'$ and $\Omega_1\supset\Omega_1'\supset\Omega_0$. Then it is enough to show

$$\mathbb{R}(F|_{\Omega_1})_*\mathbb{R}\,\mathrm{Hom}\,(\mathfrak{M}_., \mathbb{R}\Gamma_{\Omega_1-\Omega_1'}(\varphi_*\mathcal{O}_X))|_W=0,$$

because we have the triangle

$$\begin{array}{ccc} & \mathbb{R}(F|_{\Omega_1})_*\mathbb{R}\,\mathrm{Hom}\,(\mathfrak{M}_., \mathbb{R}\Gamma_{\Omega_1-\Omega_1'}(\varphi_*\mathcal{O}_X))|_W & \\ \swarrow & & \nwarrow {\scriptstyle +1} \\ \mathbb{R}(F|_{\Omega_1})_*\mathbb{R}\,\mathrm{Hom}\,(\mathfrak{M}_., \mathbb{R}\Gamma_{\Omega_1-\Omega_0}(\varphi_*\mathcal{O}_X))|_W & \longrightarrow & \mathbb{R}(F|_{\Omega_1'})_*\mathbb{R}\mathrm{Hom}\,(\mathfrak{M}_., \mathbb{R}\Gamma_{\Omega_1'-\Omega_0}(\varphi_*\mathcal{O}_X))|_W. \end{array}$$

For any open subset W' of W, we have

$$\mathbb{R}\,\mathrm{Hom}\,(\mathfrak{M}_., \mathbb{R}\Gamma_{F^{-1}(W')\cap(\Omega_1-\Omega_1')}(F^{-1}(W')\cap\Omega_1;\ \varphi_*\mathcal{O}_X))=0.$$

In fact, since $\overline{F^{-1}(W')\cap(\Omega_1-\Omega_1')}\subset U$ and $(F^{-1}(W')\cap(\Omega_1-\Omega_1'))\cap S=\emptyset$, we can apply Proposition 4.4.2.

Secondly, we shall consider the case where $\Omega_1=\Omega_1'\supset\Omega_0\supset\Omega_0'$. Then, for any open subset W' of W, we obtain

$$\mathbb{R}\,\mathrm{Hom}\,(\mathfrak{M}_., \mathbb{R}\Gamma_{\Omega_0-\Omega_0'}(F^{-1}(W')\cap\Omega_0;\ \varphi_*\mathcal{O}_X))=0$$

by applying Proposition 4.4.2 and hence, by the same way as in the preceding discussion, we have

$$\mathbb{R}(F|_{\Omega_1})_*\mathbb{R}\,\mathrm{Hom}\,(\mathfrak{M}_., \mathbb{R}\Gamma_{\Omega_0-\Omega_0'}(\varphi_*\mathcal{O}_X))|_W=0$$

and hence

$$\begin{aligned}&\mathbb{R}(F|_{\Omega_1})_*\mathbb{R}\,\mathrm{Hom}\,(\mathfrak{M}_., \mathbb{R}\Gamma_{\Omega_1-\Omega_0}(\varphi_*\mathcal{O}_X))|_W\\ &\quad=\mathbb{R}(F|_{\Omega_1})_*\mathbb{R}\,\mathrm{Hom}\,(\mathfrak{M}_., \mathbb{R}\Gamma_{\Omega_1-\Omega_0'}(\varphi_*\mathcal{O}_X))|_W.\end{aligned}$$

Now we shall prove (b) in general. We may assume $\Omega_1\supset\Omega_0$ and $\Omega_1'\supset\Omega_0'$. Set $\Omega_1''=\Omega_1\cap\Omega_1'$ and $\Omega_0''=\Omega_0\cup\Omega_0'$. Then we have

$$\begin{aligned}&\mathbb{R}(F|_{\Omega_1})_*\mathbb{R}\,\mathrm{Hom}\,(\mathfrak{M}_., \mathbb{R}\Gamma_{\Omega_1-\Omega_0}(\varphi_*\mathcal{O}_X))|_W\\ &\quad=\mathbb{R}(F|_{\Omega_1})_*\mathbb{R}\,\mathrm{Hom}\,(\mathfrak{M}_., \mathbb{R}\Gamma_{\Omega_1-\Omega_0''}(\varphi_*\mathcal{O}_X))|_W\\ &\quad=\mathbb{R}(F|_{\Omega_1''})_*\mathbb{R}\,\mathrm{Hom}\,(\mathfrak{M}_., \mathbb{R}\Gamma_{\Omega_1''-\Omega_0''}(\varphi_*\mathcal{O}_X))|_W\end{aligned}$$

by using the preceding results. In the same way

$$\begin{aligned}&\mathbb{R}(F|_{\Omega_1'})_*\mathbb{R}\,\mathrm{Hom}\,(\mathfrak{M}_., \mathbb{R}\Gamma_{\Omega_1'-\Omega_0'}(\varphi_*\mathcal{O}_X))|_W\\ &\quad=\mathbb{R}(F|_{\Omega_1''})_*\mathbb{R}\,\mathrm{Hom}\,(\mathfrak{M}_., \mathbb{R}\Gamma_{\Omega_1''-\Omega_0''}(\varphi_*\mathcal{O}_X))|_W.\end{aligned}$$

Thus we obtain the desired result (b).

Next we shall show (a). The question being local on W, we may assume that $S\cap F^{-1}(\overline{W})\subset\Omega_1-\Omega_0$ by replacing W with its relatively compact open subset. Set $E=\{(t, x);\ \mathrm{Re}\,t<-\varepsilon_0\}$. Then E is a G_0-open subset disjoint from

U. Replacing Ω_1 and Ω_0 with $\Omega_1 \cup E$ and $\Omega_0 \cup E$, respectively we may assume that $\Omega_1 \cap \Omega_0 \supset E$. Let U' be an open subset of U such that

$$S \cap F^{-1}(\overline{W}) \subset U' \quad \text{and} \quad \overline{U}' \subset \Omega_1 \cap U.$$

Set $G(b) = \{(t, x);\ \mathrm{Re}\, t \leqq -b(|\mathrm{Im}\, t| + |x|)\}$. Since $(\overline{U}' + G_0) \cap (\overline{U}' + G_0^a) \subset (U + G_0) \cap (U + G_0^a) = U$, we have

$$(\overline{U}' + G(b)) \cap (\overline{U}' + G(b)^a) \subset U \quad \text{for} \quad b \gg 1. \tag{4.4.4}$$

Then $\Omega_1' = (U' + G(b)) \cup E$ and $\Omega_0' = \boldsymbol{C}^{n+1} - (\overline{U}' + G(b)^a)$ satisfy the conditions (i) and (ii). Hence, by replacing Ω_1 and Ω_0 with Ω_1' and Ω_0' respectively, we may assume from the first

(0′) Ω_1 and Ω_0 are $G(b)$-open for some $b > 0$ and $\Omega_1 \cap \Omega_0 \supset E$,

(i′) $\overline{\Omega_1 - \Omega_0} \subset U$,

(ii′) $\Omega_1 - \overline{\Omega_0} \supset S \cap F^{-1}(\overline{W})$.

For $x_1 \in W$, denote by $W(x_1, r)$ the open ball centered at x_1 with radius r. We shall show that

$$\begin{aligned} &\mathbf{R}\mathrm{Hom}(\mathfrak{M}_{.}, \mathbf{R}\Gamma_{\Omega_1 - \Omega_0}(F^{-1}(W(x_1, r)) \cap \Omega_1;\ \mathcal{O}_X)) \\ &\cong \mathbf{R}\mathrm{Hom}(\mathfrak{M}_{.}, \mathbf{R}\Gamma_{\Omega_1 - \Omega_0}(F^{-1}(W(x_1, r') \cap \Omega_1;\ \mathcal{O}_X))) \end{aligned} \tag{4.4.5}$$

for $0 < r' \leqq r$ and $W(x_1, r) \subset W$.

Set, for $a > 0$,

$$U(x_1, a, r) = \{(t, x);\ \mathrm{Re}\, t < ar - \varepsilon_0 - a|x - x_1|\}.$$

Then we have

$$\begin{aligned} &U(x_1, a, r) \cap U \subset F^{-1}(W(x_1, r)), \\ &U(x_1, a, r') \cap U \subset U(x_1, a', r'') \cap U \quad \text{for} \quad 0 < a \leqq a',\ 0 < r' \leqq r'', \end{aligned}$$

and

$$F^{-1}(x_1) \cap \overline{U} \subset U(x_1, a, r') \quad \text{for} \quad ar' > 2\varepsilon_0.$$

Note that $|\mathrm{Re}\, t| < \varepsilon_0$ on U. We now have the following

Lemma 4.4.4. $\mathbf{R}\mathrm{Hom}(\mathfrak{M}_{.}, \mathbf{R}\Gamma_{\Omega_1 - \Omega_0}(\Omega_1 \cap U(x_1, a, r');\ \mathcal{O}_X))$
$\cong \mathbf{R}\mathrm{Hom}(\mathfrak{M}_{.}, \mathbf{R}\Gamma_{\Omega_1 - \Omega_0}(\Omega_1 \cap U(x_1, a, r'');\ \mathcal{O}_X))$
$\cong \mathbf{R}\mathrm{Hom}(\mathfrak{M}_{.}, \mathbf{R}\Gamma_{\Omega_1 - \Omega_0}(\Omega_1 \cap U(x_1, a', r');\ \mathcal{O}_X))$

if $0 < r' \leqq r''$, $W(x_1, r'') \subset W$, $1 < a \leqq a'$ *and* $ar' > 2\varepsilon_0$.

Proof. It is enough to prove

$$\text{(4.4.6)}\quad \mathbb{R}\operatorname{Hom}(\mathfrak{M}_{\cdot},\ \mathbb{R}\Gamma_{U(x_1,a,r'')\cap\Omega_1-(\Omega_0\cup U(x_1,a,r'))}(\Omega_1\cap U(x_1,\ a,\ r'');\ \mathcal{O}_X))=0,$$

and

$$\text{(4.4.7)}\quad \mathbb{R}\operatorname{Hom}(\mathfrak{M}_{\cdot},\ \mathbb{R}\Gamma_{U(x_1,a',r')\cap\Omega_1-(\Omega_0\cup U(x_1,a,r'))}(\Omega_1\cap U(x_1,\ a',\ r');\ \mathcal{O}_X))=0.$$

Take an open set U' such that $S\cap F^{-1}(\overline{W})\subset U'$ and $\overline{U}'\subset(\Omega_1-\overline{\Omega})\cap U$. Choose $a''>a',\ b,\ 1$ and define $R=U\times\{(w,\ v)\in \boldsymbol{C}^{1+n};\ \operatorname{Re} w<-2a''(|\operatorname{Im} w|+|v|)\}$, $g(t,\ x)=\operatorname{Re} t-a''|x-x_1|$ and Q by $R\cup\{(t,\ x;\ w,\ v)\in TU;\ (t,\ x)\in U',\ x\neq x_1$ and $\operatorname{Re}(<(w,\ v),\ \partial g(t,\ x)>)=\operatorname{Re}\left(w\frac{\partial g}{\partial t}(t,\ x)+\sum_{j=1}^{n} v_j\frac{\partial g}{\partial x_j}(t,\ x)\right)<-\varepsilon(|\operatorname{Im} w|+|v|)\}$ for $0<\varepsilon\ll 1$. Then Q is a convex cone. If $p\in U'-U(x_1,\ a,\ r')$ and if $(\tau,\ \xi)\in\boldsymbol{C}^{1+n}$ satisfies $\operatorname{Re}(\langle(\tau,\ \xi),\ Q\cap\tau^{-1}(p)\rangle)<0$, then $(\tau,\ \xi)$ is very near to $\mathbb{R}^+\partial g(p)$ $(0<\varepsilon\ll 1)$. Since

$$\partial g(p)=\left(\frac{1}{2},\ \frac{-a''}{2}\frac{\bar{x}-\bar{x}_1}{|x-x_1|}\right)\qquad \text{for}\quad p=(t,\ x),$$

we have $|\tau|<|\xi|$. (In fact, we have $|\xi|\geqq(a''-\varepsilon)\operatorname{Re}\tau$ and $|\operatorname{Im}\tau|\leqq\varepsilon\operatorname{Re}\tau$. Hence, if $a''>\varepsilon+(1+\varepsilon^2)^{1/2}$, we have $|\tau|<|\xi|$). Therefore $(p;\ (\tau,\ \xi))$ is not contained in Λ by (4.2.8). Thus the conditions in Proposition 4.4.1 are satisfied for $U(x_1,\ a,\ r)\cap\Omega_1-(\Omega_0\cup U(x_1,\ a,\ r'))$ and $U(x_1,\ a',\ r)\cap\Omega_1-(\Omega_0\cup U(x_1,\ a,\ r))$ for these Q, R and g. This implies (4.4.6) and (4.4.7) and completes the proof of Lemma 4.4.4.

Now let us return to the proof of (4.4.5). Since $\bigcup_{a>0}(U\cap U(x_1,\ a,\ r))=U\cap F^{-1}(W(x_1,\ r))$, we can apply Mittag-Leffler's theorem, and we obtain

$$\begin{aligned}
&\operatorname{Ext}^j(\mathfrak{M}_{\cdot},\ \mathbb{R}\Gamma_{\Omega_1-\Omega_0}(F^{-1}(W(x_1,\ r)\cap\Omega_1);\ \mathcal{O}_X))\\
&\quad=\varprojlim_a \operatorname{Ext}^j(\mathfrak{M}_{\cdot},\ \mathbb{R}\Gamma_{\Omega_1-\Omega_0}(U(x_1,\ a,\ r)\cap\Omega_1;\ \mathcal{O}_X))\\
&\quad=\varprojlim_a \operatorname{Ext}^j(\mathfrak{M}_{\cdot},\ \mathbb{R}\Gamma_{\Omega_1-\Omega_0}(U(x_1,\ a,\ r')\cap\Omega_1;\ \mathcal{O}_X))\\
&\quad=\operatorname{Ext}^j(\mathfrak{M}_{\cdot},\ \mathbb{R}\Gamma_{\Omega_1-\Omega_0}(F^{-1}(W(x_1,\ r')\cap\Omega_1);\ \mathcal{O}_X)).
\end{aligned}$$

Thus we find (4.4.5). Hence, if we put

$$\mathscr{F}^{\cdot}=\mathbb{R}(F|_{\Omega_1})_*\mathbb{R}\operatorname{Hom}(\mathfrak{M}_{\cdot},\ \mathbb{R}\Gamma_{\Omega_1-\Omega_0}(\varphi_*\mathcal{O}_X))$$

then we have

$$H^j(W(x_1,\ r);\ \mathscr{F}^{\cdot})\xrightarrow{\sim} H^j(W(x_1,\ r');\ \mathscr{F}^{\cdot})\quad \text{for any}\quad j.$$

Note that we can take a representative $\mathscr{F}^{\cdot}$ such that $\mathscr{F}^j=0$ for $j<0$. Then the property (a) in Proposition 4.4.3 follows from the following lemma.

Lemma 4.4.5. *Let $\mathscr{F}^{\cdot}$ be a complex of sheaves on an open set A of $\mathbf{R}^n$ such that $\mathscr{F}^j=0$ for $j\ll 0$. Suppose that for any $y_0\in A$ and r such that*

$$A(y_0, r)\underset{\text{def}}{=}\{y\in\mathbf{R}^n;\ |y-y_0|<r\}\subset A,$$

we have

$$H^i(A(y_0, r);\ \mathscr{F}^{\cdot})\xrightarrow{\sim} H^i(A(y_0, r');\ \mathscr{F}^{\cdot})$$

for $0<r'\leqq r$ and any i. Then any cohomology group of $\mathscr{F}^{\cdot}$ is a locally constant sheaf on A.

Proof. First we shall consider the case where $\mathscr{F}^{\cdot}$ is a single complex, i.e., $\mathscr{H}^j(\mathscr{F}^{\cdot})=0$ for $j\neq 0$. In this case, the lemma follows from the following sublemma.

Sublemma 4.4.6. *Let $\mathscr{F}$ be a sheaf on A. Suppose that $\mathscr{F}(A(y_0, r))\to\mathscr{F}(A(y_0, r'))$ is an isomorphism for any y_0, r and r' such that $A(y_0, r)\subset A$ and $0<r'\leqq r$. Then $\mathscr{F}$ is a locally constant sheaf on A.*

Proof of the sublemma. The question being local on A, we can assume $A=\{y;\ |y|<1\}$. It is enough to show that $\mathscr{F}(A)\to\mathscr{F}_{y_0}$ is an isomorphism for any $y_0\in A$, or $\mathscr{F}(A)\to\mathscr{F}(A(y_0, r))$ is an isomorphism for any $y_0\in A$ and any r such that $0<r<1-|y_0|$. Set $A_t=A(ty_0, 1-t|y_0|)$. Then $A_0=A$, $A_1=A(y_0, 1-|y_0|)$ and $A_t\supset A_{t'}$ for $0\leqq t\leqq t'\leqq 1$. If $1-(3t'-2t)|y_0|>0$, then $A_t\supset A_{t'}\supset A(ty_0, 1-(2t'-t)|y_0|)\supset A(t'y_0, 1-(3t'-2t)|y_0|)$. Hence we have a diagram

$$\mathscr{F}(A_t)\xrightarrow{i}\mathscr{F}(A_{t'})\xrightarrow{j}\mathscr{F}(A(ty_0, 1-(2t'-t)|y_0|))$$
$$\xrightarrow{k}\mathscr{F}(A(t'y_0, 1-(3t'-2t)|y_0|)).$$

Since $j\circ i$ and $k\circ j$ are isomorphisms by the assumption, i is an isomorphism. This shows that $\mathscr{F}(A_1)\cong\mathscr{F}(A_0)$. Since $\mathscr{F}(A_1)=\mathscr{F}(A(y_0, 1-|y_0|))\cong\mathscr{F}(A(y_0, r))$, we obtain the desired result. Q.E.D.

We resume the proof of Lemma 4.4.5. Suppose that $\mathscr{H}^j(\mathscr{F}^{\cdot})=0$ for $j<k$. Then there is a triangle

$$\begin{array}{ccc} & \mathscr{H}^k(\mathscr{F}^{\cdot})[-k] & \\ \swarrow & & \nwarrow{\scriptstyle +1} \\ \mathscr{F}^{\cdot} & \longrightarrow & \mathscr{G}^{\cdot} \end{array}$$

with $\mathscr{G}^{\cdot}$ such that $\mathscr{H}^j(\mathscr{G}^{\cdot})=0$ for $j\leqq k$ and $\mathscr{H}^j(\mathscr{G}^{\cdot})=\mathscr{H}^j(\mathscr{F}^{\cdot})$ for $j>k$. Then we have

$$H^k(A(y_0, r);\ \mathscr{F}^{\cdot})=\Gamma(A(y_0, r);\ \mathscr{H}^k(\mathscr{F}^{\cdot})).$$

Hence $\mathscr{H}^k(\mathscr{F}^\cdot)$ is a locally constant sheaf by Sublemma 4.4.6. Hence

$$\mathbf{R}\Gamma(A(y_0, r); \mathscr{H}^k(\mathscr{F}^\cdot)) \cong \mathbf{R}\Gamma(A(y_0, r'); \mathscr{H}^k(\mathscr{F}^\cdot))$$

This implies that

$$\mathbf{R}\Gamma(A(y_0, r); \mathscr{G}^\cdot) \cong \mathbf{R}\Gamma(A(y_0, r'); \mathscr{G}^\cdot).$$

Hence $\mathscr{H}^{k+1}(\mathscr{F}^\cdot) = \mathscr{H}^{k+1}(\mathscr{G}^\cdot)$ is a locally constant sheaf on A. By repeating this procedure, we can prove Lemma 4.4.5.

Thus we have completed the proof of Proposition 4.4.3.

§ 5. Multi-valued Holomorphic Function Solutions of $\mathscr{M}$

Now we shall investigate holomorphic solutions of $\mathscr{M}$ and we shall prove that they are prolonged to multi-valued holomorphic functions with finite determination property. In order to make our discussion smooth, we shall introduce the following module, which is similar to the space of microfunctions. (See S-K-K [24] Chapter I for the theory of microfunction.)

Let $\mathscr{Z}$ be the set of closed subsets Z of $\mathbb{C}^{1+n}$ such that the normal cone $C_{q_0}(Z)$ of Z is contained in $\{(t, x); \operatorname{Re} t \geqq 0\}$. We define

$$C = \varinjlim_{Z \in \mathscr{Z}} \mathscr{H}^1_Z(\mathscr{O}_{\mathbb{C}^{1+n}})_{q_0} \tag{4.5.1}$$

Clearly any $Z \in \mathscr{Z}$ is contained in some $Z' \in \mathscr{Z}$ such that $\mathbb{C}^{1+n} - Z'$ is convex in a neighborhood of q_0. Hence we have

$$\varinjlim_{Z \in \mathscr{Z}} \mathscr{H}^k_Z(\mathscr{O}_{\mathbb{C}^{1+n}})_{q_0} = 0 \qquad \text{for} \quad k \neq 1 \tag{4.5.2}$$

by Oka-Cartan's theorem.

Set $W(a) = \{(t, x); \operatorname{Re} t < -a(|\operatorname{Im} t| + |x|)\}$ for $a > 0$. Then $W(a) \cap Z = \phi$ in a neighborhood of q_0 for any $a > 0$ and $Z \in \mathscr{Z}$.

Lemma 4.5.1. *The vector space C is canonically endowed with a structure of $\mathscr{E}^{\mathbf{R}}_{p_0}$-module.*

Proof. Set $G(a) = \{(t, x); \operatorname{Re} t \leqq -a(|\operatorname{Im} t| + |x|)\}$. Then $\mathscr{E}^{\mathbf{R}}_{p_0}$ is, by the definition, the inductive limit of $\mathfrak{E}(G(a); D')$, where D' runs over the set of $G(a)$-open neighborhoods of q_0. For any Z in $\mathscr{Z}$, there is $Z' \in \mathscr{Z}$ containing Z and $G(a)$-closed in a neighborhood of q_0. Hence $\mathscr{H}^1_{Z'}(\mathscr{O}_{\mathbb{C}^{1+n}})_{q_0}$ has a structure of $\mathfrak{C}(G(a); D')$-module. Taking the inductive limit, we obtain the desired result. Q.E.D.

Each element of C is a modulo class η of a holomorphic function defined on $V-Z$ for some open neighborhood V of q_0 and some $Z\in\mathscr{Z}$. We shall call this holomorphic function a *representative* of η.

We shall make explicit the action of operators in $\tilde{\mathscr{E}}_{q_0}$ on C. Let P be an element of $\tilde{\mathscr{E}}_{q_0}$ and η an element of C. Let $(P_0(t, x, D_x), K(t_1, t_2, x, D_x))$ the representative of P and φ a representative of η. Then, for $0<-\lambda\ll 1$, $(P)_\lambda\varphi = P_0(t, x, D_x)\varphi(t, x)+\int_\lambda^t K(t, s, z, D_x)\varphi(s, x)ds$ gives an element of C. This element equals $P\eta$ by Proposition 3.1.5 and Proposition 3.2.1 in Chapter III. See also Case 1 discussed in Section 3, Chapter III.

Theorem 4.5.2. *Let ϕ be an element of* $\mathrm{Hom}_{\mathscr{E}_{p_0}}(\mathscr{M}_{p_0}, C)$, *$s$ an element of $\mathscr{M}_{p_0}$ and φ a representative of $\phi(s)$. Then there are an open neighborhood V of q_0 and multi-valued holomorphic function $\tilde{\varphi}$ on $V-S$ such that a branch of $\tilde{\varphi}$ coincides with φ on $V-Z$ for some $Z\in\mathscr{Z}$.*

Now we note that we can choose $Z\in\mathscr{Z}$ and an open neighborhood V of q_0 so that φ is defined on $V-Z$. Moreover we can assume that $\mathscr{M}$ is generated by s. Hence by Corollary 3.5.7 in Chapter III, Section 5, $\mathscr{M}_{p_0}=\mathscr{E}/(\mathscr{E}P_1+\cdots+P\mathscr{E}_N)$ with $P_j\in\tilde{\mathscr{E}}_{q_0}$. Further we can assume that P_j are of order <0, and hence P_j is represented by the kernel $K_j(t_1, t_2, x, D_x)$. Then, as has already been noted, the integral

$$\int_\lambda^t K_j(t, s, x, D_x)\varphi(s, x)ds \qquad (0<-\lambda\ll 1)$$

gives P_j's and hence this function is a holomorphic function defined on a neighborhood of q_0 for $j=1,\dots, N$.

When these additional conditions are satisfied, we can employ the same arguments as in [13]. Here we do not repeat the detailed arguments but leave them to the reader.

Set $\pi=\pi_1(B(\varepsilon, \delta)-S)$ for $0<\varepsilon\ll\delta\ll 1$. Then π does not depend on the choice of ε and δ. By replacing ε_0 and δ_0 with smaller ones, we may assume

$$\pi=\pi_1(B(\varepsilon, \delta)-S) \text{ if } 0<\varepsilon\leqq\varepsilon_0,\ 0<\delta\leqq\delta_0 \text{ and } \varepsilon<\delta. \tag{4.5.3}$$

Theorem 4.5.3. *Let ϕ, s, φ and $\tilde{\varphi}$ be as in Theorem* 4.5.2. *Let $\mathfrak{a}$ be the ideal of $\mathbb{C}[\pi]$ consisting of $\sigma\in\mathbb{C}[\pi]$ such that $\sigma(\tilde{\varphi})$ is holomorphic on a neighborhood of q_0. Let P be a micro-differential operator* (*of infinite order*) *defined on a neighborhood of* p_0 *and* ψ *a representative of* $\phi(Ps)$. *Then* ψ *is also continued to a multi-valued holomorphic function* $\tilde{\psi}$ *defined on* $V'-S$ *for some neighborhood* V' *of* q_0. *Moreover* $\sigma(\tilde{\psi})$ *is holomorphic on a neighborhood of* q_0 *for any* $\sigma\in\mathfrak{a}$.

Proof. We can assume from the beginning that $\mathcal{M}=\mathcal{E}s$. Then $\mathcal{M}_{p_0}$ is generated by s over $\tilde{\mathcal{E}}_{q_0}$ by Corollary 3.5.7 in Chapter III, Section 5. By Corollary 3.5.6 there, $\mathcal{M}^{\infty}_{p_0}=\tilde{\mathcal{E}}^{\infty}_{p_0}\mathcal{M}_{p_0}$, and hence $\mathcal{M}^{\infty}_{p_0}=\tilde{\mathcal{E}}^{\infty}_{q_0}s$. Thus we can assume that P belongs to $\tilde{\mathcal{E}}^{\infty}_{q_0}$. If P is a linear differential operator, then this theorem is obvious because $\psi-P\varphi$ is holomorphic on a neighborhood of q_0. Since any element of $\tilde{\mathcal{E}}^{\infty}_{q_0}$ is a sum of a linear differential operator and an integro-differential operator $K(t_1, t_2, x, D_x)$, we may assume from the first that P is given by $K(t_1, t_2, x, D_x)$.

Assume that φ is holomorphic on $B(\varepsilon, \delta)-W(1)$ for $0\ll\varepsilon\ll\delta\ll 1$ and $K(t_1, t_2, x, D_x)$ is defined for $|t_1|, |t_2|<\delta$, $|x|<\varepsilon$. Take a real λ such $0<-\lambda<\varepsilon$. Then, by the definition of the action of P on C (cf. Propositions 3.1.5 and 3.2.1 in Chapter III), $\psi(t, x)-\int_{\lambda}^{t} K(t, s, z, D_x)\varphi(s, x)dx$ is holomorphic on a neighborhood of the origin. Hence we can assume, without loss of generality,

$$\psi(t, x)=\int_{\lambda}^{t} K(t, s, x, D_x)\varphi(s, x)ds.$$

Hence, as shown in Chapter III, Section 4, ψ is continued to a multi-valued holomorphic function on $B(\varepsilon, \delta)-F^{-1}(H)-S$.

Hence, in order to show that ψ is continued to a multi-valued function on $B(\varepsilon, \delta)-S$, it is enough to show that ψ does not have singularities on $F^{-1}(H)$.

We shall now employ the same method as in p. 127 of [13]. We take another fibering F', where F' is the map from X to $\mathbb{C}^n$ defined by $(t, x)\mapsto(x_1+t^2, x_2+t^3,\dots, x_n+t^{n+1})$. Let H' be the image of the points of S where $F'|_S\colon S\to\mathbb{C}^n$ is not a local isomorphism. Then, by the same argument as above, ψ is continued to a multi-valued holomorphic function on $V-S-F'^{-1}(H')$ for a neighborhood V. Hence, if we can prove that $F^{-1}(H)\cap F'^{-1}(H')$ has codimension $\geqq 2$, then $F^{-1}(H)\cap F'^{-1}(H')$ is a removable singularity and we can continue ψ to a multi-valued holomorphic function on $V-S$ (Proposition 3.4.2 in Chapter III, § 4.).

Lemma 4.5.4. $F^{-1}(H)\cap F'^{-1}(H')$ *has codimension* $\geqq 2$.

Proof. If not, there is a hypersurface T which is an irreducible component of $F^{-1}(H)$ and $F'^{-1}(H')$ at once. T is a union of fiber of F (resp., F'). Therefore the vector fields $\partial/\partial t$ and $\frac{\partial}{\partial t}-2t\frac{\partial}{\partial x_1}-3t^2\frac{\partial}{\partial x_2}-\cdots-(n+1)t^n\frac{\partial}{\partial x_n}$ are tangent to V. Hence any vector field in the Lie algebra generated by these two vector fields is tangent to V. However, this Lie algebra contains $\partial/\partial t$, $\partial/\partial x_1,\ldots,$ $\partial/\partial x_n$. This contradicts the fact that T is a hypersurface. Q.E.D.

Thus we have proved that ψ is continued to a multi-valued holomorphic function on $V-S$ for a neighborhood V of q_0.

In order to prove that $\sigma(\psi)$ is holomorphic on a neighborhood of q_0 for $\sigma\in\mathfrak{a}$, we shall use the results in Section 4 of Chapter III. Let $[\varphi]$ and $[\psi]$ be the elements of F/G (given in (3.4.6) and (3.4.7)) corresponding to φ and ψ, respectively. Then we have $[\psi]=P[\varphi]$. Since the action of $\boldsymbol{C}[\pi]$ on F/G commutes with that of $\tilde{\mathscr{E}}^{\infty}_{q_0}$, we have $\sigma([\psi])=P(\sigma([\varphi]))=0$. This means that $\sigma(\psi)$ is a multi-valued holomorphic function defined on $V-F^{-1}(H)$ for a neighborhood V of q_0. On the other hand, $\sigma(\psi)$ is continued to a multi-valued function on $V-S$. Hence $\sigma(\psi)$ is holomorphic on a neighborhood of q_0 by Proposition 3.4.2. This completes the proof of Theorem 4.5.3.

§6. Proof of Theorem 4.1.1.

Now, let us investigate the structure of the group $\mathrm{Hom}_{\mathscr{E}_{p_0}}(\mathscr{M}_{p_0}, C)$.

Since $\mathscr{E}^{\boldsymbol{R}}_{p_0}\underset{\mathfrak{E}(G_0;D)}{\otimes}\mathfrak{M}_{\cdot}$ is quasi-isomorphic to $\mathscr{M}^{\boldsymbol{R}}_{p_0}$ by (4.3.4), we have

$$\mathrm{Hom}_{\mathscr{E}_{p_0}}(\mathscr{M}_{p_0}, C)=\mathrm{Hom}_{\mathfrak{E}(G_0;D)}(\mathfrak{M}_{\cdot}, C)\,. \tag{4.6.1}$$

The module C is, by the definition, $\varinjlim_{Z\in\mathscr{Z},V} H^1_Z(V;\mathscr{O}_X)$, where V ranges over a neighborhood system of q_0. Since $S+G^a_0\in\mathscr{Z}$, $Z\cup(S+G^a_0)$ belongs to $\mathscr{Z}$ for any $Z\in\mathscr{Z}$. Hence we may assume that Z contains S and that Z is G_0-closed. Hence we have

$$\mathbf{R}\,\mathrm{Hom}\,\mathscr{E}_{p_0}(\mathscr{M}_{p_0}, C)=\varinjlim_{\substack{Z\in\mathscr{Z}\\ Z\supset S\\ 0<\varepsilon<\delta}}\mathbf{R}\,\mathrm{Hom}_{\mathfrak{E}(G_0;D)}(\mathfrak{M}_{\cdot}, \mathbf{R}\Gamma_{B(\varepsilon,\delta)\cap Z}(B(\varepsilon,\delta);\mathscr{O}))[1]\,.$$

By using Proposition 4.4.3, we find that, if $\overline{B(\varepsilon,\delta)}\subset U$

$$\begin{aligned}&\mathbf{R}\,\mathrm{Hom}_{\mathfrak{E}(G_0;D)}(\mathfrak{M}_{\cdot}, \mathbf{R}\Gamma_{B(\varepsilon,\delta)\cap Z}(B(\varepsilon,\delta);\mathscr{O}))\\ &\quad=\mathbf{R}\,\mathrm{Hom}_{\mathfrak{E}(G_0;D)}(\mathfrak{M}_{\cdot}, \mathbf{R}\Gamma_{B(\varepsilon,\delta)-W(1)}(B(\varepsilon,\delta);\mathscr{O}))\,,\end{aligned} \tag{4.6.2}$$

where $W(1)=\{(t,x);\ \mathrm{Re}\,t<-(|\mathrm{Im}\,t|+|x|)\}$.

Take $0<\varepsilon_1<\delta_1$ such that $\overline{B(\varepsilon_1,\delta_1)}\subset U$. Then Proposition 4.4.3 shows that the both sides of (4.6.2) are independent of ε and δ if $0<\varepsilon<\delta$, $\varepsilon\leqq\varepsilon_1$ and $\delta\leqq\delta_1$. Thus we obtain

$$\text{(4.6.3)} \qquad \mathrm{Hom}_{\mathscr{E}_{p_0}}(\mathscr{M}, C)=\mathrm{Ext}^1(\mathfrak{M}_., \mathbf{R}\Gamma_{B(\varepsilon_1,\delta_1)-W(1)}(B(\varepsilon_1, \delta_1); \mathscr{O})).$$

Let us take a point x_1 of $B(\varepsilon_1, \delta_1)-W(1)$ such that the condition (4.2.10) is satisfied. For $0<r<\varepsilon_1-|x_1|$, we denote by $V(r)$ the open ball centered at x_1 with radius r. Then, by Proposition 4.4.3, we have

$$\begin{aligned}&\mathrm{Ext}^1(\mathfrak{M}_., \mathbf{R}\Gamma_{B(\varepsilon_1,\delta_1)-W(1)}(B(\varepsilon_1, \delta_1); \mathscr{O}))\\ &=\mathrm{Ext}^1(\mathfrak{M}_., \mathbf{R}\Gamma_{F^{-1}(V(r))\cap B(\varepsilon_1,\delta_1)-W(1)}(F^{-1}(V(r))\cap B(\varepsilon_1, \delta_1); \mathscr{O})).\end{aligned}$$

There are positive numbers r_0, ρ_0 and holomorphic functions h_j $(1\leqq j\leqq N)$ defined on $V(r_0)$ such that $0<r_0<\varepsilon_1-|x_1|$, $F^{-1}(V(r_0))\cap B(\varepsilon_1, \delta_1)\cap S=\bigsqcup_{j=1}^{N}\{(t, x); x\in V(r_0), t=h_j(x)\}$ and that $|\mathrm{Im}\,(h_j(x)-h_k(x))|>2\rho_0$ for $x\in V(r_0)$ $(j\neq k)$. We denote by S_j the hypersurface $\{(t, x); x\in V(r_0), t=h_j(x)\}$ $(j=1,\dots, N)$. For $0<\rho<\rho_0$ and $0<r<r_0$, set

$$\Omega_1(r, \rho)=\bigsqcup_{j=1}^{N}\{(t, x); |x-x_1|<r, |t-h_j(x)|<\rho\}+G_0$$

and define $\Omega_0(r, \rho)$ by

$$\Omega_0(r, \rho)=\Omega_1(r, \rho)-\bigcup_{j=1}^{N}(\{(t, x); |x-x_1|<r, |t-h_j(x)|<\rho\}+G_0^a).$$

The open sets $\Omega_1(r, \rho)$ and $\Omega_0(r, \rho)$ are the union of disjoint N open sets. Then again by Proposition 4.4.3, we have

$$\text{(4.6.4)} \qquad \begin{aligned}&\mathrm{Ext}^1(\mathfrak{M}_., \mathbf{R}\Gamma_{F^{-1}(V(r))\cap B(\varepsilon_1,\delta_1)-W(1)}(F^{-1}(V(r))\cap B(\varepsilon_1, \delta_1); \mathscr{O}))\\ &=\mathrm{Ext}^1(\mathfrak{M}_., \mathbf{R}\Gamma_{\Omega_1(r,\rho)-\Omega_0(r,\rho)}(\Omega_1(r, \rho); \mathscr{O})).\end{aligned}$$

Let p_j denote the point $(h_j(x_1), x_1; 1, -\mathrm{grad}_x h_j(x_1))$ of T_S^*X $(j=1,\dots, N)$. Then we have

$$\varinjlim_{r,\rho}\mathbf{R}\Gamma_{\Omega_1(r,\rho)-\Omega_0(r,\rho)}(\Omega_1(r, \rho); \mathscr{O}_X))=\bigoplus_{j=1}^{N}\mathscr{C}_{S|X,p_j}^{\mathbf{R}}[-1].$$

Finally we obtain the following

Proposition 4.6.1. *$\mathrm{Hom}_{\mathscr{E}_{p_0}}(\mathscr{M}_{p_0}, C)=\bigoplus_{j=1}^{N}\mathscr{H}om_{\mathscr{E}}(\mathscr{M}, \mathscr{C}_{S|X}^{\mathbf{R}})_{p_j}$ and $\mathrm{Ext}^k_{\mathscr{E}_{p_0}}(\mathscr{M}_{p_0}, C)=0$ for $k\neq 0$.*

This implies, in particular, that $\mathrm{Hom}_{\mathscr{E}_{p_0}}(\mathscr{M}_{p_0}, C)$ is finite-dimensional because so is each $\mathscr{H}om_{\mathscr{E}}(\mathscr{M}, \mathscr{C}_{S|X}^{\mathbf{R}})_{p_j}$.

