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The First Cohomology Groups of Infinite Dimensional Lie Algebras[1)]

By

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Introduction

Let V be a finite dimensional vector space. We denote by $D(V)$ the Lie algebra consisting of all formal vector fields over V. Let L be a Lie subalgebra of $D(V)$. We are interested in the first cohomology group $H^1(L)$ of a Lie algebra L with adjoint representation.

Let L be an infinite dimensional transitive simple Lie algebra, that is, L is one of $D(V)$, $L_{\mathfrak{sl}}$, $L_{\mathfrak{sp}}$, or L_{ct}. (For a notation, see § 2.) It is known in T. Morimoto [5] that $H^1(D(V))=H^1(L_{ct})=0$, and $\dim H^1(L_{\mathfrak{sl}})=\dim H^1(L_{\mathfrak{sp}})=1$.

In this paper we will treat the following two types of infinite dimensional Lie algebras:

(1) Infinite dimensional transitive graded Lie algebras $\mathfrak{g}=\sum_{p=-1}^{\infty}\mathfrak{g}_p$. (For a precise definition, see § 1.)

(2) Infinite dimensional intransitive Lie algebras $L[W^*]$ whose transitive parts L are infinite and simple. (In this case W is a subspace of V.)

In Section 3 and Section 4, we will give two criteria for $H^1(\mathfrak{g})$ to be of finite dimension. More precisely we will prove

Theorem A. *Let $\mathfrak{g}=\sum_{p=-1}^{\infty}\mathfrak{g}_p$ be an infinite transitive graded Lie algebra with a semi-simple linear isotropy algebra $\mathfrak{g}_0$. Then $H^1(\mathfrak{g})$ is finite dimensional.*

Theorem B. *Let $\mathfrak{g}=\sum_{p=-1}^{\infty}\mathfrak{g}_p$ be an infinite transitive graded Lie algebra whose linear isotropy algebra $\mathfrak{g}_0$ contains an element e which satisfies $[e, x_p]=$*

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px_p for all $x_p \in \mathfrak{g}_p$. Then $H^1(\mathfrak{g})$ is finite dimensional. Furthermore if $\mathfrak{g}$ is derived from $\mathfrak{g}_0$, then $H^1(\mathfrak{g})$ is isomorphic to $n(\mathfrak{g}_0)/\mathfrak{g}_0$, where $n(\mathfrak{g}_0)$ denotes the normalizer of $\mathfrak{g}_0$ in $\mathfrak{gl}(\mathfrak{g}_{-1})$.

It may well be doubted if every infinite transitive graded Lie algebra $\mathfrak{g}$ has the finite dimensional cohomology group $H^1(\mathfrak{g})$. But unfortunately this presumption is false. In Section 5 we will give an easy condition for $\mathfrak{g}$ to be $\dim H^1(\mathfrak{g}) = \infty$. (For such a Lie algebra $\mathfrak{g}$, we can construct derivations of arbitrarily large negative degree.)

That is, we will prove

Theorem C. *Let $\mathfrak{g} = \sum_{p=-1}^{\infty} \mathfrak{g}_p$ be an infinite transitive graded Lie algebra which satisfies $\mathfrak{g}^{(2)} = [\mathfrak{g}^{(1)}, \mathfrak{g}^{(1)}] = 0$, where $\mathfrak{g}^{(1)} = [\mathfrak{g}, \mathfrak{g}]$. Then $H^1(\mathfrak{g})$ is infinite dimensional.*

In Section 6 our objects are infinite intransitive Lie algebras $L[W^*]$. Let $V = U + W$ (direct sum). We denote by $S(W^*)$ the ring of formal power series over W. Let L be an infinite transitive simple Lie algebra over U. Then a Lie algebra $L[W^*]$ is obtained as a topological completion of $L \otimes S(W^*)$. These Lie algebras $L[W^*]$ are obtained as the result of the classification theorem of infinite intransitive Lie algebras [6]. In determining $H^1(L[W^*])$, V. Guillemin's work is essential. Using his results we will prove

Theorem D. *Let $D(W)$ be a Lie algebra of all formal vector fields over W and let e be a basis of one dimensional center of $\mathfrak{gl}(U)$. Then we have*

$$H^1(L[W^*]) \cong \begin{cases} D(W) & \text{for } L = D(U) \text{ or } L_{ct}(U), \\ D(W) + S(W^*) \otimes e & \text{for } L = L_{\mathfrak{sl}}(U) \text{ or } L_{\mathfrak{sp}}(U). \end{cases}$$

Above results can be considered as a formal version of Y. Kanie [3]. In a forthcoming paper, we will give an example of an infinite intransitive Lie algebra L such that $H^1(L) = 0$.

Throughout this paper, all vector spaces and Lie algebras are assumed to be defined over the field $\mathbb{C}$ of complex numbers.

§1. Infinite Transitive Graded Lie Algebras

In this section, we define transitive graded Lie algebras which we will study in the subsequent sections.

Definition 1.1. Let $\mathfrak{g}$ be a Lie algebra. Assume that there is given a family $\{\mathfrak{g}_p\}_{p\geqq -1}$ of subspaces of $\mathfrak{g}$ which satisfies the following conditions:

a) $\mathfrak{g}=\sum_{p=-1}^{\infty}\mathfrak{g}_p$ (direct sum);

b) $\dim \mathfrak{g}_p<\infty$;

c) $[\mathfrak{g}_p, \mathfrak{g}_q]\subset \mathfrak{g}_{p+q}$;

d) For every non-zero $x_p\in \mathfrak{g}_p$, $p\geqq 0$, there is an element $x_{-1}\in \mathfrak{g}_{-1}$ such that $[x_p, x_{-1}]\neq 0$. Under these conditions, we say that the direct sum $\mathfrak{g}=\sum_{p=-1}^{\infty}\mathfrak{g}_p$ or simply $\mathfrak{g}$ is *a transitive graded Lie algebra*.

By conditions c) and d), $\mathfrak{g}_0$ is considered as a Lie subalgebra of $\mathfrak{gl}(\mathfrak{g}_{-1})$. The Lie algebra $\mathfrak{g}_0$ is called *the linear isotropy algebra* of $\mathfrak{g}$. A graded Lie algebra $\mathfrak{g}$ is said to be *irreducible* if the representation of $\mathfrak{g}_0$ on the vector space $\mathfrak{g}_{-1}$ given by $[\mathfrak{g}_0, \mathfrak{g}_{-1}]\subset \mathfrak{g}_{-1}$ is irreducible.

Definition 1.2. The space $\mathfrak{g}_0^{(p)}$ which is called *the p-th prolongation* of $\mathfrak{g}_0$ is defined by

$$\mathfrak{g}_0^{(p)}=\mathfrak{g}_0\otimes S^p(\mathfrak{g}_{-1}^*)\cap \mathfrak{g}_{-1}\otimes S^{p+1}(\mathfrak{g}_{-1}^*),$$

where $S^p(g_{-1}^*)$ denotes the p-times symmetric tensor of the dual space $\mathfrak{g}_{-1}^*$ of $\mathfrak{g}_{-1}$.

