Publ. RIMS, Kyoto Univ.
18 (1982), 35–48

Centralizer of an Ergodic Measure Preserving Transformation

By

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§1. Introduction

Let T be an ergodic measure preserving transformation of a Lebesgue measure space $(\Omega, \mathfrak{B}, P)$, $P(\Omega)=1$, that is, T is a one to one mapping from Ω onto itself, bimeasurable ($T\mathfrak{B}=\mathfrak{B}$), measure preserving ($P(T^{-1}A)=P(A)$ for A in $\mathfrak{B}$) and ergodic (every measurable function $f(\omega)$ with $f(T\omega)=f(\omega)$ a.e. is constant a.e.). For measure preserving transformations U and U' we write $U=U'$ if $P(U\omega\neq U'\omega)=0$ and $U\neq U'$ otherwise. A measure preserving transformation U of $(\Omega, \mathfrak{B}, P)$ is called a *p-th root* of $T(p\geqq 2)$ if $U^p=T$. A 1-parameter group $\{U_t\}$ of measure preserving transformations of $(\Omega, \mathfrak{B}, P)$ (i.e. $U_{t+s}=U_tU_s$ for $-\infty<t, s<+\infty$) is called a measurable flow if $(\omega, t)\to U_t\omega$ is a measurable mapping from $\Omega\times\mathbb{R}$ onto Ω. If there exists a measurable flow $\{U_t\}$ with $U_1=T$, T is said to be *embeddable* in a measurable flow. The existence of a p-th root or an embedding measurable flow has been one of problems in ergodic theory.

It is obvious that the existence of an embedding measurable flow of T implies the existence of a p-th root of T for every $p\geqq 2$ and also that a p-th root of T (if exists) and an embedding measurable flow of T (if exists) are ergodic.

A measure preserving transformation U of $(\Omega, \mathfrak{B}, P)$ is said to commute with T if $UT=TU$. We denote by $C(T)$ the group consisting of all measure preserving transformations each of which commutes with T and call it the *centralizer* of T. Since a p-th root of T (if exists) and a transformation U_t for fixed t in an embedding measurable flow $\{U_t\}$ of T (if exists) are in $C(T)$, we may expect to solve the existence problem of roots and an embedding measurable

Communicated by H. Araki, September 8, 1980.
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flow of some kind of ergodic transformation by determining its centralizer. A transformation U in $C(T)$ is called to *commute irrationally* with T if $T^n \neq U^m$ except for $n=m=0$. Whether an ergodic transformation commuting irrationally with given ergodic transformation exists or does not is a new problem.

Blum and Friedman [1], [2] discussed the existence problem of roots for ergodic transformations with discrete spectrum and constructed, for instance, an example of such transformation without any roots. Chacon [3], [4] gave an example of an ergodic transformation with continuous spectrum having no square root and one having no roots. Ornstein [10] gave an example of ergodic transformation T with $C(T)=\{T^n: n \in \boldsymbol{Z}\}$ which, in consequence, has no roots nor an ergodic transformation commuting irrationally with it. In [11] he showed that every Bernoulli shift is embeddable in a measurable flow.

In Section 2 using the property of an invariant χ_U of a transformation U in $C(T)$ which was introduced by T. Hamachi [6] we study the existence problem of a p-th root and an embedding measurable flow for ergodic transformations with pure point spectra. We show also that for an ergodic transformation with pure point spectrum there exists an ergodic transformation which commutes irrationally with it. In Section 3 we determine the $C(T)$ for a kind of transformations and give the following examples of ergodic transformations: (1) one not having any root, (2) one having only square root but no others, (3) one having a 2^n-th root for any positive integer n but no others, (4) one having a n-th root for any positive integer n but not embeddable in a measurable flow, and (5) one whose commutant is not commutative. In Section 4 we give two examples of normalizers of ergodic non-singular transformations; one of them is one not having a square root and the other is one having any root but not embeddable in a measurable flow.

The author would like to thank I. Miyawaki for his contribution to the proof of Theorem 6 (2).

§2. Invariant χ_U and Transformation with Pure Point Spectra

Let T be an ergodic measure preserving transformation of a Lebesgue measure space $(\Omega, \mathfrak{B}, P)$, $P(\Omega)=1$. A number γ in the unit interval $[0, 1)$ and a function $f_\gamma(\omega)$ are called a *point spectrum* and an *eigenfunction*, respectively, of T if $f_\gamma(T\omega)=\exp(2\pi i\gamma) f_\gamma(\omega)$ for a.e. ω in Ω and $|f_\gamma(\omega)|=1$. We denote by $S_p(T)$ the set of all point spectra of T. It is a countable subgroup of $[0, 1)$

with respect to the additive operation modulo 1.

Lemma 1 ([6]). *Let T be an ergodic measure preserving transformation of a Lebesgue measure space $(\Omega, \mathfrak{B}, P)$, $P(\Omega)=1$, then for a transformation U in $C(T)$ there is a homomorphism χ_U from $S_p(T)$ into $[0, 1)$ such that $f_\gamma(U\omega) =\exp(2\pi i\chi_U(\gamma)) f_\gamma(\omega)$ for a.e. ω, for γ in $S_p(T)$ and an eigenfunction $f_\gamma(\omega)$.*

Proof. Since $f_\gamma(U\omega)/f_\gamma(\omega)$ is a T-invariant function and T is ergodic, there is a constant $\chi_U(\gamma)$ such that $f_\gamma(U\omega)/f_\gamma(\omega)=\exp(2\pi i\chi_U(\gamma))$ for a.e. ω. This constant $\chi_U(\gamma)$ does not depend on the choice of an eigenfunction $f_\gamma(\omega)$ for γ and χ_U is a homomorphism from $S_p(T)$ into $[0, 1)$.

Corollary 2. *If $S_p(T)=\{0\}$ then for any ergodic transformation U in $C(T)$ $S_p(U)=\{0\}$.*

Proof. Obvious from Lemma 1.

Lemma 3. (1) *For U and V in $C(T)$*

$$\chi_{UV}(\gamma)=\chi_U(\gamma)+\chi_V(\gamma) \pmod 1 \quad \text{for} \quad \gamma \quad \text{in} \quad S_p(T).$$

Especially, $\chi_{U^n}(\gamma)=n\chi_U(\gamma)$ (mod 1) for γ in $S_p(T)$ and n in Z.