Proposition 4.6.2. *Let s be an element of $\mathscr{M}_{p_0}^{\infty}$. If $\phi(s)=0$ for any ϕ*

$\in \mathrm{Hom}_{\mathscr{E}^\infty_{p_0}}(\mathscr{M}^\infty_{p_0}, C)$, *then* $s=0$.

Proof. Since $\mathscr{M}^\infty_{p_0}=\mathscr{E}^\infty_{p_0}\underset{\tilde{\mathscr{E}}^\infty_{q_0}}{\otimes}\mathscr{M}^\infty_{p_0}$ (Chapter III, §5, Proposition 3.5.5), we have

$$\mathrm{Hom}_{\mathscr{E}^\infty_{p_0}}(\mathscr{M}^\infty_{p_0}, C)=\mathrm{Hom}_{\tilde{\mathscr{E}}^\infty_{q_0}}(\mathscr{M}^\infty_{p_0}, C).$$

By (4.6.3), we have

$$\mathrm{Hom}_{\tilde{\mathscr{E}}^\infty_{q_0}}(\mathscr{M}^\infty_{p_0}, C)=\mathrm{Hom}_{\tilde{\mathscr{E}}^\infty_{q_0}}(\mathscr{M}^\infty_{p_0}, C'),\ \text{where}\ C'=\mathscr{H}^1_{C^{1+n}-W(1)}(\mathscr{O}_X)_{q_0}.$$

Note that C' is an $\tilde{\mathscr{E}}^\infty_{q_0}$-module because $W(1)$ is G_0-open. Since C' contains C as a submodule, we have $\phi(s)=0$ for any $\phi\in\mathrm{Hom}_{\tilde{\mathscr{E}}^\infty_{q_0}}(\mathscr{M}^\infty_{p_0}, C')$. Let $u_1,\dots, u_{N_0}$ be the system of generators given in Section 2. Then $\mathscr{M}^\infty_{p_0}=\sum_j \tilde{\mathscr{E}}^\infty_{q_0}u_j$ by Corollary 3.5.6. Hence we can express $s=\sum_j P_ju_j$ with $P_j\in\tilde{\mathscr{E}}^\infty_{q_0}$. There is a G_0-round open neighborhood D' of q_0 such that P_j is considered as an element of $\mathfrak{E}(G_0; D')$ and also as a section of $\mathscr{E}^\infty$ on $\{(t, x; \tau, \xi); \tau\neq 0, (t, x)\in D'\}$. Now, set $\mathfrak{M}'=H_0(\mathfrak{E}(G_0; D')\underset{\mathfrak{E}(G_0;D)}{\otimes}\mathfrak{M}_\cdot)$ and denote by s_j $(j=1,\dots, N_0)$ the elements of $\mathfrak{M}'$ corresponding to the base of $\mathfrak{M}_0=\mathfrak{E}(G_0, D)^{N_0}$. Let $\tilde{s}$ be the section $\sum_j P_ju_j$ of $\mathscr{M}^\infty$ defined on $\{(t, x; \tau, \xi); \tau\neq 0, (t, x)\in D'\}$. Since $\mathrm{Hom}_{\tilde{\mathscr{E}}^\infty_{q_0}}(\mathscr{M}^\infty_{p_0}, C')$ is finite-dimensional and $C'=\varinjlim_{0<\varepsilon<\delta} H^1_{B(\varepsilon,\delta)-W(1)}(B(\varepsilon, \delta); \mathscr{O}_X)$, there is $0<\varepsilon<\delta$ such that $\sum P_j\phi(s_j)=0$ for $\phi\in\mathrm{Hom}_{\mathfrak{E}(G_0,D')}(\mathfrak{M}', H^1_{B(\varepsilon,\delta)-W(1)}(B(\varepsilon, \delta); \mathscr{O}_X))$.

Now let us take $x_1\in B(\delta)$, S_j, $\Omega_1(r, \rho)$, $\Omega_0(r, \rho)$ and p_j as before. Then, by (4.6.4), the map

$$H^1_{B(\varepsilon,\delta)-W(1)}(B(\varepsilon, \delta); \mathscr{O}_X)\longrightarrow H^1_{\Omega_1(r,\rho)-\Omega_0(r,\rho)}(\Omega_1(r, \rho); \mathscr{O}_X)$$

induces an isomorphism

$$\begin{aligned}&\mathrm{Hom}_{\mathfrak{E}(G_0;D')}(\mathfrak{M}', H^1_{B(\varepsilon,\delta)-W(1)}(B(\varepsilon, \delta); \mathscr{O}_X)\\ &\quad\xrightarrow{\sim}\mathrm{Hom}_{\mathfrak{E}(G_0;D')}(\mathfrak{M}', H^1_{\Omega_1(r,\rho)-\Omega_0(r,\rho)}(\Omega_1(r, \rho); \mathscr{O}_X)).\end{aligned}$$

Hence we obtain $\sum_j P_j\phi(s_j)=0$ for any ϕ in

$$\mathrm{Hom}_{\mathfrak{E}(G_0;D')}(\mathfrak{M}', H^1_{\Omega_1(r,\rho)-\Omega_0(r,\rho)}(\Omega_1(r, \rho); \mathscr{O}_X)).$$

Hence, by taking the inductive limit on r and ρ, we have $\sum P_j\phi(s_j)=0$ for any $\phi\in\mathrm{Hom}_{\mathfrak{E}(G_0,D)}(\mathfrak{M}', \bigoplus_j \mathscr{C}^{\mathbf{R}}_{S_j|X,p_j})$, or equivalently, $\phi(\tilde{s})=0$ for any $\phi\in \mathrm{Hom}_{\mathscr{E}^\infty_{p_j}}(\mathscr{M}^\infty_{p_j}, \mathscr{C}^{\mathbf{R}}_{S_j|X,p_j})$. By Theorem 1.3.1 of Chapter I, $\mathscr{M}^{\mathbf{R}}=\mathscr{E}^{\mathbf{R}}\underset{\mathscr{E}}{\otimes}\mathscr{M}$ is locally isomorphic to a direct sum of copies of $\mathscr{C}^{\mathbf{R}}_{S_j|X}$ at p_j, and hence we obtain $\tilde{s}=0$ as an element of $\mathscr{M}^{\mathbf{R}}_{p_j}$. Since $\mathscr{M}^\infty_{p_j}$ is contained in $\mathscr{M}^{\mathbf{R}}_{p_j}$, we have $\tilde{s}_{p_j}=0$, i.e., $\mathrm{supp}\,\tilde{s}\not\ni p_j$. Since $\mathrm{supp}\,\tilde{s}$ is a union of irreducible components of $\Lambda\cap\pi^{-1}(D')$

by Proposition 1.3.8 of Chapter I, supp $\tilde{s}=\phi$ and hence $s=0$. Q.E.D.

Let $s_1,\dots, s_{N_0}$ be the system of generators of $\mathcal{M}$ given in Section 3 of this chapter. Let ϕ be an element of $\mathrm{Hom}_{\mathscr{E}_{p_0}}(\mathcal{M}_{p_0}, C)$, and φ_j a representative of $\phi(s_j)$. Then, as is shown in Theorem 4.5.2, there is an open neighborhood V of q_0 such that φ_j can be continued to a multi-valued holomorphic function defined on $V-S$. Since ϕ ranges only over a finite-dimensional vector space, we may assume that V does not depend on ϕ (for a suitable choice of φ_j). We can take V as $B(\varepsilon_1, \delta_1)$ for $0<\varepsilon_1<\delta_1\ll 1$. Let ι be the projection from the universal covering $\widetilde{B(\varepsilon_1, \delta_1)-S}$ of $B(\varepsilon_1, \delta_1)-S$ onto $B(\varepsilon_1, \delta_1)-S$ and let us fix a section β of ι over $B(\varepsilon_1, \delta_1)-W(1)$. Let $\tilde{\varphi}_j$ be an analytic continuation of $\varphi_j\circ\beta^{-1}$. Then we have the following

Proposition 4.6.3. *For $\sigma\in\pi=\pi_1(B(\varepsilon_0, \delta_0)-S)=\pi_1(B(\varepsilon_1, \delta_1)-S)$, the map $s_j\mapsto(\sigma(\tilde{\varphi}_j))\circ\beta$ defines an $\mathscr{E}^{\infty}_{p_0}$-linear homomorphism from $\mathcal{M}^{\infty}_{p_0}$ into C.*

Proof. By Section 2, s_j satisfies the fundamental relations

$$\sum_j P_{ij}s_j=0 \quad (i=1,\dots, N_1)$$

with $P_{ij}\in\tilde{\mathscr{E}}_{q_0}$ such that $\mathrm{ord}\, P_{ij}<0$. Since ϕ is $\mathscr{E}^{\infty}_{p_0}$-linear, for $0<-\lambda\ll 1$, $\sum_j (P_{ij})_\lambda\tilde{\varphi}_j$ is holomorphic on a neighborhood of q_0. (Proposition 3.1.5 and Proposition 3.2.1 in Chapter III.) On the other hand, $\sigma(\sum_j (P_{ij})\tilde{\varphi}_j)$ is equal to $\sum_j (P_{ij})_\lambda\sigma(\tilde{\varphi}_j)$ modulo multi-valued holomorphic functions defined on $V'-F^{-1}(H)$ for a neighborhood V' of q_0. (Chapter III, §4.) Hence $\sum_j (P_{ij})_\lambda\sigma(\tilde{\varphi}_j)$ is a multi-valued holomorphic function defined on $V'-F^{-1}(H)$ for a neighborhood V' of q_0.

If we apply the same argument for the fibering F' (cf. the proof of Theorem 4.5.3), $\sum_j (P_{ij})_\lambda\sigma(\tilde{\varphi}_j)$ is seen to be homomorphic on $V'-F'^{-1}(H')$. Therefore $\sum_j (P_{ij})_\lambda\sigma(\tilde{\varphi}_j)$ is holomorphic at the origin. Q.E.D.

In what follows, we denote by ϕ^σ the map defined in the above proposition.

§7. Proof of Theorem 4.1.1 (Continued)

In order to prove Theorem 4.1.1, we shall make full use of the results of Chapter III, Section 4.

Let $\mathfrak{c}$ be the subset of $\boldsymbol{C}[\pi]$ consisting of $\sigma\in\boldsymbol{C}[\pi]$ such that $\sigma(\varphi)$ is holomorphic on a neighborhood of q_0 for any $\phi\in\mathrm{Hom}_{\mathscr{E}_{p_0}}(\mathcal{M}_{p_0}, C)$ and any representative φ of any element of $\phi(\mathscr{M}_{p_0})$. Then $\mathfrak{c}$ is a two-sided ideal of $\boldsymbol{C}[\pi]$, and the vector space $\boldsymbol{C}[\pi]/\mathfrak{c}$ has a finite dimension. In fact, $\mathfrak{c}$ is the kernel of the homomorphism

$$\boldsymbol{C}[\pi]\ni\sigma\longmapsto(\phi\mapsto\phi^{\sigma})\in\mathrm{End}_{\boldsymbol{C}}(\mathrm{Hom}_{\mathscr{E}_{p_0}}(\mathscr{M}_{p_0}, C)).$$

Set $\mathfrak{a}=\sum_{\gamma\in\pi}(\gamma-1)\mathfrak{c}$. Then, by Lemma 3.4.6, $\boldsymbol{C}[\pi]/\mathfrak{a}$ is also finite-dimensional. Let $\mathscr{L}$ be the holonomic system of D-type with singularities along S and with the monodromy type $\mathfrak{a}$, i.e., $\mathscr{L}^{\infty}_{q_0}$ (resp., $\mathscr{L}_{q_0}$) is the set of multi-valued holomorphic functions φ defined on $V-S$ for a neighborhood V of q_0 such that $\sigma(\varphi)=0$ for any $\sigma\in\mathfrak{a}$ (resp., and in the Nilsson class). Let $\mathscr{P}$ be the coherent $\mathscr{D}_X$-sub-Module of $\mathscr{L}$ such that $\mathscr{P}_{q_0}$ is the image of $\mathscr{O}_{q_0}\underset{\boldsymbol{C}}{\otimes}\mathscr{H}om_{\mathscr{D}}(\mathscr{O}, \mathscr{L})_{q_0}\to\mathscr{L}_{q_0}$ defined by $f\otimes\chi\longmapsto\chi(f)$.

Since $\mathscr{H}om_{\mathscr{D}}(\mathscr{O}, \mathscr{L})_{q_0}=\boldsymbol{C}$, such a $\mathscr{P}$ exists and $\mathscr{P}$ is isomorphic to $\mathscr{O}$. Moreover $\mathscr{H}om_{\mathscr{D}}(\mathscr{O}, \mathscr{P})_{q_0}\xrightarrow{\sim}\mathscr{H}om_{\mathscr{D}}(\mathscr{O}, \mathscr{L})_{q_0}$. Set $\mathscr{N}=\mathscr{L}/\mathscr{P}$. Then we have

$$\mathscr{H}om_{\mathscr{D}}(\mathscr{O}, \mathscr{N})_{q_0}=0. \tag{4.7.1}$$

Let ϕ be an element of $\mathrm{Hom}_{\mathscr{E}_{p_0}}(\mathscr{M}_{p_0}, C)$. Let s be an element of $\mathscr{M}^{\infty}_{p_0}$ and let φ be a representative of $\phi(s)$. Since $\sigma(\varphi)$ is holomorphic at q_0 for any $\sigma\in\mathfrak{c}$, $\mathfrak{a}\varphi=0$. Hence φ belongs to $\mathscr{L}^{\infty}_{q_0}$. Since φ is determined up to holomorphic function defined on a neighborhood of q_0, the homomorphism $s\mapsto(\varphi \bmod \mathscr{P}^{\infty}_{q_0})$ is a well-defined $\boldsymbol{C}$-linear map from $\mathscr{M}^{\infty}_{p_0}$ into $\mathscr{N}^{\infty}_{q_0}$. We shall denote this map by $E(\phi)$. This homomorphism $E(\phi)$ is evidently $\mathscr{D}^{\infty}_{q_0}$-linear.

Let $F(\phi)$ be the $\mathscr{D}^{\infty}_{q_0}$-linear homomorphism from $\mathscr{M}^{\infty}_{p_0}$ into $\mathscr{E}^{\infty}_{p_0}\underset{\mathscr{D}^{\infty}_{q_0}}{\otimes}\mathscr{N}^{\infty}_{q_0}$ defined by $s\mapsto 1\otimes E(\phi)(s)$.

We shall now prove

Proposition 4.7.1. *$F(\phi)$ is $\mathscr{E}^{\infty}_{p_0}$-linear.*

At first sight, this proposition might seem obvious. However, this is far from obvious, and, as a matter of fact, this proposition is one of the most essential steps of the proof of Theorem 4.1.1.

In order to prove Proposition 4.7.1, we prepare some lemmas.

Let X' (resp., Y') be the universal covering of $B(\varepsilon_1, \delta_1)-S-F^{-1}(H)$ (resp., $B(\varepsilon_1)-H$) and ι (resp., κ) the projection from X' (resp., $X''\underset{\mathrm{def}}{=}B(\varepsilon_1, \delta_1)\underset{B(\varepsilon_1)}{\times}Y'$) onto $B(\varepsilon_1, \delta_1)-S-F^{-1}(H)$ (resp., $B(\varepsilon_1, \delta_1)-F^{-1}(H)$). Set $F=\varinjlim_{U}\mathscr{O}_{X'}(\iota^{-1}(U-S-F^{-1}(H))$ and $G=\varinjlim_{U}\mathscr{O}_{X''}(\kappa^{-1}(U-F^{-1}(H)))$, where U ranges over a neighborhood system of q_0. Then, as was shown in Chapter III, Section 4, the quotient F/G is an $\tilde{\mathscr{E}}^{\infty}_{q_0}$-module. For any left ideal $\mathfrak{b}$ of $\mathbf{C}[\pi]$ contained in $\mathfrak{a}$ such that $\dim \mathbf{C}[\pi]/\mathfrak{b}<\infty$, let $\mathscr{L}(\mathfrak{b})$ be a holonomic system of D-type with singularities along $S \cup F^{-1}(H)$ and the monodromy type $\mathfrak{b}$. (See §4.3 of Chapter III). Then $\mathscr{L}$ is a sub-Module of $\mathscr{L}(\mathfrak{b})$ and $\mathscr{L}(\mathfrak{b})^{\infty}_{q_0}$ is contained in F. Let $\mathscr{G}(\mathfrak{b})$ be the $\mathscr{E}_X$-sub-Module of $\mathscr{L}(\mathfrak{b})$ consisting of sections which do not have singularities on S, i.e., $\mathscr{G}(\mathfrak{b})$ is a holonomic $\mathscr{D}_X$-Module of D-type with singularities along $F^{-1}(H)$ and with the monodromy type $\mathfrak{b}$. We have $\mathscr{G}(\mathfrak{b})^{\infty}_{q_0} \cap \mathscr{L}^{\infty}_{q_0} = \mathscr{P}_{q_0}$, because a function in $\mathscr{G}(\mathfrak{b})^{\infty}_{q_0} \cap \mathscr{L}^{\infty}_{q_0}$ has singularities neither on S nor on $F^{-1}(H)$. Therefore we have

$$\mathscr{L}^{\infty}_{q_0}/\mathscr{P}_{q_0} \overset{i}{\hookrightarrow} \mathscr{L}(\mathfrak{b})^{\infty}_{q_0}/\mathscr{G}(\mathfrak{b})^{\infty}_{q_0} \overset{j}{\hookrightarrow} F/G. \tag{4.7.2}$$

Since $P\in\tilde{\mathscr{E}}^{\infty}_{q_0}$ operates by $(P)_\lambda$ on C and on F/G, we have the following:

(4.7.3) The homomorphism $j\circ i\circ E(\phi)\colon \mathscr{M}^{\infty}_{p_0}\to F/G$ is $\tilde{\mathscr{E}}^{\infty}_{q_0}$-linear.

Lemma 4.7.2. *There exists a left ideal* $\mathfrak{b}\subset\mathfrak{a}$ *of* $\mathbf{C}[\pi]$ *satisfying the following conditions*

$$\dim_{\mathbf{C}} \mathbf{C}[\pi]/\mathfrak{b}<\infty \tag{4.7.4}$$

$$\mathscr{M}^{\infty}_{p_0} \longrightarrow \tilde{\mathscr{E}}^{\infty}_{q_0} \underset{\mathscr{D}^{\infty}_{q_0}}{\otimes} (\mathscr{L}(\mathfrak{b})/\mathscr{G}(\mathfrak{b}))^{\infty}_{q_0} \tag{4.7.5}$$

defined by $s\mapsto 1\otimes i\circ E(\phi)(s)$ *is* $\tilde{\mathscr{E}}^{\infty}_{q_0}$*-linear, where* i *is the homomorphism from* $\mathscr{N}^{\infty}_{q_0}$ *into* $(\mathscr{L}(\mathfrak{b})/\mathscr{G}(\mathfrak{b}))^{\infty}_{q_0}$.

Proof. By Proposition 3.4.9, there exists a left ideal $\mathfrak{b}\subset\mathfrak{a}$ satisfying (4.7.4) and the following condition:

(4.7.6) The image of the homomorphism from $\tilde{\mathscr{E}}^{\infty}_{q_0} \underset{\mathscr{D}^{\infty}_{q_0}}{\otimes} \mathscr{L}(\mathfrak{a})^{\infty}_{q_0}$ into $\tilde{\mathscr{E}}^{\infty}_{q_0} \underset{\mathscr{D}^{\infty}_{q_0}}{\otimes} \mathscr{L}(\mathfrak{b})^{\infty}_{q_0}$ is contained in the image of the homomorphism $\mathscr{L}(\mathfrak{b})^{\infty}_{q_0}$ into $\tilde{\mathscr{E}}^{\infty}_{q_0} \underset{\mathscr{D}^{\infty}_{q_0}}{\otimes} \mathscr{L}(\mathfrak{b})^{\infty}_{q_0}$.

Let s be an element of $\mathscr{M}^{\infty}_{p_0}$ and P an element of $\tilde{\mathscr{E}}^{\infty}_{q_0}$. By (4.7.6), there exists $u\in(\mathscr{L}(\mathfrak{b})/\mathscr{G}(\mathfrak{b}))^{\infty}_{q_0}$ such that $P\otimes i\circ E(\phi)s=1\otimes u$ holds as an equality in $\tilde{\mathscr{E}}^{\infty}_{q_0} \underset{\mathscr{D}^{\infty}_{q_0}}{\otimes} (\mathscr{L}(\mathfrak{b})/\mathscr{G}(\mathfrak{b}))^{\infty}_{q_0}$. Therefore we have $P\otimes j\circ i\circ E(\phi)(s)=1\otimes j(u)$ in $\tilde{\mathscr{E}}^{\infty}_{q_0} \underset{\mathscr{D}^{\infty}_{q_0}}{\otimes} (F/G)$. By applying the homomorphism $\tilde{\mathscr{E}}^{\infty}_{q_0} \underset{\mathscr{D}^{\infty}_{q_0}}{\otimes} (F/G)\ni Q\otimes v\mapsto Qv\in F/G$, we obtain $Pj\circ i\circ E(\phi)(s)=j(u)$. Since $j\circ i\circ E(\phi)$ is $\tilde{\mathscr{E}}^{\infty}_{q_0}$-linear, this shows $j\circ i\circ E(\phi)(Ps)=j(u)$, and hence $i\circ E(\phi)(Ps)=u$. Thus we obtain $P\otimes i\circ E(\phi)s=1\otimes i\circ E(\phi)(Ps)$, and Lemma 4.7.2 is proved. Q.E.D.

Now, let us prove Proposition 4.7.1. Since $\mathscr{M}^{\infty}_{p_0}\to\tilde{\mathscr{E}}^{\infty}_{q_0} \underset{\mathscr{D}^{\infty}_{q_0}}{\otimes} (\mathscr{L}(\mathfrak{b})/\mathscr{G}(\mathfrak{b}))^{\infty}_{q_0}$ is

$\tilde{\mathscr{E}}^\infty_{q_0}$-linear by the preceding lemma, the homomorphism obtained by tensoring $\mathscr{E}^\infty_{p_0}$, $\mathscr{E}^\infty_{p_0} \underset{\tilde{\mathscr{E}}^\infty_{q_0}}{\otimes} \mathscr{M}^\infty_{p_0} \longrightarrow \mathscr{E}^\infty_{p_0} \underset{\tilde{\mathscr{E}}^\infty_{q_0}}{\otimes} (\tilde{\mathscr{E}}^\infty_{q_0} \underset{\mathscr{D}^\infty_{q_0}}{\otimes} (\mathscr{L}(\mathrm{b})/\mathscr{G}(\mathrm{b})^\infty_{q_0}) = \mathscr{E}^\infty_{p_0} \underset{\mathscr{D}^\infty_{q_0}}{\otimes} (\mathscr{L}(\mathrm{b})/\mathscr{G}(\mathrm{b}))^\infty_{q_0}$ (defined by $P\otimes s\mapsto P\otimes E(\phi)(s)$) is $\mathscr{E}^\infty_{p_0}$-linear. On the other hand,

$$\mathscr{E}^\infty_{p_0} \underset{\tilde{\mathscr{E}}^\infty_{q_0}}{\otimes} \mathscr{M}^\infty_{p_0} \longrightarrow \mathscr{M}^\infty_{p_0} \qquad (P\otimes s\mapsto Ps)$$

is an $\mathscr{E}^\infty_{p_0}$-linear isomorphism by Proposition 3.5.5, and hence the homomorphism

$$\mathscr{M}^\infty_{p_0} \longrightarrow \mathscr{E}^\infty_{p_0} \underset{\tilde{\mathscr{E}}^\infty_{q_0}}{\otimes} \mathscr{M}^\infty_{p_0} \quad (\text{defined by } s\mapsto 1\otimes s)$$

is also $\mathscr{E}^\infty_{p_0}$-linear. This implies that

$$\mathscr{M}^\infty_{p_0} \longrightarrow \mathscr{E}^\infty_{p_0} \underset{\mathscr{D}^\infty_{q_0}}{\otimes} (\mathscr{L}(\mathrm{b})/\mathscr{G}(\mathrm{b}))^\infty_{q_0} \quad (\text{defined by } s\mapsto 1\otimes E(\phi)s)$$

is an $\mathscr{E}^\infty_{p_0}$-linear homomorphism. On the other hand, $\mathscr{N}_{q_0}$ is a submodule of $(\mathscr{L}(\mathrm{b})/\mathscr{G}(\mathrm{b}))_{q_0}$ by (4.7.2), and hence $\mathscr{E}^\infty_{p_0} \underset{\mathscr{D}^\infty_{q_0}}{\otimes} \mathscr{N}^\infty_{q_0}$ is a submodule of $\mathscr{E}^\infty_{p_0} \underset{\mathscr{D}^\infty_{q_0}}{\otimes} (\mathscr{L}(\mathrm{b})/\mathscr{G}(\mathrm{b}))^\infty_{q_0}$. This shows that $F(\phi)$ is $\mathscr{E}^\infty_{p_0}$-linear, which completes the proof of Proposition 4.7.1.

Now we are ready to prove Theorem 4.1.1. Let $\{\phi_1,\ldots, \phi_r\}$ be a base of $\mathrm{Hom}_{\mathscr{E}_{p_0}}(\mathscr{M}_{p_0}, C)$. Set

$$\Phi = E(\phi_1)\oplus\cdots\oplus E(\phi_r):\ \ \mathscr{M}^\infty_{p_0}\to\mathscr{N}^r_{q_0}.$$

Then Φ is a $\mathscr{D}^\infty_{q_0}$-linear-homomorphism. Let Ψ denote the $\mathscr{E}^\infty_{p_0}$-linear homomorphism $\mathscr{M}^\infty_{p_0}\to\mathscr{E}^\infty_{p_0} \underset{\mathscr{D}^\infty_{q_0}}{\otimes} \mathscr{N}^r_{q_0}$ defined by $s\mapsto 1\otimes\Phi(s)$. We shall show Φ is injective. Let s be an element of $\mathscr{M}^\infty_{p_0}$ such that $\Phi(s)=0$. Hence, for any $\phi\in \mathrm{Hom}_{\mathscr{E}^\infty_{p_0}}(\mathscr{M}^\infty_{p_0}, C)$, $\phi(s)=0$. Therefore, by Proposition 4.6.2, $s=0$. Therefore Ψ is injective. This completes the proof of Theorem 4.1.1.

Chapter V. Basic Properties of Holonomic Systems with R.S.

In this chapter we prove several basic properties of holonomic systems with R.S. In Section 1 we derive several important properties of holonomic systems with R.S. from the embedding theorem proved in Chapter IV. In Section 2 by the embedding theorem we prove our main theorem which asserts that $\mathscr{E}^\infty_X \underset{\mathscr{E}_X}{\otimes} \mathscr{M} = \mathscr{E}^\infty_X \underset{\mathscr{E}_X}{\otimes} \mathscr{M}_{\mathrm{reg}}$ holds for any holonomic $\mathscr{E}$-Module $\mathscr{M}$. In Section 3 we show that the application of the integration procedure and the restriction procedure to holonomic systems with R.S. yields holonomic systems with R.S. under moderate conditions. Here we essentially use the embedding theorem again. In Section 4 we discuss the restriction of holonomic $\mathscr{D}$-Modules with R.S. to an arbitrary submanifold. In the course of the proof we prove some results which are basic for the proof of comparison theorems given in Chapter VI.

§1.

In this section we first prove the following Theorem 5.1.1. This result is interesting and basic and it will be often used in our later arguments. Next we construct a special $\mathscr{E}(0)$-sub-Module of a holonomic $\mathscr{E}$-Module with R.S. by using the notion of orders. (Theorem 5.1.6). Then using this result we prove Corollary 5.1.7, which clarifies the relationship between the notion of holonomic systems with R.S. and the general notion of systems with regular singularities along an involutory variety. We also use Theorem 5.1.6 to prove that a holonomic $\mathscr{D}$-Module with R.S. has a good filtration.

Theorem 5.1.1. *Let $\mathscr{M}$ be a holonomic $\mathscr{E}_X$-Module with* R.S. *defined on a neighborhood of a point p in $T^*X - T^*_XX$. Suppose that the characteristic variety Λ of $\mathscr{M}$ is in a generic position at p. Then $\mathscr{M}_p$ is a finitely generated $\mathscr{D}_{X,\pi(p)}$-module and we have*

$$\begin{aligned}\mathscr{E}_{X,p'} \underset{\pi^{-1}\mathscr{D}_{X,\pi(p)}}{\otimes} \mathscr{M}_p &= \mathscr{M}_p \quad \textit{for} \quad p'=p \\ &= 0 \quad \textit{for} \quad p' \in \pi^{-1}\pi(p) - T^*_XX - \mathbb{C}^\times p.\end{aligned}$$

Proof. Set $q=\pi(p)$ and $Y=\pi(\Lambda)$. By Theorem 4.1.1, there exist a holonomic system $\mathscr{L}$ of D-type with singularities along Y, a holonomic sub-Module $\mathscr{P}$ of $\mathscr{L}$ isomorphic to a direct sum of $\mathcal{O}$ and a $\mathscr{D}^\infty$-linear homomorphism

$$\psi : \mathscr{M}_p^\infty \longrightarrow (\mathscr{L}/\mathscr{P})_q^\infty$$

such that the composite of ψ with $(\mathscr{L}/\mathscr{P})_q^\infty \to \mathscr{E}_{X,p}^\infty \underset{\mathscr{D}_{X,q}^\infty}{\otimes} (\mathscr{L}/\mathscr{P})_q^\infty = \mathscr{E}_{X,p}^\infty \underset{\mathscr{D}_{X,q}}{\otimes} \mathscr{L}_q$ is an injective $\mathscr{E}_{X,p}^\infty$-linear homomorphism from $\mathscr{M}_p^\infty$ into $\mathscr{E}_{X,p}^\infty \underset{\mathscr{D}_{X,q}}{\otimes} \mathscr{L}_q$. We shall show that $\psi(\mathscr{M}_p)$ is contained in $\mathscr{L}/\mathscr{P}$. Let u be a section of $\mathscr{M}$ and v a section of $\mathscr{L}^\infty$ such that $\psi(u)$ is v modulo $\mathscr{P}=\mathscr{P}^\infty$. Since u satisfies a system of micro-differential equations with R.S., v also satisfies a system of micro-differential equations with R.S. in a neighborhood of p. Hence v belongs to $\mathscr{L}$ at a generic point of Y. Therefore v is a section of $\mathscr{L}$. (Chapter II, §3, Proposition 2.3.5.)

Thus $\mathscr{M}_p$ is a finitely generated $\mathscr{D}_{X,q}$-module. Let $\psi'\colon \mathscr{M}_p \to (\mathscr{L}/\mathscr{P})_q$ be the restriction of ψ. Since ψ' is injective, $1\otimes\psi'\colon \mathscr{E}_p \underset{\mathscr{D}_{X,q}}{\otimes} \mathscr{M}_p \to \mathscr{E}_p \underset{\mathscr{D}_{X,q}}{\otimes} (\mathscr{L}/\mathscr{P})_q$ is injective. Consider the diagram

(5.1.1)
$$\begin{array}{ccc} \mathscr{M}_p & & \\ \uparrow{\scriptstyle\varphi} & \searrow^{\phi} & \\ \mathscr{E}_p \underset{\mathscr{D}_{X,q}}{\otimes} \mathscr{M}_p & \xrightarrow{1\otimes\psi'} & \mathscr{E}_p \underset{\mathscr{D}_{X,q}}{\otimes} (\mathscr{L}/\mathscr{P})_q , \end{array}$$

where φ and ϕ are defined by $\varphi(P\otimes u)=Pu$ and $\phi(u)=1\otimes\psi'(u)$, respectively.

Since ϕ is $\mathscr{E}_p$-linear, the diagram (5.1.1) is commutative. Hence φ is injective because $1\otimes\psi'$ is injective. On the other hand, the surjectivity of φ is clear. Therefore φ is an isomorphism, namely, $\mathscr{E}_p \underset{\mathscr{D}_{X,q}}{\otimes} \mathscr{M}_p = \mathscr{M}_p$.

Lastly we shall show that

$$\mathscr{E}_{p'} \underset{\mathscr{D}_{X,q}}{\otimes} \mathscr{M}_p = 0 \qquad \text{for} \quad p' \in \pi^{-1}(q) - \boldsymbol{C}^\times p - T^*_X X.$$

Let $\mathscr{F}$ be a coherent $\mathscr{D}_X$-Module defined in a neighborhood of q such that $\mathscr{F}_q = \mathscr{M}_p$. It is enough to show that the characteristic variety Λ' of $\mathscr{F}$ is contained in $\Lambda \cup T^*_X X$. Since $\mathscr{E}\underset{\mathscr{D}}{\otimes}\mathscr{F}$ is a sub-Module of $\mathscr{E}\underset{\mathscr{D}}{\otimes}(\mathscr{L}/\mathscr{P})$, Λ' is contained in $T^*_X X \cup \pi^{-1}(Y)$. Since $\mathscr{E}\underset{\mathscr{D}}{\otimes}\mathscr{F}$ is $\mathscr{M}$ in a neighborhood of p, we can write $\Lambda' = \Lambda \cup T^*_X X \cup \Lambda''$, where Λ'' is a closed Lagrangian variety such that $p \notin \Lambda''$ and $\Lambda'' \subset \pi^{-1}(Y)$. Then Lemma 5.1.2 proved below implies that Λ'' is void, i.e., $\Lambda' = \Lambda \cup T^*_X X$. Q.E.D.

Lemma 5.1.2. *Let Λ be a closed homogeneous Lagrangian variety of $T^*X - T^*_X X$ and let p be a point of Λ. Assume that Λ satisfies the following condition*:

(5.1.2)
$$\pi^{-1}(\pi(p)) \cap \Lambda = \boldsymbol{C}^\times p\,.$$

Let Λ' be another closed Lagrangian variety such that $p \notin \Lambda'$ and that $\pi(\Lambda') \subset \pi(\Lambda)$. Then $\Lambda' \cap \pi^{-1}(\pi(p)) = \emptyset$.