We say that $\mathfrak{g}_0$ is of *finite type* if $\mathfrak{g}_0^{(p)}=0$ for some (and hence for all larger) p. Otherwise we say that $\mathfrak{g}_0$ is of *infinite type*. Put $\mathfrak{g}_0^{(-1)}=\mathfrak{g}_{-1}$, $\mathfrak{g}_0^{(0)}=\mathfrak{g}_0$ and $\tilde{\mathfrak{g}}=\sum_{p=-1}^{\infty}\mathfrak{g}_0^{(p)}$. Then $\tilde{\mathfrak{g}}$ has a Lie algebra structure with respect to a canonical bracket operation. We say that the transitive graded Lie algebra $\tilde{\mathfrak{g}}=\sum_{p=-1}^{\infty}\mathfrak{g}_0^{(p)}$ thus obtained is *derived* from $\mathfrak{g}_0$. If $\mathfrak{g}$ is an abstract transitive graded Lie algebra with a linear isotropy algebra $\mathfrak{g}_0$, then $\mathfrak{g}$ is considered as a graded Lie subalgebra of $\tilde{\mathfrak{g}}$. It is clear that if a transitive graded Lie algebra $\mathfrak{g}$ is of infinite dimension, its linear isotropy algebra $\mathfrak{g}_0$ must be of infinite type.

Let A be a Lie algebra. A derivation c of A is a linear mapping from A to itself satisfying $c[x, y]=[c(x), y]+[x, c(y)]$ for all $x, y\in A$. We denote by $\mathrm{Der}(A)$ (resp. $\mathrm{ad}(A)$) the derivation algebra (resp. the algebra of inner derivations of A). Then, by definition, the first cohomology group $H^1(A)$ of A with adjoint representation is equal to the space $\mathrm{Der}(A)/\mathrm{ad}(A)$. A derivation c of a graded Lie algebra $\mathfrak{g}=\sum_{p=-1}^{\infty}\mathfrak{g}_p$ is said to be of degree r or $\deg c=r$ if it satisfies $c(\mathfrak{g}_p)\subset \mathfrak{g}_{p+r}$ for all p.

§2. Infinite Transitive Simple Lie Algebras

It is well-known that there are the following four classes of infinite transitive simple Lie algebras over $\boldsymbol{C}$ (see [5]).

(1) $L_{\mathfrak{gl}}(n)$: the Lie algebra of all formal (or better, formal power series) vector fields in n-variables $x_1, x_2,\dots, x_n$.

(2) $L_{\mathfrak{sl}}(n)$: the Lie algebra of formal vector fields in n-variables $x_1, x_2,\dots, x_n$, preserving the volume form $dx_1 \wedge dx_2 \wedge \cdots \wedge dx_n$.

(3) $L_{\mathfrak{sp}}(2n)$: the Lie algebra of formal vector fields in $2n$-variables $x_1, x_2,\dots, x_n, y_1, y_2,\dots, y_n$, preserving the symplectic form $\sum_{i=1}^{n} dx_i \wedge dy_i$.

(4) $L_{\mathfrak{ct}}(2n+1)$: the Lie algebra of formal vector fields in $(2n+1)$-variables $z, x_1, x_2,\dots, x_n, y_1, y_2,\dots, y_n$, preserving the contact form $dz+2^{-1}\sum_{i=1}^{n} x_i\cdot dy_i - y_i dx_i$, up to functional factors.

We will often write $D(V)$ for $L_{\mathfrak{gl}}(n)$, where V is an n-dimensional vector space with a basis $\partial/\partial x_1,\dots, \partial/\partial x_n$. Let L be one of Lie algebras $D(V)$, $L_{\mathfrak{sl}}$ and $L_{\mathfrak{sp}}$. Each L has the natural filtration $\{L_p\}_{p\in \boldsymbol{Z}}$ defined as follows.

$$L_p = L \quad \text{for} \quad p \leqq -1;$$
$$L_0 = \{X \in L;\ \text{the value } X(0) \text{ of } X \text{ at the origin} = 0\};$$
$$L_p = \{X \in L_{p-1};\ [X, L] \subset L_{p-1}\} \quad \text{for} \quad p \geqq 1.$$

Then the decreasing sequence of subspaces: $L = L_{-1} \supset L_0 \supset L_1 \supset L_2 \supset \cdots$ satisfies

(a) $\bigcap_{p=-1}^{\infty} L_p = 0$;

(b) $[L_p, L_q] \subset L_{p+q}$;

(c) $\dim L_p/L_{p+1} < \infty$.

Put $\mathfrak{g}_p(L) = L_p/L_{p+1}$. Then by (b), (c) and the definition of L_p, $p \geqq 1$, we have the transitive graded Lie algebra $\mathrm{gr}(L) = \sum_{p=-1}^{\infty} \mathfrak{g}_p(L)$. We also have the Lie algebra $L' = \prod_{p=-1}^{\infty} \mathfrak{g}_p(L)$, which is the completion of $\mathrm{gr}(L)$.

Under these notations we will summarize a few useful properties of L. (For the proof, see Kobayashi-Nagano [4].)

(1) Each L is an infinite transitive irreducible Lie algebra and moreover L is isomorphic to L', where the word "irreducible" means that the action of $\mathfrak{g}_0(L)$ on $\mathfrak{g}_{-1}(L)$ is irreducible.

(2) The linear isotropy algebras $\mathfrak{g}_0(L)$ of $D(V)$, $L_{\mathfrak{sl}}$ and $L_{\mathfrak{sp}}$ are $\mathfrak{gl}(n, \boldsymbol{C})$,

$\mathfrak{sl}(n, \boldsymbol{C})$ and $\mathfrak{sp}(n, \boldsymbol{C})$ respectively, and for $p \geqq 1$ $\mathfrak{g}_p(L)$ is isomorphic to the p-th prolongation $\mathfrak{g}_0(L)^{(p)}$ of $\mathfrak{g}_0(L)$.

(3) For $\mathfrak{g}_0(L) = \mathfrak{sl}(n, \boldsymbol{C})$ or $\mathfrak{sp}(n, \boldsymbol{C})$, it holds that

(i) $[\mathfrak{g}_0(L)^{(r)}, \mathfrak{g}_0(L)^{(s)}] = \mathfrak{g}_0(L)^{(r+s)}$ for $r, s \geqq 0$,

(ii) $\mathfrak{g}_0(L)$ acts irreducibly on $\mathfrak{g}_0(L)^{(r)}$ for $r \geqq -1$.

By the classification theorem of Kobayashi-Nagano [4], we know that there are only three classes of transitive simple irreducible Lie algebras of infinite type over $\boldsymbol{C}$, that is, they are $D(V)$, $L_{\mathfrak{sl}}$ and $L_{\mathfrak{sp}}$.

For the contact Lie algebra $L_{ct}(2n+1)$ (or simply L_{ct}), we must define another filtration.

$$
\begin{aligned}
&L_p = L_{ct} \quad \text{for} \quad p \leqq -2; \\
&L_{-1} = \{X \in L_{ct};\ \langle X, \theta \rangle_0 = 0, \quad \text{where } \theta \text{ is the contact form}\}; \\
&L_0 = \{X \in L_{ct};\ X(0) = 0\}; \\
&L_p = \{X \in L_{p-1};\ [X, L_{-1}] \subset L_{p-1}\} \quad \text{for} \quad p \geqq 1.
\end{aligned}
$$

Using this filtration, L_{ct} is isomorphic to $\prod_{p=-2}^{\infty} \mathfrak{g}_p(L)$. For the subsequent discussion about L_{ct}, we have only to recall that

(4) $L_{-1} = [L_{ct}, L_1]$.