(2) $\chi_T(\gamma)=\gamma$ *for γ in $S_p(T)$.*

(3) *If U in $C(T)$ is ergodic, $\chi_U(\gamma)=0$ implies $\gamma=0$.*

Proof. Obvious.

Theorem 4 ([2]). *If $1/n$ is in $S_p(T)$ for some integer n with $(n, p)\neq 1$ then there is no p-th root of T, where (n, p) is the greatest common measure of n and p.*

Proof. Assume that there is a p-th root U of T. Since $p\chi_U(\gamma)=\gamma$ (mod 1) for γ in $S_p(T)$, there is an integer q such that $\chi_U(1/n)=(1+nq)/np$. Since $n\chi_U(1/n)=0$, $(1+nq)/p$ is an integer, contradiction to $(n, p)\neq 1$.

An ergodic measure preserving transformation T of a Lebesgue measure space $(\Omega, \mathfrak{B}, P)$, $P(\Omega)=1$ is said to have *pure point spectra* if there is a complete orthonormal system of $L^2(\Omega, P)$ consisting of eigenfunctions of T.

Lemma 5. *Let T be an ergodic measure preserving transformation of a Lebesgue measure space $(\Omega, \mathfrak{B}, P)$, $P(\Omega)=1$ with pure point spectra.*

(1) *The mapping $U\to\chi_U$ is one to one from $C(T)$ onto the group consisting of all homomorphisms from $S_p(T)$ into $[0, 1)$.*

(2) *If $\chi_U(\gamma)=0$ implies $\gamma=0$, then U is ergodic.*

(3) *If there is a homomorphism* ϕ *from* $S_p(T)$ *into* $[0, 1)$ *such that* $p\phi(\gamma) = \gamma$ *for* γ *in* $S_p(T)$, *then there is a p-th root of T.*

(4) *If there is a 1-parameter group* ϕ_t *of homomorphisms from* $S_p(T)$ *into* $[0, 1)$ *such that* $\phi_t(\gamma)$ *is a measurable with respect to t and* $\phi_1(\gamma)=\gamma$ *for* γ *in* $S_p(T)$, *then T is embeddable in a measurable flow.*

(5) *If there is a homomorphism* ϕ *from* $S_p(T)$ *into* $[0, 1)$ *such that* $n\phi(\gamma) = m\gamma$ *for* γ *in* $S_p(T)$ *implies* $n=m=0$, *and such that* $\phi(\gamma)=0$ *implies* $\gamma=0$, *then there is an ergodic transformation irrationally commuting with T.*

Proof. (1) We denote by $\widehat{S_p(T)}$ the character group of $S_p(T)$ and by $\langle\gamma, g\rangle$, γ in $S_p(T)$, g in $\widehat{S_p(T)}$ an inner product which is a bilinear form with absolute value 1. Let a be an element in $\widehat{S_p(T)}$ defined by $\langle\gamma, a\rangle = \exp(2\pi i\gamma)$ for γ in $S_p(T)$ and identify it with the translation $g \to g+a$ of $S_p(T)$. By Halmos-Neumann theorem [5] there is a measure preserving mapping Ψ from $(\Omega, \mathfrak{B}, P)$ onto $\widehat{S_p(T)}$ with the Haar measure such that $g+a=\Psi T\Psi^{-1}g$ for a.e. g in $\widehat{S_p(T)}$. For a homomorphism ϕ from $S_p(T)$ into $[0, 1)$ identify the element b in $\widehat{S_p(T)}$ defined by $\langle\gamma, b\rangle = \exp(2\pi i\phi(\gamma))$ for γ in $S_p(T)$ with the translation $g \to g+b$ of $\widehat{S_p(T)}$. Then the transformation U of $(\Omega, \mathfrak{B}, P)$ defined by $U\omega = \Psi^{-1}b\Psi\omega$ for ω in Ω is in $C(T)$ with $\chi_U(\gamma)=\phi(\gamma)$ for γ in $S_p(T)$. This means the mapping is onto. If $\chi_U(\gamma)=0$ for any γ in $S_p(T)$, then $f_\gamma(U\omega)=f_\gamma(\omega)$ for every eigenfunction $f_\gamma(\omega)$ all of which span $L^2(\Omega, P)$. Hence, U is the identity transformation and it follows that the mapping is one to one.

(2) Let $f(\omega)$ be a U-invariant function in $L^2(\Omega, P)$. Then we have

$$\int f(\omega)\overline{f_\gamma(\omega)}dP(\omega) = \int f(U\omega)\overline{f_\gamma(U\omega)}dP(\omega)$$
$$= \overline{\exp(2\pi i\chi_U(\gamma))}\int f(\omega)\overline{f_\gamma(\omega)}dP(\omega).$$

Since $\chi_U(\gamma)\neq 0$ for $\gamma\neq 0$, $\int f(\omega)\overline{f_\gamma(\omega)}dP(\omega)=0$ for $\gamma\neq 0$. Hence, $f(\omega)$ is constant a.e. and U is ergodic.

(3), (4) and (5) follow easily from (1) and Lemma 3.

An infinite sequence $\lambda_1, \lambda_2, \dots, \lambda_n, \dots$ of real numbers is called to be *rationally independent* if for any positive integer n $q_1\lambda_1+q_2\lambda_2+\cdots+q_n\lambda_n=0$, q_i in $\boldsymbol{Q}$, $i=1, 2, \dots, n$ implies $q_1=q_2=\cdots=q_n=0$, where $\boldsymbol{Q}$ is the set of all rational numbers.

Theorem 6. *Let T be an ergodic measure preserving transformation of a Lebesgue space* $(\Omega, \mathfrak{B}, P)$, $P(\Omega)=1$ *with pure point spectrum.*

(1) *If there is a rationally independent sequence* $1, \lambda_1, \lambda_2, \dots, \lambda_n, \dots$, *such that* $S_p(T) \subset \{m + q_1\lambda_1 + q_2\lambda_2 + \cdots + q_n\lambda_n : n \in \mathbb{N}, m \in \mathbb{Z}, q_i \in \mathbb{Q}\}$, *then* T *is embeddable in a measurable flow.*

(2) *If* n/p *is not in* $S_p(T)$ *for* $n = 1, 2, \dots, p-1$, *then there is a* p*-th root of* T.

