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Gravity Waves on the Free Surface of an Incompressible Perfect Fluid of Finite Depth

By

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§ 1. Introduction

We consider the nonstationary waves on the surface of an incompressible perfect fluid of finite depth above the almost horizontal bottom in the case of two dimensional irrotational motion.

We assume that the density of mass is equal to one, the gravitational field to $(0, -1)$ and at the time $t \geqq 0$ the fluid occupies the domain

$$\Omega(t)=\{(y_1, y_2) \mid y_1 \in \boldsymbol{R}^1, \ -h+b(y_1) \leqq y_2 \leqq \eta(t, y_1)\}$$

where h is a positive constant. We denote by Γ_b the bottom $y_2=-h+b(y_1)$ and by Γ_s the free surface $y_2=\eta(t, y_1)$. The motion of the fluid occupying at $t=0$ the given domain Ω is described by the velocity $v=(v_1, v_2)$, the pressure p of the fluid and η satisfying the equations

$$\frac{\partial}{\partial t} v+(v \cdot \nabla) v=-(0, 1)-\nabla p \qquad \text{for} \quad t \geqq 0, \ y \in \Omega(t) \tag{1.1}$$

$$\frac{\partial}{\partial y_1} v_1+\frac{\partial}{\partial y_2} v_2=\frac{\partial}{\partial y_1} v_2-\frac{\partial}{\partial y_2} v_1=0 \qquad \text{for} \quad t \geqq 0, \ y \in \Omega(t) \tag{1.2}$$

$$\left(\frac{\partial}{\partial t}+v \cdot \nabla\right)(\eta-y_2)=0, \quad p=p_0 \qquad \text{on} \quad \Gamma_s \tag{1.3}$$

$$v \cdot N=0 \qquad \text{on} \quad \Gamma_b \tag{1.4}$$

and taking the prescribed values

$$\eta(0, y_1)=\eta_0(y_1), \quad v(0, y)=v_0(y) \tag{1.5}$$

where $\nabla=\text{grad}$, $v \cdot \nabla=v_1(\partial/\partial y_1)+v_2(\partial/\partial y_2)$, p_0 is a constant, N is the outer normal to Γ_b and v_0 satisfies (1.2) for $y \in \Omega$ and (1.4).

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For the investigation of the solvability of this problem, it is convenient to use the Lagrangian coordinates. Let

$$y_1 = x + X_1(t, x), \quad y_2 = X_2(t, x), \qquad x \in \boldsymbol{R}^1$$

be the parameter-representation of the free surface $y_2 = \eta(t, y_1)$ such that

$$\frac{\partial}{\partial t} X(t, x) = v(t, x + X_1(t, x), X_2(t, x)).$$

We see that on the free surface $X_{tt} = v_t + (v \cdot \nabla) v = -(0, 1) - \nabla p$. On the other hand, differentiating $p(t, x + X_1(t, x), X_2(t, x)) = p_0$ with respect to x, we have $(1 + X_{1x}, X_{2x}) \cdot \nabla p = 0$. Hence we have $(1 + X_{1x}) X_{1tt} + X_{2x}(1 + X_{2tt}) = 0$. It follows from (1.2) and (1.4) that under the appropriate assumptions on v and $\Omega(t)$, $v_2 |_{\Gamma_s}$ is uniquely determined by $v_1 |_{\Gamma_s}$. Therefore we conclude that there exists the operator $K = K(X, b, h)$ such that $X_{2t} = K X_{1t}$. In Section 3 we shall give the operator K the explicit form which enables us to investigate how the operator K depends on X, b and h. In Section 4 the properties of the operator K will be shown. Thus the problem is reduced to the initial value problem

(1.6) $(1 + X_{1x}) X_{1tt} + X_{2x}(1 + X_{2tt}) = 0, \quad X_{2t} = K X_{1t}, \qquad 0 \leqq t \leqq T,$