In Section 3 and Section 6, we essentially use the following facts which were proved by T. Morimoto [5].

Theorem 2.1. *Let L be an infinite transitive simple Lie algebra over $\boldsymbol{C}$. Then*

$$
H^1(L) \cong \begin{cases} 0 & \text{for } L = D(V) \text{ or } L_{ct} \\ \boldsymbol{C} & \text{for } L = L_{\mathfrak{sl}} \text{ or } L_{\mathfrak{sp}}. \end{cases}
$$

Remark 1. Let L be one of Lie algebras $L_{\mathfrak{sl}}$ or $L_{\mathfrak{sp}}$. Since L is isomorphic with the Lie algebra $L' = \prod_{p=-1}^{\infty} \mathfrak{g}_p(L)$, their isotropy algebras $\mathfrak{sl}(V)$ and $\mathfrak{sp}(V)$ are considered to be subalgebras of them. Let e denote a unit matrix in $\mathfrak{gl}(V)$. Then the above theorem asserts that $\mathrm{ad}(e)$ yields a basis of one dimensional space $H^1(L)$.

Remark 2. Let $\mathrm{gr}(L)$ be a graded Lie algebra associated with an infinite transitive simple Lie algebra L. Then we also have $H^1(\mathrm{gr}(L)) = 0$ for $L = D(V)$ or L_{ct}, and $H^1(\mathrm{gr}(L)) \cong \boldsymbol{C}$ for $L = L_{\mathfrak{sl}}$ or $L_{\mathfrak{sp}}$. These facts are particularly used in Section 3.

§3. The First Cohomology Groups of Infinite Transitive Graded Lie Algebras (I)

Throughout this section, let $\mathfrak{g}=\sum_{p=-1}^{\infty}\mathfrak{g}_p$ be an infinite (dimensional) transitive graded Lie algebra over C and let its linear isotropy algebra $\mathfrak{g}_0$ be semi-simple. Put $\mathfrak{g}_{-1}=V$. Then $\mathfrak{g}_0$ is considered as a Lie subalgebra of $\mathfrak{gl}(V)$. First we will determine the type of $\mathfrak{g}$.

Since $\mathfrak{g}_0$ is semi-simple, we can decompose V into $V=V_1+V_2+\cdots+V_k$ (vector space direct sum), where each V_i $(i=1, 2,\dots, k)$ is a $\mathfrak{g}_0$-invariant subspace and $\mathfrak{g}_0$ acts irreducibly on V_i. We denote by $\mathfrak{h}_i$ the Lie algebra of linear transformations of V_i induced by $\mathfrak{g}_0$. By the natural inclusion, $\mathfrak{h}_i$ is considered as a Lie subalgebra of $\mathfrak{gl}(V)$. We also denote it by the same letter $\mathfrak{h}_i$ if there is no confusion. Put $\mathfrak{n}_i=\{t\in\mathfrak{g}_0;\ t(V_j)=0 \text{ for all } j\neq i\}$. Then each $\mathfrak{n}_i$ is an ideal of $\mathfrak{g}_0$ and $\mathfrak{n}_1+\cdots+\mathfrak{n}_k$ is a direct sum as Lie algebras. It clearly holds

$$\mathfrak{n}_1+\cdots+\mathfrak{n}_k\subset\mathfrak{g}_0\subset\mathfrak{h}_1+\cdots+\mathfrak{h}_k. \tag{3.1}$$

Lemma 3.1. *For $p\geqq 1$, $\mathfrak{g}_0^{(p)}=\mathfrak{n}_1^{(p)}+\cdots+\mathfrak{n}_k^{(p)}$ (direct sum).*

Proof. Let $t\colon V\times\cdots\times V\to V$ be an element of $\mathfrak{g}_0^{(p)}$. First note that $t(v_1,\dots, v_{p+1})=0$ if $v_i\in V_i$, $v_j\in V_j$ for $i\neq j$. (It is easy to see that $t(v_1,\dots, v_{p+1})\in V_i\cap V_j$.) Let $v_1,\dots, v_{p+1}\in V$ and $v_i=v_i^1+\cdots+v_i^k$ with $v_i^1\in V_1$, $v_i^2\in V_2,\dots, v_i^k\in V_k$ for $i=1,\dots, p+1$. Then by the above remark, we have

$$\begin{aligned} t(v_1,\dots, v_{p+1}) &= t(v_1^1,\dots, v_{p+1}^1)+\cdots+t(v_1^k,\dots, v_{p+1}^k)\\ &= t_1(v_1^1,\dots, v_{p+1}^1)+\cdots+t_k(v_1^k,\dots, v_{p+1}^k)\\ &= t_1(v_1,\dots, v_{p+1})+\cdots+t_k(v_1,\dots, v_{p+1}), \end{aligned} \tag{3.2}$$

where t_i denotes an element of $\mathfrak{n}_i^{(p)}$ induced by t. (Since $t_i(*, v_1',\dots, v_p')\in\mathfrak{n}_i$ for $v_1',\dots, v_p'\in V$, t_i is an element of $\mathfrak{n}_i^{(p)}$.) Since $\mathfrak{n}_1+\cdots+\mathfrak{n}_k$ is a direct sum, our assertion is obvious. q.e.d.

Since $\mathfrak{g}$ is infinite dimensional and $\mathfrak{g}_p$ is a subspace of $\mathfrak{g}_0^{(p)}$, $\mathfrak{g}_0$ must be of infinite type by Lemma 3.1. From now on, without loss of generality, we assume that $\mathfrak{n}_1,\dots, \mathfrak{n}_l$ $(l\leqq k)$ are of infinite type and $\mathfrak{n}_{l+1},\dots, \mathfrak{n}_k$ are of finite type.

Lemma 3.2. *Let $\mathfrak{g}_0$ be a linear isotropy algebra of an infinite transitive graded Lie algebra $\mathfrak{g}$. Then there exists a Lie subalgebra $\mathfrak{g}_b$ of finite type of $\mathfrak{h}_{l+1}+\cdots+\mathfrak{h}_k$ and $\mathfrak{g}_0$ is written as*

(3.3) $\mathfrak{g}_0=\mathfrak{n}_1+\cdots+\mathfrak{n}_l+\mathfrak{g}_b$ *(Lie algebra direct sum)*,

where each ideal $\mathfrak{n}_i$ $(i=1,\dots,l)$ *is isomorphic to either* $\mathfrak{sl}(V_i)$ *or* $\mathfrak{sp}(V_i)$.