Proof. We shall prove this by a reduction to absurdity. Denote $\pi(p)$ by q and denote $\pi(\Lambda)$ (resp., $\pi(\Lambda')$) by Y (resp., Z). Suppose that $\Lambda' \cap \pi^{-1}(q) \neq \emptyset$. Then Z should contain $\pi(p)$. Since Λ' is a homogeneous Lagrangian variety, Λ' contains $T^*_Z X$. On the other hand, since Y contains Z by the assumption, $T^*_Z X$ contains $W \underset{\text{def}}{=} T^*_Y X \cap \pi^{-1}(Z - Z_1)$ for some nowhere dense analytic subset Z_1 of Z. Here we note that $(\overline{W} - T^*_X X) \cap \pi^{-1}(q) \neq \emptyset$. Since $\overline{W} - T^*_X X$ is contained in Λ, $(\overline{W}-T_X^*X)\cap\pi^{-1}(q)$ is contained in $\mathbb{C}^\times p$. Hence $\overline{W}$ should contain p. Therefore Λ' should contain p. This is a contradiction. Q. E. D.

In the course of the proof of Theorem 5.1.1, we have obtained the following

Theorem 5.1.3. *Let $\mathscr{M}$ and Λ be as in Theorem 5.1.1. Then there exist a holonomic system $\mathscr{L}$ of D-type, a holonomic $\mathscr{D}$-sub-Module $\mathscr{P}$ of $\mathscr{L}$ with $\mathrm{SS}(\mathscr{P})\subset T_X^*X$ and an injective $\mathscr{D}_{X,\pi(p)}$-linear homomorphism $\phi: \mathscr{M}_p\to(\mathscr{L}/\mathscr{P})_{\pi(p)}$.*

The following theorem follows immediately from Theorem 5.1.1.

Theorem 5.1.4. *Let $\mathscr{M}$ and Λ be as in Theorem 5.1.1. Then there exists a holonomic $\mathscr{D}$-Module $\mathscr{F}$ with R. S. which satisfies the following conditions:*

(5.1.3) $\mathscr{F}_{\pi(p)}=\mathscr{M}_p$

(5.1.4) $\mathscr{E}\underset{\pi^{-1}\mathscr{D}}{\otimes}\mathscr{F}=\mathscr{M}$ *holds in a neighborhood of* p.

(5.1.5) $\mathrm{SS}(\mathscr{F})$ *is contained in* $\Lambda\cup T_X^*X$ *is a neighborhood of* $\pi^{-1}(\pi(p))$.

In what follows, we shall prove several coherency properties of the sheaves related to holonomic systems with R. S. The key point in our arguments is the coherency over $\mathscr{O}_X$ of $\mathscr{L}_0$, the sub-Module of $\mathscr{L}$ consisting of the sections in the strict Nilsson class. (Chapter II, §2.)

Theorem 5.1.5. *Let $\mathscr{M}$ and Λ be as in Theorem 5.1.1. Let $\mathscr{M}_0$ be a coherent $\mathscr{E}(0)$-sub-Module of $\mathscr{M}$. Then $\mathscr{M}_{0,p}$, the stalk of $\mathscr{M}_0$ at p, is a finitely generated $\mathscr{O}_{X,\pi(p)}$-module.*

Proof. Let us use the same notations as in the proof of Theorem 5.1.1. Let $\mathscr{L}_0$ be the sub-Module of $\mathscr{L}$ consisting of the sections in the strict Nilsson class. Let f be a holomorphic function on X such that $f^{-1}(0)=Y$. Then there exists $m\in\mathbb{Z}$ such that $\mathrm{Re}\,(\mathrm{ord}\,u)<m$ for any section u of $\mathscr{M}_0$ defined on an open subset of Λ_{reg}. Let u be a section of $\mathscr{M}_0$ and let v be a section of $\mathscr{L}$ whose modulo class in $\mathscr{L}/\mathscr{P}$ is $\psi(u)$. Then $\mathrm{Re}\,(\mathrm{ord}\,f^{m+1}v)<-1$ on $T^*_{Y_{\mathrm{reg}}}X$. Hence Proposition 2.3.7 in Chapter II entails that $f^{m+1}v$ belongs to $\mathscr{L}_0$. Hence $\psi(u)$ belongs to $f^{-m-1}\mathscr{L}_0/\mathscr{P}$. Therefore we see that $\mathscr{M}_{0,p}$ is a submodule of $(f^{-m-1}\mathscr{L}_0/\mathscr{P})_{\pi(p)}$. Since both $\mathscr{L}_0$ and $\mathscr{P}$ are of finite type over $\mathscr{O}_X$, we obtain the required result. Q. E. D.

Theorem 5.1.6. *Let c be a real number and $\mathscr{M}$ a holonomic $\mathscr{E}_X$-Module with R. S. defined on an open subset Ω of $T^*X-T_X^*X$. Let Λ be the support of*

$\mathscr{M}$. *Let* $\mathscr{M}_0$ *denote the subsheaf of* $\mathscr{M}$ *given by assigning* $\{s\in\mathscr{M}_0(U);\ \mathrm{ord}_p s \subset\{\lambda\in\boldsymbol{C};\ \mathrm{Re}\,\lambda\leqq c\}$ *for any point* p *of* $U\cap\Lambda_{\mathrm{reg}}\}$ *to* U. *Then the following hold*:

(i) $\mathscr{M}_0$ *is a coherent* $\mathscr{E}(0)|_\Omega$*-Module.*

(ii) $\mathscr{M}=\mathscr{E}\mathscr{M}_0$ *and* $\mathscr{M}_0=\mathscr{E}_\Lambda\mathscr{M}_0$.

(iii) *For any closed analytic subset* W *of an open subset* U *of* T^*X *such that* codim $W\geqq n+1$, *we have* $\mathscr{H}^0_W(\mathscr{M}/\mathscr{M}_0)=0$.

Proof. The property (ii) of $\mathscr{M}_0$ is clear. The property (iii) follows from the fact that $\mathrm{ord}_p\, u$ is locally constant in $p\in\Lambda_{\mathrm{reg}}$. To prove (i), we shall employ a quantized contact transformation. There is a finite subset F of $\boldsymbol{C}$ such that $\mathrm{ord}\, s\subset F+\boldsymbol{Z}$ for any section s of $\mathscr{M}$. By a quantized contact transformation, we may assume that $c=-1/2$ and for any $\lambda\in F+\boldsymbol{Z}$, $\mathrm{Re}\,\lambda\neq-1/2$. Let p_0 be a point of Λ. Again by a quantized contact transformation, we may assume that Λ is in a generic position at p_0. Set $q_0=\pi(p_0)$. Then we may assume that $\Omega=\pi^{-1}(U)-T^*_XX$ for an open neighborhood U of q_0 and $\Lambda\cap\pi^{-1}(q_0)=\boldsymbol{C}^\times p_0$. Let $\mathscr{F}$ be a coherent $\mathscr{D}_X$-Module such that $\mathscr{F}_{q_0}=\mathscr{M}_{p_0}$. Then, by Theorem 5.1.4, $\mathscr{F}$ is a holonomic $\mathscr{D}_X$-Module with R.S., $\mathrm{SS}(\mathscr{F})\subset\Lambda\cup T^*_XX$, and $\mathscr{M}=\mathscr{E}\underset{\mathscr{D}}{\otimes}\mathscr{F}$. Set $S=\pi(\Lambda)$. Then we have

$$\Lambda\cap\pi^{-1}(S_{\mathrm{reg}})=T^*_{S_{\mathrm{reg}}}X-T^*_XX\,. \tag{5.1.6}$$

We have

$$\mathscr{H}^0_S(\mathscr{F})=0\,. \tag{5.1.7}$$

In fact, $\mathscr{F}'=\mathscr{H}^0_S(\mathscr{F})$ is a holonomic $\mathscr{D}_X$-Module such that $\mathrm{SS}(\mathscr{F}')\subset(\Lambda\cup T^*_XX)\cap\pi^{-1}(S)$. Hence $\mathscr{F}'|_{S_{\mathrm{reg}}}$ is locally isomorphic to a direct sum of copies of $\mathscr{B}_{S_{\mathrm{reg}}|X}$. Since an order of a section of $\mathscr{E}\underset{\mathscr{D}}{\otimes}\mathscr{B}_{S_{\mathrm{reg}}|X}|_\Lambda$ is a half integer, $\mathscr{E}\underset{\mathscr{D}}{\otimes}\mathscr{F}'=0$ on $T^*_{S_{\mathrm{reg}}}X-T^*_XX$. Hence we obtain

$$\mathrm{SS}(\mathscr{F}')\subset(\pi^{-1}(S)\cap T^*_XX)\cup(\Lambda\cap\pi^{-1}(S_{\mathrm{sing}}))\,.$$

On the other hand, since

$$\dim(\pi^{-1}(S)\cap T^*_XX)\cup(\Lambda\cap\pi^{-1}(S_{\mathrm{sing}}))\leqq n-1\,,$$

we have $\mathrm{SS}(\mathscr{F}')=\emptyset$. Thus (5.1.7) has been proved. Let $\mathscr{L}$ denote $\mathscr{H}^0_{[X|S]}(\mathscr{F})$. Then $\mathscr{L}$ is a holonomic $\mathscr{D}_X$-Module of D-type by Proposition 2.3.4 in Chapter II, and $\mathscr{L}$ contains $\mathscr{F}$ as a sub-Module. Let $\mathscr{L}_0$ be the subsheaf of $\mathscr{L}$ consisting of the sections of $\mathscr{L}$ in the strict Nilsson class. Then we have

$$\mathscr{L}_0=\{u\in\mathscr{L};\ \mathrm{ord}_{T^*_{S_{\mathrm{reg}}}X-T^*_XX}(u)\subset H\}\,, \tag{5.1.8}$$

where $H=\left\{\lambda\in\boldsymbol{C};\ \mathrm{Re}\,\lambda\leqq -\frac{1}{2}\right\}$. Since $\mathscr{L}_0$ is a coherent $\mathscr{O}_X$-Module, $\mathscr{F}\cap\mathscr{L}_0$ is also a coherent $\mathscr{O}_X$-Module. Let $\mathscr{M}'_0$ be the $\mathscr{E}(0)$-sub-Module of $\mathscr{M}$ generated by $\mathscr{F}\cap\mathscr{L}_0$. Then $\mathscr{M}'_0$ is clearly a coherent $\mathscr{E}(0)$-Module. We shall prove $\mathscr{M}_0=\mathscr{M}'_0$ on $\pi^{-1}(S_{\mathrm{reg}})$. Let q be a point of S_{reg} and we shall take a local coordinate system $(x_1,\dots,x_n)$ around q such that S_{reg} is given by $x_1=0$. By the definition of $\mathscr{L}_0$, $\mathscr{L}_0$ contains $x_1D_1\mathscr{L}_0$, $D_2\mathscr{L}_0,\dots, D_n\mathscr{L}_0$ and hence we have

$$\mathscr{L}_0\cap\mathscr{F}\supset x_1D_1(\mathscr{L}_0\cap\mathscr{F})+D_2(\mathscr{L}_0\cap\mathscr{F})+\cdots+D_n(\mathscr{L}_0\cap\mathscr{F}).$$

Since $\mathscr{E}_\Lambda$ is generated by x_1D_1, $D_2,\dots, D_n$ over $\mathscr{E}(0)$, $\mathscr{M}'_0$ is an $\mathscr{E}_\Lambda$-Module on $\pi^{-1}(S_{\mathrm{reg}})$. By Proposition 2.3.8 in Chapter II, there exists a polynomial $b(\lambda)$ satisfying

(5.1.9) $$D_1b\Big(-x_1D_1-\frac{1}{2}\Big)\mathscr{L}_0\subset\mathscr{L}_0$$

and

(5.1.10) any root λ of $b(\lambda)=0$ satisfies $-\frac{3}{2}<\mathrm{Re}\,\lambda\leqq-\frac{1}{2}$.

This shows that

$$D_1b\Big(-x_1D_1-\frac{1}{2}\Big)\mathscr{L}_0\subset\mathscr{L}_0$$

and hence

$$D_1b\Big(-x_1D_1-\frac{1}{2}\Big)(\mathscr{L}_0\cap\mathscr{F})\subset\mathscr{L}_0\cap\mathscr{F}.$$

Hence we obtain

$$b\Big(-x_1D_1-\frac{1}{2}\Big)\mathscr{M}'_0\subset\mathscr{M}'_0(-1).^{(*)}$$

Therefore we obtain $\mathscr{M}'_0=\mathscr{M}_0$ on Λ_{reg} by Lemma 1.5.7 in Chapter I. Set $W=\Lambda\cap\pi^{-1}(S_{\mathrm{sing}})$. We shall show

(5.1.11) $$\mathscr{H}^0_W(\mathscr{M}'_0(1)/\mathscr{M}'_0)_{p_0}=0.$$

By the definition of $\mathscr{L}_0$, we have

(*) In general, for an $\mathscr{E}(0)$-Module $\mathscr{M}$, we denote by $\mathscr{M}(k)$ the $\mathscr{E}(0)$-Module $\mathscr{E}(k)\underset{\mathscr{E}(0)}{\otimes}\mathscr{M}$. If $\mathscr{M}$ is an $\mathscr{E}(0)$-sub-Module of an $\mathscr{E}$-Module $\mathscr{N}$, then $\mathscr{M}(k)$ coincides with $\mathscr{E}(k)\mathscr{M}$.

$$(\mathscr{L}_0 \cap \mathscr{F})_{q_0} = \varinjlim_V \{s \in \mathscr{F}(V);\ \mathrm{ord}_p s \in H \text{ for any } p \in \pi^{-1}(V) \cap T^*_{Y_{\mathrm{reg}}}X\}$$
$$= \varinjlim_V \{s \in \mathscr{M}(\pi^{-1}(V));\ \mathrm{ord}_p s \in H \text{ for any } p \in \pi^{-1}(V) \cap \Lambda_{\mathrm{reg}}\},$$

where V ranges over a neighborhood system of q_0. Hence $(\mathscr{L}_0 \cap \mathscr{F})_{q_0}$ is an $\mathscr{E}(0)_{p_0}$-module. Therefore we have $\mathscr{M}'_{0,p_0} = (\mathscr{L}_0 \cap \mathscr{F})_{q_0}$, and $\mathscr{M}'_{0,p_0} = \varinjlim_{U \ni p_0} \{s \in \mathscr{M}(U);\ \mathrm{ord}_p s \in H$ for any $p \in U \cap \Lambda_{\mathrm{reg}}\}$, where U runs over a neighborhood system of p_0. Let P be a micro-differential operator of order 0 such that $\sigma_0(P)|_{W \cap \Lambda} = 0$ and $\sigma_0(P)$ is not identically zero on any irreducible component of Λ. In order to show (5.1.11), let us take a section s of $\mathscr{M}'_0(1)$ on a neighborhood U of p_0 such that $s|_{U-W} \in \mathscr{M}'_0$. Since $\mathscr{M}'_0(1)/\mathscr{M}_0$ is a coherent $\mathscr{O}(0)$-Module, there is an integer m such $P^m u \in \mathscr{M}'_0$. Hence we find that $\mathrm{ord}\, P^m u \in H$ for any $p \in U \cap \Lambda_{\mathrm{reg}}$. On the other hand, $\mathrm{ord}_p P^m u = \mathrm{ord}_p u + \mathrm{ord}_p P^m = \mathrm{ord}_p u$. This implies $u \in \mathscr{M}'_{0,p_0}$ and hence we obtain (5.1.11).

Since $\mathscr{H}^0_W(\mathscr{M}'_0(1)/\mathscr{M}_0)$ is a coherent $\mathscr{O}_{T^*X}$-Module, $\mathscr{H}^0_W(\mathscr{M}'_0(1)/\mathscr{M}'_0) = 0$ holds on a neighborhood U of p_0. This entails further that $\mathscr{H}^0_W(\mathscr{M}'_0(k)/\mathscr{M}'_0(k-1))|_U = 0$ for any k, because $\mathscr{M}'_0(k)/\mathscr{M}'_0(k-1)$ is locally isomorphic to $\mathscr{M}'_0(1)/\mathscr{M}'_0$. This implies $\mathscr{H}^0_W(\mathscr{M}'_0(k)/\mathscr{M}'_0)|_U = 0$ and hence

(5.1.12) $$\mathscr{H}^0_W(\mathscr{M}/\mathscr{M}'_0)|_U = 0.$$

Now we are ready to prove $\mathscr{M}_0 = \mathscr{M}'_0$. In fact, $\mathscr{M}'_0 = \mathscr{M}_0$ outside W and $\mathscr{M}'_0 \subset \mathscr{M}_0$. Therefore we find $\mathscr{M}_0/\mathscr{M}'_0 \subset \mathscr{H}^0_W(\mathscr{M}/\mathscr{M}'_0) = 0$.

Since $\mathscr{M}'_0$ is a coherent $\mathscr{E}(0)$-Module, this proves (i). Q.E.D.

As an immediate consequence of Theorem 5.1.6, we obtain the following important corollary.

Corollary 5.1.7. *Let $\mathscr{M}$ be a holonomic $\mathscr{E}_X$-Module with* R.S. *and V a homogeneous involutory analytic set containing* Supp $\mathscr{M}$. *Then $\mathscr{M}$ has regular singularities along $V - T^*_X X$.*

Proof. The sub-Module $\mathscr{M}_0$ of $\mathscr{M}$ given in Theorem 5.1.6 is an $\mathscr{E}_V$-Module which is coherent over $\mathscr{E}(0)$ and generates $\mathscr{M}$ as an $\mathscr{E}$-Module. Hence $\mathscr{M}$ has regular singularities along $V - T^*_X X$. Q.E.D.

By the aid of Theorem 5.1.6, we can also prove the following

Theorem 5.1.8. *Let $\mathscr{M}$ be a holonomic $\mathscr{E}_X$-Module with* R.S. *Let W be a closed subvariety of an open subset $\Omega \subset T^*X - T^*_X X$ such that* codim $W \geq \dim X + 1$. *Let $\mathscr{N}$ be a coherent $\mathscr{E}(0)$-Module and let $\mathscr{N}'$ denote the subsheaf* *given by assigning* $\{s\in\mathscr{M}(U);\ s|_{U-W}\in\mathscr{N}(U)\}$ *to* U. *Then* $\mathscr{N}'$ *is a coherent* $\mathscr{E}(0)$*-Module.*

Proof. First note that $\mathscr{N}'\subset\mathscr{E}\mathscr{N}$. In fact, we have $\mathscr{H}^0_W(\mathscr{M}/\mathscr{E}\mathscr{N})=0$ and hence we have

$$(\mathscr{N}'+\mathscr{E}\mathscr{N})/\mathscr{E}\mathscr{N}\subset\mathscr{H}^0_W((\mathscr{N}'+\mathscr{E}\mathscr{N})/\mathscr{E}\mathscr{N})\subset\mathscr{H}^0_W(\mathscr{M}/\mathscr{E}\mathscr{N})=0\,.$$

Therefore, by replacing $\mathscr{M}$ with $\mathscr{E}\mathscr{N}$, we may assume from the first that $\mathscr{M}=\mathscr{E}\mathscr{N}$. Let $\mathscr{M}_0$ be the subsheaf of $\mathscr{M}$ given in Theorem 5.1.6. By considering $\mathscr{M}_0(m)$ for sufficiently large m, we may assume without loss of generality that $\mathscr{N}$ is contained in $\mathscr{M}_0$. Let J be the defining Ideal of W and denote $\sigma_0^{-1}(J)$ by $\mathscr{J}$. Here σ_0 designates the symbol map from $\mathscr{E}(0)$ to $\mathscr{O}_{T^*X}$. Define $\mathscr{N}_0$ by $\mathscr{N}$ and $\mathscr{N}_k$ $(k\geqq 1)$ inductively by

$$\{u\in\mathscr{N}_{k-1}(1);\ \mathscr{J}u\subset\mathscr{N}_{k-1}\}\,.$$

Evidently $\{\mathscr{N}_k\}_{k\geqq 0}$ is an increasing sequence of $\mathscr{E}(0)$-sub-Modules. Denote $\bigcup_{k\geqq 0}\mathscr{N}_k$ by $\mathscr{N}''$. Then it is clear that

$$\mathscr{N}''=\{u\in\mathscr{N}''(1);\ \mathscr{J}u\subset\mathscr{N}''\} \tag{5.1.13}$$

holds.

Let us first prove that $\mathscr{N}''$ is a coherent $\mathscr{E}(0)$-Module. In order to prove this we show, by the induction on k, that $\mathscr{N}_k$ is coherent. In fact, choosing $P_j\in\mathscr{E}(0)$ so that

$$\mathscr{J}=\mathscr{E}(0)P_1+\cdots+\mathscr{E}(0)P_l$$

holds, we see that the following sequence is exact.

$$0\longrightarrow\mathscr{N}_k/\mathscr{N}_{k-1}\longrightarrow\mathscr{N}_{k-1}(1)/\mathscr{N}_{k-1}\xrightarrow{(P_1,\dots,P_l)}(\mathscr{N}_{k-1}(1)/\mathscr{N}_{k-1})^l. \tag{5.1.14}$$

(5.1.14) combined with the induction on k clearly entails that $\mathscr{N}_k$ is coherent. Since $\mathscr{N}$ is contained in $\mathscr{M}_0$, $\mathscr{N}_k$ $(k\geqq 0)$ is contained in $\mathscr{M}_0$. Therefore $\mathscr{N}''=\bigcup_{k\geqq 0}\mathscr{N}_k$ is a union of coherent $\mathscr{E}(0)$-sub-Modules of $\mathscr{M}_0$. This implies that $\mathscr{N}''$ is coherent over $\mathscr{E}(0)$.

On the other hand, it is clear from the definition of $\mathscr{N}_k$ that $\mathscr{N}_k=\mathscr{N}$ holds outside W. Hence $\mathscr{N}''=\mathscr{N}$ holds outside W.

Now we shall show $\mathscr{N}'=\mathscr{N}''$. It follows from (5.1.13) that

$$\mathscr{H}^0_W(\mathscr{N}''/\mathscr{N}''(-1))=0\,. \tag{5.1.15}$$

Then, by the same argument as that used in the proof of Theorem 5.1.6 for the proof of (5.1.12), we can conclude from (5.1.15) that $\mathscr{H}^0_W(\mathscr{M}/\mathscr{N}'')=0$ holds. Hence we find

$$\begin{aligned}\mathscr{N}'' &= \{u \in \mathscr{M};\ u|_{A-W} \in \mathscr{N}''\}\\ &= \{u \in \mathscr{M};\ u|_{A-W} \in \mathscr{N}\} = \mathscr{N}'.\end{aligned}$$

Thus we conclude that $\mathscr{N}''=\mathscr{N}'$. Therefore $\mathscr{N}'$ is a coherent $\mathscr{E}(0)$-Module.

Q.E.D.

We next apply Theorem 5.1.6 to prove the existence of a good filtration for a holonomic $\mathscr{D}$-Module $\mathscr{M}$ with R.S. (See Section A.5 in Appendix A for the definition of a good filtration.) Before proving the result (Corollary 5.1.11) we prepare the following elementary lemma.

Lemma 5.1.9. *Define a complex manifold X' by $\boldsymbol{C}\times X$ and identify X with the submanifold $\{0\}\times X$ of X'. Let ρ be the canonical projection from $X \underset{X'}{\times} T^*X'$ onto T^*X. Let $\mathscr{M}$ be an $\mathscr{E}_X$-Module defined on an open subset Ω of T^*X. Denote $\mathscr{E}_{X'\leftarrow X} \underset{\rho^{-1}\mathscr{E}_X}{\otimes} \rho^{-1}\mathscr{M}$ by $\mathscr{N}$. Then we have the following:*

(i) *$\mathscr{N}$ is a coherent $\mathscr{E}_{X'}$-Module on $\rho^{-1}(\Omega)$ if and only if $\mathscr{M}$ is a coherent $\mathscr{E}_X$-Module on Ω.*

(ii) *$\mathscr{N}$ is a holonomic $\mathscr{E}_{X'}$-Module on $\rho^{-1}(\Omega)$ if and only if $\mathscr{M}$ is a holonomic $\mathscr{E}_X$-Module.*

(iii) *Let V be a homogeneous involutory subvariety of $\Omega - T^*_XX$. If $\mathscr{N}$ is a coherent $\mathscr{E}_{X'}$-Module with regular singularities along $\rho^{-1}(V)$ on a neighborhood of $p \in X \underset{X'}{\times} T^*X'$, then $\mathscr{M}$ has regular singularities along V on a neighborhood of $\rho(p)$. Conversely, if $\mathscr{M}$ has regular singularities along V, then $\mathscr{N}$ has regular singularities along $\rho^{-1}(V)$ on $\rho^{-1}(\Omega - T^*_XX)$.*

(iv) *$\mathscr{N}$ is a holonomic $\mathscr{E}_{X'}$-Module with* R.S. *if and only if $\mathscr{M}$ is a holonomic $\mathscr{E}_X$-Module with* R.S.

Proof. The proof of the assertion (i) is given in Appendix A. (Proposition A.2.) In order to prove (ii), it suffices to show

$$\operatorname{Supp}(\mathscr{E}_{X'\leftarrow X} \underset{\rho^{-1}\mathscr{E}_X}{\otimes} \rho^{-1}\mathscr{M}) = \rho^{-1}(\operatorname{Supp}\mathscr{M}).$$

This immediately follows from the fact that $\mathscr{E}_{X'\leftarrow X}$ is faithfully flat over $\rho^{-1}\mathscr{E}_X$. (S-K-K [24] Chapter II, § 3.)

(iii) Let us prove the first assertion. Let t denote a coordinate of $\boldsymbol{C}$. Then $\mathscr{E}_{X'\leftarrow X}=\mathscr{E}_{X'}/\mathscr{E}_{X'}t$. We denote by $1_{X'\leftarrow X}$ the section of $\mathscr{E}_{X'\leftarrow X}$ given by

$1 \in \mathscr{E}_{X'}$. Let $\mathscr{M}_0$ be a coherent $\mathscr{E}(0)$-sub-Module of $\mathscr{M}$. Define a coherent $\mathscr{E}_{X'}(0)$-Module $\mathscr{N}_0$ by $\mathscr{E}_{X'\leftarrow X}(0) \underset{\mathscr{E}_X(0)}{\otimes} \mathscr{M}_0$, where $\mathscr{E}_{X'\leftarrow X}(0) \underset{\mathrm{def}}{=} \mathscr{E}_{X'}(0)/\mathscr{E}_X(0)t$ $(=\mathscr{E}_{X'}(0)1_{X'\leftarrow X})$. Then it follows from the assumption that $\mathscr{E}_{\rho^{-1}(V)}\mathscr{N}_0$ is coherent over $\mathscr{E}_{X'}(0)$. We shall prove

$$\mathscr{E}_{\rho^{-1}(V)}\mathscr{N}_0 = \mathscr{E}_{X'\leftarrow X}(0) \underset{\mathscr{E}_X(0)}{\otimes} \mathscr{E}_V\mathscr{M}_0\,. \tag{5.1.16}$$

If (5.1.16) is proved, then the coherence of $\mathscr{E}_V\mathscr{M}_0$ over $\mathscr{E}_X(0)$ can be derived from that of $\mathscr{E}_{\rho^{-1}(V)}\mathscr{N}_0$ over $\mathscr{E}_{X'}(0)$ by the same reasoning as in the proof of Proposition A.2 in Appendix A. The assertion (5.1.16) follows from Sublemma 5.1.10 proved below. In fact, it is clear that $\mathscr{E}_{X'\leftarrow X}(0) \underset{\mathscr{E}_X(0)}{\otimes} \mathscr{E}_V\mathscr{M}_0$ is contained in $\mathscr{E}_{\rho^{-1}(V)}\mathscr{N}_0$, while Lemma 5.1.10 entails

$$\mathscr{I}^k_{\rho^{-1}(V)}(\mathscr{E}_{X'\leftarrow X}(0) \underset{\mathscr{E}_X(0)}{\otimes} \mathscr{M}_0) = \mathscr{E}_{X'\leftarrow X}(0) \underset{\mathscr{E}_X(0)}{\otimes} \mathscr{I}^k_V\mathscr{M}_0 \tag{5.1.17}$$

for $k=1, 2, \ldots$. Here $\mathscr{I}_V$ (resp., $\mathscr{I}_{\rho^{-1}(V)}$) denotes $\{P \in \mathscr{E}_X(1);\ \sigma_1(P) \in I_V\}$ (resp., $\{P \in \mathscr{E}_{X'}(1);\ \sigma_1(P) \in I_{\rho^{-1}(V)}\}$), where I_V (resp., $I_{\rho^{-1}(V)}$) is the sheaf of holomorphic functions on T^*X (resp., T^*X') vanishing on V (resp., $\rho^{-1}(V)$). Clearly (5.1.17) implies that $\mathscr{E}_{\rho^{-1}(V)}\mathscr{N}_0$ is contained in $\mathscr{E}_{X'\leftarrow X}(0) \underset{\mathscr{E}_X(0)}{\otimes} \mathscr{E}_V\mathscr{M}_0$. Thus we shall be finished if we prove the following

Sublemma 5.1.10. $\mathscr{I}_{\rho^{-1}(V)}\mathscr{E}_{X'\leftarrow X}(0) = \mathscr{E}_{X'\leftarrow X}(0)\mathscr{I}_V$.

Proof. It is obvious that $\mathscr{E}_{X'\leftarrow X}(0)\mathscr{I}_V$ is contained in $\mathscr{I}_{\rho^{-1}(V)}\mathscr{E}_{X'\leftarrow X}(0)$. Hence it suffices to show that $\mathscr{I}_{\rho^{-1}(V)}\mathscr{E}_{X'\leftarrow X}(0)$ is contained in $\mathscr{E}_{X'\leftarrow X}(0)\mathscr{I}_V$. Take a local coordinate system x of X, (x, ξ) of T^*X and $(t, x; \tau, \xi)$ of T^*X'. Then for $P \in \mathscr{I}_{\rho^{-1}(V)}$, we can find $f(t, x; \tau, \xi)$, $g_k(t, x; \tau, \xi)$ and $h_k(x; \xi)$ $(1 \leq k \leq N)$ which are of homogeneous degree of 1, 0 and 1 with respect to (τ, ξ), respectively, so that $h_k \in I_V$ and $\sigma_1(P) = ft + \sum_{k=1}^{N} g_k h_k$ holds. Hence there exist F, $G_k \in \mathscr{E}_{X'}$ and $H_k \in \mathscr{E}_X$ with $\sigma_1(F) = f$, $\sigma_0(G_k) = g_k$ and $\sigma_1(H_k) = h_k$ so that

$$P - Ft - \sum_{k=1}^{N} G_k H_k \in \mathscr{E}_{X'}(0) \tag{5.1.18}$$

holds. Since it follows from the definition that $\mathscr{E}_{X'\leftarrow X}(0) = \mathscr{E}_{X'}(0)1_{X'\leftarrow X}$ holds, $\mathscr{I}_{\rho^{-1}(V)}\mathscr{E}_{X'\leftarrow X}(0) = \mathscr{I}_{\rho^{-1}(V)}1_{X'\leftarrow X}$ holds. Since H_k is in $\mathscr{I}_V$, (5.1.18) entails

$$P\,1_{X'\leftarrow X} = \sum_{k=1}^{N} (G_k 1_{X'\leftarrow X}) H_k\,.$$

Therefore we conclude that $P\,1_{X'\leftarrow X}$ is contained in $\mathscr{E}_{X'\leftarrow X}(0)\mathscr{I}_V$. This completes the proof of Sublemma 5.1.10. Q.E.D.

Let us return to the proof of the second assertion in (iii). Suppose that $\mathscr{M}$ has regular singularities along V. Then there exists locally a coherent $\mathscr{E}_V$-sub-Module $\mathscr{M}_0$ of $\mathscr{M}$ such that $\mathscr{M}=\mathscr{E}_X\mathscr{M}_0$ and $\mathscr{M}_0=\mathscr{E}_V\mathscr{M}_0$. Then $\mathscr{N}_0 \underset{\text{def}}{=} \mathscr{E}_{X'\leftarrow X}(0)(1_{X'\leftarrow X} \underset{\mathscr{E}_X(0)}{\otimes} \mathscr{M}_0)$ is a coherent $\mathscr{E}_{X'}(0)$-Module which generates $\mathscr{N}$ as an $\mathscr{E}_{X'}$-Module. Furthermore Sublemma 5.1.10 asserts that $\mathscr{N}_0$ is an $\mathscr{E}_{\rho^{-1}(V)}$-sub-Module. Therefore $\mathscr{N}$ has regular singularities along $\rho^{-1}(V)$. This completes the proof of the second assertion of (iii).

(iv) Denote Supp $\mathscr{M}$ by Λ. Then $\mathscr{M}$ is isomorphic to $\mathscr{O}_X^m$ for some integer m on a neighborhood of $T_X^*X-\Lambda$. Hence $\mathscr{N}$ is with R.S. on $\rho^{-1}(T_X^*X)$. Let us next prove that $\mathscr{M}$ is with R.S. on $\Lambda-T_X^*X$ if and only if $\mathscr{N}$ is with R.S. on $\rho^{-1}(\Lambda-T_X^*X)$. It follows from (iii) that $\rho^{-1}\,\mathrm{IR}(\mathscr{M};\Lambda-T_X^*X)=\mathrm{IR}(\mathscr{N};\rho^{-1}(\Lambda-T_X^*X))$. Hence $\mathrm{IR}(\mathscr{N};\rho^{-1}(\Lambda-T_X^*X))$ is a nowhere dense subset of $\rho^{-1}(\Lambda-T_X^*X)$ if and only if $\mathrm{IR}(\mathscr{M};\Lambda-T_X^*X)$ is a nowhere dense subset of $\Lambda-T_X^*X$. Therefore $\mathscr{N}$ is with R.S. on $\rho^{-1}(\Lambda-T_X^*X)$ if and only if $\mathscr{M}$ is with R.S. on $\Lambda-T_X^*X$. Q.E.D.

Now we have the following

Corollary 5.1.11. *Let $\mathscr{M}$ be a holonomic $\mathscr{D}_X$-Module with* R.S. *defined on X. Then there exists a good filtration $\{\mathscr{M}_k\}_{k\in\mathbf{Z}}$ of $\mathscr{M}$ defined on X.*

Proof. We shall use the results and the notations of Appendix A. By Lemma 5.1.9, $\mathscr{N}=\Phi(\mathscr{M})$ is a holonomic $\mathscr{E}_X$-Module with R.S. defined on a neighborhood of V. Let $\mathscr{N}_0$ be the sub-Module of $\mathscr{N}$ consisting of the sections s of $\mathscr{N}$ such that $\operatorname{ord} s\subset\{\lambda\in\mathbf{C};\ \operatorname{Re}\lambda<0\}$. Then, by Theorem 5.1.6, $\mathscr{N}_0$ is a coherent $\mathscr{E}(0)$-sub-Module of $\mathscr{N}$ satisfying $\mathscr{E}_V\mathscr{N}_0\subset\mathscr{N}_0$ and $\mathscr{N}=\mathscr{E}\mathscr{N}_0$. Hence we can apply Proposition A.8 in Appendix A to see that $\{j_0^{-1}\mathscr{N}_0(k)\cap\mathscr{M}\}_{k\in\mathbf{Z}}$ is a good filtration of $\mathscr{M}$. Q.E.D.

§ 2.

The purpose of this section is to derive the following Theorem 5.2.1 from Theorem 4.1.1 proved in Chapter IV.

Theorem 5.2.1. *Let $\mathscr{M}$ be a holonomic $\mathscr{E}_X$-Module defined on a neighborhood of $p_0\in T^*X$. Then $\mathscr{M}_{\mathrm{reg}}$ is a holonomic* (*in particular, coherent*) *$\mathscr{E}_X$-Module and*

$$\mathscr{E}_X^\infty \underset{\mathscr{E}_X}{\otimes} \mathscr{M} = \mathscr{E}_X^\infty \underset{\mathscr{E}_X}{\otimes} \mathscr{M}_{\mathrm{reg}} \tag{5.2.1}$$

holds on a neighborhood of p_0.