Proof. (1) Define $\phi_t(\gamma) = t(q_1\lambda_1 + q_2\lambda_2 + \cdots + q_n\lambda_n) \pmod 1$ for $\gamma = m + q_1\lambda_1 + q_2\lambda_2 + \cdots + q_n\lambda_n$ and $-\infty < t < +\infty$, then $\{\phi_t\}$ is a 1-parameter measurable group of homomorphisms from $S_p(T)$ into $[0, 1)$ with $\phi_1(\gamma) = \gamma$ for γ in $S_p(T)$. Hence, by Lemma 5 (4) T is embeddable in a measurable flow.

(2) We show first a proof of (2) in case of prime p (Blum and Friedman [2]) for completeness of the discussion, and next, one in case of not prime p.

Let Γ be a maximal subgroup of $[0, 1)$ such that Γ includes $S_p(T)$ and that n/p is not in Γ for $n = 1, 2, \dots, p-1$. We show that there exists a homomorphism ϕ from Γ into itself such that $p\phi(\gamma) = \gamma$ for γ in Γ. It is enough to see that for any γ in Γ there exists uniquely an integer n, $n = 0, 1, \dots, p-1$ such that $(\gamma + n)/p$ is in Γ. If there is γ in Γ such that $(\gamma + n)/p$ is not in Γ for any $n = 0, 1, \dots, p-1$, then $\{m\gamma/p + \eta : m \in \mathbb{Z}, \eta \in \Gamma\}$ is a subgroup of $[0, 1)$ including Γ and does not contain n/p for any $n = 1, 2, \dots, p-1$. This contradicts to the maximality of Γ. The uniqueness follows easily from that n/p is not in Γ for any $n = 1, 2, \dots, p-1$.

Let $p = p_1 p_2 \cdots p_k$, where p_i's are prime numbers. Since $1/p_1$ is not in $S_p(T)$, from the above discussion there is a homomorphism ϕ_1 from $S_p(T)$ into $[0, 1)$ such that $p_1\phi_1(\gamma) = \gamma$ for γ in $S_p(T)$. Denote by Γ_1 the ϕ_1-image of $S_p(T)$. If $\phi_1(\gamma) = 1/p_2$ for some γ in $S_p(T)$, then $\gamma = p_1\phi_1(\gamma) = p_1/p_2$, which contradicts the assumption of $S_p(T)$ if $p_1 \neq p_2$, and $\gamma \neq 0$ if $p_1 = p_2$, respectively. Therefore, $1/p_2$ is not in Γ_1. By the same way we can obtain a sequence $\phi_2, \phi_3, \dots, \phi_k$ of homomorphisms and a sequence $\Gamma_2, \Gamma_3, \dots, \Gamma_k$ of subgroups of $[0, 1)$ such that ϕ_i is a homomorphism from Γ_{i-1} onto Γ_i and $p_i\phi_i(\gamma) = \gamma$ for γ in Γ_{i-1}, for $i = 2, 3, \dots, k$. Define a homomorphism ϕ from $S_p(T)$ into $[0, 1)$ by $\phi(\gamma) = \phi_k(\cdots(\phi_3(\phi_2(\phi_1(\gamma))\cdots)))$ for γ in $S_p(T)$. Then we have $p\phi(\gamma) = \gamma$ for γ in $S_p(T)$ and by Lemma 5 (3) there is a p-th root of T.

Lemma 7. *Let* p' *and* q' *be positive integers with* $(p', q') = 1$, *then for a prime number* p *and a positive integer* q

$$(q' + p'q, p'p) = 1 \quad \text{or} \quad (q' + p'(q+1), p'p) = 1.$$

Proof. Assume that $a=(q'+p'q, p'p)>1$ and $a'=(p', a)>1$. Then since a is a divisor of $q'+p'q$, a' is a divisor of q', and hence, $(p', q')\geqq a'>1$, contradiction. Thus $(q'+p'q, p'p)>1$ implies that $(p', a)=1$, that $p=a=(q'+p'q, p'p)$ because p is prime, and hence, that p is a divisor of $q'+p'q$ but not of p'. Therefore the assumption that $(q'+p'q, p'p)>1$ and $(q'+p'(q+1), p'p)>1$ implies that p is a divisor both of $q'+p'q$ and $q'+p'(q+1)$ but not of p'. It is impossible. This proves the lemma.

Theorem 8. *For an ergodic measure preserving T with pure point spectra there exists an ergodic measure preserving transformation which commutes irrationally with T.*

Proof. By Lemma 5 (5) it is enough to construct a homomorphism ϕ from $S_p(T)$ into $[0, 1)$ such that $\phi(\gamma)=0$ implies $\gamma=0$ and such that $l\phi(\gamma)=m\gamma$ for γ in $S_p(T)$ implies $l=m=0$.

(1) First we construct it in case that $S_p(T)\subset \boldsymbol{Q}$. In this case there is a sequence $p_0, p_1, p_2,\dots, p_n,\dots$ of prime integers such that $S_p(T)=\{m/p_0p_1\cdots p_n: n\in \boldsymbol{N}, m\in \boldsymbol{Z}\}$. Let $l_1, l_2,\dots, l_n,\dots$ be a sequence of positive integers in which every positive integer appears infinitely often.

(1–1) In case that there are infinitely many distinct prime numbers in the sequence $p_0, p_1,\dots, p_n,\dots$. Let $n_0=0$ and $q_0=1$. Having chosen $n_0, n_1,\dots, n_{k-1}$ and $q_0, q_1,\dots, q_{n_{k-1}}$, n_k can be chosen such that there is an integer q with $1/3l_k<(q-1)/p_{n_k}<(q+1)/p_{n_k}<2/3l_k$, and q_i for $i=n_{k-1}+1, n_{k-1}+2,\dots, n_k-1, n_k$ can be chosen by Lemma 7 such that $(q_0+p_0q_1+\cdots+p_0p_1\cdots p_{i-1}q_i, p_0p_1\cdots p_i)=1$ for $n_{k-1}+1\leqq i\leqq n_k$ and $1/3l_k<q_{n_k}/p_{n_k}<(q_{n_k}+1)/p_{n_k}<2/3l_k$. For the obtained $q_0, q_1,\dots, q_n,\dots$ we define

$$\phi(m/p_0p_1\cdots p_n)=m(q_0+p_0q_1+\cdots+p_0p_1\cdots p_{n-1}q_n)/p_0p_1\cdots p_n \pmod 1.$$