Proof. Let $\pi: \mathfrak{g}_0\to\mathfrak{h}_i$ $(i=1,\dots,l)$ be a natural projection. Since π is a Lie algebra homomorphism, $\mathfrak{g}_0/\mathrm{Ker}\,\pi$ is isomorphic to $\mathfrak{h}_i$. Recall that the quotient space of a semi-simple Lie algebra is also semi-simple. Thus $\mathfrak{h}_i$ is semi-simple and its center is zero. Moreover each $\mathfrak{h}_i$ acts irreducibly on V_i and is of infinite type. Hence by the classification theorem of transitive irreducible Lie algebras of infinite type, we know that $\mathfrak{h}_i$ must be equal to either $\mathfrak{sl}(V_i)$ or $\mathfrak{sp}(V_i)$. Since $\mathfrak{n}_i$ is an ideal of $\mathfrak{h}_i$, we have $\mathfrak{n}_i=\mathfrak{sl}(V_i)$ or $\mathfrak{sp}(V_i)$. ($\mathfrak{sl}(V_i)$ and $\mathfrak{sp}(V_i)$ are naturally imbedded in $\mathfrak{gl}(V)$.) Note that $\mathfrak{n}_1=\mathfrak{h}_1, \mathfrak{n}_2=\mathfrak{h}_2,\dots,\mathfrak{n}_l=\mathfrak{h}_l$. Then we can find a subspace $\mathfrak{g}_b$ such that $\mathfrak{n}_{l+1}+\cdots+\mathfrak{n}_k\subset\mathfrak{g}_b\subset\mathfrak{h}_{l+1}+\cdots+\mathfrak{h}_k$. Considering (3.1), we obtain the expression of $\mathfrak{g}_0$ as (3.3). By Lemma 3.1, we also have $\mathfrak{g}_b^{(p)}=\mathfrak{n}_{l+1}^{(p)}+\cdots+\mathfrak{n}_k^{(p)}$. Thus $\mathfrak{g}_b$ is of finite type. q.e.d.

Next we will determine the type of $\mathfrak{g}_1$. From (3.3) in Lemma 3.2, we have $\mathfrak{g}_0^{(1)}=\mathfrak{n}_1^{(1)}+\cdots+\mathfrak{n}_l^{(1)}+\mathfrak{g}_b^{(1)}$, and $\mathfrak{g}_1$ is a subspace of $\mathfrak{g}_0^{(1)}$. Without loss of generality, we assume that $\mathfrak{g}_1\cap\mathfrak{n}_1^{(1)}\neq 0,\dots,\mathfrak{g}_1\cap\mathfrak{n}_m^{(1)}\neq 0$ and $\mathfrak{g}_1\cap\mathfrak{n}_{m+1}^{(1)}=0,\dots,\mathfrak{g}_1\cap\mathfrak{n}_l^{(1)}=0$. Then we have

Lemma 3.3. $\mathfrak{g}_1$ *has the following form*:

$$\mathfrak{g}_1=\mathfrak{n}_1^{(1)}+\cdots+\mathfrak{n}_m^{(1)}+H_1,$$

where H_1 *is a subspace of* $\mathfrak{g}_b^{(1)}$.

Proof. For $i=1,\dots,m$, $\mathfrak{g}_1\cap\mathfrak{n}_i^{(1)}$ is an n_i-invariant subspace of $\mathfrak{n}_i^{(1)}$. By the property (3) (ii) in Section 2, we have $\mathfrak{g}_1\supset\mathfrak{n}_1^{(1)}+\cdots+\mathfrak{n}_m^{(1)}$. Hence there exists a subspace H_1 of $\mathfrak{n}_{m+1}^{(1)}+\cdots+\mathfrak{n}_l^{(1)}+\mathfrak{g}_b^{(1)}$ such that $\mathfrak{g}_1=\mathfrak{n}_1^{(1)}+\cdots+\mathfrak{n}_m^{(1)}+H_1$ and $H_1\cap\mathfrak{n}_{m+1}^{(1)}=0,\dots,H_1\cap\mathfrak{n}_l^{(1)}=0$. For $j=m+1,\dots,l$, decompose $t\in H_1$ into $t=t_{m+1}+\cdots+t_l+t_b$ with $t_{m+1}\in\mathfrak{n}_{m+1}^{(1)},\dots,t_l\in\mathfrak{n}_l^{(1)}$, $t_b\in\mathfrak{g}_b^{(1)}$. Define a subspace A_j of $\mathfrak{n}_j^{(1)}$ by

$$A_j=\{t_j\in\mathfrak{n}_j^{(1)};\ t=t_{m+1}+\cdots+t_l+t_b\in H_1\}.$$

For all $x_j\in\mathfrak{n}_j$ and $t_j\in A_j$, it holds that $[x_j, t_j]=[x_j, t]\in\mathfrak{n}_j^{(1)}\cap\mathfrak{g}_l=\{0\}$. This means that $[\mathfrak{n}_j, A_j]=0$. Using the property (3) (ii), we have $A_j=0$ for $j=m+1,\dots,l$. Hence $H_1\subset\mathfrak{g}_b^{(1)}$. q.e.d.

Since $\mathfrak{g}_p\supset[\mathfrak{g}_1,[\mathfrak{g}_1,[\dots,[\mathfrak{g}_1,\mathfrak{g}_1]\dots]$ for $p>1$, $\mathfrak{g}_p$ contains $\mathfrak{n}_1^{(p)}+\cdots+\mathfrak{n}_m^{(p)}$ by Lemma 3.3 and the property (3) (i) in Section 2. By the same argument as Lemma 3.3, we get

Lemma 3.4. *For $p>1$, $\mathfrak{g}_p$ has the following form:*

$$\mathfrak{g}_p=\mathfrak{n}_1^{(p)}+\cdots+\mathfrak{n}_m^{(p)}+H_p,$$

where H_p is a subspace of $\mathfrak{g}_b^{(p)}$. (For sufficiently large p, $H_p=0$ since $\mathfrak{g}_b$ is of finite type.)

By Lemma 3.3 and Lemma 3.4, we can easily determine the form of the given infinite transitive graded Lie algebra $\mathfrak{g}=\sum_{p=-1}^{\infty}\mathfrak{g}_p$. That is, we have

Proposition 3.5. *Let $\mathfrak{g}=\sum_{p=-1}^{\infty}\mathfrak{g}_p$ be an infinite transitive graded Lie algebra. Then $\mathfrak{g}$ has the following form:*

$$\mathfrak{g}=G_1+\cdots+G_m+G'_{m+1}+\cdots+G'_l+G'_b \quad (\textit{direct sum}),$$

where G_i $(i=1,\dots,m)$ is of the form $\mathfrak{gr}(L_{\mathfrak{sl}}(V_i))$ or $\mathfrak{gr}(L_{\mathfrak{sp}}(V_i))$, and G'_j $(j=m+1,\dots,l)$ is of the form $V_j+\mathfrak{sl}(V_j)$ or $V_j+\mathfrak{sp}(V_j)$, and G'_b is a finite dimensional Lie algebra. (From now on, we put $G'=G'_{m+1}+\cdots+G'_l+G'_b$. Then G' is a finite dimensional ideal of $\mathfrak{g}$.)

For computing $H^1(\mathfrak{g})$, we need some lemmas.

Lemma 3.6. *Let A be an abstract Lie algebra and let A_i $(i=1,\dots,k)$ be perfect ideals of A. If $A=A_1+\cdots+A_k$ (direct sum), then $H^1(A)\cong H^1(A_1)+\cdots+H^1(A_k)$ (direct sum).*

Proof. Let $c\in \mathrm{Der}(A)$. We denote by c_{ij} the $\mathrm{Hom}(A_i, A_j)$-component of c. For $x, y\in A_i$, we have

$$\begin{aligned}(3.4)\qquad c[x, y]&=[c(x), y]+[x, c(y)]\\ &=\sum_{j=1}^{k}[c_{ij}(x), y]+\sum_{j=1}^{k}[x, c_{ij}(y)]\\ &=[c_{ii}(x), y]+[x, c_{ii}(y)]\in A_i.\end{aligned}$$

Combined this with $A_i=[A_i, A_i]$, we obtain $c_{ij}=0$ for $i\neq j$. Put $c_{ii}=c_i$. By (3.4), c_i induces a derivation of A_i. Hence $\mathrm{Der}(A)=\mathrm{Der}(A_1)+\cdots+\mathrm{Der}(A_k)$ (direct sum). Our assertion is now evident. q.e.d.