Proof. First we consider the case where $p_0 \notin T^*_X X$. In this case, applying a suitable quantized contact transformation (Chapter I, § 6, Corollary 1.6.4), we may assume without loss of generality that $\Lambda \underset{\text{def}}{=} \operatorname{Supp} \mathscr{M}$ is in a generic position at p_0. Let q_0 denote $\pi(p_0)$ and let S denote $\pi(\Lambda)$. In what follows, n designates dim X. Here we note the following

Lemma 5.2.2. *There exist a holonomic* $\mathscr{E}_X$*-Module* $\mathscr{N}$ *with* R.S. *defined on a neighborhood of* p_0 *and an* $\mathscr{E}_X^\infty$*-linear homomorphism* $h: \mathscr{N}^\infty \to \mathscr{M}^\infty$ *such that* $h_{p_0}: \mathscr{N}^\infty_{p_0} \to \mathscr{M}^\infty_{p_0}$ *and* $h^*_{p_0} \underset{\text{def}}{=} \mathscr{E}xt^n_{\mathscr{E}^\infty_X}(h, \mathscr{E}^\infty_X) \otimes \Omega^{\otimes -1}: \mathscr{M}^{*\infty}_{p_0} \to \mathscr{N}^{*\infty}_{p_0}$ *are injective.*

Proof. We apply Theorem 4.1.1 in Chapter IV to the dual system $\mathscr{M}^* \underset{\text{def}}{=} \mathscr{E}xt^n_{\mathscr{E}_X}(\mathscr{M}, \mathscr{E}_X) \otimes \Omega^{\otimes -1}$ of $\mathscr{M}$. Then we can find a holonomic system $\mathscr{L}'$ of D-type along S and an $\mathscr{E}^\infty$-linear homomorphism $\varphi': \mathscr{M}^{*\infty} \to (\mathscr{E} \underset{\mathscr{D}}{\otimes} \mathscr{L}')^\infty$ defined on a neighborhood of p_0 such that $\varphi'_{p_0}: \mathscr{M}^{*\infty}_{p_0} \to \mathscr{E}^\infty_{p_0} \underset{\mathscr{D}_{q_0}}{\otimes} \mathscr{L}'_{q_0}$ is injective. Applying Theorem 4.1.1 to $\mathscr{M}$, we can also find a holonomic system $\mathscr{L}$ of D-type, a $\mathscr{D}$-sub-Module $\mathscr{P}$ of $\mathscr{L}$ isomorphic to a direct sum of copies of $\mathscr{O}_X$ and a $\mathscr{D}^\infty_{X,q_0}$-linear homomorphism $\phi: \mathscr{M}^\infty_{p_0} \to (\mathscr{L}/\mathscr{P})^\infty_{q_0}$ such that $E(\phi): \mathscr{M}^\infty_{p_0} \to \mathscr{E}^\infty_{p_0} \underset{\mathscr{D}^\infty_{q_0}}{\otimes} (\mathscr{L}/\mathscr{P})^\infty_{q_0}$ is an injective $\mathscr{E}^\infty_{p_0}$-linear homomorphism. Hence $E(\phi)$ is the germ of an $\mathscr{E}^\infty$-linear homomorphism $\tilde{\phi}: \mathscr{M}^\infty \to \mathscr{E}^\infty \underset{\mathscr{D}}{\otimes} (\mathscr{L}/\mathscr{P})$ defined on a neighborhood of p_0. Let us now consider the map χ from $\mathscr{O}_{q_0} \underset{\mathbf{C}}{\otimes} \mathscr{H}om_{\mathscr{D}}(\mathscr{O}, \mathscr{L})_{q_0}$ to $\mathscr{L}_{q_0}$ by assigning $\varphi(s)$ to (s, φ). Let $\mathscr{Q}$ be the coherent $\mathscr{D}$-sub-Module of $\mathscr{L}$ such that $\mathscr{Q}_{q_0}$ is the image of χ. Since $\dim_{\mathbf{C}} \mathscr{H}om_{\mathscr{D}}(\mathscr{O}, \mathscr{L})_{q_0}$ is finite, such a $\mathscr{Q}$ exists and it is isomorphic to a direct sum of copies of $\mathscr{O}$. Furthermore it follows from the definition of $\mathscr{Q}$ that $\mathscr{H}om_{\mathscr{D}}(\mathscr{O}, \mathscr{Q})_{q_0}$ is isomorphic to $\mathscr{H}om_{\mathscr{D}}(\mathscr{O}, \mathscr{L})_{q_0}$. Hence, by replacing $\mathscr{P}$ with $\mathscr{Q}$ if necessary, we may assume from the first that $\mathscr{H}om_{\mathscr{D}}(\mathscr{O}, \mathscr{P})_{q_0}$ is isomorphic to $\mathscr{H}om_{\mathscr{D}}(\mathscr{O}, \mathscr{L})_{q_0}$.

By taking the dual of φ', we obtain an $\mathscr{E}^\infty$-linear homomorphism

$$\varphi = \varphi'^*: \mathscr{E}^\infty \underset{\mathscr{D}}{\otimes} \mathscr{L}'^* \to \mathscr{M}^\infty .$$

We shall now prove

$$\phi \circ \varphi_{p_0}(1 \otimes \mathscr{L}'^*_{q_0}) \subset (\mathscr{L}/\mathscr{P})_{q_0} . \tag{5.2.2}$$

Let s be a section of $\mathscr{L}'^*$ defined on a neighborhood of q_0. Then s satisfies a holonomic system of linear differential equations which has R.S. on $T^*_{S_{\text{reg}}} X$. Since assigning $\phi\varphi(1 \otimes s)$ to s defines a $\mathscr{D}^\infty$-linear map, $\phi\varphi_{p_0}(s) \in \mathscr{L}^\infty_{q_0}/\mathscr{P}^\infty_{q_0}$ also satisfies a holonomic system of linear differential equations which has R.S. on

$T^*_{S_{\mathrm{reg}}}X$. Hence, by Proposition 2.3.5 in Chapter II, s is contained in $\mathscr{L}_{q_0}/\mathscr{P}_{q_0}$. Thus we have verified (5.2.2). Hence we can find a $\mathscr{D}$-linear homomorphism $\psi: \mathscr{L}'^* \to \mathscr{L}/\mathscr{P}$ such that $\mathscr{E}^\infty \otimes \psi: \mathscr{E}^\infty \underset{\mathscr{D}}{\otimes} \mathscr{L}'^* \to \mathscr{E}^\infty \underset{\mathscr{D}}{\otimes} (\mathscr{L}/\mathscr{P})$ coincides with $\tilde{\phi}\varphi$. Denote by $\mathscr{N}$ the image of ψ. Then $(\mathscr{E}^\infty \underset{\mathscr{D}}{\otimes} \mathscr{N})_{p_0}$ is contained in $\mathscr{M}^\infty_{p_0}$ as a submodule of $(\mathscr{E}^\infty \underset{\mathscr{D}}{\otimes} (\mathscr{L}/\mathscr{P}))_{p_0}$. Hence we find that $\mathscr{E}^\infty \underset{\mathscr{D}}{\otimes} \mathscr{N}$ is a sub-Module of $\mathscr{M}^\infty$ near p_0. On the other hand, $\mathrm{SS}(\mathscr{N}) \subset \mathrm{SS}(\mathscr{L}')$. Let Λ' be the closure of $\mathrm{SS}(\mathscr{N}) - \Lambda - T^*_X X$. Since $\mathrm{SS}(\mathscr{N})$ is contained in $\Lambda = \mathrm{Supp}\,\mathscr{M}$ on a neighborhood of p_0, the Lagrangian variety Λ' does not contain p_0. On the other hand, $\pi(\Lambda')$ is contained in the closure of $\pi(\mathrm{SS}(\mathscr{L}') - T^*_X X)$, and hence in S. Hence by Lemma 5.1.2 in Section 1, we see that $\Lambda' = \emptyset$. This implies that $\mathrm{SS}(\mathscr{N})$ is contained in $\Lambda \cup T^*_X X$. Therefore $\mathscr{N}$ is with R.S. Furthermore we know that the homomorphism $h: \mathscr{E}^\infty \otimes \mathscr{N} \to \mathscr{M}^\infty$ is injective at p_0. Since $\mathscr{L}'^* \to \mathscr{N}$ is surjective, $\mathscr{L}' \leftarrow \mathscr{N}^*$ is injective. Hence $\mathscr{E}^\infty \underset{\mathscr{E}}{\otimes} \mathscr{L}' \leftarrow \mathscr{E}^\infty \underset{\mathscr{E}}{\otimes} \mathscr{N}^*$ is injective. Therefore $h^*: \mathscr{M}^{*\infty} \to (\mathscr{E}^\infty \otimes \mathscr{N})^{*\infty}$ is injective at p_0. This completes the proof of Lemma 5.2.2.

We now resume proving Theorem 5.2.1. Applying Lemma 5.2.2 to $\mathscr{M}^*$ instead of $\mathscr{M}$, we can find a holonomic $\mathscr{E}$-Module $\mathscr{N}'$ with R.S. and an $\mathscr{E}^\infty_X$-linear homomorphism $k: \mathscr{M}^\infty \to \mathscr{N}'^\infty$ such that k_{p_0} and $k^*_{p_0}$ are injective. It follows from Proposition 1.1.21 and Proposition 1.3.6 in Chapter I that $kh(\mathscr{N}) \subset \mathscr{N}'$. Since $(kh)_{p_0}$ is injective, $\varphi \underset{\mathrm{def}}{=} kh|_{\mathscr{N}}: \mathscr{N} \to \mathscr{N}'$ is injective on a neighborhood of p_0. In the same way, $\varphi^*: \mathscr{N}'^* \to \mathscr{N}^*$ is seen to be injective on a neighborhood of p_0. Since $*$ is an exact functor which is an involution on the category of holonomic systems, φ is an isomorphism. This implies that both k_{p_0} and h_{p_0} are isomorphisms. Since $\mathscr{N}^\infty$, $\mathscr{N}'^\infty$ and $\mathscr{M}^\infty$ are $\mathscr{E}^\infty$-Modules locally of finite presentation, k and h are isomorphisms on a neighborhood of p_0. Then, by Proposition 1.3.6 in Chapter I, we find $\mathscr{M}_{\mathrm{reg}} \cong \mathscr{N}$. This completes the proof of Theorem 5.2.1 when $p_0 \notin T^*_X X$.

Now we consider the case where $p_0 \in T^*_X X$. Denote $\boldsymbol{C} \times X$ by X' and identify X with the subset $\{0\} \times X$ of X'. Denote by ρ the canonical projection from $X \underset{X'}{\times} T^*X'$ onto T^*X. Let V be the involutory submanifold $\{(t, x; \tau, \xi) \in T^*X'; t=0, \tau \neq 0\}$ and denote by j_0 the inclusion map from X into V defined by $j_0(x) = (0, x; 1, 0)$. Let $\mathscr{L}$ be the $\mathscr{E}_{X'}$-Module $\mathscr{E}_{X'}/\mathscr{E}_{X'}t$ and denote by $\tilde{\mathscr{M}}$ the $\mathscr{E}_{X'}$-Module $\Phi(\mathscr{M}) \underset{\mathrm{def}}{=} \mathscr{L} \underset{\rho^{-1}\mathscr{E}_X}{\otimes} \rho^{-1}\mathscr{M}$(*). By what has been proved so far, $\tilde{\mathscr{M}}_{\mathrm{reg}}$ is a holonomic $\mathscr{E}_{X'}$-Module with R.S. on a neighborhood of V and it

(*) Here we use the same notations as in Appendix A.

satisfies $\mathscr{E}^\infty_{X'} \underset{\mathscr{E}_{X'}}{\otimes} \tilde{\mathscr{M}} = \mathscr{E}^\infty_{X'} \underset{\mathscr{E}_{X'}}{\otimes} \tilde{\mathscr{M}}_{\text{reg}}$ there. Since Corollary 5.1.7 guarantees that $\tilde{\mathscr{M}}_{\text{reg}}$ has regular singularities along V, $\mathscr{M}' = j_0^{-1}\Psi(\tilde{\mathscr{M}}_{\text{reg}})^{(*)}$ $(= \mathscr{H}om_{\mathscr{E}_{X'}}(\mathscr{L}, \tilde{\mathscr{M}}_{\text{reg}}))$ is a coherent $\mathscr{D}_X$-Module. Since $\mathscr{M}|_{T^*_X X}$ is a coherent $\mathscr{D}_X$-Module by the assumption, the monodromy of $\tilde{\mathscr{M}}$ in the sense of Appendix is the identity. (Proposition A.5 of Appendix A.) Therefore it follows from Proposition A.9 of Appendix A that the monodromy of $\tilde{\mathscr{M}}_{\text{reg}}$ is also the identity. Hence Proposition A.6 of Appendix A asserts that

$$\mathscr{L} \underset{\mathscr{D}_X}{\otimes} \mathscr{M}' = \tilde{\mathscr{M}}_{\text{reg}}. \tag{5.2.3}$$

Hence it follows from Lemma 5.1.9 in Section 1 that $\mathscr{M}'$ is a holonomic $\mathscr{D}_X$-Module with R. S. Furthermore (5.2.3) implies that $\mathscr{L}^\infty \underset{\mathscr{D}_X}{\otimes} \mathscr{M}' = \mathscr{E}^\infty_{X'} \underset{\mathscr{E}_{X'}}{\otimes} \tilde{\mathscr{M}} = \mathscr{L}^\infty \underset{\mathscr{E}_X}{\otimes} \mathscr{M}$ holds. On the other hand, Proposition A.10 of Appendix A asserts that

$$\mathscr{H}om_{\mathscr{D}^\infty_X}(\mathscr{M}^\infty|_{T^*_X X}, \mathscr{M}'^\infty) \cong j_0^{-1}\mathscr{H}om_{\mathscr{E}^\infty_{X'}}(\mathscr{L}^\infty \underset{\mathscr{E}_X}{\otimes} \mathscr{M}, \mathscr{L}^\infty \underset{\mathscr{D}_X}{\otimes} \mathscr{M}') \quad \text{and}$$
$$\mathscr{H}om_{\mathscr{D}^\infty_X}(\mathscr{M}'^\infty, \mathscr{M}^\infty|_{T^*_X X}) = j_0^{-1}\mathscr{H}om_{\mathscr{E}^\infty_{X'}}(\mathscr{L}^\infty \underset{\mathscr{D}_X}{\otimes} \mathscr{M}', \mathscr{L}^\infty \underset{\mathscr{E}_X}{\otimes} \mathscr{M}).$$

Hence $\mathscr{M}^\infty$ and $\mathscr{M}'^\infty$ are isomorphic. On the other hand, $\mathscr{M}'$ is with R. S. Therefore $\mathscr{E}_X \underset{\mathscr{D}_X}{\otimes} \mathscr{M}' \cong \mathscr{M}_{\text{reg}}$ holds. This completes the proof of Theorem 5.2.1 when $p_0 \in T^*_X X$. Q. E. D.

As an important consequence of Theorem 5.2.1, we find the following

Theorem 5.2.3. *Let $\mathscr{L}$ be a holonomic system of D-type along a hypersurface S. Then $\mathscr{L}$ is a holonomic $\mathscr{D}$-Module with* R. S.

Proof. It follows from Proposition 2.3.5 in Chapter II, $\mathscr{L}_{\text{reg}}$ is a sub-Module of $\mathscr{L}$. On the other hand, Theorem 5.2.1 asserts that $\mathscr{D}^\infty \underset{\mathscr{D}}{\otimes} \mathscr{L}_{\text{reg}} = \mathscr{D}^\infty \underset{\mathscr{D}}{\otimes} \mathscr{L}$ holds. Since $\mathscr{D}^\infty$ is faithfully flat over $\mathscr{D}$ (S-K-K [24] Remark 2, (2) in p. 406), we conclude that $\mathscr{L}_{\text{reg}} = \mathscr{L}$ holds. Therefore $\mathscr{L}$ is with R. S.

Q. E. D.

§3.

In this section we first show that the restriction of a holonomic $\mathscr{E}$-Module with R. S. to a non-characteristic submanifold yields a holonomic system with R. S. By a quantized contact transformation, this result proves the correspond-

(*) Here we use the same notations as in Appendix A.

ing statement for the integration along finite fiber. These results are also extended to $\mathscr{D}$-Modules.

Theorem 5.3.1. *Let φ: $Y \to X$ be a holomorphic map and $\mathscr{M}$ a holonomic $\mathscr{E}_X$-Module with* R.S. *defined on an open set U in $T^*X - T^*_X X$. Let V be an open set in $T^*Y - T^*_Y Y$ such that $\varpi^{-1}(\mathrm{Supp}\,\mathscr{M}) \cap \rho^{-1}(V) \to V$ is finite. Then $\varphi^*\mathscr{M} \underset{\mathrm{def}}{=} \rho_*(\mathscr{E}_{Y\to X} \underset{\varpi^{-1}\mathscr{E}_X}{\otimes} \varpi^{-1}\mathscr{M})$ is a holonomic $\mathscr{E}_Y|_V$-Module with* R.S.

Proof. It suffices to prove the results when φ is imbedding and when φ is smooth, since the general case can be dealt with as a combination of these two cases. In case φ is smooth, the result is obvious, because $\varphi^*\mathscr{M}$ is a system obtained by adding the de Rham equations along fibers to $\mathscr{M}$. Hence it suffices to show the theorem when φ is imbedding. Therefore we may assume without loss of generality that Y is a submanifold of X of codimension 1. Furthermore, by a suitable quantized contact transformation (Chapter I, § 6, Corollary 1.6.4), we may assume that X is an open set in $\boldsymbol{C}^n$, $Y = \{x \in X; x_1 = 0\}$ and that $\mathrm{Supp}\,\mathscr{M}$ is in a generic position at $p \in \mathrm{Supp}\,\mathscr{M}$. Let S denote $\pi(\mathrm{Supp}\,\mathscr{M} - T^*_X X)$ and let q denote $\pi(p)$. Let $\mathscr{N}$ be a coherent $\mathscr{D}_X$-Module such that $\mathscr{N}_q = \mathscr{M}_p$. (Theorem 5.1.1 in § 1.) Then $\mathscr{N}$ is a holonomic $\mathscr{D}_X$-Module with R.S. such that $\mathrm{SS}(\mathscr{N}) \subset \mathrm{Supp}\,\mathscr{M} \cup T^*_X X$. Let ϕ: $\mathscr{M}_p \to (\mathscr{L}/\mathscr{P})_q$ be an injective $\mathscr{D}_{X,q}$-linear homomorphism satisfying the conditions in Theorem 5.1.3. Then we have an injection $\tilde{\phi}$: $\mathscr{N} \to \mathscr{L}/\mathscr{P}$. On the other hand, it follows from the definition that

$$\varphi^*\mathscr{M} = \mathscr{E}_Y \underset{\mathscr{D}_Y}{\otimes} (\mathscr{D}_{Y\to X} \underset{\mathscr{D}_X}{\otimes} \mathscr{N}) .$$

Hence it suffices to show that $\mathscr{N}|_Y \underset{\mathrm{def}}{=} \mathscr{D}_{Y\to X} \underset{\mathscr{D}_X}{\otimes} \mathscr{N}$ is a holonomic $\mathscr{D}_Y$-Module with R.S. In view of Lemma 5.1.9 in Section 1 combined with the following isomorphism ([8] Propositions 4.2 and 4.3.)

$$\mathscr{D}_{X\leftarrow Y} \underset{\mathscr{D}_Y}{\otimes} (\mathscr{N}|_Y) \cong \mathscr{N}_{x_1}/\mathscr{N}, \tag{5.3.1}$$

it suffices to show that $\mathscr{N}_{x_1}$ is with R.S. Here and in what follows, $\mathscr{N}_{x_1}$ etc. denotes, by definition, $\mathscr{H}^0_{[X|Y]}(\mathscr{N})$ etc. Since the localization procedure is an exact functor and since the localization $\mathscr{L}_{x_1}$ of $\mathscr{L}$ is of D-type (Theorem 2.3.3 in Chapter II), $\mathscr{L}_{x_1}$ is with R.S. (Theorem 5.2.3 in § 2.) Since the quotient of holonomic systems with R.S. is with R.S., $\mathscr{L}_{x_1}/\mathscr{P}_{x_1}$ is also with R.S. (Proposition 1.1.17 in Chapter I.) Therefore $\mathscr{N}_{x_1}$ is with R.S. This completes the proof of the theorem. Q.E.D.

By applying a quantized contact transformation we obtain following

Theorem 5.3.2 as an immediate consequence of Theorem 5.3.1.

Theorem 5.3.2. *Let φ: $Y \to X$ be a holomorphic map, U an open set in $T^*X - T^*_XX$ and V an open set in $T^*Y - T^*_YY$. Let $\mathcal{N}$ be a coherent $\mathcal{E}_Y$-Module defined on V. Assume that $\mathcal{N}$ is a holonomic $\mathcal{E}_Y$-Module with* R.S. *Assume that $\rho^{-1}(\operatorname{Supp}\mathcal{N}) \cap \varpi^{-1}(U) \to U$ is a finite map. Then $\varphi_*\mathcal{N} \underset{\text{def}}{=} \varpi_*(\mathcal{E}_{X\leftarrow Y} \underset{\rho^{-1}\mathcal{E}_Y}{\otimes} \rho^{-1}\mathcal{N})$ is a holonomic $\mathcal{E}_X|_V$-Module with* R.S.

Theorem 5.3.1 and Theorem 5.3.2 also hold at the zero-section of T^*X, namely, the following theorems hold:

Theorem 5.3.3. *Theorem* 5.3.1 *holds for a pair* (U, V), *where U* (*resp.*, V) *is an open set of T^*X* (*resp.*, T^*Y).

Theorem 5.3.4. *Theorem* 5.3.2 *holds for a pair* (U, V), *where U* (*resp.*, V) *is an open set of T^*X* (*resp.*, T^*Y).

Since the proofs of these theorems are the same, we prove Theorem 5.3.3. Take a coordinate system x (resp., y) on X (resp., Y) and let (t, x) be a coordinate system on $X' \underset{\text{def}}{=} \mathbf{C} \times X$. Set $Y' = \mathbf{C} \times Y$ and define a map $\tilde{\varphi}$: $Y' \to X'$ by $\tilde{\varphi}(t, y) = (t, \varphi(y))$. Denote by $\tilde{\mathcal{M}}$ the $\mathcal{E}_{X'}$-Module $\Phi(\mathcal{M}) \underset{\text{def}}{=} (\mathcal{E}_{X'}/\mathcal{E}_{X'}t) \otimes \mathcal{M}$. Then, by Lemma 5.1.9 in Section 1, $\tilde{\mathcal{M}}$ is with R.S. Hence it follows from Theorem 5.3.1 that $\tilde{\mathcal{N}} \underset{\text{def}}{=} \tilde{\varphi}^*\tilde{\mathcal{M}}$ is with R.S. near $\{(t, y; \tau, \eta) \in T^*Y'; \tau \neq 0\}$. Since $\tilde{\mathcal{N}} = (\mathcal{E}_{Y'}/\mathcal{E}_{Y'}t) \underset{\mathcal{E}_Y}{\otimes} (\varphi^*\mathcal{N})$, again by Lemma 5.1.9 in Section 1 we have $\varphi^*\mathcal{N}$ is with R.S. Q.E.D.

§4.

4.1. In [8] it is proved that for a holonomic $\mathcal{D}$-Module $\mathcal{M}$ on X, $\mathcal{H}^k_{[T]}(\mathcal{M})$ and $\mathcal{H}^k_{[X|T]}(\mathcal{M})$ are holonomic for any k and any analytic subset T of X. Using this result, we proved there that $\mathcal{O}_Y \underset{\mathcal{O}_X}{\otimes} \mathcal{M}$ is holonomic for any submanifold Y of X and any holonomic $\mathcal{D}$-Module $\mathcal{M}$ on X.

We shall prove in this section that these cohomology groups have R.S., if $\mathcal{M}$ has R.S.

Theorem 5.4.1. *Let T be an analytic subset of a complex manifold X and $\mathcal{M}$ a holonomic $\mathcal{D}_X$-Module with* R.S. *Then*

(5.4.1) $\mathcal{H}^k_{[T]}(\mathcal{M})$ *and* $\mathcal{H}^k_{[X|T]}(\mathcal{M})$ *have* R.S. *for any* k.

(5.4.2) $$(\mathcal{H}^k_{[T]}(\mathcal{M}))^\infty = \mathcal{H}^k_T(\mathcal{M}^\infty)$$

and

$$(\mathscr{H}^k_{[X|T]}(\mathscr{M}))^\infty = \mathscr{H}^k_{X|T}(\mathscr{M}^\infty),$$

where $\mathscr{M}^\infty = \mathscr{D}^\infty_X \underset{\mathscr{D}_X}{\otimes} \mathscr{M}$.

We shall prove this theorem by reducing it to the case where $\mathscr{M}$ is of D-type. First let us recall the exact sequences

$$0 \longrightarrow \mathscr{H}^0_{[T]}(\mathscr{M}) \longrightarrow \mathscr{M} \longrightarrow \mathscr{H}^0_{[X|T]}(\mathscr{M}) \longrightarrow \mathscr{H}^1_{[T]}(\mathscr{M}) \longrightarrow 0$$

and

$$0 \longrightarrow \mathscr{H}^0_T(\mathscr{M}^\infty) \longrightarrow \mathscr{M}^\infty \longrightarrow \mathscr{H}^0_{X|T}(\mathscr{M}^\infty) \longrightarrow \mathscr{H}^1_T(\mathscr{M}^\infty) \longrightarrow 0.$$

Hence the isomorphisms

$$\mathscr{H}^k_{[X|T]}(\mathscr{M}) = \mathscr{H}^{k+1}_{[T]}(\mathscr{M}) \qquad \text{for} \quad k \geqq 1$$

and

$$\mathscr{H}^k_{X|T}(\mathscr{M}^\infty) = \mathscr{H}^{k+1}_T(\mathscr{M}^\infty) \qquad \text{for} \quad k \geqq 1$$

show immediately that the statement on $\mathscr{H}^k_{[T]}(\mathscr{M})$ and the statement on $\mathscr{H}^k_{[X|T]}(\mathscr{M})$ in Theorem 5.4.1 are equivalent. Hence we will concentrate our attention to $\mathscr{H}^k_{[T]}(\mathscr{M})$ in what follows.

Next we reduce the situation of the theorem to the case where T is a hypersurface.

Lemma 5.4.2. *Suppose that for a holonomic $\mathscr{D}$-Module $\mathscr{M}$ with* R.S. *on X Theorem* 5.3.1 *holds for any hypersurface T. Then Theorem* 5.4.1 *holds for $\mathscr{M}$ and any analytic subset T.*

Proof. Any analytic subset T is locally an intersection of finite number of hypersurfaces $T_1, \ldots, T_l$. We shall prove by the induction on l. If $l=0$ (i.e., $X=T$), there is nothing to prove. If $l=1$, then this is nothing but the assumption. Suppose that $l>1$. Set $T' = T_2 \cap \cdots \cap T_l$. Then, by the hypothesis of induction, $\mathbf{R}\Gamma_{[T']}(\mathscr{M})$ and $\mathbf{R}\Gamma_{[T_1 \cup T']}(\mathscr{M})$ have holonomic systems with R.S. as their cohomologies and

$$\mathbf{R}\Gamma_{[T']}(\mathscr{M})^\infty = \mathbf{R}\Gamma_{T'}(\mathscr{M}^\infty).$$

and

$$\mathbf{R}\Gamma_{[T_1 \cup T']}(\mathscr{M})^\infty = \mathbf{R}\Gamma_{T_1 \cup T'}(\mathscr{M}^\infty)$$

hold. On the other hand, we have triangles

$$\begin{array}{ccc} & \mathbf{R}\Gamma_{[T]}(\mathcal{M})^{\infty} & \\ \swarrow & & \nwarrow^{+1} \\ \mathbf{R}\Gamma_{[T_1]}(\mathcal{M})^{\infty} \oplus \mathbf{R}\Gamma_{[T']}(\mathcal{M})^{\infty} & \longrightarrow & \mathbf{R}\Gamma_{[T_1 \cup T']}(\mathcal{M})^{\infty} \end{array}$$

and

$$\begin{array}{ccc} & \mathbf{R}\Gamma_{T}(\mathcal{M}^{\infty}) & \\ \swarrow & & \nwarrow^{+1} \\ \mathbf{R}\Gamma_{T_1}(\mathcal{M}^{\infty}) \oplus \mathbf{R}\Gamma_{T'}(\mathcal{M}^{\infty}) & \longrightarrow & \mathbf{R}\Gamma_{T_1 \cup T'}(\mathcal{M}^{\infty})\,. \end{array}$$

Therefore $\mathbf{R}\Gamma_{[T]}(\mathcal{M})^{\infty} = \mathbf{R}\Gamma_{T}(\mathcal{M}^{\infty})$ holds and $\mathbf{R}\Gamma_{[T]}(\mathcal{M})$ has holonomic systems with R. S. as its cohomologies. Q. E. D.

By this reduction, we immediately find the following

Lemma 5.4.3. *For a holonomic system $\mathcal{M}$ of D-type the statement of Theorem* 5.3.1 *holds.*

This is an immediate consequence of the preceding lemma Proposition 2.3.3 in Chapter II and Theorem 5.2.3 in Section 2.

Lemma 5.4.4. *Let $\mathcal{L}$ be a holonomic system of D-type along a hypersurfaces S and Z an analytic set. Assume that Z is locally of complete intersection of codimension l outside S. Then we have*

$$\mathcal{H}^k_{[Z]}(\mathcal{L}) = 0 \qquad \textit{for} \quad k \neq l$$

and the statement of Theorem 5.4.1 *holds for $\mathcal{M} = \mathcal{H}^l_{[Z]}(\mathcal{L})$ and for any analytic subset T. Furthermore, for any proper analytic subset Z' of Z, $\mathcal{H}^0_{[Z']}(\mathcal{M}) = 0$ holds.*

Proof. Since $\mathcal{L} = \mathcal{O}_X^N$ outside S, $\mathcal{H}^k_{[Z]}(\mathcal{L})$ vanishes outside S for $k \neq l$. Hence we have

$$\begin{aligned} \mathbf{R}\Gamma_{[X|S]}\mathbf{R}\Gamma_{[Z]}(\mathcal{L}) &= \mathbf{R}\Gamma_{[X|S]}\mathcal{H}^l_{[Z]}(\mathcal{L})\,[-l] \\ &= \mathcal{H}^0_{[X|S]}\mathcal{H}^l_{[Z]}(\mathcal{L})\,[-l]\,. \end{aligned}$$

Since $\mathbf{R}\Gamma_{[X|Z]}(\mathcal{L}) = \mathcal{L}$ and $\mathbf{R}\Gamma_{[Z]}\mathbf{R}\Gamma_{[X|S]} = \mathbf{R}\Gamma_{[X|S]}\mathbf{R}\Gamma_{[Z]}$ hold,

$$\mathbf{R}\Gamma_{[Z]}(\mathcal{L}) = \mathcal{H}^0_{[X|S]}\mathcal{H}^l_{[Z]}(\mathcal{L})\,[-l]\,.$$

This proves the first statement of the lemma.

Next let us prove the second statement. It follows from the definition of $\mathcal{M}$ that

$$\mathscr{H}^k_{[T]}(\mathscr{M}) = \mathscr{H}^{k+l}_{[T\cap Z]}(\mathscr{L})$$

holds, and hence $\mathscr{H}^k_{[T]}(\mathscr{M})$ has R.S. by the preceding lemma.

Now we shall prove (5.4.2) for $\mathscr{M} = \mathscr{H}^l_{[Z]}(\mathscr{L})$. The preceding lemma entails

$$\mathscr{H}^k_Z(\mathscr{L}^\infty) = (\mathscr{H}^k_{[Z]}(\mathscr{L}))^\infty = \begin{cases} \mathscr{M}^\infty & \text{for } k=l \\ 0 & \text{for } k \neq l. \end{cases}$$

Hence we have

$$\begin{aligned} \mathbf{R}\Gamma_T(\mathscr{M}^\infty) &= \mathbf{R}\Gamma_T(\mathbf{R}\Gamma_Z(\mathscr{L}^\infty)[l]) \\ &= \mathbf{R}\Gamma_{T\cap Z}(\mathscr{L}^\infty)[l] \\ &= (\mathbf{R}\Gamma_{[T\cap Z]}(\mathscr{L}))^\infty [l] \\ &= (\mathbf{R}\Gamma_{[T]}(\mathbf{R}\Gamma_{[Z]}(\mathscr{L})[l]))^\infty \\ &= (\mathbf{R}\Gamma_{[T]}(\mathscr{M}))^\infty . \end{aligned}$$

Therefore, by considering the cohomology groups, we obtain (5.4.2).

Now let us prove the last statement, i.e., $\mathscr{H}^0_{[Z']}(\mathscr{H}^l_{[Z]}(\mathscr{L})) = 0$. Let Z'' be an analytic subset of Z which contains Z' and is of locally complete intersection of codimension $l+1$ outside S. Then we have

$$\begin{aligned} &\mathscr{H}^0_{[Z']}\mathscr{H}^l_{[Z]}(\mathscr{L}) \subset \mathscr{H}^0_{[Z'']}\mathscr{H}^l_{[Z]}(\mathscr{L}) \\ &= \mathscr{H}^l_{[Z'']}(\mathscr{L}) = 0 . \end{aligned}$$

Q.E.D.

4.2. We shall prove Theorem 5.4.1 by the induction on the codimension of the support of $\mathscr{M}$. In order to facilitate the induction, we shall consider the following situation. Let X be a complex manifold of dimension n, Y a complex manifold of dimension $n-l$ and F a smooth map from X into Y. Let Z be an analytic subset of X such that the restriction of F to Z gives a finite map from Z to Y. Let $\mathscr{M}$ be a coherent $\mathscr{D}_X$-Module such that $\mathrm{SS}(\mathscr{M}) \subset T^*_Z X$. Since $\mathscr{M}$ is (locally) a union of increasing sequence of coherent $\mathscr{O}_X$-Modules, we have a homomorphism

$$\mathscr{M} \longrightarrow \mathscr{O}_X \underset{F^{-1}\mathscr{O}_Y}{\otimes} F^{-1}F_*(\Omega_{X/Y} \underset{\mathscr{O}_X}{\otimes} \mathscr{M})[l],$$

where $\Omega_{X/Y} = \Omega_X \underset{F^{-1}\mathscr{O}_Y}{\otimes} F^{-1}(\Omega_Y^{\otimes -1})$ ([5]). Applying the functor $\mathbf{R}\Gamma_{[Z]}$ to it and considering its 0-th cohomology group, we get a map

$$\mathscr{M} \longrightarrow \mathscr{H}^l_{[Z]}(\mathscr{O}_X \underset{F^{-1}\mathscr{O}_Y}{\otimes} F^{-1}F_*(\Omega_{X/Y} \underset{\mathscr{O}_X}{\otimes} \mathscr{M})).$$

We have the canonical homomorphism $\mathscr{O}_X \to \mathscr{D}_{X\to Y}$ and $\Omega_{X/Y} \to \mathscr{D}_{Y\leftarrow X}$. Hence we obtain a homomorphism

(5.4.3) $$\mathscr{M} \longrightarrow \mathscr{H}^l_{[Z]}(\mathscr{D}_{X\to Y} \underset{F^{-1}\mathscr{D}_Y}{\otimes} F^{-1}F_*(\mathscr{D}_{Y\leftarrow X} \underset{\mathscr{D}_X}{\otimes} \mathscr{M})).$$

Then we have the following

Lemma 5.4.5. *When $Z \to Y$ is an isomorphism, then* (5.4.3) *is an isomorphism.*

Proof. Since $\mathscr{H}^k_{[Z]}(\mathscr{D}_{X\to Y})=0$ for $k\neq l$ and since $\mathscr{H}^l_{[Z]}(\mathscr{D}_{X\to Y})=\mathscr{D}_X \underset{\mathscr{O}_X}{\otimes} \mathscr{O}_Z$ is flat over $F^{-1}\mathscr{D}_Y$, $\mathscr{M}\mapsto \mathscr{H}^l_{[Z]}(\mathscr{D}_{X\to Y} \underset{F^{-1}\mathscr{D}_Y}{\otimes} F^{-1}(\mathscr{D}_{Y\leftarrow X} \underset{\mathscr{D}_X}{\otimes} \mathscr{M}))$ is an exact functor from the category of coherent $\mathscr{D}_X$-Modules with support in Z to the category of $\mathscr{D}_X$-Modules.