Then ϕ is a homomorphism from $S_p(T)$ into $[0, 1)$. If $\phi(m/p_0p_1\cdots p_n)=0 \pmod 1$, then from $(q_0+p_0q_1+\cdots+p_0p_1\cdots p_{n-1}q_n, p_0p_1\cdots p_n)=1$ m is a multiple of $p_0p_1\cdots p_n$, and hence, $m/p_0p_1\cdots p_n=0 \pmod 1$. For a positive integer l and infinitely many positive integers k such that $l_k=l$ we have $1/3<l\phi(1/p_0p_1\cdots p_{n_k})=l_k(q_0+p_0q_1+\cdots+p_0p_1\cdots p_{n_k-1}q_{n_k})/p_0p_1\cdots p_{n_k}<2/3$. Since, on the other hand, $m/p_0p_1\cdots p_{n_k}$ converges to 0 as $k\to\infty$ for an integer m, $l\phi(\gamma)=m\gamma \pmod 1$ for γ in $S_p(T)$ does not hold for positive integers l and m.

(1–2) In case that there are only finite number of distinct prime numbers in the sequence $p_0, p_1,\dots, p_n,\dots$, we may assume that $S_p(T)=\{s/p_0^n+t/p_1p_2\cdots p_m:$

$n, m \in \boldsymbol{N}, s, t \in \boldsymbol{Z}\}$, where p_0 does not appear in the sequence $p_1, p_2, \ldots, p_n, \ldots$. Let $n_0=0$ and $q_0=1$. Having chosen $n_0, n_1, \ldots, n_{k-1}$ and $q_0, q_1, \ldots, q_{k-1}$, we take $n_k\ (>n_{k-1})$ and q_k such that $1/3l_k < q_k/p_0^{n_k-n_{k-1}} < (q_k+1)/p_0^{n_k-n_{k-1}} < 2/3l_k$. For the obtained $n_0, n_1, \ldots, n_k, \ldots$ and $q_0, q_1, \ldots, q_n, \ldots$ define

$$\begin{aligned}&\phi(s/p_0^{n_k}+t/p_1p_2\cdots p_m)\\&\quad=s(q_0+p_0^{n_1}q_1+\cdots+p_0^{n_{k-1}}q_k)/p_0^{n_k}+t/p_1p_2\cdots p_m \pmod 1.\end{aligned}$$

Then ϕ is a homomorphism from $S_p(T)$ into $[0, 1)$. From $q_0=1$, $(q_0+p_0^{n_1}q_1+\cdots+p_0^{n_{k-1}}q_k, p_0^{n_k})=1$, and hence, $\phi(\gamma)=0$ implies $\gamma=0$. Since $1/3<l_k(q_0+p_0^{n_1}q_1+\cdots+p_0^{n_{k-1}}q_k)/p_0^{n_k}<2/3$, by the same way as one in (1–1) we can show that $l\phi(\gamma)=m\gamma$ for γ in $S_p(T)$ holds only for $l=m=0$.

(2) In case that an irrational number λ_1 is in $S_p(T)$, there is a rationally independent sequence $1, \lambda_1, \lambda_2, \ldots, \lambda_n, \ldots$ such that any number in $S_p(T)$ has a form $q_0+q_1\lambda_1+\cdots+q_n\lambda_n$, $n \in \boldsymbol{N}$, $q_i \in \boldsymbol{Q}$, $i=0, 1, \ldots, n$. Let $1, \eta_1, \eta_2, \ldots, \eta_n, \ldots$ be another rationally independent sequence such that η_1 and λ_1 are rationally independent and define

$$\phi(q_0+q_1\lambda_1+\cdots+q_n\lambda_n)=q_0+q_1\eta_1+\cdots+q_n\eta_n \pmod 1,\quad n \in \boldsymbol{N},\ q_i \in \boldsymbol{Q},$$

$i=0, 1, \ldots, n$. Then ϕ is a homomorphism from $S_p(T)$ into $[0, 1)$. It is obvious that $\phi(\gamma)=0$ implies $\gamma=0$. If $l\phi(\gamma)=m\gamma$ for γ in $S_p(T)$, we have $l\phi(\lambda_1)=l\eta_1=m\eta_1$ and $l=m=0$ follows from that λ_1 and η_1 are rationally independent. The proof is complete.

§ 3. Examples

Let Γ be a torsionfree countable abelian group and G the character group of Γ which is a separable compact abelian group. We denote by $\langle\gamma, g\rangle$, $\gamma \in \Gamma$, $g \in G$ their inner product and by dg the Haar measure on G. We note that an endomorphism (automorphism) σ of G determines uniquely an endomorphism (automorphism, respectively) $\hat{\sigma}$ of Γ by $\langle\gamma, \sigma(g)\rangle=\langle\hat{\sigma}(\gamma), g\rangle$, $\gamma \in \Gamma$, $g \in G$, and vice versa. We denote by End(G) and End(Γ) the sets of all endomorphisms of G and Γ, respectively.

Theorem 9. *For elements λ and η in G and σ in* End(G) *let $U_{\sigma,\lambda,\eta}$ be a measure preserving transformation of the direct product measure space $(G \times G, dg \times dg)$ defined by $U_{\sigma,\lambda,\eta}(g, g')=(g+\lambda, g'+\sigma(g)+\eta)$ for (g, g') in $G \times G$. If $\langle\gamma, \lambda\rangle=1$ implies $\gamma=0$ and if $\hat{\sigma}(\gamma)=0$ implies $\gamma=0$, then* (1) *$U_{\sigma,\lambda,\eta}$ is ergodic,*

(2) $\exp(2\pi i S_p(U_{\sigma,\lambda,\eta}))=\{\langle\gamma, \lambda\rangle : \gamma\in\Gamma\}$ *and* (3) $C(U_{\sigma,\lambda,\eta})=\{U_{\delta,\alpha,\beta}: \delta\in\mathrm{End}\,(G),$ $\alpha, \beta\in G$ *with* $\sigma(\alpha)=\delta(\lambda)\}$, *where the left side of* (2) *is the set consisting of all* $\exp(2\pi i\zeta)$ *for* ζ *in* $S_p(U_{\sigma,\lambda,\eta})$.