Lemma 3.7. *Let A be an abstract Lie algebra such that $A=A_1+A_2$ (direct sum) with $A_1=[A_1, A_1]$. Moreover assume that the center of A_1 is zero. Then $H^1(A)\cong H^1(A_1)+H^1(A_2)$ (direct sum).*

Proof. We can write $c=c_{11}+c_{12}+c_{21}+c_{22}$ by using same notations as Lemma 3.6. Since A_1 is perfect, we have $c_{12}=0$. Let $x\in A_1$ and $y\in A_2$. By the equation $0=c[x, y]=[c(x), y]+[x, c(y)]$, we get $[x, c_{21}(y)]=0$. This means that $c_{21}(y) \in \{\text{center of } A_1\}$. Since the center of A_1 is zero, we have $c_{21}=0$. Now it is easy to verify the assertion. q.e.d.

Combined with Theorem 2.1, we obtain finally the following theorem.

Theorem 3.8. *Let* $\mathfrak{g}=\sum_{p=-1}^{\infty}\mathfrak{g}_p$ *be an infinite transitive graded Lie algebra with a semi-simple linear isotropy algebra* $\mathfrak{g}_0$. *Then* $H^1(\mathfrak{g})$ *is finite dimensional.*

Proof. By Proposition 3.5, $\mathfrak{g}$ has the following form: $\mathfrak{g}=G_1+\cdots+G_m+G'$, where $\dim G'<\infty$. Since $G_1+\cdots+G_m$ is perfect and has no non-trivial center, we have $H^1(\mathfrak{g})\cong H^1(G_1+\cdots+G_m)+H^1(G')$ by Lemma 3.7. On the other hand, $H^1(G_1+\cdots+G_m)\cong H^1(G_1)+\cdots+H^1(G_m)$ by Lemma 3.6, and $\dim H^1(G_i)=1$ or 0 for $i=1,\dots, m$ by Theorem 2.1. (See also Remark 2, and recall that $G_i=\mathrm{gr}(L_{\mathfrak{sl}}(V_i))$ or $\mathrm{gr}(L_{\mathfrak{sp}}(V_i))$.) Thus we obtain $\dim H^1(\mathfrak{g})<\infty$.

q.e.d.

§4. The First Cohomology Groups of Infinite Transitive Graded Lie Algebras (II)

In this section, we assume that the linear isotropy algebra $\mathfrak{g}_0$ of $\mathfrak{g}=\sum_{p=-1}^{\infty}\mathfrak{g}_p$ contains an element e which satisfies $[e, x_p]=px_p$ for all $x_p\in\mathfrak{g}_p$. Put $\mathfrak{g}_{-1}=V$. We can write $c(e)=\sum_{p=-1}^{\infty}x_p$ with $x_p\in\mathfrak{g}_p$. For all $v\in V$, we have

$$[c(e), v]+[e, c(v)]=c[e, v]=-c(v).$$

Comparing the V-components of this equation, we obtain $[x_0, v]=0$, and hence $x_0=0$ by the transitivity condition of $\mathfrak{g}$. We now define a new derivation c' derived from c by

$$c'=c+\mathrm{ad}\Big(\sum_{p\neq 0}\frac{1}{p}x_p\Big). \tag{4.1}$$

It is clear that $c'(e)=0$.

Lemma 4.1. $\deg c'=0$. (*For the definition of "degree" of a derivation, see* § 1.)

Proof. We must show that $c'(\mathfrak{g}_p)\subset\mathfrak{g}_p$ for all $p\geqq -1$. Put $c'(x)=\sum_{q=-1}^{\infty}y_q$ ($y_q\in\mathfrak{g}_q$) for $x\in\mathfrak{g}_p$. Then we have

$$c'[e, x]=pc'(x)=p\sum_{q=-1}^{\infty}y_q=[e, c'(x)]=\sum_{q=-1}^{\infty}qy_q.$$

Hence $y_q=0$ for $q\neq p$ and thus $c'(x)=y_p\in\mathfrak{g}_p$. q.e.d.

Lemma 4.2. *If $c'=0$ on V, then $c'=0$ on $\mathfrak{g}$.*

Proof. For $x\in\mathfrak{g}_0$ and $v\in V$, it holds that $[c'(x), v]+[x, c'(v)]=c'[x, v]$. By the assumption of c', we obtain $[c'(x), v]=0$. Combining $c'(\mathfrak{g}_0)\subset\mathfrak{g}_0$ with the transitivity of $\mathfrak{g}$, we obtain $c'(x)=0$. Repeating this procedure for all $p\geqq 1$, we can also obtain that $c'(\mathfrak{g}_p)=0$. Hence $c'=0$ on $\mathfrak{g}$. q.e.d.

Let $[c]\in H^1(\mathfrak{g})$ denote an equivalence class of a derivation c of $\mathfrak{g}$. Since $c'=c+\mathrm{ad}\left(\sum_{p\neq 0}\frac{1}{p}x_p\right)$, we have $[c]=[c']$. By Lemma 4.1, a restriction of c' to V is an element of $\mathfrak{gl}(V)$. We denote this linear mapping by $c'|_V$.

Theorem 4.3. *Let $\mathfrak{g}=\sum_{p=-1}^{\infty}\mathfrak{g}_p$ be an infinite transitive graded Lie algebra whose linear isotropy algebra $\mathfrak{g}_0$ contains an element e which satisfies $[e, x_p]=px_p$ for all $x_p\in\mathfrak{g}_p$. Then $\dim H^1(\mathfrak{g})\leqq(\dim V)^2$.*

Proof. Let c be any derivation of $\mathfrak{g}$. We define a linear mapping ψ: $\mathrm{Der}(\mathfrak{g})\to\mathfrak{gl}(V)$ by $\psi(c)=c'|_V$. By Lemma 4.2, we obtain that if c is contained in $\mathrm{Ker}\,\psi$, then c is an inner derivation. Hence our assertion is obvious.

q.e.d.

In case that $\mathfrak{g}$ is derived from $\mathfrak{g}_0$, we can get the more precise result. Let $n(\mathfrak{g}_0)$ denote the normalizer of $\mathfrak{g}_0$ in $\mathfrak{gl}(V)$. Then we have

Lemma 4.4. *Let $\mathfrak{g}$ be a Lie algebra derived from $\mathfrak{g}_0$. Then for all $x\in n(\mathfrak{g}_0)$, $\mathrm{ad}(x)$ is a derivation of $\mathfrak{g}$.*

Proof. It is sufficient to prove that $\mathrm{ad}(x)(\mathfrak{g}_0^{(p)})\subset\mathfrak{g}_0^{(p)}$ for all $p\geqq 1$. Let $z\in\mathfrak{g}_0^{(1)}$ and $v\in V$. With respect to the bracket operation in $D(V)$, we have

$$[\mathrm{ad}(x)(z), v]=\mathrm{ad}(x)[z, v]+[z, [v, x]]\in\mathfrak{g}_0.$$

Hence we have $\mathrm{ad}(x)(z)\in\mathfrak{g}_0^{(1)}$, that is, $\mathrm{ad}(x)(\mathfrak{g}_0^{(1)})\subset\mathfrak{g}_0^{(1)}$. Since $\mathfrak{g}_0^{(p+1)}=(\mathfrak{g}_0^{(p)})^{(1)}$, it can be inductively proved that $\mathrm{ad}(x)(\mathfrak{g}_0^{(p)})\subset\mathfrak{g}_0^{(p)}$ for all $p\geqq 1$. q.e.d.