The question being local, we may assume that $\mathscr{M}=\mathscr{D}_X \underset{\mathscr{O}_X}{\otimes} \mathscr{O}_Z$. Then $F_*(\mathscr{D}_{Y\leftarrow X} \underset{\mathscr{D}_X}{\otimes} \mathscr{M})=\mathscr{D}_Y$ holds and hence $\mathscr{H}^l_{[Z]}(\mathscr{D}_{X\to Y} \underset{F^{-1}\mathscr{D}_Y}{\otimes} F^{-1}F_*(\mathscr{D}_{Y\leftarrow X} \underset{\mathscr{D}_X}{\otimes} \mathscr{M})) =\mathscr{D}_X \underset{\mathscr{O}_X}{\otimes} \mathscr{O}_Z$ holds.

Remark. We can prove that (5.4.3) is injective.

4.3. We now prove Theorem 5.4.1 by the induction on the codimension of the support of $\mathscr{M}$.

Set $l=\operatorname{codim}\operatorname{Supp}\mathscr{M}$. Then we can find locally an analytic subset Z of complete intersection of codimension l and a smooth map $F: X\to Y$ to an $(n-l)$ dimensional complex manifold Y such that $Z\to Y$ is a finite map and $Z\supset \operatorname{Supp}\mathscr{M}$. Set $\mathscr{N}=F_*(\mathscr{D}_{Y\leftarrow X} \underset{\mathscr{D}_X}{\otimes} \mathscr{M})$. Then by Theorem 5.3.4 we find that $\mathscr{N}$ has R.S. There is a hypersurface S of Y such that $\mathrm{SS}(\mathscr{N})=T^*_YY$ on $Y-S$. Set $\mathscr{L} =\mathscr{H}^0_{[Y|S]}(\mathscr{N})$. Then $\mathscr{L}$ is of D-type along S. Let φ be the composition of the homomorphisms

$$\mathscr{M} \longrightarrow \mathscr{H}^l_{[Z]}(\mathscr{D}_{X\to Y} \underset{\mathscr{D}_Y}{\otimes} \mathscr{N}) \longrightarrow \mathscr{H}^l_{[Z]}(\mathscr{D}_{X\to Y} \underset{\mathscr{D}_Y}{\otimes} \mathscr{L}).$$

Let Z_0 be the union of $Z\cap F^{-1}(S)$ and the subset of Z where $Z\to Y$ is not a local isomorphism. Then $\operatorname{codim} Z_0 \geqq l+1$. Since $\mathscr{N}=\mathscr{L}$ outside S, φ is injective and $\mathscr{D}_X$-linear outside Z_0 by Lemma 5.4.5. On the other hand, it is obvious that $\mathscr{D}_{X\to Y} \underset{\mathscr{D}_Y}{\otimes} \mathscr{L}$ is a holonomic system of D-type along $F^{-1}(S)$. Hence $\mathscr{H}^0_{[Z_0]}\mathscr{H}^l_{[Z]}(\mathscr{D}_{X\to Y} \underset{\mathscr{D}_Y}{\otimes} \mathscr{L})=0$ holds. Therefore φ is $\mathscr{D}_X$-linear. Let $\mathscr{M}'$ and $\mathscr{M}''$ be the kernel and the image of φ, respectively. Since φ is injective outside Z_0 by Lemma 5.4.5, $\operatorname{Supp}\mathscr{M}'$ is contained in Z_0. On the other hand, we have an exact sequence

$$(5.4.4)\qquad \cdots \longrightarrow \mathscr{H}^{k-1}_{[T]}(\mathscr{M}'') \longrightarrow \mathscr{H}^{k}_{[T]}(\mathscr{M}') \longrightarrow \mathscr{H}^{k}_{[T]}(\mathscr{M}) \longrightarrow \mathscr{H}^{k}_{[T]}(\mathscr{M}'') \longrightarrow \mathscr{H}^{k+1}_{[T]}(\mathscr{M}') \longrightarrow \cdots,$$

and a commutative diagram

(5.4.5)

$$\begin{array}{ccccccccccc}
\cdots\to & \mathscr{H}^{k-1}_{[T]}(\mathscr{M}'')^\infty & \to & \mathscr{H}^{k}_{[T]}(\mathscr{M}')^\infty & \to & \mathscr{H}^{k}_{[T]}(\mathscr{M})^\infty & \to & \mathscr{H}^{k}_{[T]}(\mathscr{M}'') & \to & \mathscr{H}^{k+1}_{[T]}(\mathscr{M}')^\infty & \to\cdots \\
& \downarrow{\scriptstyle \alpha''_{k-1}} & & \downarrow{\scriptstyle \alpha'_{k}} & & \downarrow{\scriptstyle \alpha_{k}} & & \downarrow{\scriptstyle \alpha''_{k}} & & \downarrow{\scriptstyle \alpha'_{k+1}} & \\
\cdots\to & \mathscr{H}^{k-1}_{T}(\mathscr{M}''^\infty) & \to & \mathscr{H}^{k-1}_{T}(\mathscr{M}'^\infty) & \to & \mathscr{H}^{k}_{T}(\mathscr{M}^\infty) & \to & \mathscr{H}^{k}_{T}(\mathscr{M}''^\infty) & \to & \mathscr{H}^{k+1}_{T}(\mathscr{M}'^\infty) & \to\cdots.
\end{array}$$

Set $\mathscr{K}=\mathscr{H}^{l}_{[Z]}(\mathscr{D}_{X\to Y}\underset{\mathscr{D}_Y}{\otimes}\mathscr{L})$ and let $\mathscr{K}'$ be the cokernel of φ. Since $\mathscr{K}$ has R.S. by Lemma 5.4.4, $\mathscr{K}'$ has also R.S. Since $\mathscr{D}_{X\to Y}\underset{\mathscr{D}_Y}{\otimes}\mathscr{L}$ is of D-type, $\mathscr{H}^{k}_{[T]}(\mathscr{K})$ has R.S. for any k and $\mathscr{H}^{k}_{[T]}(\mathscr{K})^\infty\to\mathscr{H}^{k}_{T}(\mathscr{K}^\infty)$ is an isomorphism. We have also exact sequences

$$(5.4.6)\qquad \cdots \longrightarrow \mathscr{H}^{k-1}_{[T]}(\mathscr{K}) \longrightarrow \mathscr{H}^{k-1}_{[T]}(\mathscr{K}') \longrightarrow \mathscr{H}^{k}_{[T]}(\mathscr{M}'') \longrightarrow \mathscr{H}^{k}_{[T]}(\mathscr{K}) \longrightarrow \mathscr{H}^{k}_{[T]}(\mathscr{K}') \longrightarrow \cdots,$$

and

(5.4.7)

$$\begin{array}{ccccccccccc}
\cdots\to & \mathscr{H}^{k-1}_{[T]}(\mathscr{K})^\infty & \to & \mathscr{H}^{k-1}_{[T]}(\mathscr{K}')^\infty & \to & \mathscr{H}^{k}_{[T]}(\mathscr{M}'')^\infty & \to & \mathscr{H}^{k}_{[T]}(\mathscr{K})^\infty & \to & \mathscr{H}^{k}_{[T]}(\mathscr{K}')^\infty & \to\cdots \\
& \downarrow{\scriptstyle \beta_{k-1}} & & \downarrow{\scriptstyle \beta'_{k-1}} & & \downarrow{\scriptstyle \alpha''_{k}} & & \downarrow{\scriptstyle \beta_{k}} & & \downarrow{\scriptstyle \beta'_{k}} & \\
\cdots\to & \mathscr{H}^{k-1}_{T}(\mathscr{K}^\infty) & \to & \mathscr{H}^{k-1}_{T}(\mathscr{K}'^\infty) & \to & \mathscr{H}^{k}_{T}(\mathscr{M}''^\infty) & \to & \mathscr{H}^{k}_{T}(\mathscr{K}^\infty) & \to & \mathscr{H}^{k}_{T}(\mathscr{K}'^\infty) & \to\cdots.
\end{array}$$

We shall prove that $\mathscr{H}^{k}_{[T]}(\mathscr{M})$ has R.S. by the induction on k. Suppose that $\mathscr{H}^{k}_{[T]}(\mathscr{M})$ has R.S. for $k<k_0$. Then $\mathscr{H}^{k}_{[T]}(\mathscr{K}')$ has R.S. for $k<k_0$ because $\operatorname{Supp}\mathscr{K}'\subset Z$. Hence it follows from the exact sequence (5.4.6) that $\mathscr{H}^{k_0}_{[T]}(\mathscr{M}'')$ has R.S. On the other hand, $\operatorname{Supp}\mathscr{M}'\subset Z_0$. Therefore, by the induction on l, $\mathscr{H}^{k_0}_{[T]}(\mathscr{M}')$ has R.S. Then the exact sequence (5.4.4) entails that $\mathscr{H}^{k_0}_{[T]}(\mathscr{M})$ has R.S.

Next we shall prove that

$$\alpha_k\colon \mathscr{H}^{k}_{[T]}(\mathscr{M})^\infty \longrightarrow \mathscr{H}^{k}_{T}(\mathscr{M}^\infty)$$

is an isomorphism by the induction on k. By the induction on l, we may assume that α'_k is an isomorphism for any k.

Suppose that α_k is an isomorphism for $k<k_0$. Since $\operatorname{Supp}(\mathscr{K}')\subset Z$, $\beta'_k\colon \mathscr{H}^{k}_{[T]}(\mathscr{K}')^\infty\to\mathscr{H}^{k}_{T}(\mathscr{K}'^\infty)$ is an isomorphism for $k<k_0$. Since $\beta_k\colon\mathscr{H}^{k}_{[T]}(\mathscr{K})^\infty\to\mathscr{H}^{k}_{T}(\mathscr{K}^\infty)$ is an isomorphism for any k, it follows from (5.4.7) that α''_{k_0} is injective. Since $\alpha''_{k_0-1}\colon\mathscr{H}^{k_0-1}_{[T]}(\mathscr{M}'')^\infty\to\mathscr{H}^{k_0-1}_{T}(\mathscr{M}''^\infty)$ is an isomorphism by the

hypothesis of the induction on k, α_{k_0} is injective by the diagram (5.4.5). Thus we have shown that α_{k_0} is injective for any holonomic system $\mathcal{M}$ with R.S. supported in Z. Applying this result to $\mathcal{K}'$, β'_{k_0} is seen to be injective. Thus by the exact sequence (5.4.7) and the bijectivity of β'_{k_0-1}, we find that α''_{k_0} is surjective. On the other hand, Supp $\mathcal{M}'\subset Z_0$, α'_k is bijective for any k by the hypothesis of the induction on l. Hence it follows from the diagram (5.4.7) that α_{k_0} is surjective. Q.E.D.

Corollary 5.4.6. *Let Y be a submanifold of X and $\mathcal{M}$ a holonomic $\mathcal{D}_X$-Module with* R.S. *Then $\mathcal{T}or_k^{\mathcal{O}_X}(\mathcal{O}_Y, \mathcal{M})$ is a holonomic $\mathcal{D}_Y$-Module with* R.S.

Proof. Set $l=\operatorname{codim} Y$. Then

$$\mathcal{D}_{X\leftarrow Y}\underset{\mathcal{D}_Y}{\otimes}\mathcal{T}or_k^{\mathcal{O}_X}(\mathcal{O}_Y, \mathcal{M})=\mathcal{H}_{[Y]}^{l-k}(\mathcal{M})$$

and this has R.S. by Theorem 5.4.1. Therefore $\mathcal{T}or_k^{\mathcal{O}_X}(\mathcal{O}_Y, \mathcal{M})$ has R.S.

Corollary 5.4.7. *Let $\mathcal{M}$ and $\mathcal{N}$ be two holonomic $\mathcal{D}_X$-Modules with* R.S. *Then $\mathcal{T}or_k^{\mathcal{O}_X}(\mathcal{M}, \mathcal{N})$ has* R.S.

Proof. Since $\mathcal{T}or_k^{\mathcal{O}_X}(\mathcal{M}, \mathcal{N})=\mathcal{T}or_k^{\mathcal{O}_{X\times X}}(\mathcal{O}_X, \mathcal{M}\hat{\otimes}\mathcal{N})$, this is an immediate consequence of Corollary 5.4.6.

Corollary 5.4.8. *Let $F: Y\to X$ be a holomorphic map and $\mathcal{M}$ a holonomic $\mathcal{D}_X$-Module with* R.S. *Then $\mathcal{T}or_k^{F^{-1}\mathcal{O}_X}(\mathcal{O}_Y, F^{-1}\mathcal{M})$ has* R.S.

Proof. Since $\mathcal{T}or_k^{F^{-1}\mathcal{O}_X}(\mathcal{O}_Y, F^{-1}\mathcal{M})=\mathcal{T}or_k^{\mathcal{O}_{X\times Y}}(\mathcal{O}_Y, \mathcal{M}\hat{\otimes}\mathcal{O}_Y)$, this follows from Corollary 5.4.6.

Chapter VI. Comparison Theorems

The purpose of this chapter is to prove several comparison theorems for holonomic systems with R.S. As a by-product, we prove in Section 2 that holonomic $\mathcal{D}$-Modules with R.S. remain holonomic $\mathcal{D}$-Module with R.S. under the integration procedure with projective fibers and the general restriction procedure. In Section 4 we also show that the validity of comparison theorems is a characteristic property of holonomic system with R.S.

§1.

1.1. We first show the following Theorem 6.1.1 as a consequence of

Theorem 5.4.1. In the next subsection we generalize the result to $\mathscr{E}$-Modules.

Theorem 6.1.1. *Let* $\mathscr{M}$ *and* $\mathscr{N}$ *be two holonomic* $\mathscr{D}_X$*-Modules with* R.S. *Then*

$$\mathbf{R}\mathscr{H}om_{\mathscr{D}_X}(\mathscr{M}, \mathscr{N}) \xrightarrow{\sim} \mathbf{R}\mathscr{H}om_{\mathscr{D}_X}(\mathscr{M}, \mathscr{N}^\infty) .$$

Proof. We shall use the diagonal process.

We have

$$\mathbf{R}\mathscr{H}om_{\mathscr{D}_X}(\mathscr{M}, \mathscr{N}) = \mathbf{R}\mathscr{H}om_{\mathscr{D}_X}(\mathscr{M}\hat{\otimes}\mathscr{N}^*, \mathscr{B}_{X|X\times X})[n],$$

where

$$\mathscr{B}_{X|X\times X} = \mathscr{H}^n{}_{[X]}(\mathscr{O}_{X\times X})$$

and

$$\mathbf{R}\mathscr{H}om_{\mathscr{D}_X}(\mathscr{M}, \mathscr{N}^\infty) = \mathbf{R}\mathscr{H}om_{\mathscr{D}_X}(\mathscr{M}\hat{\otimes}\mathscr{N}^*, \mathscr{B}^\infty_{X|X\times X})[n] .$$

Set $\mathscr{L} = \mathscr{M}\hat{\otimes}\mathscr{N}^*$.

Since

$$\mathscr{B}^\infty_{X|X\times X} = \mathbf{R}\Gamma_X(\mathscr{O}_{X\times X})[n] ,$$

we have

$$\begin{aligned}\mathbf{R}\mathscr{H}om_{\mathscr{D}_X}(\mathscr{M}, \mathscr{N}^\infty) &= \mathbf{R}\Gamma_X\mathbf{R}\mathscr{H}om_{\mathscr{D}_X}(\mathscr{L}, \mathscr{O}_{X\times X})[2n]\\ &= \mathbf{R}\Gamma_X\mathbf{R}\mathscr{H}om_{\mathscr{D}_X}(\mathscr{O}_{X\times X}, \mathscr{L}^*)[2n]\\ &= \mathbf{R}\Gamma_X\mathbf{R}\mathscr{H}om_{\mathscr{D}_X}(\mathscr{O}_{X\times X}, \mathscr{L}^{*\infty})[2n]\\ &= \mathbf{R}\mathscr{H}om_{\mathscr{D}_X}(\mathscr{O}_{X\times X}, \mathbf{R}\Gamma_X(\mathscr{L}^{*\infty}))[2n] .\end{aligned}$$

Since $\mathscr{L}^*$ has R.S., $\mathbf{R}\Gamma_X(\mathscr{L}^{*\infty}) = \mathbf{R}\Gamma_{[X]}(\mathscr{L}^*)^\infty$ holds by Theorem 5.4.1. Hence we have

$$\begin{aligned}(6.1.1)\qquad \mathbf{R}\mathscr{H}om_{\mathscr{D}_{X\times X}}(\mathscr{M}, \mathscr{N}^\infty) &= \mathbf{R}\mathscr{H}om_{\mathscr{D}_X}(\mathscr{O}_{X\times X}, (\mathbf{R}\Gamma_{[X]}(\mathscr{L}^*))^\infty)[2n]\\ &= \mathbf{R}\mathscr{H}om_{\mathscr{D}_{X\times X}}(\mathscr{O}_{X\times X}, \mathbf{R}\Gamma_{[X]}(\mathscr{L}^*))[2n] .\end{aligned}$$

In order to calculate the right hand side of (6.1.1) we prepare the following

Lemma 6.1.2. *Let* $\mathscr{M}$ *and* $\mathscr{N}$ *be two holonomic* $\mathscr{D}_X$*-Modules. Then for any analytic subset* Z *of* X*, we have*

$$\mathbf{R}\mathscr{H}om_{\mathscr{D}_X}(\mathscr{M}, \mathbf{R}\Gamma_{[Z]}(\mathscr{N})) = \mathbf{R}\mathscr{H}om_{\mathscr{D}_X}(\mathscr{N}^*, \mathbf{R}\Gamma_{[Z]}(\mathscr{M}^*)) .$$

Proof. By Theorem 1.2 of [8], we have

$$
\begin{aligned}
&\mathbb{R}\mathscr{H}om_{\mathscr{D}_X}(\mathscr{M}, \mathbb{R}\Gamma_{[Z]}(\mathscr{N})) \\
&= \mathbb{R}\mathscr{H}om_{\mathscr{D}_X}(\mathscr{M}, \mathscr{D}_X) \overset{\mathbb{L}}{\underset{\mathscr{D}_X}{\otimes}} \mathbb{R}\Gamma_{[Z]}(\mathscr{N}) \\
&= \mathbb{R}\Gamma_{[Z]}\mathbb{R}\mathscr{H}om(\mathscr{M}, \mathscr{D}_X) \overset{\mathbb{L}}{\underset{\mathscr{D}_X}{\otimes}} \mathscr{N} \\
&= \mathbb{R}\mathscr{H}om_{\mathscr{D}_X}(\mathbb{R}\mathscr{H}om_{\mathscr{D}_X}(\mathscr{N}, \mathscr{D}_X), \mathbb{R}\Gamma_{[Z]}\mathbb{R}\mathscr{H}om(\mathscr{M}, \mathscr{D}_X)) .
\end{aligned}
$$

Q.E.D.

Proof of Theorem 6.1.1 *continued.* Applying this lemma to (6.1.1), we have

$$
\begin{aligned}
\mathbb{R}\mathscr{H}om_{\mathscr{D}_X}(\mathscr{M}, \mathscr{N}^{\infty}) &= \mathbb{R}\mathscr{H}om_{\mathscr{D}_{X\times X}}(\mathscr{L}, \mathbb{R}\Gamma_{[X]}(\mathscr{O}_{X\times X}))[2n] \\
&= \mathbb{R}\mathscr{H}om_{\mathscr{D}_{X\times X}}(\mathscr{L}, \mathscr{B}_{X|X\times X})[n] \\
&= \mathbb{R}\mathscr{H}om_{\mathscr{D}_X}(\mathscr{M}, \mathscr{N}) .
\end{aligned}
$$

This completes the proof of Theorem 6.1.1.

1.2. We shall generalize the result in subsection 1.1 to $\mathscr{E}_X$-Modules.

Theorem 6.1.3. *Let $\mathscr{M}$ and $\mathscr{N}$ be two holonomic $\mathscr{E}_X$-Modules with* R.S. *Then*

$$
\mathbb{R}\mathscr{H}om_{\mathscr{E}_X}(\mathscr{M}, \mathscr{N}) = \mathbb{R}\mathscr{H}om_{\mathscr{E}_X}(\mathscr{M}, \mathscr{N}^{\infty}) .
$$

Proof. At the zero section, this theorem is nothing but Theorem 6.1.1. Therefore we shall prove this theorem outside the zero section. Since

$$
\mathbb{R}\mathscr{H}om_{\mathscr{E}_X}(\mathscr{M}, \mathscr{N}) = \mathbb{R}\mathscr{H}om_{\mathscr{E}_{X\times X}}(\mathscr{M}\hat{\otimes}\mathscr{N}^*, \mathscr{C}_{X|X\times X})[n]
$$

and

$$
\mathbb{R}\mathscr{H}om_{\mathscr{E}_X}(\mathscr{M}, \mathscr{N}^{\infty}) = \mathbb{R}\mathscr{H}om_{\mathscr{E}_{X\times X}}(\mathscr{M}\hat{\otimes}\mathscr{N}^*, \mathscr{C}^{\infty}_{X|X\times X})[n]
$$

holds, we may assume without loss of generality that the support of $\mathscr{N}$ is a non-singular Lagrangian manifold and that $\mathscr{N}$ has multiplicity 1.

By a quantized contact transformation, we may assume that Supp $\mathscr{M}$ is in a generic position and Supp $\mathscr{N}$ is a conormal bundle of a smooth hypersurface Y of X and $\mathscr{N} \cong \mathscr{C}_{Y|X}$. Now, there is a holonomic $\mathscr{D}_X$-Module $\tilde{\mathscr{M}}$ with R.S. that $\mathscr{M} = \mathscr{E}_X \underset{\mathscr{D}_X}{\otimes} \tilde{\mathscr{M}}$. Therefore it is enough to show

$$
\mathbb{R}\mathscr{H}om_{\mathscr{D}_X}(\tilde{\mathscr{M}}, \mathscr{C}_{Y|X}) \xrightarrow{\sim} \mathbb{R}\mathscr{H}om_{\mathscr{D}_X}(\tilde{\mathscr{M}}, \mathscr{C}^{\infty}_{Y|X}) .
$$

Since $\mathscr{C}^{\infty}_{Y|X}/\mathscr{C}_{Y|X} = \mathscr{B}^{\infty}_{Y|X}/\mathscr{B}_{Y|X}$, this follows from the isomorphism

$$
\mathbb{R}\mathscr{H}om_{\mathscr{D}_X}(\tilde{\mathscr{M}}, \mathscr{B}_{Y|X}) \xrightarrow{\sim} \mathbb{R}\mathscr{H}om_{\mathscr{D}_X}(\tilde{\mathscr{M}}, \mathscr{B}^{\infty}_{Y|X}) .
$$

Q.E.D.

§2.

The purpose of this section is to extend the results obtained in Chapter V, Section 3 under the additional assumption that $\mathscr{M}$ is a $\mathscr{D}$-Module.

2.1. Let $F: X \to Y$ be a projective map (i.e., F can be embedded in $Y \times \mathbf{P}^N \to Y$). Let $\mathscr{M}$ be a holonomic $\mathscr{D}_X$-Module with R.S. By Corollary 5.1.11 in Chapter V, Section 1, $\mathscr{M}$ has a good filtration $\{\mathscr{M}_k\}_{k\in\mathbf{Z}}$ defined on X. (See also Appendix A.) Therefore $\mathbf{R}F_*(\mathscr{D}_{Y\leftarrow X} \overset{\mathbf{L}}{\underset{\mathscr{D}_X}{\otimes}} \mathscr{M})$ has holonomic systems as cohomologies. ([11] Lemma 5, [7] Theorem 4.2.) In this section we will show that they are actually with R.S. if so is $\mathscr{M}$, namely we will prove

Theorem 6.2.1. *Under the condition above* $\mathbf{R}^k F_*(\mathscr{D}_{Y\leftarrow X} \overset{\mathbf{L}}{\underset{\mathscr{D}_X}{\otimes}} \mathscr{M})$ *is a holonomic* $\mathscr{D}_Y$*-Module with* R.S.

2.2. Let $F: X \to Y$ be a smooth projective map with fiber dimension l. Let $\mathscr{M}$ be a $\mathscr{D}_X$-Module and $\mathscr{L}$ a coherent $\mathscr{D}_Y$-Module. Then we have the following

Proposition 6.2.2.

(i) $$\mathbf{R}F_*\mathbf{R}\mathscr{H}om_{\mathscr{D}_X}(\mathscr{D}_{X\to Y} \underset{\mathscr{D}_Y}{\otimes} \mathscr{L}, \mathscr{M}) = \mathbf{R}\mathscr{H}om_{\mathscr{D}_Y}(\mathscr{L}, \mathbf{R}F_*(\mathscr{D}_{Y\leftarrow X} \overset{\mathbf{L}}{\underset{\mathscr{D}_X}{\otimes}} \mathscr{M}))[-l],$$

(ii) $$\mathbf{R}F_*\mathbf{R}\mathscr{H}om_{\mathscr{D}_X}(\mathscr{D}_{X\to Y} \overset{\mathbf{L}}{\underset{\mathscr{D}_Y}{\otimes}} \mathscr{L}, \mathscr{M}^\infty) = \mathbf{R}\mathscr{H}om_{\mathscr{D}_Y}(\mathscr{L}, \mathbf{R}F_*(\mathscr{D}_{Y\leftarrow X} \overset{\mathbf{L}}{\underset{\mathscr{D}_X}{\otimes}} \mathscr{M}^\infty))[-l].$$

Proof. Since the problem is of local character on Y, we may assume that $\mathscr{L}$ has a free resolution. Thus we may assume that $\mathscr{L} = \mathscr{D}_Y$. Therefore it suffices to show

$$\mathbf{R}F_*\mathbf{R}\mathscr{H}om_{\mathscr{D}_X}(\mathscr{D}_{X\to Y}, \mathscr{M}) = \mathbf{R}F_*(\mathscr{D}_{Y\leftarrow X} \overset{\mathbf{L}}{\underset{\mathscr{D}_X}{\otimes}} \mathscr{M})[-l].$$

Since

$$\mathbf{R}\mathscr{H}om_{\mathscr{D}_X}(\mathscr{D}_{X\to Y}, \mathscr{M}) = \mathbf{R}\mathscr{H}om_{\mathscr{D}_X}(\mathscr{D}_{X\to Y}, \mathscr{D}_X) \overset{\mathbf{L}}{\underset{\mathscr{D}_X}{\otimes}} \mathscr{M}$$

and

$$\mathbf{R}\mathcal{H}om_{\mathcal{D}_X}(\mathcal{D}_{X\to Y}, \mathcal{D}_X)=\mathcal{D}_{Y\leftarrow X}[-l]\,,$$

we get the isomorphism (i). The second one can be obtained exactly in the same way as above. Q.E.D.

Proposition 6.2.3. *For a coherent $\mathcal{D}_X$-Module $\mathcal{M}$ such that there is a coherent $\mathcal{O}_X$ sub-Module $\mathcal{M}_0$ of $\mathcal{M}$ with $\mathcal{M}=\mathcal{D}_X\mathcal{M}_0$, we have*

$$\mathbf{R}F_*(\mathcal{D}_{Y\leftarrow X}\underset{\mathcal{D}_X}{\otimes}\mathcal{M})^\infty=\mathbf{R}F_*(\mathcal{D}_{Y\leftarrow X}\underset{\mathcal{D}_X}{\otimes}\mathcal{M}^\infty)\,.$$

This result can be proved in the same way as in [7], Section 4. We omit the details.

2.3. Let X be a complex manifold and $\mathcal{M}^\cdot$ a bounded complex of $\mathcal{D}_X$-Modules. Assume that all cohomology groups $\mathcal{H}^k(\mathcal{M}^\cdot)$ are holonomic.

Proposition 6.2.4. *Assume that*

$$\mathbf{R}\mathcal{H}om_{\mathcal{D}_X}(\mathcal{L}, \mathcal{M}^\cdot)\xrightarrow{\sim}\mathbf{R}\mathcal{H}om_{\mathcal{D}_X}(\mathcal{L}, \mathcal{M}^{\cdot\infty})$$

holds for any holonomic $\mathcal{D}_X$-Module $\mathcal{L}$ with R.S. *Then $\mathcal{H}^k(\mathcal{M}^\cdot)$ is with* R.S. *for any k.*

Proof. Suppose that $\mathcal{H}^k(\mathcal{M}^\cdot)=0$ holds for $k>k_1$ or $k<k_0$. We shall prove the proposition by the induction on k_1-k_0. If $k_1<k_0$, then the proposition is trivial. Suppose that $k_1\geqq k_0$. Set $\mathcal{L}=\mathcal{H}^{k_0}(\mathcal{M}^\cdot)_{\mathrm{reg}}$. We have

$$\mathcal{H}^{k_0}(\mathbf{R}\mathcal{H}om(\mathcal{L}, \mathcal{M}^\cdot))=\mathcal{H}om(\mathcal{L}, \mathcal{H}^{k_0}(\mathcal{M}^\cdot))$$

and

$$\mathcal{H}^{k_0}(\mathbf{R}\mathcal{H}om(\mathcal{L}, \mathcal{M}^{\cdot\infty}))=\mathcal{H}om(\mathcal{L}, \mathcal{H}^{k_0}(\mathcal{M}^\cdot)^\infty)\,.$$

Thus we have

$$\mathcal{H}om(\mathcal{L}, \mathcal{H}^{k_0}(\mathcal{M}^\cdot))\xrightarrow{\sim}\mathcal{H}om(\mathcal{L}, \mathcal{H}^{k_0}(\mathcal{M}^\cdot)^\infty)\,.$$

It is clear that

$$\mathcal{H}om_{\mathcal{D}}(\mathcal{L}, \mathcal{H}^{k_0}(\mathcal{M}^\cdot)^\infty)\xrightarrow{\sim}\mathcal{H}om_{\mathcal{D}^\infty}(\mathcal{L}^\infty, \mathcal{H}^{k_0}(\mathcal{M}^\cdot)^\infty)$$

holds.

We shall apply these isomorphisms to $\mathcal{L}=\mathcal{H}^{k_0}(\mathcal{M}^\cdot)_{\mathrm{reg}}$. The isomorphism $\mathcal{L}^\infty\xrightarrow{\sim}\mathcal{H}^{k_0}(\mathcal{M}^\cdot)^\infty$ comes from the homomorphism $i\colon \mathcal{L}\to\mathcal{H}^{k_0}(\mathcal{M}^\cdot)$. Thus $\mathcal{L}\cong\mathcal{H}^{k_0}(\mathcal{M}^\cdot)$ holds. This implies $\mathcal{H}^{k_0}(\mathcal{M}^\cdot)$ has R.S. Let $\mathcal{N}^\cdot$ be a complex such that there exists a triangle

$$\begin{array}{ccc} & \mathcal{H}^{k_0}(\mathcal{M}^{\cdot})[-k_0] & \\ \swarrow & & \nwarrow^{+1} \\ \mathcal{M}^{\cdot} & \longrightarrow & \mathcal{N}^{\cdot} \end{array}.$$

Then $\mathcal{N}^{\cdot}$ satisfies the condition of the proposition. Furthermore

$$\mathcal{H}^{k}(\mathcal{N}^{\cdot})\cong\mathcal{H}^{k}(\mathcal{M}^{\cdot}) \qquad \text{for} \quad k\neq k_0$$

and

$$\mathcal{H}^{k_0}(\mathcal{N}^{\cdot})=0$$

hold. Thus, by the hypothesis of the induction, $\mathcal{H}^{k}(\mathcal{N}^{\cdot})$ has R. S. Q. E. D.

2.4. We now embark on the proof of Theorem 6.2.1. We first embed $F: X\to Y$ into a smooth projective map $F': X'\to Y$. Then

$$\begin{aligned}&\mathbf{R}F_*(\mathcal{D}_{Y\leftarrow X}\overset{\mathbf{L}}{\underset{\mathcal{D}_X}{\otimes}}\mathcal{M})\\ &\qquad =\mathbf{R}F_*(\mathcal{D}_{Y\leftarrow X'}\overset{\mathbf{L}}{\underset{\mathcal{D}_{X'}}{\otimes}}(\mathcal{D}_{X'\leftarrow X}\underset{\mathcal{D}_X}{\otimes}\mathcal{M})).\end{aligned}$$

Since $\mathcal{D}_{X'\leftarrow X}\underset{\mathcal{D}_X}{\otimes}\mathcal{M}$ has R. S. (Lemma 5.2.9), we may assume from the first that F is smooth and projective. Set $\mathcal{N}^{\cdot}=\mathbf{R}F_*(\mathcal{D}_{Y\leftarrow X}\overset{\mathbf{L}}{\underset{\mathcal{D}_X}{\otimes}}\mathcal{M})$. In order to show that $\mathcal{N}^{\cdot}$ has holonomic $\mathcal{D}_Y$-Modules with R. S. as its cohomologies, it is enough to show that for any holonomic system $\mathcal{L}$ with R. S.

$$\begin{aligned}&\mathbf{R}\mathcal{H}om_{\mathcal{D}_Y}(\mathcal{L},\mathcal{N}^{\cdot})\\ &\qquad =\mathbf{R}\mathcal{H}om_{\mathcal{D}_Y}(\mathcal{L},\mathcal{N}^{\cdot\infty}). \qquad \text{(Proposition 6.2.4.)}\end{aligned}$$

On the other hand, we see by Proposition 6.2.2 that

$$\begin{aligned}&\mathbf{R}\mathcal{H}om_{\mathcal{D}_Y}(\mathcal{L},\mathcal{N}^{\cdot})\\ &\qquad =\mathbf{R}F_*\mathbf{R}\mathcal{H}om_{\mathcal{D}_X}(\mathcal{D}_{X\to Y}\underset{\mathcal{D}_Y}{\otimes}\mathcal{L},\mathcal{M})\end{aligned}$$

and

$$\begin{aligned}&\mathbf{R}\mathcal{H}om_{\mathcal{D}_Y}(\mathcal{L},\mathcal{N}^{\cdot\infty})\\ &\qquad =\mathbf{R}F_*\mathbf{R}\mathcal{H}om_{\mathcal{D}_X}(\mathcal{D}_{X\to Y}\underset{\mathcal{D}_Y}{\otimes}\mathcal{L},\mathcal{M}^{\infty}).\end{aligned}$$

Since $\mathcal{D}_{X\to Y}\underset{\mathcal{D}_Y}{\otimes}\mathcal{L}$ and $\mathcal{M}$ have R. S., they are isomorphic.

This completes the proof of Theorem 6.2.1.

Remark. Theorem 6.2.1 enables us to improve several results obtained earlier by using the integration of $\mathcal{D}$-Modules along projective fibers so that we may conclude that the resulting system is with R. S. As an example of such results, we state the following

Theorem 6.2.5. *Let f_j $(j=1,\dots,n)$ be real-valued real analytic functions defined on a real analytic manifold M. Let s_j $(j=1,\dots,n)$ be complex numbers with non-negative real part. Then there exists a holonomic $\mathscr{D}$-Module $\mathscr{M}$ with* R.S. *which the hyperfunction $\prod_{j=1}^{n} f_{j+}^{s_j}$ solves.* (*Cf. Theorem* 1 *and Lemma* 5 *of* [11].)

§3.

In this section we use Theorem 6.1.1 to prove that cohomology groups considered for formal power series coincide with those considered for convergent power series, if the holonomic system $\mathscr{M}$ in question is with R.S. Theorem 6.3.1 below is, actually, a dual statement of a special case of Theorem 6.1.1. (See [17] and [28] for related topics.)