Proof. Since $\{\langle\gamma, g\rangle\langle\gamma', g'\rangle : \gamma, \gamma'\in\Gamma\}$ is a complete orthonormal system of $L^2(G\times G, dg\times dg)$, any function in it is represented as $f(g, g')=\sum_{\gamma,\gamma'} a_{\gamma,\gamma'}$ $\langle\gamma, g\rangle\langle\gamma', g'\rangle$ for (g, g') in $G\times G$ and we have $f(U_{\sigma,\lambda,\eta}(g, g'))=\sum_{\gamma,\gamma'} a_{\gamma,\gamma'}$ $\langle\gamma, \lambda\rangle\langle\gamma', \eta\rangle\langle\gamma+\hat\sigma(\gamma'), g\rangle\langle\gamma', g'\rangle$ for (g, g') in $G\times G$.

(1) From $f(U_{\sigma,\lambda,\eta}(g, g'))=f(g, g')$ for (g, g') in $G\times G$ we have $a_{\gamma,\gamma'}\langle\gamma, \lambda\rangle$ $\langle\gamma', \eta\rangle=a_{\gamma+\hat\sigma(\gamma'),\gamma'}$ for γ, γ' in Γ. Hence, $|a_{\gamma,\gamma'}|=|a_{\gamma+\hat\sigma(\gamma'),\gamma'}|=|a_{\gamma+2\hat\sigma(\gamma'),\gamma'}|=\cdots$. Since $f(g, g')$ is in $L^2(G\times G, dg\times dg)$, Γ is torsionfree and since $\hat\sigma(\gamma)=0$ implies $\gamma=0$, we have $a_{\gamma,\gamma'}=0$ for $\gamma'\neq 0$. Since $a_{\gamma,0}\langle\gamma, \lambda\rangle=a_{\gamma,0}$ and since $\langle\gamma, \lambda\rangle=1$ implies $\gamma=0$ we have $a_{\gamma,0}=0$ for $\gamma\neq 0$. Hence, $f(g, g')=a_{0,0}$ and $U_{\sigma,\lambda,\eta}$ is ergodic.

(2) Let $f(g, g')$ be an eigenfunction of $U_{\sigma,\lambda,\eta}$ for a point spectrum ζ, then we have $a_{\gamma,\gamma'}\langle\gamma, \lambda\rangle\langle\gamma', \eta\rangle=\exp(2\pi i\zeta)\,a_{\gamma+\hat\sigma(\gamma'),\gamma'}$, $\gamma, \gamma'\in\Gamma$. By the same way as one in (1) we have $a_{\gamma,\gamma'}=0$ for $\gamma'\neq 0$. Since $f(g, g')$ is not zero function there is γ with $a_{\gamma,0}\neq 0$. For such γ it follows from $a_{\gamma,0}\langle\gamma, \lambda\rangle=\exp(2\pi i\zeta)\,a_{\gamma,0}$ that $\langle\gamma, \lambda\rangle=\exp(2\pi i\zeta)$, which means $\exp(2\pi i S_p(U_{\sigma,\lambda,\eta}))=\{\langle\gamma, \lambda\rangle : \gamma\in\Gamma\}$. It is obvious that every $\langle\gamma, \lambda\rangle$ is in $\exp(2\pi i S_p(U_{\sigma,\lambda,\eta}))$.

(3) Let V be a transformation in $C(U_{\sigma,\lambda,\eta})$ and put $f_\gamma(g, g')=\langle\gamma, g\rangle$ for γ in Γ and (g, g') in $G\times G$. Since f_γ is an eigenfunction of $U_{\sigma,\lambda,\eta}$, by Lemma 1 there is an element α in G such that $f_\gamma(V(g, g'))=\langle\gamma, \alpha\rangle f_\gamma(g, g')$ for (g, g') in $G\times G$. From $f_\gamma(V(g, g'))=\langle\gamma, g+\alpha\rangle$ there is a mapping Φ from $G\times G$ onto G such that $V(g, g')=(g+\alpha, \Phi(g, g'))$ for (g, g') in $G\times G$. From $VU_{\sigma,\lambda,\eta}=U_{\sigma,\lambda,\eta}V$ we have $\Phi(U_{\sigma,\lambda,\eta}(g, g'))=\Phi(g, g')+\sigma(g+\alpha)+\eta$ for (g, g') in $G\times G$. Put $\Psi(g, g')=$ $\Phi(g, g')-g'$, then $\Psi(U_{\sigma,\lambda,\eta}(g, g'))=\Psi(g, g')+\sigma(\alpha)$ for (g, g') in $G\times G$. Hence, $\langle\gamma, \sigma(\alpha)\rangle$ is in $\exp(2\pi i S_p(U_{\sigma,\lambda,\eta}))$ for any γ in Γ. By (2) there is an element $\hat\sigma(\gamma)$ in Γ such that $\langle\gamma, \sigma(\alpha)\rangle=\langle\hat\delta(\gamma), \lambda\rangle$ for γ in Γ. From the property of λ, $\hat\delta$ is an endomorphism of Γ and $\sigma(\alpha)=\delta(\lambda)$. Put $\Theta(g, g')=\Phi(g, g')-g'-\delta(g)$ for (g, g') in $G\times G$, we have $\Theta(U_{\sigma,\lambda,\eta}(g, g'))=\Theta(g, g')$ for (g, g') in $G\times G$. From (1) there is an element β in G such that $\Theta(g, g')=\beta$ for (g, g') in $G\times G$. Hence, $V(g, g')=(g+\alpha, g'+\delta(g)+\beta)$ for (g, g') in $G\times G$, that is, $V=U_{\delta,\alpha,\beta}$. Conversely, $U_{\delta,\alpha,\beta}$ with $\sigma(\alpha)=\delta(\lambda)$ is in $C(U_{\sigma,\lambda,\eta})$.