(1.7) $X = U, \quad X_{1t} = V, \qquad t = 0.$

In this paper we shall show that this problem is uniquely solvable in a Sobolev space when U, V, T and b are small. The proof is based on the quasi-linearization of (1.6) and the successive approximation for the obtained quasi-linear system. Our proof follows that of Nalimov [1] with the modifications caused by the fact that the operator

$$K(0, 0, h) = -i \tanh(hD), \quad D = \frac{1}{i} \frac{d}{dx}$$

is not an isomorphism of H^s. In Section 5 we shall show that by putting

$$Y = X_{tt}, \quad Z = X_x, \quad W = (X, Y, Z), \quad W' = (X, Y_1),$$

we can reduce the problem (1.6), (1.7) to the problem

$$(1.8) \qquad \begin{cases} X_{tt} = Y, \quad Y_{1tt} + a(W) |D| Y_1 = f_1(W, W_t'), \\ Y_{2t} = f_2(W, W_t'), \qquad Z_{jt} = f_{2+j}(W, W_t'), \quad j = 1, 2, \end{cases}$$

$$(1.9) \qquad W = \tilde{W}, \quad W_t' = \tilde{W}_t', \quad t = 0.$$

In applying the successive approximation to the problem (1.8), (1.9), the following initial value problem is fundamental.

$$u_{tt}+a(W)|D|u=g, \qquad 0\leqq t\leqq T, \tag{1.10}$$

$$u=u_0, \qquad u_t=u_1, \quad t=0. \tag{1.11}$$

In Section 6 we shall deal with the initial value problems for these linear and nonlinear equations.

In the case of the infinite depth, i.e., $h=\infty$, V. I. Nalimov [1] showed the unique solvability of (1.6), (1.7) in a Sobolev space. The unique solvability of the problem on the irrotational motion of the incompressible perfect fluid with the free surface has been proved in the class of functions analytic with respect to space variables; in the case of the finite depth in two dimensions, see [2], [3], where the shallow water theory is treated; in three dimensions, see [4], [5].

We turn the reader's attention to that we do not distinguish the inessential positive constants occurring in proofs and use the same symbol C.

Finally I wish to thank T. Nishida who communicated the problem to me and T. Kano for the fruitful discussion with him.

§2. Operators in Sobolev Spaces

In this section we give the results of the functional analysis which will be required in later sections. In solving the problems stated in Section 1, we use only the spaces of real-valued functions of one variable, but here we deal with complex-valued functions of several variables except the last article.

2.1. Notations and Definitions. Let $k\geqq 0$ be an integer, $0<T<\infty$ and B be a Banach space. We say that $u\in C^k([0, T], B)$ if u is a B-valued k-times continuously differentiable function on $[0, T]$. Let Ω be an open set in $\mathbb{R}^n$. By $C^k(\Omega)$, $0\leqq k\leqq\infty$, we denote the set of all functions defined in Ω, which have continuous partial derivatives of order $\leqq k$. By $C_0^k(\Omega)$ we denote the totality of $u\in C^k(\Omega)$ whose support is compact in Ω. By $\mathscr{B}^{k+r}(\Omega)$, ($k\geqq 0$ is an integer, $0\leqq r<1$), we denote the set of all $u\in C^k(\Omega)$ with

$$\|u\|_{\mathscr{B}^{k+r}(\Omega)}=\sup_{|\alpha|\leqq k, x\in\Omega}|D^\alpha u(x)|+\sup_{|\alpha|=k, x, y\in\Omega}|D^\alpha u(x)-D^\alpha u(y)||x-y|^{-r}<\infty$$

where $\alpha=(\alpha_1,\dots, \alpha_n)$, $\alpha_j\geqq 0$ is an integer, $|\alpha|=\alpha_1+\cdots+\alpha_n$, $D_j=(1/i)\partial/\partial x_j$, $D^\alpha=D_1^{\alpha_1}\cdots D_n^{\alpha_n}$. By $\mathscr{S}(\mathbb{R}^n)$ we denote the set of $u\in C^\infty(\mathbb{R}^n)$ such that

$$\sup_x|x^\alpha D^\beta u(x)|<\infty$$

for all α and β, where $x^\alpha=x_1^{\alpha_1}\cdots x_n^{\alpha_n}$. For the details of distribution theory