Theorem 4.5. *Let $\mathfrak{g}$ be an infinite transitive graded Lie algebra derived from $\mathfrak{g}_0$. Moreover assume that $\mathfrak{g}_0$ contains an element e which satisfies $[e, x_p]=px_p$ for all $x_p\in\mathfrak{g}_p$. Then $H^1(\mathfrak{g})$ is isomorphic to $n(\mathfrak{g}_0)/\mathfrak{g}_0$.*

Proof. By Lemma 4.4, we can define a linear mapping f: $n(\mathfrak{g}_0)\to H^1(\mathfrak{g})$ by $f(x)=[\mathrm{ad}(x)]$. We prove that f is surjective. Let c be any derivation of $\mathfrak{g}$. Recall that $[c]=[c']$. Since c' satisfies $c'(V)\subset V$, there exists an element x of $\mathfrak{gl}(V)$ such that $c'=\mathrm{ad}(x)$ on V. Let $v\in V$ and $y\in\mathfrak{g}_0$. By the Jacobi identity in $D(V)$, we have

$$\mathrm{ad}(x)[v, y]=[\mathrm{ad}(x)(v), y]+[v, \mathrm{ad}(x)(y)]. \tag{4.2}$$

On the other hand, c' satisfies

$$c'[v, y]=[c'(v), y]+[v, c'(y)]. \tag{4.3}$$

Note that $\mathrm{ad}(x)[v, y]=c'[v, y]$ and $\mathrm{ad}(x)(v)=c'(v)$. From equations (4.2) and (4.3), it holds that $[v, (\mathrm{ad}(x)-c')(y)]=0$. By the transitivity condition of $\mathfrak{g}$, we obtain that $\mathrm{ad}(x)=c'$ on $\mathfrak{g}_0$ and hence $x\in n(\mathfrak{g}_0)$. By Lemma 4.4, $\mathrm{ad}(x)$ is a derivation of $\mathfrak{g}$, and $c'-\mathrm{ad}(x)$ vanishes on V. Now by Lemma 4.2, we clearly have $c'=\mathrm{ad}(x)$ on $\mathfrak{g}$. Thus we have proved that f is surjective. Since $\mathrm{Ker} f=\mathfrak{g}_0$, we obtain that $H^1(\mathfrak{g})$ is isomorphic to $n(\mathfrak{g}_0)/\mathfrak{g}_0$. q.e.d.

§5. Example of Infinite Transitive Graded Lie Algebra $\mathfrak{g}$ with $\dim H^1(\mathfrak{g})=\infty$

As stated in Introduction, we give an example of $\mathfrak{g}$ such that $H^1(\mathfrak{g})$ is of infinite dimension. Note that a derivation c of degree $\leqq -2$ is necessarily an outer derivation. We define a sequence of derived ideals $\mathfrak{g}^{(p)}$ of $\mathfrak{g}$ inductively by $\mathfrak{g}^{(1)}=[\mathfrak{g}, \mathfrak{g}],\dots, \mathfrak{g}^{(p)}=[\mathfrak{g}^{(p-1)}, \mathfrak{g}^{(p-1)}]$. Then we prove

Theorem 5.1. *Let $\mathfrak{g}=\sum_{p=-1}^{\infty}\mathfrak{g}_p$ be an infinite transitive graded Lie algebra which satisfies $\mathfrak{g}^{(2)}=0$. Then $\dim H^1(\mathfrak{g})=\infty$.*

Proof. Put $\varphi_k=\mathrm{ad}(v_1)\,\mathrm{ad}(v_2)\ldots\mathrm{ad}(v_k)$ for $v_1, v_2,\dots, v_k\in\mathfrak{g}_{-1}$. We show that φ_k ($k\geqq 1$) is a derivation of $\mathfrak{g}$ by induction. In the case of $k=1$, $\varphi_1=\mathrm{ad}(v_1)$ is an "inner" derivation. Let $k\geqq 1$. Assume that $\varphi_k[x, y]=[\varphi_k(x), y]+[x, \varphi_k(y)]$ for any $x, y\in\mathfrak{g}$. Put $\varphi_{k+1}=\varphi_k\circ\mathrm{ad}(v_{k+1})$ for $v_{k+1}\in\mathfrak{g}_{-1}$. Then by the Jacobi identity, we have

$$\varphi_{k+1}[x, y]=\varphi_k[v_{k+1}, [x, y]]=\varphi_k[[v_{k+1}, x], y]+\varphi_k[x, [v_{k+1}, y]].$$

By the assumption of induction and by $\mathfrak{g}^{(2)}=0$, this element is equal to

$$\begin{aligned}\varphi_k[[v_{k+1}, x], y]+\varphi_k[x, [v_{k+1}, y]]&=[\varphi_k[v_{k+1}, x], y]+[x, \varphi_k[v_{k+1}, y]]\\&=[\varphi_{k+1}(x), y]+[x, \varphi_{k+1}(y)].\end{aligned}$$

Hence φ_k is a derivation for all $k\geqq 1$. Now if $\varphi_k=\mathrm{ad}(v_1)\,\mathrm{ad}(v_2)\cdots\mathrm{ad}(v_k)=0$ on $\mathfrak{g}$ for all $v_1, v_2,\dots, v_k\in\mathfrak{g}_{-1}$, we would have $\underbrace{[\mathfrak{g}_{-1}, [\mathfrak{g}_{-1}, [\dots, [\mathfrak{g}_{-1},}_{k\text{-times}} \mathfrak{g}_k]\cdots]=0$. By the transitivity condition of $\mathfrak{g}$, we must have $\mathfrak{g}_k=0$. This is a contradiction.

Thus there exist $v_1, v_2, \ldots, v_k \in \mathfrak{g}_{-1}$ for arbitrarily large k such that $\varphi_k = \mathrm{ad}(v_1) \cdot \mathrm{ad}(v_2) \cdots \mathrm{ad}(v_k) \neq 0$. Since $\deg \varphi_k = -k$, φ_k is a non-trivial outer derivation of $\mathfrak{g}$, and hence $\dim H^1(\mathfrak{g}) = \infty$. q.e.d.

A typical example. Let $\mathfrak{g}_{-1}$ be a two dimensional vector space with a basis $\partial/\partial x$, $\partial/\partial y$, and let $\mathfrak{g}_p$ be a one dimensional vector space with a basis $x^{p+1}\partial/\partial y$ for $p \geqq 0$. Then we have an infinite transitive graded Lie algebra $\mathfrak{g} = \sum_{p=-1}^{\infty} \mathfrak{g}_p$, which satisfies $\mathfrak{g}^{(2)} = 0$. In this case, $\varphi_k = \underbrace{\mathrm{ad}(\partial/\partial x) \cdots \mathrm{ad}(\partial/\partial x)}_{k\text{-times}}$ are non-trivial derivations of $\mathfrak{g}$ for all $k \geqq 1$. Hence $H^1(\mathfrak{g})$ is of infinite dimension.