For a sequence $p_1, p_2,\ldots, p_n,\ldots$ of prime numbers let $\boldsymbol{Q}_0$ be the subgroup of $\boldsymbol{Q}$ consisting of all rational numbers of the form $m/p_1p_2\cdots p_n$, $n\in\boldsymbol{N}$, $m\in\boldsymbol{Z}$, and Ω be the infinite direct product set $[0, 1) \times \prod_{n=1}^{\infty} \{0, 1, \dots, p_n - 1\}$. Define $\langle q, \omega \rangle$ for $q = m/p_1 p_2 \cdots p_n$ in Q_0 and ω in Ω by $\langle q, \omega \rangle = \exp(2\pi i m(\omega_0 + \omega_1 + p_1\omega_2 + \cdots + p_1 p_2 \cdots p_{n-1}\omega_n)/p_1 p_2 \cdots p_n)$, where ω_n is the n-th coordinate of ω. Then Ω is the dual of Q_0 with respect to the inner product $\langle q, \omega \rangle$. The adding operation in Ω defined by $\langle q, \omega + \omega' \rangle = \langle q, \omega \rangle \langle q, \omega' \rangle$, $q \in Q_0$, $\omega, \omega' \in \Omega$ may be called a *generalized adding machine*. The Haar measure P on Ω is the infinite direct product $d\omega_0 \times \prod_{n=1}^{\infty} \{1/p_n, 1/p_n, \dots, 1/p_n\}$. Let Q_1 be the set of all rational numbers of the form $m/p(1)^{n(1)} p(2)^{n(2)} \cdots p(k)^{n(k)}$, $k \in N$, $m \in Z$, $n(i) \in N$ and $p(i)$'s are prime numbers each of which appears infinitely many times in the sequence $p_1, p_2, \dots, p_n, \dots$. For a number q_1 in Q_1 a mapping $q \to q_1 q$ is an endomorphism of Q_0 and so, for q_1 in Q_1 and ω in Ω, $q_1\omega$ is defined such that $\omega \to q_1\omega$ is an endomorphism of Ω. We can see that there are no other endomorphisms of Q_0 (or Ω) except ones determined by numbers in Q_1 as above. Let Q_2 be the set of all numbers q_1 in Q_1 such that $1/q_1$ is also in Q_1, then the endomorphism of Q_0 (or Ω) determined by q_1 in Q_1 is an automorphism if and only if q_1 is in Q_2.

Theorem 10. *Let Ω be the compact abelian group described above and λ be an element of Ω whose 0-th coordinate λ_0 is irrational. Define a measure preserving transformation T of the direct product space $(\Omega \times \Omega, P \times P)$ by $T(\omega, \omega') = (\omega + \lambda, \omega' + \omega)$ for (ω, ω') in $\Omega \times \Omega$. Then we have the followings:*

(1) *T is ergodic.*

(2) *$C(T) = \{U_{q_1,\alpha} : q_1 \in Q_1, \alpha \in \Omega\}$, where*

$$U_{q_1,\alpha}(\omega, \omega') = (\omega + q_1\lambda, \omega' + q_1\omega + \alpha) \quad \text{for} \quad (\omega, \omega') \quad \text{in} \quad \Omega \times \Omega.$$

(3) *T is not embeddable in a measurable flow.*

(4) *If $U_{q_1,\alpha}^p = T^m$, $p, m \in Z$, then $m = pq_1$ and*

$$p\alpha = m(m-1)\lambda/2 - p(p-1)q_1^2\lambda/2.$$

(5) *If there is a p-th root of T, p is in Q_2.*

(6) *There is an ergodic measure preserving transformation which commutes irrationally with T.*

Proof. (1) and (2) follow from Theorem 9.

(3) Let $\{U_t\}$ be a measurable flow in $C(T)$, then by (2) we may assume $U_t = U_{q_1(t),\alpha(t)}$ for $q_1(t)$ in Q_1 and $\alpha(t)$ in Ω. Since $\omega \to \omega + q_1(t)\lambda$ is a measurable flow of Ω, $q_1(t) = 0$ follows from the countability of Q_1. Thus we conclude that $U_1 = U_{0,\alpha(1)} \neq T$ for any measurable flow $\{U_t\}$ in $C(T)$.

(4) follows from $T^m(\omega, \omega')=(\omega+m\lambda, \omega'+m\omega+m(m-1)\lambda/2)$ and $U^p_{q_1,\alpha}(\omega, \omega')=(\omega+pq_1\lambda, \omega'+pq_1\omega+p\alpha+p(p-1)q_1^2\lambda/2)$, $(\omega, \omega')\in\Omega\times\Omega$.

(5) Let $U=U_{q_1,\alpha}$ is a p-th root of T then by (4) we have $pq_1=1$, which implies that p is in $\boldsymbol{Q}_2$.

(6) Let $U=U_{q_1,\alpha}$ for $q_1\neq 0$ in $\boldsymbol{Q}_1$ and α in Ω such that the 0-th coordinate α_0 of α is rationally independent with λ_0. Then by Theorem 9 U is ergodic. If $U^p=T^m$ then by (4) $p\alpha=\{m(m-1)/2-p(p-1)q_1^2/2\}\lambda$, and hence, $p=m=0$ follows from that α_0 and λ_0 are rationally independent.

Example 1. For an irrational number λ define a transformation T of $[0, 1)\times[0, 1)$ by

$$T(x, y)=(x+\lambda, y+x) \quad \text{for} \quad (x, y) \quad \text{in} \quad [0, 1)\times[0, 1).$$

This T is the same transformation as one in Theorem 9 in case of $p_n=1$ for every $n\in\boldsymbol{N}$, in which case $\boldsymbol{Q}_0=\boldsymbol{Q}_1=\boldsymbol{Z}$ and $\boldsymbol{Q}_2=\{1\}$. Since $C(T)=\{U_{n,\alpha}: n\in\boldsymbol{Z}, \alpha\in[0, 1)\}$, where $U_{n,\alpha}(x, y)=(x+n\lambda, y+nx+\alpha)$ for (x, y) in $[0, 1)\times[0, 1)$, it is easy to see that T has no p-th root for every $p\geqq 2$. By Theorem 9 T^2 is also ergodic and $C(T^2)=C(T)\cup\{U'_{n,\beta}: n\in\boldsymbol{Z}, \beta\in[0, 1)\}$, where $U'_{n,\beta}(x, y)=(x+n\lambda+1/2, y+nx+\beta)$ for (x, y) in $[0, 1)\times[0, 1)$. Hence, T^2 has only a square root T but no other root.

Example 2. For the transformation T of Theorem 10 in case of $p_n=2$ for every n, we have that $\boldsymbol{Q}_0=\boldsymbol{Q}_1=$the group of all 2-adic rational numbers and $\boldsymbol{Q}_2=\{2^n: n\in\boldsymbol{Z}\}$. Then T has a 2^n-th root $U_{1/2^n,\alpha}$ $(\alpha=-(2^n-1)\lambda/2^{2n+1})$ for every $n\geqq 1$ but no other root.