($\mathscr{S}(\boldsymbol{R}^n)$, $\mathscr{S}'(\boldsymbol{R}^n)$, Fourier transformation, convolution and others), see [6], [8]. We denote by $H^s(\boldsymbol{R}^n)$, $-\infty<s<+\infty$, the set of all $u\in\mathscr{S}'(\boldsymbol{R}^n)$ such that $(1+|\xi|)^s\hat{u}(\xi)\in L_2(\boldsymbol{R}^n)$. $H^s(\boldsymbol{R}^n)$ is a Hilbert space with the inner product

$$(u, v)_s=(2\pi)^{-n}\int(1+|\xi|)^{2s}\hat{u}(\xi)\overline{\hat{v}(\xi)}d\xi, \quad \hat{u}(\xi)=\int u(x)e^{-ix\xi}dx .$$

We put $\|u\|_s=\sqrt{(u, u)_s}$, $(u, v)=(u, v)_0$, $\|u\|=\|u\|_0$. Note that

$$(u, v)=(2\pi)^{-n}\int\hat{u}(\xi)\overline{\hat{v}(\xi)}d\xi=\int u(x)\overline{v(x)}dx, \quad \text{(Parseval's formula)}.$$

For $u=(u_1,\dots, u_m)$, if $u_j\in H^s(\boldsymbol{R}^n)$, $j=1,\dots, m$, then we say that $u\in H^s(\boldsymbol{R}^n)$ and write $(u, v)_s=(u_1, v_1)+\cdots+(u_m, v_m)$, $\|u\|_s=\sqrt{(u, u)_s}$ for $u, v\in H^s(\boldsymbol{R}^n)$. A pseudo-differential operator $P(D)$ with a symbol $P(\xi)$ is defined by

$$P(D)u=(2\pi)^{-n}\int P(\xi)\hat{u}(\xi)e^{ix\xi}d\xi .$$

Note that $\|u\|_s=\|(1+|D|)^su\|$. For the convolution $u*v(x)=\int u(x-y)v(y)dy$ note that $\widehat{u*v}=\hat{u}\hat{v}$, $\widehat{uv}=(2\pi)^{-n}\hat{u}*\hat{v}$.

2.2. Convolution and Mollifier.

Lemma 2.1 (Hausdorff-Young's inequality). *Let* $1\leqq p\leqq q\leqq\infty$, $1-(1/p)+(1/q)=1/r$. *Then for* $f\in L_r(\boldsymbol{R}^n)$, $g\in L_p(\boldsymbol{R}^n)$ *the inequality*

$$\|f*g\|_{L_q(\boldsymbol{R}^n)}\leqq\|f\|_{L_r(\boldsymbol{R}^n)}\|g\|_{L_p(\boldsymbol{R}^n)}$$

holds.

Lemma 2.2 (Hardy-Littlewood-Sobolev's inequality). *Let* $1<p<q<\infty$ *and put* $r=n(1-(1/p)+(1/q))$. *Then for* $f\in L_p(\boldsymbol{R}^n)$

$$\|f*|x|^{-r}\|_{L_q(\boldsymbol{R}^n)}\leqq K\|f\|_{L_p(\boldsymbol{R}^n)}$$

holds where $K=K(p, q, n)>0$. *There is the another formulation equivalent to the above. Let* $p>1$, $q>1$ *and* $(1/p)+(1/q)>1$ *and put* $r=n(2-(1/p)-(1/q))$. *Then for* $f\in L_p(\boldsymbol{R}^n)$, $g\in L_q(\boldsymbol{R}^n)$

$$\left|\int f(x)g(y)|x-y|^{-r}dxdy\right|\leqq C\|f\|_{L_p(\boldsymbol{R}^n)}\|g\|_{L_q(\boldsymbol{R}^n)}$$

holds where $C=C(p, q, n)>0$.

For proofs of Lemmas 2.1 and 2.2 we refer to [7] Section 2.