§ 6. The First Cohomology Groups of Infinite Intransitive Lie Algebras $L[W^*]$

6.1. First we explain a Lie algebra $L[W^*]$ which is a main object in this section. Let V be a finite dimensional vector space with $V = U + W$ (direct sum). We denote by $S(W^*)$ the ring of formal power series over W. Let L be an infinite transitive simple Lie algebra over U. Both L and $S(W^*)$ are complete topological vector spaces with respect to their natural topology induced by the filtrations. Then a Lie algebra $L[W^*]$ is obtained as a topological completion of $L \otimes S(W^*)$. Since $L[W^*]$ is a perfect Lie algebra, we know that each derivation c of $L[W^*]$ is continuous.

6.2. Let A be an abstract Lie algebra over $\boldsymbol{C}$. Then the commutator ring of A, which we denote by $\boldsymbol{C}_A$, is defined as follows:

$$\boldsymbol{C}_A = \{\rho \in \mathrm{Hom}_{\boldsymbol{C}}(A, A);\ \rho \circ \mathrm{ad}(x) = \mathrm{ad}(x) \circ \rho \quad \text{for all} \quad x \in A\}.$$

In this sub-section we want to determine the commutator rings $\boldsymbol{C}_L$ and $\boldsymbol{C}_{L[W^*]}$.

Proposition 6.1. *For an infinite transitive simple Lie algebra L, it holds that $\boldsymbol{C}_L = \boldsymbol{C}$.*

For the proof of Proposition 6.1, we need three lemmas. First we rewrite the some properties of L stated in Section 2 in the following lemma.

Lemma 6.2. (1) $L_0 = [L, L_1]$, *for* $L = D(U)$, $L_{\mathfrak{sl}}(U)$ *and* $L_{\mathfrak{sp}}(U)$,
(2) $L_{-1} = [L, L_1]$, *for* $L = L_{\mathfrak{ct}}(U)$.

Lemma 6.3. (V. Guillemin [1]). *$\boldsymbol{C}_L$ is a commutative field which canonically contains the field $\boldsymbol{C}$.*

Proof. For $a \in \boldsymbol{C}$, let ρ_a be a mapping such that $x \mapsto ax$ for $x \in L$. Then it is clear that ρ_a belongs to $\boldsymbol{C}_L$. Through a mapping $a \mapsto \rho_a$, we can consider $\boldsymbol{C}$ is contained in $\boldsymbol{C}_L$. Let ρ be a non-zero element of $\boldsymbol{C}_L$. Since L is simple, we have $\rho(L)=L$ and $\operatorname{Ker}\rho=0$. Hence a non-zero ρ has an inverse. Let ρ_1, $\rho_2 \in \boldsymbol{C}_L$. It is clear that $\rho_1 \circ \rho_2 \in \boldsymbol{C}_L$. Now it is sufficient to show that $\rho_1 \circ \rho_2 = \rho_2 \circ \rho_1$. For all $x, y \in L$ we have $\rho_1 \circ \rho_2 [x, y] = [\rho_1(x), \rho_2(y)] = \rho_2 \circ \rho_1 [x, y]$. Combining this equation with $L=[L, L]$, we obtain $\rho_1 \circ \rho_2 = \rho_2 \circ \rho_1$. q. e. d.

Lemma 6.4. *Each $\boldsymbol{C}_L$ has a faithful representation as a ring of endomorphisms as follows*:

(1) $\boldsymbol{C}_L \subsetneq \operatorname{Hom}_{\boldsymbol{C}}(L/L_0, L/L_0)$ *for* $L=D(U)$, $L_{\mathfrak{sl}}(U)$ *and* $L_{\mathfrak{sp}}(U)$,

(2) $\boldsymbol{C}_L \subsetneq \operatorname{Hom}_{\boldsymbol{C}}(L/L_{-1}, L/L_{-1})$ *for* $L=L_{\mathrm{ct}}(U)$.

Proof. (1) Let ρ be an element of $\boldsymbol{C}_L$. Since the filtration $\{L_p\}$ of L satisfies $L_0=[L, L_1]$ by Lemma 6.2, we obtain $\rho(L_0) \subset L_0$. Hence a linear mapping $\rho \mapsto \bar{\rho} \in \operatorname{Hom}_{\boldsymbol{C}}(L/L_0, L/L_0)$ is naturally induced. Assume $\bar{\rho}=0$. Then $\rho(L)$ is an ideal of L contained in L_0, and hence $\rho(L)=0$. Thus a linear mapping $\rho \to \bar{\rho}$ is faithful. The assertion (2) is proved by the same argument as (1).

q. e. d.

Proof of Proposition 6.1. First let L be an infinite irreducible transitive Lie algebra. Then by Remark 1 in Section 2, the linear isotropy algebra $\mathfrak{g}_0$ of L is considered as a Lie subalgebra of $\mathfrak{gl}(U)$. Recall that $\mathfrak{g}_0$ of $L=D(U)$, $L_{\mathfrak{sl}}(U)$ and $L_{\mathfrak{sp}}(U)$ are $\mathfrak{gl}(n, \boldsymbol{C})$, $\mathfrak{sl}(n, \boldsymbol{C})$ and $\mathfrak{sp}(n, \boldsymbol{C})$ respectively. By Lemma 6.3 and Lemma 6.4, $\boldsymbol{C}_L$ can be regarded as an abelian Lie subalgebra of $\mathfrak{gl}(U)$. We will show that $\boldsymbol{C}_L$ is contained in the centralizer of $\mathfrak{g}_0$ in $\mathfrak{gl}(U)$. Let $\rho \in \boldsymbol{C}_L$, $x \in \mathfrak{g}_0$ and $u \in U$. Then in $D(U)$ we clearly have

$$[[\rho, x], u] = [\rho, [x, u]] - [x, [\rho, u]] = (\rho \circ \operatorname{ad}(x) - \operatorname{ad}(x) \circ \rho)(u) = 0.$$

Since $[\rho, x] \in \mathfrak{g}_0$ and L is transitive, we obtain $[\rho, x]=0$, and hence $[\boldsymbol{C}_L, \mathfrak{g}_0]=0$. Put $\tilde{\mathfrak{g}}_0 = \mathfrak{g}_0 + \boldsymbol{C}_L$. Then $\tilde{\mathfrak{g}}_0$ yields a Lie subalgebra of $\mathfrak{gl}(U)$ and $\boldsymbol{C}_L$ is contained in the center of $\tilde{\mathfrak{g}}_0$. Since $\mathfrak{g}_0$ acts irreducibly on U, $\tilde{\mathfrak{g}}_0$ also acts irreducibly on U. Note that $\tilde{\mathfrak{g}}_0$ is of infinite type. By the classification theorem of Lie algebras of infinite type ([2] or [4]), $\tilde{\mathfrak{g}}_0$ must be equal to $\mathfrak{gl}(U)$ or $\mathfrak{csp}(U)$. Thus we have $\boldsymbol{C}_L = \boldsymbol{C}$.