We first recall some basic facts about the topological vector spaces needed here.

Let X be a complex manifold and x a point in X. We shall denote by $\mathfrak{m}$ the maximal ideal of $\mathcal{O}_{X,x}$ and by $\hat{\mathcal{O}}_{X,x}$ the completion of $\mathcal{O}_{X,x}$ by $\mathfrak{m}$-adic topology, i.e.,

$$\hat{\mathcal{O}}_{X,x} = \varprojlim_k \mathcal{O}_{X,x}/\mathfrak{m}^k .$$

First note that $\mathcal{O}_{X,x}$ (resp., $\hat{\mathcal{O}}_{X,x}$) has the natural structure of DFS- (resp., FS-) topological vector space and that $\mathscr{B}^{\infty}_{\{x\}|X}$ (resp., $\mathscr{B}_{\{x\}|X}$) has the natural structure of FS- (resp., DFS-)topological vector space. Furthermore $\mathcal{O}_{X,x}$ and $\Omega_X \underset{\mathcal{O}_X}{\otimes} \mathscr{B}^{\infty}_{\{x\}|X}$ (resp., $\hat{\mathcal{O}}_{X,x}$ and $\Omega_X \underset{\mathcal{O}_X}{\otimes} \mathscr{B}_{\{x\}|X}$) are mutually dual vector spaces.

Our main result in this section is the following

Theorem 6.3.1. *Let $\mathscr{M}$ be a holonomic $\mathscr{D}_X$-Module with* R.S. *Then, for any point x in X and any integer j, the natural homomorphism*

$$\mathscr{E}xt^j_{\mathscr{D}_X}(\mathscr{M}, \mathcal{O}_{X,x}) \longrightarrow \mathscr{E}xt^j_{\mathscr{D}_X}(\mathscr{M}, \hat{\mathcal{O}}_{X,x})$$

is an isomorphism.

In [6] we proved the following result.

Proposition 6.3.2. *For a holonomic $\mathscr{D}_X$-Module $\mathscr{M}$ $\mathscr{E}xt^j_{\mathscr{D}_X}(\mathscr{M}, \mathcal{O}_{X,x})$ is the dual vector space of $\mathscr{E}xt^{n-j}_{\mathscr{D}_X}(\mathscr{M}^*, \mathscr{B}^{\infty}_{\{x\}|X})$. Here $n=\dim X$.*

This proposition was proved by using the fact that $\Omega\otimes\mathscr{B}^{\infty}_{\{x\}|X}$ is the dual vector space of $\mathcal{O}_{X,x}$ and the fact that $\dim_{C}\mathscr{E}xt^{j}_{\mathscr{D}_X}(\mathscr{M}, \mathcal{O}_{X,x})$ is finite. We know that $\dim_{C}\mathscr{E}xt^{j}_{\mathscr{D}_X}(\mathscr{M}, \mathscr{B}_{\{x\}|X})$ is finite and that $\Omega\otimes\mathscr{B}_{\{x\}|X}$ is the dual vector space of $\hat{\mathcal{O}}_{X,x}$. Thus the same argument works and we obtain the following

Proposition 6.3.3. *For a holonomic* $\mathscr{D}_X$*-Module* $\mathscr{M}$, $\mathscr{E}xt^{j}_{\mathscr{D}_X}(\mathscr{M}, \hat{\mathcal{O}}_{X,x})$ *is the dual vector space of* $\mathscr{E}xt^{n-j}_{\mathscr{D}_X}(\mathscr{M}^*, \mathscr{B}_{\{x\}|X})$.

Then Theorem 6.3.1 immediately follows from these propositions combined with Theorem 6.1.1.

§4.

In this section we shall show the converse of Theorem 6.3.1.

Theorem 6.4.1. *Let* $\mathscr{M}$ *be a holonomic* $\mathscr{D}_X$*-Module. Assume that*

$$\mathbf{R}\mathscr{H}om_{\mathscr{D}_X}(\mathscr{M}, \mathcal{O}_X)_x \cong \mathbf{R}\mathscr{H}om_{\mathscr{D}_X}(\mathscr{M}, \hat{\mathcal{O}}_{X,x}) \tag{6.4.1}$$

holds for any x *in* X. *Then* $\mathscr{M}$ *is with* R.S.

We shall prove the theorem by the induction on the dimension of X. In the course of the proof we abbreviate $\hat{\mathcal{O}}_{X,x}$ to $\hat{\mathcal{O}}_x$ for brevity.

First let us prove the theorem when $\dim X$ is equal to one. In this case, this theorem is essentially proved by Malgrange [21]. By a result of Björk [1], $\mathscr{M}$ is generated locally by one element. Therefore we may assume that $\mathscr{M}=\mathscr{D}/\mathscr{I}$ for a coherent left Ideal $\mathscr{I}$. Since $\mathscr{M}$ is holonomic, $\mathscr{I}$ is not equal to zero. We may assume that X is a domain in $\mathbf{C}$ and $\mathrm{SS}(\mathscr{M})\subset T^*_{\{0\}}X\cup T^*_XX$. Let $P=x^nD^m+a_1(x)D^{m-1}+\cdots+a_m(x)$ be a section of $\mathscr{I}$ such that n is minimal. Set $\mathscr{N}=\mathscr{D}/\mathscr{D}P$. Then we have an exact sequence

$$0 \longrightarrow \mathscr{L} \longrightarrow \mathscr{N} \longrightarrow \mathscr{M} \longrightarrow 0,$$

where $\mathscr{L}=\mathscr{I}/\mathscr{D}P$. Furthermore we have the following

Lemma 6.4.2. $\mathrm{SS}(\mathscr{L})\subset T^*_XX$.

Proof. Set $p=(0, dx)$. It is enough to show that

$$\mathscr{I}\subset\mathscr{E}_pP.$$

Let Q be an element in $\mathscr{I}$. Then, by a division theorem (S-K-K [24], Chapter II, Theorem 2.2.1), we can write

$$Q=TP+S$$

with T, $S\in\mathscr{E}_p$ and $(\mathrm{ad}\, D)^n S=0$. Since S in $\mathscr{E}_p\mathscr{J}$, we have $S=0$. In fact, the symbol Ideal of $\mathscr{E}\mathscr{J}$ at p is x^n. Q.E.D.

Therefore $\mathscr{L}$ is isomorphic to a power of $\mathscr{O}$. Thus

$$\mathbf{R}\mathscr{H}om(\mathscr{L}, \mathscr{O}_X)_x \cong \mathbf{R}\mathscr{H}om(\mathscr{L}, \hat{\mathscr{O}}_x)$$

holds for any point x.

Therefore we have

$$\mathbf{R}\mathscr{H}om(\mathscr{N}, \mathscr{O}_X)_x \xrightarrow{\sim} \mathbf{R}\mathscr{H}om(\mathscr{N}, \hat{\mathscr{O}}_x)$$

with $x=0$.

Let us now recall the following result due to Malgrange [21].

Theorem 6.4.3. *The equation $Pu=0$ is with* R.S. *if and only if*

$$\begin{aligned}&\dim \mathrm{Ker}\,(P;\, \mathscr{O}_x\to\mathscr{O}_x)-\dim \mathrm{Coker}\,(P;\, \mathscr{O}_x\to\mathscr{O}_x)\\ &\quad =\dim \mathrm{Ker}\,(P;\, \hat{\mathscr{O}}_x\to\hat{\mathscr{O}}_x)-\dim \mathrm{Coker}\,(P;\, \hat{\mathscr{O}}_x\to\hat{\mathscr{O}}_x)\end{aligned}$$

holds.

By virtue of this result we find that $\mathscr{N}$ is with R.S. Hence $\mathscr{M}$ is also with R.S.

Now we discuss the case where $\dim X>1$.

Lemma 6.4.4. *Let $\mathscr{M}$ be a coherent $\mathscr{D}_X$-Module and Y a submanifold of X which is non-characteristic with respect to $\mathscr{M}$ at a point $x\in Y$. Then*

$$\mathbf{R}\mathscr{H}om_{\mathscr{D}_X}(\mathscr{M}, \hat{\mathscr{O}}_{X,x}) \xrightarrow{\sim} \mathbf{R}\mathscr{H}om_{\mathscr{D}_Y}(\mathscr{M}_Y, \hat{\mathscr{O}}_{Y,x}),$$

where $\mathscr{M}_Y=\mathscr{O}_Y \underset{\mathscr{O}_X}{\otimes} \mathscr{M}$, i.e., the restriction of $\mathscr{M}$ to Y.

Proof. By the technics we used in S.K.K. [24], Chapter II, Section 3.5, we may assume that Y is a hypersurface and that $\mathscr{M}=\mathscr{D}/\mathscr{D}P$ for a differential operator P. Hence we may assume that

$$Y=\{x_n=0\} \quad \text{and} \quad P=D_n^m+\sum_{j=1}^{m} p_j(x, D_1,\dots, D_{n-1})D_n^{m-j},$$

where p_j is of order $\leqq j$. Then $P\colon \hat{\mathscr{O}}_{X,x}\to\hat{\mathscr{O}}_{X,x}$ is surjective and the Cauchy problem $Pu=0$, $(\partial/\partial x_n)^j u\,|_{x_n=0}=v_j$ $(j=0,\dots, m-1)$ has a unique solution u in $\hat{\mathscr{O}}_{X,x}$ for any $v_j\in\hat{\mathscr{O}}_{Y,x}$. The same result holds for the pair $(\mathscr{O}_{X,x}, \mathscr{O}_{Y,x})$. Hence we have the required result. Q.E.D.

Now, let $\mathscr{M}$ be a holonomic $\mathscr{D}_X$-Module satisfying the condition (6.4.1) and

Y a submanifold of X non-characteristic with respect to $\mathcal{M}$. Then, by the preceding lemma, the restriction $\mathcal{M}_Y$ also satisfies the condition (6.4.1). Hence, if $\dim Y < \dim X$, $\mathcal{M}_Y$ has R.S. by the hypothesis of the induction.

We shall now show that if $\mathcal{M}_Y$ has R.S. for a generic hypersurface Y, then $\mathcal{M}$ itself has R.S.

We shall first formulate this claim in a micro-local way.

Let X be a complex manifold, V a regular involutory subset of T^*X of codimension l. Then locally the set of bicharacteristics of V is a symplectic manifold. Therefore there is an $(n-l)$ dimensional complex manifold T^*Y and a smooth map $F: V \to T^*Y$ so that any fiber of F is a bicharacteristic of V. Let us fix a right $\mathcal{D}_X$-Module $\mathcal{L} = u\mathcal{E}$ with characteristic variety V such that $\mathcal{L} = \mathcal{E}/\mathcal{I}$ and that the Ideal of symbols of operators in $\mathcal{I}$ coincides with the Ideal of holomorphic functions vanishing on V. Then $\mathcal{E}nd(\mathcal{L})$ is isomorphic to $F^{-1}\mathcal{E}_Y$. Note that $\mathcal{L}$ is uniquely determined locally (S-K-K [24], Chapter II, § 5.3), and the isomorphism $\mathcal{E}nd_{\mathcal{E}_X}(\mathcal{L}) \cong F^{-1}\mathcal{E}_Y$ is unique up to inner automorphisms of $\mathcal{E}_Y$ by a section in $\mathcal{E}_Y(\lambda)$ for some complex number λ. Let $\mathcal{M}$ be a coherent $\mathcal{E}_X$-Module. Suppose that $\operatorname{Supp} \mathcal{M} \cap V \to T^*Y$ is finite. Then $F_*(\mathcal{L} \underset{\mathcal{E}_X}{\otimes} \mathcal{M})$ has a structure of $\mathcal{E}_Y$-Module. In [24] we showed that $F_*(\mathcal{L} \underset{\mathcal{E}_X}{\otimes} \mathcal{M})$ is a coherent $\mathcal{E}_Y$-Module and $\mathcal{T}or_k^{\mathcal{E}_X}(\mathcal{L}, \mathcal{M}) = 0$ for $k \neq 0$. Note that $\mathcal{E}nd_{\mathcal{E}_X^\infty}(\mathcal{L}^\infty) \cong F^{-1}\mathcal{E}_Y^\infty$ and $F_*(\mathcal{L}^\infty \underset{\mathcal{E}_X}{\otimes} \mathcal{M}) = \mathcal{E}_Y^\infty \underset{\mathcal{E}_Y}{\otimes} F_*(\mathcal{L} \underset{\mathcal{E}_X}{\otimes} \mathcal{M})$. Therefore we can determine $F_*(\mathcal{L} \underset{\mathcal{E}_X}{\otimes} \mathcal{M})$ modulo quantized contact transform of $(T^*Y, \mathcal{E}_Y)$, which we shall denote by $\mathcal{M}_V$.

Now let $\mathcal{M}$ be a holonomic $\mathcal{E}_X$-Module with a smooth Lagrangian manifold Λ as its characteristic variety defined near $p \in \Lambda$. Let $f(x, \xi)$ be a homogeneous function on T^*X of degree 0 such that $f(p) = 0$. Suppose that

(6.4.2) $\qquad df(p)$ and $\omega(p)$ are linearly independent.

Then, for any $a \in \boldsymbol{C}$, $V_a = \{(x, \xi) \in T^*X; f(x, \xi) = a\}$ is a regular involutory hypersurface of T^*X in a neighborhood of p. Assume that $df|_\Lambda$ does not vanish at p. Then we have the following

Theorem 6.4.5. *Consider the problem in the situation described above. Assume in addition that $\mathcal{M}_{V_a}$ has* R.S. *for any a with $|a| \ll 1$. Then $\mathcal{M}$ itself has* R.S. *in a neighborhood of p.*

First let us show that we can transform the geometric situation into a very simple one by a contact transformation.

Lemma 6.4.6. *There exists a homogeneous canonical transformation of* T^*X *in a neighborhood of* p *which makes* $f=x_n$ *and* $\Lambda=\{(x, \xi); \xi_2=\cdots=\xi_n=x_1=0\}$.

Proof. By the condition (6.4.2), $n=\dim X>1$ and $p\notin T_X^*X$. Fix a fibering $F: V_0\to T^*Y$ as in Chapter I, Section 4. Then $F(V_0\cap\Lambda)=\Lambda_0$ is a Lagrangian manifold of T^*Y. Therefore by a contact transformation of T^*Y, we may assume

$$\begin{aligned}\Lambda_0=\{(y, \eta)=(y_1,\dots, y_{n-1}, \eta_1,\dots, \eta_{n-1})\in T^*Y;\\ y_1=\eta_2=\cdots=\eta_{n-1}=0\}.\end{aligned}$$

Take a homogeneous hypersurface Z containing Λ and $\{p\in V; (F\circ\eta_n)^{-1}(0)\}$ such that $H_f(p)\notin T_pZ$.

Then we solve the initial value problem

$$\begin{cases}\{f, g_n\}=-1\\ g_n|_Z=0.\end{cases}$$

Next we solve the initial value problem for $g_1,\dots, g_{n-1}$ and $f_1,\dots,f_{n-1}$

$$\begin{cases}\{g_n, g_j\}=0\\ g_j|_V=\eta_j\circ F\end{cases}\qquad (j=1,\dots, n-1),$$

$$\begin{cases}\{g_n, f_j\}=0\\ f_j|_V=y_j\circ F\end{cases}\qquad (j=1,\dots, n-1).$$

We also define f_n by f. Then g_j are homogeneous of degree 1, f_j are of degree 0 and, furthermore, they satisfy the following:

$$\begin{cases}\{g_n, \{f, g_j\}\}=0 & (j=1,\dots, n),\\ \{g_n, \{f, f_j\}\}=0 & (j=1,\dots, n).\end{cases}$$

Hence we can find a contact transformation which makes $g_j=\xi_j$ and $f_j=x_j$. Since $\Lambda\subset\xi_n^{-1}(0)$ and $\Lambda\cap\{x_n=0\}\subset\{\xi_2=\cdots=\xi_n=x_1=0\}$, $\Lambda=\{\xi_2=\cdots=\xi_n=x_1=0\}$. Q.E.D.

Now let us prove Theorem 6.4.5. We may assume that the geometric situation is as in Lemma 6.4.6. Set $Y_a=\{x; x_n=a\}$. Then, by the assumption the restriction $\mathcal{M}_{Y_a}$ has R.S. for any a. By Lemma 1.3.4 of Chapter I, Section 3, there is a system $\mathcal{L}$ given by $(x_1D_1-A)u=D_2u=\cdots=D_nu=0$, where u is a column vector of N unknown functions $u_1,\dots, u_N$ and A is a constant matrix of size $N\times N$ and $\mathcal{M}^\infty$ is isomorphic to $\mathcal{L}^\infty$. Let $\varphi: \mathcal{M}^\infty\to\mathcal{L}^\infty$ be this isomorphism. It is enough to show that $\varphi(\mathscr{M})\subset\mathscr{L}$. Any section v of $\mathscr{L}^\infty$ is written uniquely in the form

$$v=\sum_{\nu=1}^{N} P_\nu(x_2,\dots, x_n, D_1)u_\nu \tag{6.4.3}$$

with $P_\nu\in\mathscr{E}^\infty$ such that $[P_\nu, x_j]=0\ (1<j\leqq n)$ and $[P_\nu, D_1]=0$. Clearly v belongs to $\mathscr{L}$ if and only if any P_ν belongs to $\mathscr{E}_X$. Let w be a section of $\mathscr{M}$. Set $v=\varphi(w)$ and write v in the form (6.4.3). Since $w|_{Y_a}$ (i.e. $w \bmod (x_n-a)$) satisfies a holonomic system of micro-differential equations on Y with R.S., $v|_{Y_a}$ belongs to $\mathscr{L}_{Y_a}$ and hence $P_\nu(x_2,\dots, x_{n-1}, a, D_1)$ is of finite order for any a. This implies that $P_\nu(x_2,\dots, x_n, D_1)$ is of finite order. Therefore v belongs to $\mathscr{L}$. This completes the proof of Theorem 6.4.5.

Let us now resume the proof of Theorem 6.4.1. By the reduction done before, we may assume that $\dim X>1$. Set $\mathrm{SS}(\mathscr{M})=\bigcup_j T^*_{Y_j}X$. If $\dim Y_j=\dim X$, then $\mathscr{M}$ has R.S. on $T^*_{Y_j}X$ by the definition. We shall prove that $\mathscr{M}$ has R.S. on $T^*_{Y_j}X$ if $\dim Y_j\neq 0$. We can choose a local coordinate system $(x_1,\dots, x_n)$ of X around a non-singular point of Y_j so that $Y_j=\{x;\, x_1=\cdots=x_l=0\}$ with $1\leqq l<n$. Since $\mathrm{SS}(\mathscr{M})\cap\pi^{-1}(Y_{j\mathrm{reg}})$ is isotropic, $\mathrm{SS}(\mathscr{M})\cap\pi^{-1}(Y_{j\,\mathrm{reg}})$ is contained in $T^*_{Y_{j\mathrm{reg}}}X$. (Sublemma 3.3 of [6].) Set $Z_a=\{x;\, x_n=a\}$. Then Z_a is noncharacteristic with respect to $\mathscr{M}$. Therefore $\mathscr{M}_{Z_a}$ has R.S. as seen before. Applying Theorem 6.4.5, we find that $\mathscr{M}$ has R.S. along $T^*_{Y_j}X$.

Hence there is a hypersurface Z of X such that $\mathscr{M}$ has R.S. on $T^*_Z X$ and $\mathrm{SS}(\mathscr{M})\subset\pi^{-1}(Z)\cup T^*_X X$. Then $\mathscr{N}=\mathscr{H}^0_{[X|Z]}(\mathscr{M})$ is a holonomic system of D-type and hence with R.S. (Corollary 4.1.2 of Chapter IV, §1).

Let $\mathscr{M}''$ be the image of $\mathscr{M}$ in $\mathscr{N}$. Then $\mathscr{M}''$ has R.S. Let $\mathscr{M}'$ be the kernel of $\mathscr{M}\to\mathscr{M}''$. Then $\mathscr{M}'$ satisfies also the condition (6.4.1) and $\mathrm{Supp}\,\mathscr{M}'\subset Z$. Thus by replacing $\mathscr{M}$ with $\mathscr{M}'$, we may assume from the first that the support of $\mathscr{M}$ is contained in the hypersurface Z.

Lemma 6.4.7. *Let $F\colon X\to Y$ be a smooth map and let $\mathscr{M}$ be a coherent $\mathscr{D}_X$-Module such that* $\mathrm{Supp}\,\mathscr{M}\to Y$ *is finite. Then*

$$\mathbf{R}F_*\mathbf{R}\mathscr{H}om_{\mathscr{D}_X}(\mathscr{M}, \mathscr{O}_X)\cong\mathbf{R}\mathscr{H}om_{\mathscr{D}_Y}(F_*(\mathscr{D}_{Y\leftarrow X}\underset{\mathscr{D}_X}{\otimes}\mathscr{M}), \mathscr{O}_Y)\,[\dim Y-\dim X]$$

and

$$\mathbf{R}\mathscr{H}om_{\mathscr{D}_X}(\mathscr{M}, \hat{\mathscr{O}}_{X,x})\cong\mathbf{R}\mathscr{H}om_{\mathscr{D}_Y}(F_*(\mathscr{D}_{Y\leftarrow X}\underset{\mathscr{D}_X}{\otimes}\mathscr{M});\, \hat{\mathscr{O}}_{Y,F(x)})\,[\dim Y-\dim X]\,.$$

Proof. By the induction on $\dim X-\dim Y$, we may assume that $1=\dim X$

$-\dim Y$. Then choosing a local coordinate system so that $X=\mathbf{C}^n$, $Y=\mathbf{C}^{n-1}$, $F(x_1,\dots,x_n)=(x_2,\dots,x_{n-1})$ and the support of $\mathcal{M}$ is contained in $\{x\in X; f(x)=0\}$ with $f(x)=x_n^m+a_1(x_1,\dots,x_{n-1})x_n^{m-1}+\cdots$. Then we can assume that $\mathcal{M}=\mathcal{D}_X/\mathcal{D}_X f$, and we may assume $x=0$ and $f^{-1}(0)\cap F^{-1}(0)=\{0\}$. Then we have $F_*(\mathcal{D}_{Y\leftarrow X}\underset{\mathcal{D}_X}{\otimes}\mathcal{M})=\mathcal{D}_Y^m$ and hence it suffices to show that

$$\begin{aligned}&\operatorname{Ker}(f\colon \mathcal{O}_{X,x}\to\mathcal{O}_{X,x})\\&\quad=\operatorname{Ker}(f\colon \hat{\mathcal{O}}_{X.x}\to\hat{\mathcal{O}}_{X,x})\\&\quad=0\end{aligned}$$

and

$$\begin{aligned}&\operatorname{Coker}(f\colon \mathcal{O}_{X,x}\to\mathcal{O}_{X,x})=\mathcal{O}_{Y,0}^m\\&\operatorname{Coker}(f\colon \hat{\mathcal{O}}_{X,x}\to\hat{\mathcal{O}}_{X,x})=\hat{\mathcal{O}}_{Y,0}^m\,.\end{aligned}$$

These assertions immediately follow from Späth's division theorem. Q.E.D.

Now, let us resume the proof of Theorem 6.4.1. We may assume that $\operatorname{Supp}\mathcal{M}$ is contained in a hypersurface Z. Let Y be a manifold such that $\dim Y=\dim X-1$ and let $F\colon X\to Y$ be a smooth map such that $Z\to Y$ is finite. Then by the preceding lemma $\mathcal{N}=F_*(\mathcal{D}_{Y\leftarrow X}\underset{\mathcal{D}_X}{\otimes}\mathcal{M})$ satisfies the condition (6.4.1). Since $\dim Y<\dim X$, $\mathcal{N}$ has R.S. by the hypothesis of induction.

Now we shall employ Theorem 6.4.5. We can take a local coordinate system $(x_1,\dots,x_n)$ of X around p so that p is the origin 0, $\partial/\partial x_1\in C_p(Z)$ and $dx_n\notin\overline{SS(\mathcal{M})-T_p^*X}$. Let F_a be the projection $x\mapsto(x_2,\dots,x_{n-1},x_n+ax_1)$. Then $F_a|_Z\colon Z\to Y=\mathbf{C}^{n-1}$ is a finite map. Hence $\mathcal{N}\underset{\text{def}}{=}F_{a*}(\mathcal{M})$ is with R.S. (Theorem 5.3.2 in Chapter V, §3.) On the other hand, setting $f=\xi_1/\xi_n$ and defining V_a by $f^{-1}(a)$, we find $\mathcal{E}_Y\underset{\mathcal{D}_Y}{\otimes}\mathcal{N}=(\mathcal{E}\otimes\mathcal{M})|_{V_a}$. Therefore Theorem 6.4.5 asserts that $\mathcal{E}\otimes\mathcal{M}$ has R.S. on T_p^*X. This completes the proof of Theorem 6.4.1. Q.E.D.

Remark. We have the following theorem, which can be proved in the same way as Theorem 6.4.1.

Theorem 6.4.8. *Let $\mathcal{M}$ be a holonomic $\mathcal{D}_X$-Module. Suppose that*

$$\begin{aligned}&\sum_j(-1)^j\dim_{\mathbf{C}}\mathcal{E}xt^j_{\mathcal{D}_X}(\mathcal{M},\mathcal{O}_X)_x\\&\quad=\sum_j(-1)^j\dim_{\mathbf{C}}\mathcal{E}xt^j_{\mathcal{D}_X}(\mathcal{M},\hat{\mathcal{O}}_{X,x})\end{aligned}$$

for any $x\in X$. Then $\mathcal{M}$ has R.S.

Remark. We can generalize Theorem 6.4.1 as follows:

Let X be a complex manifold and $\mathcal{M}_.$ a bounded complex of $\mathcal{D}_X$-Module. Assume that $\mathcal{H}_k(\mathcal{M}_.)$ is a holonomic $\mathcal{D}_X$-Module for any k and

$$\mathbf{R}\mathcal{H}om\,(\mathcal{M}_., \mathcal{O}_X)_x \cong \mathbf{R}\mathcal{H}om(\mathcal{M}_., \hat{\mathcal{O}}_{X,x})$$

for any x. Then all cohomology groups $\mathcal{H}_k(\mathcal{M}_.)$ have R. S.

In fact, we can reduce the problem to the case where $\dim X=1$ by the same argument employed in the proof of Theorem 6.4.1.

Suppose that $\dim X=1$ and $\mathcal{H}_k(\mathcal{M}_.)=0$ for $k<k_0$. By the induction on k_0 it is sufficient to show that $\mathcal{H}_{k_0}(\mathcal{M}_.)$ has R. S. By the theory of spectral sequences, we have

$$\mathrm{Hom}\,(\mathcal{H}_{k_0}(\mathcal{M}_.), \hat{\mathcal{O}}_x/\mathcal{O}_x)=H^{k_0}(\mathbf{R}\,\mathrm{Hom}\,(\mathcal{M}_., \hat{\mathcal{O}}_x/\mathcal{O}_x))$$

and

$$\mathrm{Ext}^1\,(\mathcal{H}_{k_0}(\mathcal{M}_.), \hat{\mathcal{O}}_x/\mathcal{O}_x)\subset H^{k_0+1}(\mathbf{R}\,\mathrm{Hom}\,(\mathcal{M}_., \hat{\mathcal{O}}_x/\mathcal{O}_x))\,.$$

Since the cohomology groups of $\mathbf{R}\,\mathrm{Hom}\,(\mathcal{M}_., \hat{\mathcal{O}}_x/\mathcal{O}_x)$ vanish by the assumption, we obtain

$$\mathrm{Hom}\,(\mathcal{H}_{k_0}(\mathcal{M}_.), \hat{\mathcal{O}}_x/\mathcal{O}_x)=\mathrm{Ext}^1(\mathcal{H}_{k_0}(\mathcal{M}_.), \hat{\mathcal{O}}_x/\mathcal{O}_x)=0\,.$$

This implies that

$$\begin{aligned}&\sum_{j=0}^{1}(-1)^j \dim \mathrm{Ext}^j(\mathcal{H}_{k_0}(\mathcal{M}_.), \mathcal{O}_x)\\ &\quad=\sum_{j=0}^{1}(-1)^j \dim \mathrm{Ext}^j(\mathcal{H}_{k_0}(\mathcal{M}_.), \hat{\mathcal{O}}_x)\,.\end{aligned}$$

Hence $\mathcal{H}_{k_0}(\mathcal{M}_.)$ has R. S. by Malgrange's theorem.

Ramis [23] called a bounded complex $\mathcal{M}_.$ of $\mathcal{D}_X$-Modules Fuchsian if $\mathcal{M}_.$ satisfies the following conditions

(6.4.4) $\mathcal{H}_k(\mathcal{M}_.)$ is a holonomic $\mathcal{D}_X$-Module for any k,

(6.4.5) $\mathbf{R}\Gamma_Y(\mathcal{M}_.^\infty)=(\mathbf{R}\Gamma_{[Y]}(\mathcal{M}_.))^\infty$ for any analytic subset Y of X,

and

(6.4.6) $\mathbf{R}\mathcal{H}om(\mathcal{M}_., \mathcal{O}_X)|_Y \cong \mathbf{R}\mathcal{H}om(\mathcal{M}_., \hat{\mathcal{O}}_{X/Y})$ for any analytic set Y of X,

where $\hat{\mathcal{O}}_{X/Y}=\varprojlim_k \mathcal{O}_X/\mathcal{I}^k$ for the defining Ideal $\mathcal{I}$ of Y.

Ramis showed that (6.4.5) and (6.4.6) are equivalent under the condition (6.4.4). The above remark shows that these two conditions are also equivalent to

(6.4.7) $\mathscr{H}_k(\mathscr{M}^{\cdot})$ has R.S. for any k.

Appendix

A.1. In this paper, in proving some statement on $\mathscr{D}$-Modules, we sometimes reduce the problem to that of $\mathscr{E}$-Modules outside the zero section of the cotangent bundle by adding a dummy variable.

We shall give the detailed discussion about this method in this section.

A.2. Let X be a complex manifold and let X' denote $\mathbf{C}\times X$. We shall take local coordinate systems $x=(x_1,\dots, x_n)$, (t, x), $(x, \xi)=(x_1,\dots, x_n;\ \xi_1,\dots, \xi_n)$, $(t, x;\ \tau, \xi)$ of X, X', T^*X and T^*X', respectively.

Let us identify T^*X' with $T^*\mathbf{C}\times T^*X=\mathbf{C}\times\mathbf{C}\times T^*X$ by $(t, x;\ \tau, \xi)\leftrightarrow(t, \tau, (x, \xi))$. We embed T^*X into $T^*X'-T^*_{X'}X'$ by $j\colon p\mapsto(0, 1;\ p)$ and X into $T^*X'-T^*_{X'}X'$ by $j_0\colon x\mapsto(t, x;\ \tau, \xi)=(0, x;\ 1, 0)$.

We define $V=\{(t, x;\ \tau, \xi)\in T^*X';\ t=0, \tau\neq 0\}$ and let F be the projection from V onto T^*X. Let $\mathscr{L}$ be the sheaf $\mathscr{E}_{X'}/\mathscr{E}_{X'}t$ on V. The sheaf $\mathscr{L}$ has a structure of $(\mathscr{E}_{X'}, F^{-1}\mathscr{E}_X)$-bi-Module. Let u_0 be the section of $\mathscr{L}$ given by $1\in\mathscr{E}_{X'}$ modulo $\mathscr{E}_{X'}t$. Hence $\mathscr{L}$ is generated by u_0 as $\mathscr{E}_{X'}$-Module. Any section of $\mathscr{L}^\infty$ (resp., $\mathscr{L}$) can be written in a unique form $P(x, D_t, D_x)u_0$ for $P\in\mathscr{E}^\infty_{X'}$ (resp., $\mathscr{E}_{X'}$) such that $[D_t, P]=0$. We shall define the right $\mathscr{D}^\infty_X$-linear homomorphism $\rho_k\colon j_0^{-1}\mathscr{L}^\infty\to\mathscr{D}^\infty_X$ $(k\in\mathbf{Z})$ as follows. For a section $P(x, D_t, D_x)u_0$ of $j_0^{-1}\mathscr{L}^\infty$, we expand $P(x, D_t, D_x)=\sum_j P_j(x, D_x)D_t^j$ with $P_j\in\mathscr{D}^\infty_X$, and we define

$$\rho_k(Pu_0)=P_k(x, D_x).$$

We have the exact sequences

$$0\longrightarrow\mathscr{D}^\infty_X u_0\longrightarrow\mathscr{L}^\infty\xrightarrow{\ t\ }\mathscr{L}^\infty\xrightarrow{\rho_{-1}} j_{0*}\mathscr{D}^\infty_X\longrightarrow 0 \tag{A.2.1}$$

and

$$0\longrightarrow\mathscr{D}_X u_0\longrightarrow\mathscr{L}\xrightarrow{\ t\ }\mathscr{L}\xrightarrow{\rho_{-1}} j_{0*}\mathscr{D}_X\longrightarrow 0\,. \tag{A.2.2}$$

A.3. For an $\mathscr{E}_X$-Module $\mathscr{M}$, we define the $\mathscr{E}_{X'}$-Module $\Phi(\mathscr{M})$ by

$$\Phi(\mathscr{M})=\mathscr{L}\underset{\mathscr{E}_X}{\otimes}F^{-1}\mathscr{M} \tag{A.3.1}$$

and, for an $\mathscr{E}_{X'}$-Module $\mathscr{N}$ with support in V, we define the $\mathscr{E}_X$-Module $\Psi(\mathscr{N})$ by

(A.3.2) $$\Psi(\mathcal{N}) = j^{-1}\mathcal{H}om_{\mathcal{E}_{X'}}(\mathcal{L}, \mathcal{N}).$$

For a $\mathcal{D}_X$-Module $\mathcal{M}$, let $\rho_k(\mathcal{M})$ be the homomorphism from $j_0^{-1}\Phi(\mathcal{M})$ to $\mathcal{M}$ defined by $\rho_k \otimes \mathrm{id}_{\mathcal{M}}$.

Proposition A.1. For an $\mathcal{E}_X$-Module $\mathcal{M}$, $\Psi\Phi(\mathcal{M}) \cong \mathcal{M}$.

Proof. $\mathbf{R}\mathcal{H}om_{\mathcal{E}_{X'}}(\mathcal{L}, \Phi(\mathcal{M}))$ is quasi-isomorphic to $j^{-1}\Phi(\mathcal{M}) \xrightarrow{t} j^{-1}\Phi(\mathcal{M})$. Since $j^{-1}\mathcal{L} \xrightarrow{t} j^{-1}\mathcal{L}$ is quasi-isomorphic to $\mathcal{E}_X \xrightarrow{0} \mathcal{D}_X$ in the category of $\mathcal{D}_X$-Modules by

$$\begin{array}{ccc} j^{-1}\mathcal{L} & \xrightarrow{t} & j^{-1}\mathcal{L} \\ \downarrow{\scriptstyle \rho_0} & & \downarrow{\scriptstyle \rho_{-1}} \\ \mathcal{E}_X & \xrightarrow{0} & \mathcal{D}_X|_{T_X^*X}. \end{array}$$

Hence $j^{-1}\Phi(\mathcal{M}) \xrightarrow{t} j^{-1}\Phi(\mathcal{M})$ is quasi-isomorphic to

$$\mathcal{M} \xrightarrow{0} \mathcal{M}|_{T_X^*X}.$$

Thus we obtain the desired result by taking the 0-th cohomology. Q.E.D.

Remark. By the same reasoning as above,

$$\Psi(\Phi(\mathcal{M})^\infty) = \mathcal{M}^\infty.$$

Proposition A.2. *Let $\mathcal{M}$ be an $\mathcal{E}_X$-Module. Then $\mathcal{M}$ is an $\mathcal{E}_X$-Module locally of finite type* (*resp., a coherent $\mathcal{E}_X$-Module*) *if and only if $\Phi(\mathcal{M})$ is an $\mathcal{E}_{X'}$-Module locally of finite type* (*resp., a coherent $\mathcal{E}_{X'}$-Module*).