Example 3. For the transformation T of Theorem 10 in case that every prime number appears infinitely often in the sequence $p_1, p_2,\dots, p_n,\dots$, $\boldsymbol{Q}_0=\boldsymbol{Q}_1=\boldsymbol{Q}$ and $\boldsymbol{Q}_2=\boldsymbol{Q}-\{0\}$. Then T has a p-th root $U_{1/p,\alpha}$ $(\alpha=-(p-1)\lambda/2p^2)$ for every $p\geqq 2$, but is not embeddable in a measurable flow as seen in Theorem 10.

Example 4. We consider a dual pair $\boldsymbol{Z}^2=\boldsymbol{Z}\times\boldsymbol{Z}$, $[0, 1)^2=[0, 1)\times[0, 1)$ with an inner product $\langle(n, m), (x, y)\rangle=\exp(2\pi i(nx+my))$, $(n, m)\in\boldsymbol{Z}^2$, $(x, y)\in[0, 1)^2$. For irrational numbers λ and η which are rationally independent define a transformation T of $[0, 1)^4$ by

$$T(x, y, z, w)=(x+\lambda, y+\eta, z+x, w+y) \text{ for } (x, y, z, w) \text{ in } [0, 1)^4.$$

Then by Theorem 9 T is ergodic, $S_p(T)=\{n\lambda+m\eta \pmod 1): (n, m)\in\boldsymbol{Z}^2\}$ and $C(T)=\{U_{\sigma,\alpha}: \sigma$ is an endomorphism of $S_p(T)$, α in $[0, 1)^2\}$, where $U_{\sigma,\alpha}(x, y, z, w)$

$=(x+a\lambda+c\eta,\ y+b\lambda+d\eta,\ z+ax+cy+e,\ w+bx+dy+f)$ for (x, y, z, w) in $[0, 1)^4$ if σ is given by 2×2 matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ with integer components a, b, c, d and $\alpha=(e, f)$. Since 2×2 matrices are not commutative, $C(T)$ is not commutative. Note that this T is the product transformation of ones of Example 1, whose commutants are commutative.

Example 5. For irrational numbers λ and η which are rationally independent define a transformation T of $[0, 1)^3$ by $T(x, y, z)=(x+\lambda, y+\eta, x+y+z)$ for (x, y, z) in $[0, 1)^3$. By the same way as one of Theorem 9 we can see that T is ergodic and $C(T)=\{U_{n,m,a,b}: n, m\in \boldsymbol{Z}, a, b\in[0, 1)\}$, where $U_{n,m,a,b}(x, y, z)=(x+n\lambda+a, y+m\eta-a, z+nx+my+b)$ for (x, y, z) in $[0, 1)^3$. This $C(T)$ is also not commutative.

§4. Application to the Theory of Normalizer of Ergodic Non-singular Transformation

For an ergodic non-singular transformation τ of a Lebesgue space $(W, \mathfrak{F}, \upsilon)$ we denote by $\mathrm{Orb}_\tau(w)$ the orbit $\{\tau^n w: n\in \boldsymbol{Z}\}$ of w in W. The *full group* $[\tau]$ of τ is defined as the set consisting of all non-singular transformations ξ of $(W, \mathfrak{F}, \upsilon)$ such that ξw is in $\mathrm{Orb}_\tau(w)$ for a.e. w in W. A non-singular transformation R of $(W, \mathfrak{F}, \upsilon)$ is called a *normalizer* of $[\tau]$ if $R[\tau]R^{-1}=[\tau]$. An ergodic non-singular transformation τ determines uniquely up to conjugacy an ergodic non-singular flow $\{A_t\}$ called the associated flow of $[\tau]$ ([7], [9]) and a normalizer R of $[\tau]$ determines a non-singular transformation mod R which commutes with the associated flow $\{A_t\}$ ([8]). In this paper we call the pair $([\tau], R)$ a *non-commutative dynamics with characteristic* $(\{A_t\}, \mathrm{mod}\, R)$. T. Hamachi [6] showed that for an ergodic non-singular flow $\{T_t\}$ and a non-singular transformation U in $C(\{T_t\})$ there is a non-commutative dynamics with characteristic $(\{T_t\}, U)$. A non-commutative dynamics $([\tau], R)$ is said to have a *p-th root* if there is a p-th root of R which is also a normalizer of $[\tau]$ and to be *embeddable* in a measurable flow if R is embeddable in a measurable flow which consists of normalizers of $[\tau]$. We can easily see that a non-commutative dynamics $([\tau], R)$ has a p-th root (is embeddable in a measurable flow) mod R has a p-th root in $C(\{A_t\})$ (is embeddable in a measurable flow in $C(\{A_t\})$, respectively).

Let T be an ergodic measure preserving transformation of a Lebesgue space

$(\Omega, \mathfrak{B}, P)$ with $P(\Omega)=1$ and $c>0$. We define a measurable flow $\{\overline{T}_t\}$ on the product set $\overline{\Omega}_c=\Omega\times[0, c)$ by $\overline{T}_t(\omega, u)=(T^n\omega, u+t-cn)$ if $cn\leqq u+t<c(n+1)$, $(\omega, u)\in\overline{\Omega}_c$. $\{\overline{T}_t\}$ is ergodic and called the *special flow* with constant ceiling c with base transformation T.