Next let $L=L_{\mathrm{ct}}(U)$. Put $L/L_{-1}=U'$. Then U' is a one dimensional subspace of $\mathfrak{gl}(U')$, which contains $\boldsymbol{C}$. Hence $\boldsymbol{C}_L=\boldsymbol{C}$. q. e. d.

Using Proposition 6.1, we can verify the following proposition originally proved by V. Guillemin [1].

Proposition 6.5. *The commutator ring of* $L[W^*]$, *i.e.*, $\mathcal{C}_{L[W^*]}$, *is isomorphic to* $S(W^*)$.

Outline of proof. We will regard L as imbedded in $L[W^*]$. Let ρ be an element of $\mathcal{C}_{L[W^*]}$. We will denote by $\{f^\alpha\}$ the monomial basis in $S(W^*)$. If $x \in L$,, then we can write

$$\rho(x) = \sum_{\alpha=0}^{\infty} \rho_\alpha(x) f^\alpha, \qquad \rho_\alpha(x) \in L,$$

where ρ_α depends linearly on x. Since ρ is an element of $\mathcal{C}_{L[W^*]}$, we clearly obtain $\rho_\alpha \in \mathcal{C}_L$. By Proposition 6.1, ρ_α is an element of $\mathcal{C}$. Hence we can write

$$\rho(x) = x \otimes \prod_{\alpha=0}^{\infty} \rho_\alpha f^\alpha, \qquad \text{for all} \quad x \in L.$$

Since L is simple, we have $[L, L[W^*]] = L[W^*]$. Hence if $\rho \in \mathcal{C}_{L[W^*]}$, it is determined completely by its restriction to L. The isomorphism between $\mathcal{C}_{L[W^*]}$ and $S(W^*)$ is given by $\rho \mapsto \prod_{\alpha=0}^{\infty} \rho_\alpha f^\alpha$. This completes the proof. q. e. d.

By Proposition 6.5, $\mathrm{Der}(\mathcal{C}_{L[W^*]})$ is identified with $\mathrm{Der}(S(W^*))$. Now we have a homomorphism: l: $\mathrm{Der}(S(W^*)) \to \mathrm{Der}(L[W^*])$. Let $X \in \mathrm{Der}(L[W^*])$ and $\rho \in \mathcal{C}_{L[W^*]}$. Then $X \circ \rho - \rho \circ X$ is an element of $\mathcal{C}_{L[W^*]}$. We denote this element of $\mathcal{C}_{L[W^*]}$ by $L_X \rho$. By an easy consideration, the mapping $X \mapsto L_X$ is a homomorphism of $\mathrm{Der}(L[W^*])$ into $\mathrm{Der}(\mathcal{C}_{L[W^*]}) = \mathrm{Der}(S(W^*))$. Hence there is a natural homomorphism

$$L\colon \mathrm{Der}(L[W^*]) \longrightarrow \mathrm{Der}(S(W^*)).$$

It is easy to see that $L \circ l = $ identity, which implies that a homomorphism L is surjective. Since any elements of the kernel of L are $S(W^*)$-linear mappings, the kernel of L is identified with the set of all mappings c: $L \to L[W^*]$ satisfying the identity

$$c[x, y] = [c(x), y] + [x, c(y)] \qquad \text{for all} \quad x, y \in L.$$

We denote this set by $\mathrm{Der}(L, L[W^*])$.

Summarizing the above remarks, we have

Proposition 6.6 (V. Guillemin [1]). *There is a split exact sequence of Lie algebras*:

$$0 \longrightarrow \mathrm{Der}(L, L[W^*]) \longrightarrow \mathrm{Der}(L[W^*]) \underset{\iota}{\overset{L}{\rightleftarrows}} \mathrm{Der}(S(W^*)) \longrightarrow 0.$$

6.3. In this sub-section, we will determine the first cohomology group $H^1(L[W^*])$. By Proposition 6.6, we have a natural isomorphism:

$$\mathrm{Der}(L[W^*]) \cong \mathrm{Der}(L, L[W^*]) + \mathrm{Der}(S(W^*)) \quad \text{(direct sum)}.$$

The space $\mathrm{Der}(S(W^*))$ is canonically identified with $D(W)$, the Lie algebra of all formal vector fields over W. Hence it suffices to determine $\mathrm{Der}(L, L[W^*])$ for calculating $\mathrm{Der}(L[W^*])$.

Let $x \in L$ and $c \in \mathrm{Der}(L, L[W^*])$. We denote by f^α the basis of $S(W^*)$ consisting of monomials. Then we can write:

$$c(x) = \prod_{\alpha=0}^{\infty} x_\alpha \otimes f^\alpha, \qquad x_\alpha \in L.$$

Put $x_\alpha = c_\alpha(x)$. Then c_α is a linear mapping of L into itself. For $x, y \in L$, we have

$$\begin{aligned} c[x, y] &= \prod_{\alpha=0}^{\infty} c_\alpha[x, y] \otimes f^\alpha = [c(x), y] + [x, c(y)] \\ &= [\prod_{\alpha=0}^{\infty} c_\alpha(x) \otimes f^\alpha, y] + [x, \prod_{\alpha=0}^{\infty} c_\alpha(y) \otimes f^\alpha] \\ &= \prod_{\alpha=0}^{\infty} ([c_\alpha(x), y] + [x, c_\alpha(y)]) \otimes f^\alpha. \end{aligned}$$

Hence $c_\alpha[x, y] = [c_\alpha(x), y] + [x, c_\alpha(y)]$, which implies that c_α is an element of $\mathrm{Der}(L)$. By Theorem 2.1, there exists a unique element z_α of L (resp. $L + \mathbb{C}e$) such that $c_\alpha = \mathrm{ad}(z_\alpha)$ for $L = D(U)$ or $L_{ct}(U)$ (resp. $L = L_{\mathfrak{sl}}(U)$ or $L_{\mathfrak{sp}}(U)$). Thus we have $c = \mathrm{ad}(\prod_{\alpha=0}^{\infty} z_\alpha \otimes f^\alpha)$. Here the symbol e denotes a unit matrix, i.e. a basis of one dimensional center of $\mathfrak{gl}(U)$. Now we easily obtain the following isomorphism:

$$\mathrm{Der}(L[W^*]) \cong \begin{cases} L[W^*] + D(W) & \text{for } L = D(U) \text{ or } L_{ct}(U) \\ (L[W^*] + S(W^*) \otimes e) + D(W) & \text{for } L = L_{\mathfrak{sl}}(U) \text{ or } L_{\mathfrak{sp}}(U). \end{cases}$$

Since $L[W^*]$ has no non-trivial center, the space $\mathrm{ad}(L[W^*])$ of all inner derivations of $L[W^*]$ is naturally isomorphic to $L[W^*]$.

Summarizing the above results, we have proved:

Theorem 6.7. *Let $D(W)$ be a Lie algebra of all formal vector fields over W and let e be a basis of one dimensional center of* $\mathfrak{gl}(U)$. *Then we have the following isomorphism*:

$$H^1(L[W^*])\cong\begin{cases} D(W) & \text{for } L=D(U) \text{ or } L_{\mathfrak{ct}}(U) \\ D(W)+S(W^*)\otimes e & \text{for } L=L_{\mathfrak{sl}}(U) \text{ or } L_{\mathfrak{sp}}(U). \end{cases}$$

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