Proof. If $\mathcal{M}$ is locally of finite type (resp., coherent) over $\mathcal{E}_X$, then there exists locally an exact sequence $0 \leftarrow \mathcal{M} \leftarrow \mathcal{E}_X^N$ (resp., $0 \leftarrow \mathcal{M} \leftarrow \mathcal{E}_X^{N_0} \leftarrow \mathcal{E}_X^{N_1}$). By tensoring $\mathcal{L}$, we obtain an exact sequence $0 \leftarrow \Phi(\mathcal{M}) \leftarrow \mathcal{L}^N = \Phi(\mathcal{E}_X^N)$ (resp., $0 \leftarrow \Phi(\mathcal{M}) \leftarrow \mathcal{L}^{N_0} \leftarrow \mathcal{L}^{N_1}$). Since $\mathcal{L}$ is coherent over $\mathcal{E}_{X'}$, $\Phi(\mathcal{M})$ is locally of finite type (resp., coherent) over $\mathcal{E}_{X'}$.

We shall prove the converse. Suppose that $\Phi(\mathcal{M})$ is locally of finite type over $\mathcal{E}_{X'}$. Then there are sections $s_1, \ldots, s_N$ of $\Phi(\mathcal{M})$ which generate $\Phi(\mathcal{M})$ (locally). Since s_j is a finite linear combination of $u_0 \otimes v$'s ($v \in \mathcal{M}$), we may assume that there exist sections $v_1, \ldots, v_r$ of $\mathcal{M}$ such that $u_0 \otimes v_1, \ldots, u_0 \otimes v_r$ generate $\Phi(\mathcal{M})$. Let φ be the $\mathcal{E}_X$-linear homomorphism from $\mathcal{E}_X^r$ into $\mathcal{M}$ defined by $v_1, \ldots, v_r$. Then $\Phi(\varphi)$: $\Phi(\mathcal{E}_X^r) \to \Phi(\mathcal{M})$ is surjective. Since $\mathcal{L}$ is faithfully flat over $\mathcal{E}_X$, φ is surjective. Thus we have proved that $\mathcal{M}$ is locally of finite type.

We shall next prove that $\mathcal{M}$ is coherent if $\Phi(\mathcal{M})$ is coherent. Since $\Phi(\mathcal{M})$ is locally of finite type, $\mathcal{M}$ is locally of finite type as we have already shown.

Therefore $\mathcal{M}$ is a quotient of $\mathcal{E}_X^N$. Let $\mathcal{M}'$ be the kernel of the homomorphism $\varphi\colon \mathcal{E}_X^N \to \mathcal{M}$. Then $\Phi(\mathcal{M}')$ is the kernel of $\Phi(\varphi)\colon \mathcal{L}^N \to \Phi(\mathcal{M})$. Therefore $\Phi(\mathcal{M}')$ is locally of finite type. This implies that $\mathcal{M}'$ is locally of finite type, and hence $\mathcal{M}$ is coherent. Q.E.D.

Proposition A.3. *If $\mathcal{M}$ is a coherent $\mathcal{E}_X$-Module, then $\Phi(\mathcal{M})$ is a coherent $\mathcal{E}_{X'}$-Module with regular singularities along V.*

Proof. A quotient of a coherent $\mathcal{E}_{X'}$-Module with regular singularities along V has regular singularities along V. Since $\mathcal{M}$ is a quotient of $\mathcal{E}_X^N$, $\Phi(\mathcal{M})$ is a quotient of $\Phi(\mathcal{E}_X^N) = \mathcal{L}^N$. Q.E.D.

Proposition A.4. *Let $\mathcal{N}$ be a coherent $\mathcal{E}_{X'}$-Module with regular singularities along V. Then, for any integer k, $j_0^{-1}\mathcal{E}xt^k_{\mathcal{E}_{X'}}(\mathcal{L}, \mathcal{N})$ is a coherent $\mathcal{D}_X$-Module and*

$$\mathcal{D}_X^\infty \underset{\mathcal{D}_X}{\otimes} j_0^{-1}\mathcal{E}xt^k_{\mathcal{E}_X}(\mathcal{L}, \mathcal{N}) = j_0^{-1}\mathcal{E}xt^k_{\mathcal{E}^\infty_{X'}}(\mathcal{L}^\infty, \mathcal{N}^\infty).$$

Proof. Let us prove this proposition at a point x_0 of X. As proved in Theorem 3.2 [18], $\mathcal{N}$ is a quotient of $\mathcal{N}_0$, where $\mathcal{N}_0$ has the form $\mathcal{E}_{X'}^N / \mathcal{E}_{X'}^N (tD_t - A(x, D_x))$ for an $N \times N$ matrix $A(x, D_x)$ of linear differential operators on X, which satisfies the following condition.

(A.4) $A(x, D_x)$ has the form $(A_{ij}(x, D_x))_{1 \leq i,j \leq r}$, where $A_{ij}(x, D_x)$ is an $N_i \times N_j$ matrix of linear differential operators. If $i > j$, $A_{ij} = 0$ and A_{ii} is a matrix of functions on x, and $A_{ii}(x_0)$ is $\lambda_i I_{N_i}$ for $\lambda_i \in \boldsymbol{C}$. Furthermore we have $A_{ij} = 0$ if $\lambda_i \neq \lambda_j$.

Suppose first that the claims are true for such $\mathcal{N}_0$. Let $\mathcal{N}_1$ be the kernel of $\mathcal{N}_0 \to \mathcal{N}$. Then we have the exact sequence

$$\begin{aligned} j_0^{-1}\mathcal{E}xt^k(\mathcal{L}, \mathcal{N}_1) &\longrightarrow j_0^{-1}\mathcal{E}xt^k(\mathcal{L}, \mathcal{N}_0) \longrightarrow j_0^{-1}\mathcal{E}xt^k(\mathcal{L}, \mathcal{N}) \\ &\longrightarrow j_0^{-1}\mathcal{E}xt^{k+1}(\mathcal{L}, \mathcal{N}_1) \longrightarrow j_0^{-1}\mathcal{E}xt^{k+1}(\mathcal{L}, \mathcal{N}_0). \end{aligned}$$

We shall prove the first assertion on coherency by the descending induction on k. If $k \neq 0, 1$, then $\mathcal{E}xt^k(\mathcal{L}, \mathcal{N}) = 0$. Hence the claim is evident. By the hypothesis of the induction, $j_0^{-1}\mathcal{E}xt^{k+1}(\mathcal{L}, \mathcal{N}_1)$ is coherent. Since $j_0^{-1}\mathcal{E}xt^k(\mathcal{L}, \mathcal{N}_0)$ and $j_0^{-1}\mathcal{E}xt^{k+1}(\mathcal{L}, \mathcal{N}_0)$ are coherent by the assumption, $j_0^{-1}\mathcal{E}xt^k(\mathcal{L}, \mathcal{N})$ is locally of finite type, and hence so is $j_0^{-1}\mathcal{E}xt^k(\mathcal{L}, \mathcal{N}_1)$. Therefore $j_0^{-1}\mathcal{E}xt^k(\mathcal{L}, \mathcal{N})$ is a coherent $\mathcal{D}_X$-Module.

Let us prove the second claim also by the descending induction on k. If $k \neq 0, 1$, then $\mathcal{E}xt^k(\mathcal{L}, \mathcal{N}^\infty) = \mathcal{E}xt^k(\mathcal{L}, \mathcal{N}) = 0$ and hence the claim is evident.

Consider the following diagram:

$$\begin{array}{ccccc}
(j_0^{-1}\mathscr{E}xt^k(\mathscr{L},\mathscr{N}_1))^\infty & \longrightarrow & (j_0^{-1}\mathscr{E}xt^k(\mathscr{L},\mathscr{N}_0))^\infty & \longrightarrow & (j_0^{-1}\mathscr{E}xt^k(\mathscr{L},\mathscr{N}))^\infty \\
{\scriptstyle\alpha}\downarrow & & {\scriptstyle\beta}\downarrow & & {\scriptstyle\gamma}\downarrow \\
j_0^{-1}\mathscr{E}xt^k(\mathscr{L},\mathscr{N}_1^\infty) & \longrightarrow & j_0^{-1}\mathscr{E}xt^k(\mathscr{L},\mathscr{N}_0^\infty) & \longrightarrow & j_0^{-1}\mathscr{E}xt^k(\mathscr{L},\mathscr{N}^\infty)
\end{array}$$

$$\begin{array}{ccccc}
\longrightarrow & (j_0^{-1}\mathscr{E}xt^{k+1}(\mathscr{L},\mathscr{N}_1))^\infty & \longrightarrow & (j_0^{-1}\mathscr{E}xt^{k+1}(\mathscr{L},\mathscr{N}_0))^\infty \\
 & {\scriptstyle\delta}\downarrow & & {\scriptstyle\varepsilon}\downarrow \\
\longrightarrow & j_0^{-1}\mathscr{E}xt^{k+1}(\mathscr{L},\mathscr{N}_1^\infty) & \longrightarrow & j_0^{-1}\mathscr{E}xt^{k+1}(\mathscr{L},\mathscr{N}_0^\infty)\,.
\end{array}$$

By the hypothesis of the induction, δ is an isomorphism. We assumed that β and ε are also isomorphisms. Hence γ is surjective. By applying this to $\mathscr{N}_1$ instead of $\mathscr{N}$, α is also surjective. Hence γ is an isomorphism.

Let us prove the claims for $\mathscr{N}_0$. By decomposing $\mathscr{N}_0$ into the direct sum, we may assume that $\lambda_1=\cdots=\lambda_r$.

Then, by Theorem 3.11 [18], if λ_1 is not an integer, then $\mathscr{E}xt^k(\mathscr{L},\mathscr{N}_0)=\mathscr{E}xt^k(\mathscr{L},\mathscr{N}_0^\infty)=0$. Hence the claim holds. Let us assume that λ_1 is an integer. Then by the same theorem, we have

$$\begin{aligned}
\mathscr{H}om(\mathscr{L},\mathscr{N}_0) &=\mathrm{Ker}\,(\mathscr{D}_X^N \xrightarrow{\lambda_1-A} \mathscr{D}_X^N)\\
\mathscr{H}om(\mathscr{L},\mathscr{N}_0^\infty) &=\mathrm{Ker}\,(\mathscr{D}_X^{\infty N} \xrightarrow{\lambda_1-A} \mathscr{D}_X^{\infty N})\\
\mathscr{E}xt^1(\mathscr{L},\mathscr{N}_0) &=\mathrm{Coker}\,(\mathscr{D}_X^N \xrightarrow{\lambda_1-A} \mathscr{D}_X^N)\\
\mathscr{E}xt^1(\mathscr{L},\mathscr{N}_0^\infty) &=\mathrm{Coker}\,(\mathscr{D}_X^{\infty N} \xrightarrow{\lambda_1-A} \mathscr{D}_X^{\infty N})\,.
\end{aligned}$$

Hence the proposition immediately follows from these equalities. Q.E.D.

A.4. In Theorem 3.8 [18], we defined the monodromy of a coherent $\mathscr{E}_{X'}$-Module with regular singularities along V. Let us recall the definition. Let Λ be the submanifold $(T^*_{\{0\}}\boldsymbol{C}-T^*_{\boldsymbol{C}}\boldsymbol{C})\times T^*_XX$ of V.

For any coherent $\mathscr{E}_{X'}$-Module $\mathscr{N}$ with regular singularities along V, we can define the $\mathscr{E}_{X'}|_\Lambda$-linear automorphism $M(\mathscr{N})\colon \mathscr{N}|_\Lambda\to\mathscr{N}|_\Lambda$ functorially. If $\mathscr{N}$ is the cokernel of $tD_t-A(x,D_x)\colon \mathscr{E}_{X'}^N\to\mathscr{E}_{X'}^N$, where $A(x,D_x)$ is an $N\times N$ matrix of micro-differential operators such that A has the highest order $\leqq 0$. Then we have the commutative diagram

$$\begin{array}{ccccccc}
0 & \longleftarrow & \mathscr{N}|_\Lambda & \longleftarrow & \mathscr{E}_{X'}^N|_\Lambda & \xleftarrow{tD_t-A} & \mathscr{E}_{X'}^N|_\Lambda \\
 & & {\scriptstyle M(\mathscr{N})}\downarrow & & \downarrow{\scriptstyle e^{2\pi i A}} & & \downarrow{\scriptstyle e^{2\pi i A}} \\
0 & \longleftarrow & \mathscr{N}|_\Lambda & \longleftarrow & \mathscr{E}_{X'}^N|_\Lambda & \xleftarrow{tD_t-A} & \mathscr{E}_{X'}^N|_\Lambda\,.
\end{array}$$

Proposition A.5. *If $\mathscr{M}$ is a coherent $\mathscr{D}_X$-Module, then the monodromy of*

$\Phi(\mathcal{M})$ is the identity.

Proof. $\mathcal{M}$ is a quotient of a free Module $\mathcal{D}^N$ and hence $\Phi(\mathcal{M})$ is a quotient of $\Phi(\mathcal{D}^N)=\mathcal{L}^N$. Since the monodromy of $\mathcal{L}^N$ is the identity, we obtain the desired result.

Proposition A.6. *If $\mathcal{N}$ is a coherent $\mathcal{E}_{X'}$-Module with regular singularities along V with the identity as its monodromy. Then $\Phi\Psi(\mathcal{N})\simeq\mathcal{N}$.*

Proof. We shall prove first the following lemma.

Lemma A.7. *If $\mathcal{N}$ is as in Proposition A.6, then $\mathcal{N}|_{\Lambda}$ is locally a quotient of a direct sum of the copies of $\mathcal{L}|_{\Lambda}$.*

Proof. We shall consider on a neighborhood of $j_0(0)$. By Theorem 3.2 [18], $\mathcal{N}$ is a quotient of

$$\mathcal{N}_0=\mathcal{E}_{X'}^N/\mathcal{E}_{X'}^N(tD_t-B),$$

where $B(x, D)$ has the form described in (A.4). $\mathcal{N}=\mathcal{N}_0/M(\mathcal{N}_0)\mathcal{N}_0=\mathcal{E}_{X'}^N/(\mathcal{E}_{X'}^N(tD_t-B)+\mathcal{E}_{X'}^N(e^{2\pi iB}-1))$. By the assumption on $\mathcal{N}$, $\mathcal{N}$ is a quotient of $\mathcal{N}_0/M(\mathcal{N}_0)\mathcal{N}_0$. Hence we may assume without loss of generality that $\mathcal{N}=\mathcal{N}_0/M(\mathcal{N}_0)\mathcal{N}_0$. By decomposing $\mathcal{N}_0$ into a direct sum, we may assume that all the diagonal components of B are λ at $x=0$.

If $\lambda\notin\mathbf{Z}$, then $e^{2\pi iB}-1$ is invertible and hence $\mathcal{N}=0$. Suppose that $\lambda\in\mathbf{Z}$. Then $(e^{2\pi iB}-1)/(B-\lambda)$ is invertible. Hence $\mathcal{E}_{X'}^N(e^{2\pi iB}-1)=\mathcal{E}_{X'}^N(B-\lambda)$. Therefore $\mathcal{N}=\mathcal{E}_{X'}^N/(\mathcal{E}_{X'}^N(tD_t-\lambda)+\mathcal{E}_{X'}^N(B-\lambda))$. Thus $\mathcal{N}$ is a quotient of $\mathcal{E}_{X'}^N/\mathcal{E}_{X'}^N(tD_t-\lambda)$, which is isomorphic to $\mathcal{L}^N$. Q.E.D.

Now, let us prove Proposition A.6. By Lemma A.7, we have an exact sequence

$$\mathcal{L}^{N_1}\xrightarrow{\varphi}\mathcal{L}^{N_0}\longrightarrow\mathcal{N}\longrightarrow 0.$$

Let $\mathcal{M}$ be the cokernel of $\Psi(\varphi)\colon \Psi(\mathcal{L}^{N_1})\to\Psi(\mathcal{L}^{N_0})$. Since $\Phi\Psi(\mathcal{L})=\mathcal{L}$ and Φ is an exact functor, $\Phi(\mathcal{M})$ is the cokernel of $\Phi\Psi(\varphi)=\varphi\colon \mathcal{L}^{N_1}\to\mathcal{L}^{N_0}$. Therefore $\mathcal{N}$ is isomorphic to $\Phi(\mathcal{M})$. Hence $\Psi(\mathcal{N})$ is isomorphic to $\mathcal{M}$ and $\mathcal{N}=\Phi\Psi(\mathcal{N})$.
Q.E.D.

A.5. Let $\mathcal{M}$ be a coherent $\mathcal{D}_X$-Module. An increasing sequence $\{\mathcal{M}_k\}_{k\in\mathbf{Z}}$ of $\mathcal{O}_X$-sub-Modules of $\mathcal{M}$ is called a filtration if

(A.5.1) $\qquad\mathcal{M}_k=0$ for $k\ll 0$ (locally),

(A.5.2) $\qquad\mathcal{M}_k$ is coherent $\mathcal{O}_X$-Modules, (A.5.3) $\mathscr{D}_l\mathscr{M}_k \subset \mathscr{M}_{k+l}$ for any k and l,

(A.5.4) $\mathscr{M} = \cup \mathscr{M}_k$.

If, in addition, $\{\mathscr{M}_k\}$ satisfies

(A.5.5) $\mathscr{D}_l\mathscr{M}_k = \mathscr{M}_{k+l}$ for $k \gg 0$ and $l \geqq 0$,

then the filtration $\{\mathscr{M}_k\}$ is called good. We shall prove that the notion of a good filtration of $\mathscr{M}$ is equivalent to that of a coherent $\mathscr{E}_{X'}(0)$-sub-Module of $\Phi(\mathscr{M})$.

Proposition A.8. *Let $\mathscr{M}$ be a coherent $\mathscr{D}_X$-Module, and let $\mathscr{N}$ be $\Phi(\mathscr{M})$. We shall identify $\mathscr{M}$ with the sub-Module of $j_0^{-1}\mathscr{N}$ by $s \mapsto u_0 \otimes s$.*

(i) *If $\{\mathscr{M}_k\}_{k\in\mathbf{Z}}$ is a good filtration, then $\mathscr{N}_0 = \sum \mathscr{E}_{X'}(-k)(u_0 \otimes \mathscr{M}_k)$ is a coherent $\mathscr{E}_{X'}(0)$-Module, $\mathscr{N} = \mathscr{E}\mathscr{N}_0$, $t\mathscr{N}_0 \subset \mathscr{N}_0(-1)$ and $\mathscr{M}_k = j_0^{-1}\mathscr{N}_0(k) \cap \mathscr{M} = \rho_k(\mathscr{M})(\mathscr{N}_0)$. Here $\mathscr{N}_0(k) = \mathscr{E}(k)\mathscr{N}_0$.*

(ii) *Let $\mathscr{N}_0$ be a coherent $\mathscr{E}_{X'}(0)$-Module such that $\mathscr{N} = \mathscr{E}\mathscr{N}_0$ and $t\mathscr{N}_0 \subset \mathscr{N}_0(-1)$. Let $\mathscr{M}_k$ be $j_0^{-1}\mathscr{N}_0(k) \cap \mathscr{M}$. Then $\{\mathscr{M}_k\}$ is a good filtration of $\mathscr{M}$, and $\mathscr{N}_0 = \sum \mathscr{E}(-k)(u_0 \otimes \mathscr{M}_k)$.*

Proof. We shall prove first (i). We have $t\mathscr{N}_0 = \sum t\mathscr{E}(-k)(u_0 \otimes \mathscr{M}_k) \subset \sum [t, \mathscr{E}(-k)](u_0 \otimes \mathscr{M}_k) + \sum \mathscr{E}(-k)(tu_0 \otimes \mathscr{M}_k) \subset \sum \mathscr{E}(-k-1)(u_0 \otimes \mathscr{M}_k) = \mathscr{N}_0(-1)$. Let us prove that $\mathscr{N}_0$ is coherent. $\mathscr{E}(-k)(u_0 \otimes \mathscr{M}_k)$ are coherent $\mathscr{E}_{X'}(0)$-Module. Moreover, for $k \gg 0$, $\mathscr{E}(-k-1)(u_0 \otimes \mathscr{M}_{k+1}) = \mathscr{E}(-k-1)(u_0 \otimes \mathscr{D}_1\mathscr{M}_k) \subset \mathscr{E}(-k)(u_0 \otimes \mathscr{M}_k)$. Hence we have $\mathscr{N}_0 = \sum_{k=-N}^{N} \mathscr{E}(-k)(u_0 \otimes \mathscr{M}_k)$ for $N \gg 0$. Therefore $\mathscr{N}_0$ is a coherent $\mathscr{E}_{X'}(0)$-Module.

It is clear that $\mathscr{M}_k \subset j_0^{-1}\mathscr{N}_0(k) \cap \mathscr{M}$. We shall prove $j_0^{-1}\mathscr{N}_0(k) \subset \mathscr{M}_k$. Let $\rho_k(\mathscr{M})$ be the homomorphism from $j_0^{-1}\mathscr{N}$ into $\mathscr{M}$ defined by $\rho_k \otimes \mathrm{id}_{\mathscr{M}}$. Then $\rho_0(\mathscr{M})|_{\mathscr{M}}$ is the identity. We have $\mathscr{N}_0(k) = \sum_{l=-N}^{N} \mathscr{E}(k-l)(u_0 \otimes \mathscr{M}_l)$ for $N \gg 0$. Any element of $\mathscr{E}(k-l)(u_0 \otimes \mathscr{M}_l)$ is a combination of elements of the type $P(x, D_t, D_x)u_0 \otimes s$ with $P \in \mathscr{E}(k-l)$. Expanding $P = \sum P_j(x, D_x)D_t^j$, we have $P_j \in \mathscr{D}_{k-l-j}$. Thus we have $\rho_0(Pu_0 \otimes s) = P_{k-l}s \in \mathscr{D}_{X,k-l}\mathscr{M}_l$. Hence we obtain

$$\begin{aligned}\mathscr{M} \cap j_0^{-1}\mathscr{N}_0(k) &\subset \rho_0(j_0^{-1}\mathscr{N}_0(k)) = \rho_k(\mathscr{M})(\mathscr{N}_0) \subset \sum_{l=-N}^{N} \mathscr{D}_{X,k-l}\mathscr{M}_l \\ &= \sum_{l=-\infty}^{k} \mathscr{D}_{X,k-l}\mathscr{M}_l \subset \mathscr{M}_k.\end{aligned}$$

This implies $\mathscr{M}_k = j_0^{-1}\mathscr{N}_0(k) \cap \mathscr{M} = \rho_k(\mathscr{M})(\mathscr{N}_0)$.

We shall prove (ii). The property (A.5.4) is evident, because $\mathscr{N} = \mathscr{E}\mathscr{N}_0$. Let us take a good filtration $\{\mathscr{M}'_k\}$ of $\mathscr{M}$, and we define $\mathscr{N}'_0 = \sum \mathscr{E}(-k)(u_0 \otimes \mathscr{M}'_k)$. Then by (i), $\mathscr{N}'_0$ is a coherent $\mathscr{E}_{X'}(0)$-Module such that $\mathscr{N} = \mathscr{E}_{X'}\mathscr{N}'_0$. Hence there is N such that $\mathcal{N}'_0 \supset \mathcal{N}_0(-N)$. We may assume that $N=0$ by replacing $\mathcal{N}_0$ with $\mathcal{N}_0(-N)$. Then $\mathcal{M}_k \subset \mathcal{M}'_k$. Hence (A.5.1) is clear. We have $\mathcal{D}_l \mathcal{M}_k \subset j_0^{-1}(\mathcal{E}(l)\mathcal{N}_0(k)) \cap \mathcal{M} = j_0^{-1}(\mathcal{N}_0(k+l)) \cap \mathcal{M} = \mathcal{M}_{k+l}$. Thus (A.5.3) is proved. Let us take a point x_0 in X, and we shall prove the other claims on a neighborhood of x_0. Since $\mathcal{M}_{k,x_0} \subset \mathcal{M}'_{k,x_0}$ and $\mathcal{D}_{1,x_0}\mathcal{M}_{k,x_0} \subset \mathcal{M}_{k+1,x_0}$, $\mathcal{M}_{k,x_0}$ is a finitely generated $\mathcal{O}_{X,x_0}$-module and $\mathcal{D}_{l,x_0}\mathcal{M}_{k,x_0} = \mathcal{M}_{l+k,x_0}$ for $k \gg 0$ and $l \geqq 0$. Hence there are a neighborhood U of x and a good filtration $\{\mathcal{M}''_k\}$ of $\mathcal{M}$ defined on U such that $\mathcal{M}''_{k,x_0} = \mathcal{M}_{k,x_0}$ and $\mathcal{M}''_k \subset \mathcal{M}_k$. Set $\mathcal{N}''_0 = \sum \mathcal{E}(-k)(u_0 \otimes \mathcal{M}''_k)$. Then by (i), $\mathcal{N}''_0$ is a coherent $\mathcal{E}_{X'}(0)$-Module.

We shall prove $\mathcal{N}''_{0,j_0(x_0)} = \mathcal{N}_{0,j_0(x_0)}$. It is evident that $\mathcal{N}''_0 \subset \mathcal{N}_0$.

Let $u_1,\dots, u_N$ be a system of generators of $\mathcal{N}_0$, and let u be the column vector with $u_1,\dots, u_N$ as components. Then there is an $N \times N$ matrix $A(t, x, D_t, D_x)$ of micro-differential operators of order $\leqq 0$ such that $tD_t u = A(t, x, D_t, D_x)u$.

The proof of Theorem 3.2 [18] shows the following: there are invertible matrices of $U(t, x, D_t, D_x)$ and $U'(t, x, D_t, D_x)$ of micro-differential operators of order 0 such that

$$U(t, x, D_t, D_x)(tD_t - A(t, x, D_t, D_x))U'(t, x, D_t, D_x) = tD_t - C(t, x, D_t, D_x).$$

Moreover, $C(t, x, D_t, D_x)$ has the following form (A.5.10) for $N_i \times N_i$-matrices $B_i(x, D_x) = (b_{i,\mu\nu}(x, D_x))_{1 \leqq \mu,\nu \leqq N_i}$ $(i=1,\dots, r)$ which satisfy the following conditions:

(A.5.6) $$\sum_{i=1}^{r} N_i = N.$$

(A.5.7) $$b_{i,\mu\nu} = 0 \quad (\mu < \nu).$$

(A.5.8) $b_{i,\mu\mu}$ depends only on x.

(A.5.9) $$b_{i,\mu\mu}(x_0) = \lambda.$$

(A.5.10) $$R(tD_t - C)R^{-1} = tD_t - \begin{pmatrix} B_1 & & \\ & \ddots & \\ & & B_r \end{pmatrix}.$$

Here R is the diagonal matrix with $D_t^{m_j}$ as components. Let λ_j be the diagonal components of $\sigma_0(B_j)(x)$. By replacing u with $U'^{-1}u$, we may assume that $(tD_t - C)u = 0$.

Hence $\left(tD_t - \begin{pmatrix} B_1 & & \\ & \ddots & \\ & & B_r \end{pmatrix}\right)Ru = 0$. Set $Ru = \begin{bmatrix} w_1 \\ \vdots \\ w_r \end{bmatrix}$ with N_j-column vector w_j $(j=1,\dots, r)$. Then $(tD_t - B_j)w_j = 0$. Since the monodromy of $\mathcal{N}$ is the identity, $e^{2\pi i B_j}w_j = w_j$. If λ_j is not an integer, then $e^{2\pi i B_j} - 1$ is invertible and hence $w_j=0$. If λ_j is an integer, $(e^{2\pi i B_j}-1)/(B_j-\lambda_j)$ is invertible. Hence we have $B_j w_j=\lambda_j w_j$, which implies $(tD_t-\lambda_j)w_j=0$. Therefore u_k satisfies $(tD_t-\mu_k)u_k=0$ for some integer μ_k. Then $tD_t^{\mu_k+1}u_k=0$. Hence $D_t^{\mu_k+1}u_k=u_0\otimes u'_k$ for some $u'_k\in\mathscr{M}$. The section u'_k is contained in $\mathscr{M}_{\mu_k+1}$ by the definition. Hence u_k is contained in $\mathscr{E}(-\mu_k-1)(u_0\otimes\mathscr{M}_{\mu_k+1})$. Therefore we obtain $\mathscr{N}_{0,j_0(x_0)}\subset\mathscr{N}''_{0,j_0(x_0)}$.

Since $\mathscr{N}_0$ and $\mathscr{N}''_0$ are coherent and $\mathscr{N}_{0,j_0(x_0)}=\mathscr{N}''_{0,j_0(x_0)}$, we have $\mathscr{N}_0=\mathscr{N}''_0$ on a neighborhood of $j_0(x_0)$. Therefore (ii) follows from (i).

A.6. Proposition A.9. *Let $\mathscr{N}_1$ and $\mathscr{N}_2$ be two coherent $\mathscr{E}_{X'}$-Modules with regular singularities along V and f an $\mathscr{E}^\infty_{X'}$-linear homomorphism from $\mathscr{N}_1^\infty$ into $\mathscr{N}_2^\infty$. Then we have*

$$M(\mathscr{N}_2)^\infty\circ f=f\circ M(\mathscr{N}_1)^\infty\,,$$

where $M(\mathscr{N}_j)^\infty$ is the automorphism of $\mathscr{N}_j^\infty$ obtained from $M(\mathscr{N}_j)$ by tensoring $\mathscr{E}^\infty_{X'}$ $(j=1, 2)$.

Proof. We can take a system of generators $u_1,\dots, u_l$ (resp., $v_1,\dots, v_m$) of $\mathscr{N}_1$ (resp., $\mathscr{N}_2$) and an $l\times l$ (resp., $m\times m$) matrix of differential operators $A(x, D_x)$ (resp., $B(x, D_x)$) of the highest order $\leqq 0$ such that, if u (resp., v) denotes the column vector with $u_1,\dots, u_l$ (resp., $v_1,\dots, v_m$) **as** components, we have $(tD_t-A(x, D_x))u=0$ (resp., $(tD_t-B(x, D_x))v=0$).

Let $\mathscr{E}'$ be the sheaf of microdifferential operators which commute with D_t and set $\mathscr{E}'(m)=\mathscr{E}_{X'}(m)\cap\mathscr{E}'$. Then, for any integer m, $\mathscr{E}'(m)$ is a coherent sheaf and $\mathscr{E}'(0)$ is Noetherian (i.e., an increasing sequence of coherent Ideals is locally stationary). Let $\mathscr{I}$ be the sub-Module of $\mathscr{E}^m_{X'}$ consisting of $H\in\mathscr{E}^m_{X'}$ such that $Hv=0$. Then $\mathscr{I}\cap\mathscr{E}'(0)^m$ is a coherent $\mathscr{E}'(0)$-sub-Module of $\mathscr{I}\cap\mathscr{E}'(0)^m$. Let Φ be the operator of $\mathscr{E}'^m$ defined by $H\mapsto[tD_t, H]+HB$. Then $\Phi(\mathscr{I}\cap\mathscr{E}'^m)\subset\mathscr{I}\cap\mathscr{E}'^m$. In fact, for $H\in\mathscr{I}\cap\mathscr{E}'^m$, $\Phi(H)v=([tD_t, H]+HB)v=(HB-HtD_t)v=0$. It is easy to see that, for an $\mathscr{E}'(0)$-sub-Module $\mathscr{F}$ of $\mathscr{E}'^m$ generated by $s_1,\dots, s_p$, $\mathscr{F}+\Phi(\mathscr{F})$ is an $\mathscr{E}'(0)$-sub-Module generated by $s_1,\dots, s_p$, $\Phi(s_1),\dots,\Phi(s_p)$. Hence, if $\mathscr{F}$ is a coherent $\mathscr{E}'(0)$-sub-Module of $\mathscr{E}'^m$, then so is $\mathscr{F}+\Phi(\mathscr{F})$. There is an integer k such that any component of B^i is a linear differential operator of order $\leqq k$ for any i. Hence $\Phi^i(\mathscr{E}'(0)^m)\subset\mathscr{E}'(k)^m$ for any i. Thus $\{\sum_{i=0}^{\nu}\Phi^i(\mathscr{I}\cap\mathscr{E}'(0)^m)\}_\nu$ is an increasing sequence of coherent $\mathscr{E}'(0)$-sub-Module of $\mathscr{E}'(k)^m$ and hence $\mathscr{I}'=\sum_{i=0}^{\infty}\Phi^i(\mathscr{I}\cap\mathscr{E}'(0)^m)$ is a coherent $\mathscr{E}'(0)$-sub-Module of $\mathscr{I}\cap\mathscr{E}'^m$. Let $\{H_1,\dots, H_N\}$ be a system of generators of $\mathscr{I}'$. Then we can see easily that $\{H_1,\dots, H_N\}$ has the following properties:

(A.6.1) If we denote by H the $N\times m$ matrix of micro-differential operators whose row-vectors are $H_1,\dots, H_N$, then there is an $N\times N$ matrix R of micro-differential operators in $\mathscr{E}'(0)$ such that $HB=RH-[tD_t, H]$.

(A.6.2) If a row vector P of length m of micro-differential operators satisfies $Pv=0$ and $[D_t, P]=0$, then there is a row vector S of length N such that $P=SH$.

Let $G(x, D_x, D_t)$ be an $l\times m$ matrix of micro-differential operators of infinite order such that $f(u)=Gv$ and $[D_t, G]=0$. Since

$$\begin{aligned}0&=f((tD_t-A)u)=(tD_t-A)f(u)=(tD_t-A)Gv\\&=GtD_tv-AGv+[tD_t, G]v=(GB-AG+[tD_t, G])v,\end{aligned}$$

there is an $l\times N$ matrix $S(x, D_x, D_t)$ of micro-differential operators such that

$$AG-GB=[tD_t, G]+SH. \tag{A.6.3}$$

We define $G(\lambda)$, $H(\lambda)$, $R(\lambda)$ and $S(\lambda)$ by

$$\frac{d}{d\lambda}G(\lambda)=[tD_t, G(\lambda)],\ \frac{d}{d\lambda}H(\lambda)=[tD_t, H(\lambda)],\ \frac{dR(\lambda)}{d\lambda}=[tD_t, R(\lambda)]$$
$$\frac{d}{d\lambda}S(\lambda)=[tD_t, S(\lambda)] \text{ and } G(0)=G,\ H(0)=H,\ R(0)=R,$$
$$S(0)=S.$$

If we expand $G=\sum_j G_j(x, D_x)D_t^j$, then we have $G(\lambda)=\sum G_j(x, D_x)e^{-j\lambda}D_t^j$. Hence $G(2\pi\sqrt{-1})=G$, $H(2\pi\sqrt{-1})=H,\dots$. It is easy to verify

$$H(\lambda)B=R(\lambda)H(\lambda)-\frac{d}{d\lambda}H(\lambda). \tag{A.6.4}$$

Now, we define $K(\lambda)$ by

$$\frac{d}{d\lambda}K(\lambda)=R(\lambda)K(\lambda) \quad\text{and}\quad K(0)=1.$$

The existence of such a $K(\lambda)$ is guaranteed by Theorem 5.2.1, Chapter II of S-K-K [24].

We have

$$H(\lambda)e^{\lambda B}=K(\lambda)H. \tag{A.6.5}$$

In fact, if we set $\Phi(\lambda)=H(\lambda)e^{\lambda B}-K(\lambda)H$, then we have $\frac{d}{d\lambda}\Phi(\lambda)=H(\lambda)e^{\lambda B}B+\frac{d}{d\lambda}H(\lambda)e^{\lambda B}-\frac{d}{d\lambda}K(\lambda)H=H(\lambda)e^{\lambda B}B+(R(\lambda)H(\lambda)-H(\lambda)B)e^{\lambda B}-R(\lambda)K(\lambda)H=R(\lambda)\Phi(\lambda)$ by (A.6.4). Since $\Phi(0)=0$, we have $\Phi(\lambda)=0$ by the uniqueness of a solution of the differential equation. By (A.6.3), we have

$$AG(\lambda)-G(\lambda)B=\frac{d}{d\lambda}G(\lambda)+S(\lambda)H(\lambda). \tag{A.6.6}$$

We define $F(\lambda)$ by the equation

$$\frac{d}{d\lambda}F(\lambda)-AF(\lambda)=S(\lambda)K(\lambda) \quad \left(\text{or } \frac{d}{d\lambda}e^{-\lambda A}F(\lambda)=e^{-\lambda A}S(\lambda)K(\lambda)\right)$$

and $F(0)=0$.