Lemma 11. *Let $\{\overline{T}_t\}$ be the special flow with constant ceiling $c>0$ with an ergodic base transformation T.*

(1) *Denote by $C(\{\overline{T}_t\})$ the group consisting of all transformations each of which commutes with $\overline{T}_t$ for every t, then $C(\{\overline{T}_t\})=\{\overline{T}_\alpha\overline{U}: \alpha\in[0, 1), U\in C(T)\}$, where $\overline{U}(\omega, u)=(U\omega, u)$ for (ω, u) in $\overline{\Omega}_c$ ([8]).*

(2) *A measurable flow in $C(\{\overline{T}_t\})$ has a form $\overline{T}_{\alpha t}\overline{U}_t$, t in R, where α is a real number and $\{U_t\}$ is a measurable flow in $C(T)$.*

(3) *A transformation $\overline{T}_\alpha\overline{U}$ in $C(\{\overline{T}_t\})$ has a p-th root if and only if T^nU has a p-th root in $C(T)$ for some n.*

Proof. (1) Let Π be the mapping from $\overline{\Omega}_c$ onto $[0, c)$ defined by $\Pi(\omega, u)=u$ for (ω, u) in $\overline{\Omega}_c$ and L a transformation in $C(\{\overline{T}_t\})$. Since $\Pi(L(\omega, u))-u$ is a $\{\overline{T}_t\}$-invariant measurable function and $\{\overline{T}_t\}$ is ergodic, there is a constant α in $[0, c)$ such that $\Pi(L(\omega, u))=u+\alpha \pmod c$ for (ω, u) in $\overline{\Omega}_c$. Hence, L has the form $L(\omega, u)=(U(\omega, u), u+\alpha)$ for (ω, u) in $\overline{\Omega}_c$. From $\overline{T}_{-\alpha}L(\omega, u)=(U(\omega, u), u)$ for (ω, u) in $\overline{\Omega}_c$ and $\overline{T}_t(\overline{T}_{-\alpha}L)=(\overline{T}_{-\alpha}L)\overline{T}_t$ for t in $\boldsymbol{R}$ follows that $U(\omega, u+t)=U(\omega, u)$ if $0\leqq u+t<c$, so that $U(\omega, u)$ does not depend on u in $[0, c)$. From $\overline{T}_c(\overline{T}_{-\alpha}L)=(\overline{T}_{-\alpha}L)\overline{T}_c$ we have $U(T\omega)=TU(\omega)$, that is, U is in $C(T)$. We have $L=\overline{T}_{-\alpha}\overline{U}$.

(2) Let L_t be a measurable flow in $C(\{\overline{T}_t\})$, then by (1) L_t has the form $L_t=\overline{T}_{\alpha(t)}\overline{U}_t$, where for each t, U_t is in $C(T)$ and $\alpha(t)$ is a real number. Since $\alpha(t)=\Pi(L_t(\omega, u))-\Pi(\omega, u)$ is a measurable function and $\alpha(t+s)=\alpha(t)+\alpha(s) \pmod c$ for t, s in R there is a real number α with $\alpha(t)=\alpha t \pmod c$ for t in $\boldsymbol{R}$. Since $\overline{U}_t=\overline{T}_{-\alpha t}L_t$, t in $\boldsymbol{R}$ is a measurable flow, so is $\{U_t\}$.

(3) Let $\overline{T}_\beta\overline{V}$ be a p-th root of $\overline{T}_\alpha\overline{U}$, where U is in $C(T)$ and β is in $[0, 1)$. From $\overline{T}_{p\beta}\overline{V}^p=\overline{T}_\alpha\overline{U}$ there is an integer n such that $p\beta=\alpha-cn$ and $V^p=T^nU$. Conversely, let V be a transformation in $C(T)$ such that $V^p=T^nU$ for some integer n and $\beta=(\alpha-cn)/p$, then $\overline{T}_\beta\overline{V}$ is a p-th root of $\overline{T}_\alpha\overline{U}$.

Example 6. For an irrational number λ define transformations T and U of $[0, 1)^2$ by

$$T(x, y)=(x+2\lambda, y+2x) \text{ and } U(x, y)=(x+\lambda, y+x) \text{ for } (x, y) \text{ in } [0, 1)^2.$$

As seen in Example 1 T is ergodic and U is in $C(T)$. Since $T^nU(x, y)=(x+(2n+1)\lambda, y+(2n+1)x+2n^2\lambda)$ for (x, y) in $[0, 1)^2$ and n in Z, by Theorem 9 we have $C(T^nU)=\{U_{k,j,\beta}: k\in \mathbb{Z}, j=0, 1,\dots, 2n, \beta\in[0, 1)\}$ for n in $\mathbb{Z}$, where $U_{k,j,\beta}(x, y)=(x+k\lambda+j/(2n+1), y+kx+\beta)$ for (x, y) in $[0, 1)^2$. Hence, every T^nU for n in $\mathbb{Z}$ does not have a square root. Therefore, by Lemma 11 (3) $\bar{U}$ does not have a square root in $C(\{\bar{T}_t\})$. Then a non-commutative dynamics with characteristic $(\{\bar{T}_t\}, \bar{U})$ has no square root.

Example 7. For a sequence $p_1, p_2,\dots, p_n,\dots$ of prime numbers in which every prime number appears infinitely many times let $\Omega=[0, 1)\times\prod_{n=1}^{\infty}\{0, 1,\dots, p_n-1\}$ be the compact abelian group as same as one in Theorem 10. For an element λ in Ω such that $\langle\gamma, \lambda\rangle=1$ implies $\gamma=0$ define transformations T and U of $\Omega\times\Omega$ by $T(\omega, \omega')=(\omega+2\lambda, \omega'+2\omega)$ and $U(\omega, \omega')=(\omega+\lambda, \omega'+\omega)$ for (ω, ω') in $\Omega\times\Omega$. Then by Theorem 10 T is ergodic and U in $C(T)$. By the same discussion as one in the proof of Theorem 10 (3) a measurable flow in $C(T)$ has the form $U_{\alpha(t)}(\omega, \omega')=(\omega, \omega'+\alpha(t))$, where $\alpha(t)$ is a 1-parameter subgroup of Ω. Hence, by Lemma 11 (2) a measurable flow in $C(\{\bar{T}_t\})$ has the form $\bar{T}_{\beta t}\bar{U}_{\alpha(t)}(\omega, \omega', u)=(\omega+2n\lambda, \omega'+2n\omega+2n(n-1)\lambda+\alpha(t), u+\beta t-cn)$ if $cn \leqq u+\beta t<c(n+1)$, where β is a constant. Since $\lambda\neq 2n\lambda$ for any integer n, $\bar{U} \neq \bar{T}_{\beta t}\bar{U}_{\alpha(t)}$ for any t, that is, $\bar{U}$ which is in $C(\{\bar{T}_t\})$ is not embeddable in a measurable flow in $C(\{\bar{T}_t\})$. Since U has a p-th root in $C(T)$ for every $p\geqq 2$ as seen in Example 3, there exists a non-commutative dynamics with characteristic $(\{\bar{T}_t\}, \bar{U})$ which has a p-th root for every $p\geqq 2$ but which is not embeddable in a measurable flow.

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