Let us verify

$$e^{\lambda A}G-G(\lambda)e^{\lambda B}=F(\lambda)H. \tag{A.6.7}$$

Set $\Phi(\lambda)=e^{\lambda A}G-G(\lambda)e^{\lambda B}-F(\lambda)H$. Then we have

$$\begin{aligned} d\Phi(\lambda)/d\lambda &= Ae^{\lambda A}G-G'(\lambda)e^{\lambda B}-G(\lambda)e^{\lambda B}B-F'(\lambda)H \\ &= Ae^{\lambda A}G-(AG(\lambda)-G(\lambda)B-S(\lambda)H(\lambda))e^{\lambda B}-G(\lambda)e^{\lambda B}B \\ &\quad -(AF(\lambda)+S(\lambda)K(\lambda))H \\ &= A\Phi(\lambda)+S(\lambda)(H(\lambda)e^{\lambda B}-K(\lambda)H) \\ &= A\Phi(\lambda), \end{aligned}$$

and $\Phi(0)=0$. Thus we obtain $\Phi=0$. (A.6.7) implies, in particular,

$$e^{2\pi\sqrt{-1}A}G-Ge^{2\pi\sqrt{-1}B}=F(2\pi\sqrt{-1})H. \tag{A.6.8}$$

Hence we have

$$e^{2\pi\sqrt{-1}A}Gv=Ge^{2\pi\sqrt{-1}B}v,$$

which implies $f(M(\mathcal{N}_1)^\infty u)=M(\mathcal{N}_2)^\infty f(u)$. Since $\mathcal{N}_1^\infty$ is generated by u, we obtain $f\circ M(\mathcal{N}_1)^\infty=M(\mathcal{N}_2)^\infty\circ f$. Q.E.D.

Proposition A.10. *Let $\mathcal{M}$ be a coherent $\mathcal{D}_X$-Module and $\mathcal{N}$ a coherent $\mathcal{E}_{X'}$-Module with regular singularities. Then we have an isomorphism*

$$\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \Psi(\mathcal{N})^\infty|_{T_X^*X}) \xrightarrow{\sim} j_0^{-1}\mathcal{H}om_{\mathcal{E}_{X'}}(\Psi(\mathcal{M}), \mathcal{N}^\infty).$$

Proof. By Proposition A.4, we have $\Psi(\mathcal{N})^\infty=j_0^{-1}\mathcal{H}om_{\mathcal{E}_{X'}}(\mathcal{L}, \mathcal{N}^\infty)$. Hence, we have

$$\begin{aligned} &\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \Psi(\mathcal{N})^\infty|_{T_X^*X})=\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, j_0^{-1}\mathcal{H}om_{\mathcal{E}_{X'}}(\mathcal{L}, \mathcal{N}^\infty)) \\ &\cong \mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \mathcal{H}om_{j_0^{-1}\mathcal{E}_{X'}}(j_0^{-1}\mathcal{L}, j_0^{-1}\mathcal{N}^\infty)) \\ &\cong \mathcal{H}om_{j_0^{-1}\mathcal{E}_{X'}}(j_0^{-1}\mathcal{L}\underset{\mathcal{D}_X}{\otimes}\mathcal{M}, j_0^{-1}\mathcal{N}^\infty) \\ &\cong j_0^{-1}\mathcal{H}om_{\mathcal{E}_{X'}}(\mathcal{L}\underset{\mathcal{D}_X}{\otimes}\mathcal{M}, \mathcal{N}^\infty). \end{aligned}$$

Q.E.D.

B.1. The purpose of this section is to prove (iii) of Proposition 1.4.2 in Chapter I, Section 4.

Proposition B.1. *Let $\mathcal{F}^\cdot$ be a bounded complex on a complex manifold X whose cohomology groups are constructible and let $\mathcal{G}^\cdot$ be a complex of sheaves on X. Let p_1 and p_2 be the first and the second projection from $X \times X$ onto X, respectively. Let Δ denote the diagonal set of $X \times X$. Then we have*

$$p_{1*}\mathbf{R}\Gamma_\Delta(p_1^{-1}\mathbf{R}\mathcal{H}om_{\mathbf{C}}(\mathcal{F}^\cdot, \mathbf{C}_X) \underset{\mathbf{C}}{\otimes} p_2^{-1}\mathcal{G}^\cdot) = \mathbf{R}\mathcal{H}om_{\mathbf{C}}(\mathcal{F}^\cdot, \mathcal{G}^\cdot)[-2\dim X].$$

Lemma B.2. *Let $\mathcal{F}^\cdot$ be as in Proposition B.1 and V a $\mathbf{C}$-vector space (which is not necessarily finite-dimensional). Then we have*

$$\mathbf{R}\mathcal{H}om_{\mathbf{C}}(\mathcal{F}^\cdot, \mathbf{C}_X) \underset{\mathbf{C}}{\otimes} V_X \xrightarrow{\sim} \mathbf{R}\mathcal{H}om_{\mathbf{C}}(\mathcal{F}^\cdot, V_X)$$

where V_X is the constant sheaf on X whose stalks are V.

Proof. We may assume that $\mathcal{F}^\cdot$ is a simple complex. We identify $\mathcal{F}^\cdot$ with $\mathcal{F} = \mathcal{H}^0(\mathcal{F}^\cdot)$. There exists a triangulation of X on each of whose simplex $\mathcal{F}$ is a constant sheaf of finite rank. Hence we may assume that $\mathcal{F}$ is $\mathbf{C}_\sigma$ where σ is a simplex in X. Then the above lemma is obvious. Q.E.D.

Lemma B.3. $p_1^{-1}\mathbf{R}\mathcal{H}om_{\mathbf{C}}(\mathcal{F}^\cdot, \mathbf{C}_X) \otimes p_2^{-1}\mathcal{G}^\cdot = \mathbf{R}\mathcal{H}om_{\mathbf{C}}(p_1^{-1}\mathcal{F}^\cdot, p_2^{-1}\mathcal{G}^\cdot)$.

Proof. Let us take a point $x = (x_1, x_2)$ of $X \times X$. Let U_j be an open neighborhood of x_j $(j = 1, 2)$. Then we have

$$\begin{aligned}&\mathbf{R}\Gamma(U_1 \times U_2; \mathbf{R}\mathcal{H}om_{\mathbf{C}}(p_1^{-1}\mathcal{F}^\cdot, p_2^{-1}\mathcal{G}^\cdot))\\ &\quad = \mathbf{R}\Gamma(U_1; \mathbf{R}\mathcal{H}om_{\mathbf{C}}(\mathcal{F}^\cdot, \mathbf{R}(p_1|_{p_2^{-1}U_2})_*(p_2^{-1}\mathcal{G}^\cdot|_{p_2^{-1}U_2}))).\end{aligned}$$

On the other hand we have

$$\mathbf{R}(p_1|_{p_2^{-1}U_2})_*(p_2^{-1}\mathcal{G}^\cdot|_{p_2^{-1}U_2}) = \mathbf{R}\Gamma(U_2; \mathcal{G}^\cdot)_X.$$

Hence we obtain

$$\begin{aligned}&\mathbf{R}\Gamma(U_1 \times U_2; \mathbf{R}\mathcal{H}om_{\mathbf{C}}(p_1^{-1}\mathcal{F}^\cdot, p_2^{-1}\mathcal{G}^\cdot))\\ &\quad = \mathbf{R}\Gamma(U_1; \mathbf{R}\mathcal{H}om_{\mathbf{C}}(\mathcal{F}^\cdot, \mathbf{C}_X) \otimes \mathbf{R}\Gamma(U_2, \mathcal{G}^\cdot)_X).\end{aligned}$$

Hence, by taking an inductive limit, we obtain

$$\mathbf{R}\mathcal{H}om_{\mathbf{C}}(p_1^{-1}\mathcal{F}^\cdot, p_2^{-1}\mathcal{G}^\cdot)_x = \mathbf{R}\mathcal{H}om_{\mathbf{C}}(\mathcal{F}^\cdot, \mathbf{C}_X)_{x_1} \otimes \mathcal{G}^\cdot_{x_2}.$$

Q.E.D.

Now, we are ready to prove Proposition B.1. By Lemma B.3, we have

$$p_1^{-1}\mathbf{R}\mathcal{H}om_{\mathbf{C}}(\mathcal{F}^\cdot, \mathbf{C}_X) \otimes p_2^{-1}\mathcal{G}^\cdot = \mathbf{R}\mathcal{H}om_{\mathbf{C}}(p_1^{-1}\mathcal{F}^\cdot, p_2^{-1}\mathcal{G}^\cdot).$$

Hence we have

$$\begin{aligned}\mathbf{R}\Gamma_{\Delta}(p_1^{-1}\mathbf{R}\mathscr{H}om_{\boldsymbol{C}}(\mathscr{F}^{\cdot}, \boldsymbol{C}_X) \otimes p_2^{-1}\mathscr{G}^{\cdot})\\ =\mathbf{R}\Gamma_{\Delta}\mathbf{R}\mathscr{H}om_{\boldsymbol{C}}(p_1^{-1}\mathscr{F}^{\cdot}, p_2^{-1}\mathscr{G}^{\cdot})\\ =\mathbf{R}\mathscr{H}om_{\boldsymbol{C}}(p_1^{-1}\mathscr{F}^{\cdot}|_{\Delta}, \mathbf{R}\Gamma_{\Delta}(p_2^{-1}\mathscr{G}^{\cdot}))\,.\end{aligned}$$

On the other hand, we can easily verify

$$\mathbf{R}\Gamma_{\Delta}(p_2^{-1}\mathscr{G}^{\cdot})=\mathbf{R}\Gamma_{\Delta}(\boldsymbol{C}_{X\times X}) \otimes p_2^{-1}\mathscr{G}^{\cdot}=p_2^{-1}\mathscr{G}^{\cdot}|_{\Delta}[-2\dim X]$$

(e.g., we can use Lemma B.3 after coordinate transformations). This shows immediately Proposition B.1.

C.1. It seems that Theorem 2.2.1 and Theorem 2.2.3 are not written explicitly in [3]. In this appendix we shall give their proof.

The following proposition is proved in [3] (Proposition 5.7 in p. 96 and Theorem 4.1 in p. 85).

Proposition C.1.1. *Let X be a complex manifold, Y a hypersurface of X and j the inclusion map from $X-Y$ into X. Let L be a locally free $\boldsymbol{C}_{X-Y}$-Module of finite rank. Then there exists a coherent $\mathscr{O}_X$-sub-Module $\mathscr{L}_0$ of $j_*(\mathscr{O}_{X-Y} \underset{\boldsymbol{C}}{\otimes} L)$ satisfying the following conditions:*

(C.1.1) $\mathscr{L}_0|_{X-Y}=\mathscr{O}_{X-Y}\underset{\boldsymbol{C}}{\otimes} L$ *as a $\mathscr{D}_{X-Y}$-Module.*

(C.1.2) $\mathscr{L}=\mathscr{H}^0_{[X|Y]}(\mathscr{L}_0)$ *is a $\mathscr{D}_X$-sub-Module of $j_*(\mathscr{O}_{X-Y}\underset{\boldsymbol{C}}{\otimes} L)$.*

(C.1.3) *On Y_{reg}, $\mathscr{L}$ (resp., $\mathscr{L}_0$) is the subsheaf of $j_*(\mathscr{O}_{X-Y}\otimes L)$ consisting of sections in the Nilsson class (resp., the strict Nilsson class).*

The sheaf $\mathscr{D}_X\mathscr{L}_0$ is a coherent $\mathscr{D}_X$-Module and $\mathscr{D}_X\mathscr{L}_0|_{X-Y}=\mathscr{L}_0|_{X-Y}$ is a holonomic $\mathscr{D}_{X-Y}$-Module. Hence Theorem 3.1 [8] implies that $\mathscr{L}=\mathscr{H}^0_{[X|Y]}(\mathscr{D}_X\mathscr{L}_0)$ is a holonomic $\mathscr{D}_X$-Module. By the definition of $\mathscr{L}(L)$ and $\mathscr{L}_0(L)$ given in Chapter II, we have

$$\mathscr{L}(L)=\mathscr{H}^0_{X|Y\text{sing}}(\mathscr{L})$$

and

$$\mathscr{L}_0(L)=\mathscr{H}^0_{X|Y\text{sing}}(\mathscr{L}_0)\,.$$

Now, let us prove Theorem 2.2.1. On $X-Y_{\text{sing}}$, Theorem 2.2.1 is evident. Hence Theorem 2.2.1 follows from the following proposition. In fact, the following proposition implies $\mathscr{L}_0(L)$ is a coherent $\mathscr{O}_X$-Module and $\mathscr{L}(L)=\mathscr{L}$.

Proposition C.1.2. *Let X be a complex manifold and Y a hypersurface of X. Let $\mathscr{F}$ be a coherent $\mathscr{O}_X$-Module satisfying the following conditions:*

(C.1.4) $\mathscr{H}^0_Y(\mathscr{F})=0$.

(C.1.5) $\mathscr{F}|_{X-Y}$ *is a locally free* $\mathscr{O}_{X-Y}$*-Module.*

Let $\mathscr{L}$ denote $\mathscr{H}^0_{[X|Y]}(\mathscr{F})$. Then we have the following:

(i) *Let s be a section of $\mathscr{H}^0_{X|Y}(\mathscr{F})$ defined on an open set U of X. Assume that there is an open subset V such that $s|_V \in \mathscr{L}(V)$ and V intersects any irreducible component of $U \cap Y$. Then s belongs to $\mathscr{L}(U)$.*

(ii) *Let $\mathscr{F}'$ be the subsheaf of $\mathscr{L}$ given by $\mathscr{F}'(U)=\{s\in\mathscr{L}(U)$; there is an open subset V of U such that $s\in\mathscr{F}(V)$ and $V\cap Y$ is a dense subset of $U\cap Y\}$. Then $\mathscr{F}'$ is a coherent $\mathscr{O}_X$-Module.*

Proof. We prove this proposition in several steps.

(1) The case where Y is smooth and $\mathscr{F}$ is locally free.

In this case, we may assume $\mathscr{F}=\mathscr{O}_X$. Then $\mathscr{F}'=\mathscr{F}$ and (i) is easily proved by using the Laurent expansion with respect to the vertical direction of Y. We leave the details to the reader.

(2) The case where Y is smooth.

There is a locally free $\mathscr{O}_X$-sub-Module $\mathscr{G}$ of $\mathscr{L}$ such that $\mathscr{L}=\mathscr{H}^0_{[X|Y]}(\mathscr{G})$. We may assume further $\mathscr{G}\supset\mathscr{F}$. The property (i) is derived from the case (1). Let us prove (ii). We have $\mathscr{G}\supset\mathscr{F}'$. Then it is easy to see that $\mathscr{F}'$ is coherent over $\mathscr{O}_X$, because $\mathscr{F}'/\mathscr{F}$ is the sheaf of sections of $\mathscr{H}^0_Y(\mathscr{G}/\mathscr{F})$ whose support has codimension $\geqq 2$.

(3) The general case.

If $s\in\mathscr{H}^0_{X|Y}(\mathscr{F})$ satisfies the condition in (i), then $s|_{X-Y_{\mathrm{sing}}}\in\mathscr{L}(X-Y_{\mathrm{sing}})$ by the case (2). Therefore it is sufficient to show $\mathscr{H}^0_{X|Y_{\mathrm{sing}}}(\mathscr{L})=\mathscr{L}$. Hence the properties (i) and (ii) are local problems.

Now, let us assume that Y is defined by $f(x)=0$. We may assume that f is of the Weierstrass type:

$$f=x_n^m+a_1(x_1,\dots,x_{n-1})x_n^{m-1}+\cdots+a_m(x_1,\dots,x_{n-1}).$$

Let F be the map from X into $\boldsymbol{C}^n$ given by $x\mapsto(x_1,\dots,x_{n-1},f(x))$. Then F is locally a finite map. Hence shrinking X, we may assume that F is a finite map from X into an open subset X' of $\boldsymbol{C}^n$. Set $Y'=\{(x_1,\dots,x_{n-1},t)\in X'\subset\boldsymbol{C}^n$; $t=0\}$. Then $Y=F^{-1}(Y')$. Set $\mathscr{G}=F_*\mathscr{F}$. Then we have $\mathscr{H}^0_{Y'}(\mathscr{G})=0$, and

$\mathscr{G}|_{X'-Y'}$ is a locally free $\mathcal{O}_{X'-Y'}$-Module because F is a flat map. We have

$$F_*\mathscr{L}=\mathscr{H}^0_{[X'|Y']}(\mathscr{G}),$$

and

$$F_*(\mathscr{H}^0_{X|Y_{\rm sing}}(\mathscr{L}))\subset\mathscr{H}^0_{X|F(Y_{\rm sing})}(F_*\mathscr{L}).$$

By case (2), we have $\mathscr{H}^0_{X|F(Y_{\rm sing})}(F_*\mathscr{L})=F_*\mathscr{L}$, and hence we obtain $F_*(\mathscr{H}^0_{X|Y_{\rm sing}}(\mathscr{L}))=F_*\mathscr{L}$. This implies $\mathscr{H}^0_{X|Y_{\rm sing}}(\mathscr{L})=\mathscr{L}$.

Now, we shall prove (ii). It is easy to see

$$F_*\mathscr{F}'=\{s\in F_*\mathscr{L};\ s \text{ belongs to } F_*\mathscr{G} \text{ on an open dense subset of } Y'\}.$$

Hence Case (2) implies that $F_*\mathscr{F}'$ is a coherent $\mathcal{O}_{X'}$-Module. Therefore $\mathscr{F}'$ is a coherent $\mathcal{O}_X$-Module. Q. E. D.

C.2. Let us prove Theorem 2.2.3.

Since $\mathbf{R}\mathscr{H}om_{\mathscr{D}_X}(\mathcal{O}_X, \mathscr{L})|_{X-Y}=L$, it is sufficient to show

$$\mathbf{R}\Gamma_S\mathbf{R}\mathscr{H}om_{\mathscr{D}_X}(\mathcal{O}_X, \mathscr{L})=0. \tag{C.2.1}$$

(1st-step) Reduction to the case where Y is normally crossing.

By Hironaka's desingularization theorem ([26]), there is a monoidal transform $f\colon X'\to X$ of X satisfying the following conditions

(C.2.2) $X'-f^{-1}(Y_{\rm sing})\to X-Y_{\rm sing}$ is an isomorphism.

(C.2.3) $Y'=f^{-1}(Y)$ is normally crossing, i.e., we can choose a local coordinate system $(t_1,\dots, t_n)$ of X' around any point of Y' such that Y' is given by $t_1\cdots t_l=0$.

Let j' be the inclusion map from $X'-Y'$ into X' and let L' be the locally constant sheaf $(f|_{X-Y})^{-1}L$ on $X'-Y'$. Set $\mathscr{L}'=\mathscr{L}(L')$. Then as easily shown, we have

(C.2.4) $f_*\mathscr{L}'=\mathscr{L}$

(C.2.4′) $R^kf_*\mathscr{L}'=0$ for $k\neq 0$.

Let $\Omega^\cdot_X(\mathscr{L})$ be the de Rham complex associated with $\mathscr{L}$:

$$\Omega^0_X\underset{\mathcal{O}_X}{\otimes}\mathscr{L}\longrightarrow\Omega^1_X\underset{\mathcal{O}_X}{\otimes}\mathscr{L}\longrightarrow\cdots.$$

Let $\Omega^\cdot_{X'}(\mathscr{L}')$ be the de Rham complex associated with $\mathscr{L}'$. Then we have

$$R^kf_*\Omega^p_{X'}(\mathscr{L}')=\begin{cases}\Omega^p_X(\mathscr{L}) & \text{for } k=0\\ 0 & \text{for } k\neq 0,\end{cases} \tag{C.2.5}$$

because $\Omega^p_{X'}(\mathscr{L}')=f^{-1}\Omega^p_X \underset{f^{-1}\mathscr{O}_X}{\otimes} \mathscr{L}'$. Now, $\mathbf{R}\mathscr{H}om_{\mathscr{D}_X}(\mathscr{O}_X, \mathscr{L})$ (resp., $\mathbf{R}\mathscr{H}om_{\mathscr{D}_{X'}}(\mathscr{O}_{X'}, \mathscr{L}')$) is quasi-isomorphic to $\Omega^{\cdot}_X(\mathscr{L})$ (resp., $\Omega^{\cdot}_{X'}(\mathscr{L}')$) (see § 1, Chapter II). Hence (C.2.5) implies that

$$\mathbf{R}f_*\mathbf{R}\mathscr{H}om_{\mathscr{D}_{X'}}(\mathscr{O}_{X'}, \mathscr{L}')=\mathbf{R}\mathscr{H}om_{\mathscr{D}_X}(\mathscr{O}_X, \mathscr{L})\,.$$

Thus we obtain

$$\mathbf{R}\Gamma_Y\mathbf{R}\mathscr{H}om_{\mathscr{D}_X}(\mathscr{O}_X, \mathscr{L})=\mathbf{R}f_*(\mathbf{R}\Gamma_{Y'}\,\mathbf{R}\mathscr{H}om_{\mathscr{D}_{X'}}(\mathscr{O}_{X'}, \mathscr{L}'))\,.$$

Hence the vanishing of $\mathbf{R}\Gamma_Y\mathbf{R}\mathscr{H}om_{\mathscr{D}_X}(\mathscr{O}_X, \mathscr{L})$ is reduced to that of $\mathbf{R}\Gamma_{Y'}\mathbf{R}\mathscr{H}om_{\mathscr{D}_{X'}}(\mathscr{O}_{X'}, \mathscr{L}')$.

(2nd-step) The case where Y is normally crossing.

We have

$$\mathbf{R}\mathscr{H}om_{\mathscr{D}_X}(\mathscr{O}_X, \mathscr{L})=\mathbf{R}\mathscr{H}om_{\mathbf{C}}(\mathbf{R}\mathscr{H}om(\mathscr{L}, \mathscr{O}_X), \mathbf{C}_X)$$

and hence

$$\mathbf{R}\Gamma_Y\mathbf{R}\mathscr{H}om_{\mathscr{D}_X}(\mathscr{O}_X, \mathscr{L})=\mathbf{R}\mathscr{H}om_{\mathbf{C}}(\mathbf{R}\mathscr{H}om_{\mathscr{D}_X}(\mathscr{L}, \mathscr{O}_X)|_Y, \mathbf{C}_X)\,.$$

Hence it is sufficient to show that

(C.2.6) $\qquad \mathscr{E}xt^j_{\mathscr{D}_X}(\mathscr{L}, \mathscr{O}_X)_p=0$ for any $p\in Y$ and any j.

Let us take a local coordinate system $(x_1,\dots, x_n)$ around p such that $Y=\{x_1\cdots x_l=0\}$ and p is the origin. Set $Y_j=\{x;\, x_j=0\}$ $(1\leqq j\leqq l)$. Then $Y=\cup Y_j$. Let us take a small ball U centered at p on which L is defined. Since $\pi_1(U-Y)$ is the free abelian group generated by the l elements $\gamma_1,\dots, \gamma_l$ where γ_j is a cycle around Y_j. Recall that L is represented by the representation of $\pi_1(U-Y)$ on a finite-dimensional vector space V; hence L is determined by $B_1,\dots, B_l\in GL(V)$, which correspond to $\gamma_1,\dots, \gamma_l$. The B_j's satisfy $[B_i, B_j]=0$. Let us take $A_1,\dots, A_l\in \mathrm{End}\,(V)$ such that

(C.2.7) $\exp(2\pi\sqrt{-1}A_j)=B_j,\ [A_i, A_j]=0\,,$

and

(C.2.8) any eigenvalue λ of A_j is not $0, 1, 2,\dots$.

We define a $\mathscr{D}$-Module $\mathscr{L}'$ by

$$\mathscr{L}'=\mathscr{D}_X\underset{\mathbf{C}}{\otimes} V/(\sum_{j=1}^{l}(\mathscr{D}_X\underset{\mathbf{C}}{\otimes} V)(x_jD_j-A_j)+\sum_{j=l+1}^{n}(\mathscr{D}_X\underset{\mathbf{C}}{\otimes} V)D_j)\,.$$

Here an element of $\mathrm{End}\,(V)$ operates on V from the right. Then one can easily check that

(C.2.9) $SS(\mathscr{L}')\subset\{(x, \xi);\ x_j\xi_j=0 \ \text{ for } \ 1\leqq j\leqq l, \ \ \xi_j=0 \text{ for } l+1\leqq j\leqq n\}$.

(C.2.10) $\mathscr{L}'$ is a holonomic $\mathscr{D}_X$-Module with R. S.,

and

(C.2.11) $\mathscr{H}om_{\mathscr{D}_X}(\mathscr{O}_X, \mathscr{L}')|_{U-Y}=L|_{U-Y}$.

Lemma C.2.1. $\mathscr{H}^k_{[Y]}(\mathscr{L}')_p=0$ *for any* k.

Proof. In order to show this, it is sufficient to show that the multiplication by x_j gives an isomorphism on $\mathscr{L}'_p$.

We shall show first the multiplication by x_j gives a surjective map on $\mathscr{L}'_p$. For an element $v\in V$, we denote by the same letter the corresponding section of $\mathscr{L}'$. We first prepare the following formula:

(C.2.12) $$x_j^m D_j^m v \prod_{k=0}^{m-1}(A_j-k)^{-1}=v \quad (1\leqq j\leqq l).$$

We shall prove this formula by the induction on m. We have $x_j^{m-1}D_j^{m-1}v\prod_{k=0}^{m-2}(A_j-k)^{-1}=v$ by the hypothesis of the induction. Hence, applying this to $(A_j-m+1)^{-1}v$, we obtain

$$x_j^{m-1}D_j^{m-1}v\prod_{k=0}^{m-1}(A_j-k)^{-1}=v(A_j-m+1)^{-1}.$$

On the other hand, we have

$$x_j^m D_j^m=(x_jD_j-m+1)x_j^{m-1}D_j^{m-1}.$$

Thus we obtain

$$\begin{aligned} x_j^m D_j^m v\prod_{k=0}^{m-1}(A_j-k)^{-1}&=(x_jD_j-m+1)v(A_j-m+1)^{-1}\\ &=v(A_j-m+1)^{-1}(A_j-m+1)=v. \end{aligned}$$

Now we begin proving that x_j: $\mathscr{L}'_p\to\mathscr{L}'_p$ is surjective. The surjectivity of x_j $(l+1\leqq j\leqq n)$ is clear, because $x_j\neq 0$ at p. Hence it suffices to show the surjectivity of x_j for $j=1,\dots, l$. Let u be an element of $\mathscr{L}'_p$. It follows from the definition of $\mathscr{L}'_p$ that u has the form $\sum_v P_v v_v$ with $P_v\in\mathscr{D}_p$ and $v\in V$, we may assume from the first that $u=Pv$. Let m be an integer greater than the order of P. Then there exists a linear differential operator R such that $x_jR=Px_j^m$ holds. Let w denote the element $RD_j^m(v\prod_{k=0}^{m-1}(A_j-k)^{-1})$ of $\mathscr{L}'_p$. Then, by using (C.2.12), we find

$$x_jw=x_jRD_j^m v\prod_{k=0}^{m-1}(A_j-k)^{-1}$$

$$=Px_j^m D_j^m v \prod_{k=0}^{m-1} (A_j-k)^{-1}$$
$$=Pv=u\,.$$

This means that x_j defines a surjective map.

In order to prove the injectivity of x_j, we prepare the following

Sublemma C.2.2. *Let* $P \in (\mathscr{D}\otimes V)_p$. *Then* $P \in \sum_{j=1}^{l} (\mathscr{D}\otimes V)_p(x_jD_j-A_j)+\sum_{j=l+1}^{n}(\mathscr{D}\otimes V)_pD_j$ *if and only if* $Px_1^{A_1}\cdots x_l^{A_l}=0$.

Proof. Set $J=\sum_{j=1}^{l}(\mathscr{D}\otimes V)_p(x_jD_j-A_j)+\sum_{j=l+1}^{n}(\mathscr{D}\otimes V)_pD_j$ and $J'=\{P \in (\mathscr{D}\otimes V)_p,\ Px_1^{A_1}\cdots x_l^{A_l}=0\}$. It is easy to see that $J\subset J'$. We shall prove the converse inclusion relation. Let P be an element of J'. Then we can write

$$P=\sum_{\alpha\in \mathbf{Z}_+^n} b_\alpha(x)D^\alpha \quad \text{with} \quad b_\alpha(x)\in \mathcal{O}_p\otimes V.$$

Then there is $a_\beta(x)\in \mathcal{O}_p\otimes V$ such that

$$P\equiv \sum_{\beta\in \mathbf{Z}_+^l} a_\beta(x)D^\beta \mod J\,,$$

where $D^\beta=\partial^{|\beta|}/\partial x_1^{\beta_1}\cdots\partial x_l^{\beta_l}$ and $a_\beta(x)$ satisfies

$$(\partial/\partial x_j)a_\beta=0 \quad \text{if} \quad \beta_j\neq 0\,. \tag{C.2.13}$$

Set $A_\beta=\prod_{j=1}^{l}\prod_{k=0}^{\beta_j-1}(A_j-k)$. Then $Px_1^{A_1}\cdots x_l^{A_l}=0$ implies

$$\sum_{\beta\in \mathbf{Z}_+^l} a_\beta(x)A_\beta x^{-\beta}=0\,, \tag{C.2.14}$$

where $x^{-\beta}=x_1^{-\beta_1}\cdots x_l^{-\beta_l}$. Hence it is sufficient to show that (C.2.13) and (C.2.14) implies $a_\beta(x)=0$ for any β.

It is easy to check that (C.2.13) and (C.2.14) imply $a_\beta(x)A_\beta=0$ for any β. Since A_β is invertible, we obtain $a_\beta=0$. Thus Sublemma C.2.2 is proved.

Now we resume the proof of Lemma C.2.1. Sublemma C.2.2 shows that $\mathscr{L}'_p$ is contained in

$$(\mathscr{H}^0_{[X|Y]}(\mathcal{O})\otimes V)x_1^{A_1}\cdots x_l^{A_l}\,.$$

Hence x_j: $\mathscr{L}'_p\to\mathscr{L}'_p$ is injective. Thus Lemma C.2.1 is proved.

Lemma C.2.1 shows $\mathscr{L}'=\mathscr{L}$ and we obtain

$$\mathscr{L}=\mathscr{D}\otimes V/\Big(\sum_{j=1}^{l}(\mathscr{D}\otimes V)(x_jD_j-A_j)+\sum_{j=l+1}^{n}(\mathscr{D}\otimes V)D_j\Big)\,. \tag{C.2.15}$$

Set $\mathscr{M}=\mathscr{D}\otimes V/(\mathscr{D}\otimes V)(x_1D_1-A_1)$. Then by using the Koszul complex, we have an exact sequence of $\mathscr{D}$-Modules

$$0\longleftarrow \mathscr{L}\longleftarrow \mathscr{M}\longleftarrow \mathscr{M}^{n-1}\longleftarrow \mathscr{M}^{\binom{n-1}{2}}\longleftarrow\cdots\longleftarrow \mathscr{M}^{n-1}\longleftarrow \mathscr{M}\longleftarrow 0.$$

We can easily verify $\mathscr{E}xt^j_{\mathscr{D}}(\mathscr{M}, \mathscr{O}_X)_p=0$ for any j. Thus, we obtain $\mathscr{E}xt^j_{\mathscr{D}}(\mathscr{L}, \mathscr{O}_X)_p=0$ for any j.

References

[1] Björk, J. E., *Rings of Differential Operators*, North Holland, Amsterdam-Oxford-New York, 1979.

[2] Bony, J. M. and Schapira, P., Propagation des singularité analytiques pour les solutions des équations aux dérivées partielles, *Ann. Inst. Fourier, Grenoble*, **26** (1976), 81–140.

[3] Deligne, P., *Equations differentielles à points singuliers réguliers*, *Lecture Notes in Math.*, **163**, Springer, Berlin-Heidelberg-New York, 1970.

[4] Grothendieck, A., *Elément de géométrie algébrique III* (Première partie), *Publ. Math. IHES*, **11** (1961).

[5] Hartshorne, R., *Residues and Duality*, *Lecture Notes in Math.*, **20**, Springer, Berlin-Heidelberg-New York, 1966.

[6] Kashiwara, M., On the maximally overdetermined system of linear differential equations, I, *Publ. RIMS, Kyoto Univ.*, **10** (1975), 563–579.

[7] ———, *B*-functions and holonomic systems, *Inventiones math.*, **38** (1976), 33–53.

[8] ———, On the holonomic systems of linear differential equations, II, *Inventiones math.*, **49** (1978), 121–135.

[9] ———, *Systèmes d'équations micro-différentielles*, Départment de Mathématiques, Université de Paris-Nord, 1978, (Notes by Teresa Monteiro Fernandes).

[10] Kashiwara, M. and Kawai, T., Micro-hyperbolic pseudo-differential operators I, *J. Math. Soc. Japan*, **27** (1975), 359–404.

[11] ———, Micro-local properties of $\prod_{j=1}^{n} f_{j+}^{s_j}$, *Proc. Japan Acad.*, **51** (1975), 270–272.

[12] ———, Finiteness theorem for holonomic systems of micro-differential equations, *Proc. Japan Acad.*, **52** (1976), 341–343.

[13] ———, Holonomic character and local monodromy structure of Feynman integrals, *Commun. math. Phys.*, **54** (1977), 121–134.

[14] ———, Some applications of boundary value problems for elliptic systems of linear differential equations, *Annals of Math. Studies*, **93**, Princeton Univ. Press, Princeton, 1979, 39–61.

[15] ———, On holonomic systems with regular singularities, *Ibid.*, 113–121.

[16] Kashiwara, M., Kawai, T. and Oshima, T., Structure of cohomology groups whose coefficients are microfunction solution sheaves of systems of pseudo-differential equations with multiple characteristics, I, *Proc. Japan. Acad.*, **50** (1974), 420–425.

[17] Kashiwara, M., Kawai, T. and Sjöstrand, J., On a class of linear partial differential equations whose formal solutions always converge, *Ark. för Mat.*, **17** (1979), 83–91.

[18] Kashiwara, M. and Oshima, T., Systems of differential equations with regular singularities and their boundary value problems, *Ann. of Math.*, **106** (1977), 145–200.

[19] Kashiwara, M. and Schapira, P., Micro-hyperbolic systems, *Acta Math.*, **142** (1979), 1–55.

[20] Komatsu, H., *Relative cohomology of sheaves of solutions of differential equations, Lecture Notes in Math.*, **287**, Springer, Berlin-Heidelberg-New York, 1973.

[21] Malgrange, B., Sur les points singuliers des équations differentielles, *Enseignement Math.*, **20** (1974), 147–176.

[22] Mebkhout, Z., Local cohomology of analytic spaces, *Publ. RIMS, Kyoto Univ.*, **12 Suppl.** (1977), 247–256.

[23] Ramis, J. P., Variations sur le theme "GAGA", *Lecture Notes in Math.*, **694**, Springer, Berlin-Heidelberg-New York, 1978, 228–289.

[24] Sato, M., Kawai, T., and Kashiwara, M., Microfunctions and pseudo-differential equations, *Lecture Notes in Math.*, **287**, Springer, Berlin-Heidelberg-New York, 1973, 265–529.

[25] Verdier, J. L., Exposé 7 de *Séminaire de Geometrie Analytique de l'Ecole Normale Supérieure* (1974/75) par A. Douady et J. L. Verdier, Ecole Normale Supérieure, Paris, 1976.

[26] Aroca, J. M., Hironaka, H. and Vicente, J. L., *Desingularization Theorems*, Memorias de Matemática del Instuto Jorge Juan, **30**, Consejo Superior de Investigaciones Cientificas, Madrid, 1977.

[27] Bass, H., *Algebraic K-Theory*, Benjamin, New York-Amsterdam, 1968.

[28] Casselman, W., *Systems of analytic partial differential equations of finite codimension*, to appear.