Take $\varphi\in C_0^\infty(\boldsymbol{R}^n)$ such that $\varphi=1$ in a neighbourhood of $x=0$, $\varphi(x)\geqq 0$ and

$\int \varphi(x)dx=1$ and put $\varphi_\varepsilon(x)=\varepsilon^{-n}\varphi(\varepsilon^{-1}x)$, $\varepsilon>0$. Since $\widehat{\varphi_\varepsilon}(\xi)=\hat{\varphi}(\varepsilon\xi)$, we have $|\hat{\varphi}_\varepsilon(\xi)|\leqq 1$ and $\hat{\varphi}_\varepsilon(\xi)\to 1$, $(\varepsilon\to +0)$. Using the equality $\widehat{\varphi_\varepsilon * u}(\xi)=\hat{\varphi}_\varepsilon(\xi)\hat{u}(\xi)$ and the definition of the norm $\|\cdot\|_s$, we have

Lemma 2.3. *Let* $-\infty<s<+\infty$. *Then for* $u\in H^s(\boldsymbol{R}^n)$ *we have* $\|\varphi_\varepsilon * u\|_s \leqq \|u\|_s$ *and* $\|\varphi_\varepsilon * u-u\|_s\to 0$ *when* $\varepsilon\to +0$.

Lemma 2.4 (see [6] Lemma 6.1). *Let* $a\in\mathscr{B}^1(\boldsymbol{R}^n)$ *and define* A_ε *by*

$$A_\varepsilon u=\varphi_\varepsilon * \left(a\frac{\partial u}{\partial x_j}\right)-a\left(\varphi_\varepsilon * \frac{\partial u}{\partial x_j}\right).$$

Then for $u\in L_2(\boldsymbol{R}^n)$ *we have*

$$\|A_\varepsilon u\|\leqq C\|u\|,\quad \|A_\varepsilon u-u\|\to 0,\qquad (\varepsilon\to +0)$$

where $C>0$ *is independent of* u *and* $\varepsilon>0$.

2.3. Sobolev Spaces. Here we pick up the several facts which we shall use in estimating integral operators. For the proofs of them, see [6] Chapter 7.

Lemma 2.5.

1) $$H^s(\boldsymbol{R}^n)\subsetneq L_p(\boldsymbol{R}^n),\quad 0\leqq s<\frac{n}{2},\quad \frac{1}{p}=\frac{1}{2}-\frac{s}{n}=\frac{1}{n}\left(\frac{n}{2}-s\right),$$

i.e., there exists a constant $C>0$ *such that* $\|u\|_{L_p(\boldsymbol{R}^n)}\leqq C\|u\|_s$ *for any* $u\in H^s(\boldsymbol{R}^n)$.

2) $$H^s(\boldsymbol{R}^n)\subsetneq \mathscr{B}^r(\boldsymbol{R}^n),\ \frac{n}{2}<s,\quad 0\leqq r<s-\frac{n}{2},\quad r<1.$$

Corollary 2.6.

$$H^s(\boldsymbol{R}^n)\subsetneq L_p(\boldsymbol{R}^n),\quad \frac{n}{2}<s,\qquad 2\leqq p\leqq\infty.$$

Lemma 2.7 (see [8] Lemma 2.6.1). *For* $0<s<1$, *there exists* $A=A(s,n)>0$ *such that for any* $u\in\mathscr{S}(\boldsymbol{R}^n)$

$$(2\pi)^{-n}\int|\hat{u}(\xi)|^2(1+|\xi|^{2s})d\xi=\int|u(x)|^2dx+A\iint|u(x+y)-u(x)|^2|y|^{-n-2s}dxdy.$$

Moreover

$$2^{-2s}\|u\|_s^2\leqq(2\pi)^{-n}\int|\hat{u}(\xi)|^2(1+|\xi|^{2s})d\xi\leqq 2\|u\|_s^2.$$

Remark 2.8. Let $0<s<1$, $-\infty<r<+\infty$. Since $\|u\|_{r+s}=\|(1+|D|)^r u\|_s$ and

$$\int |y|^{-n}dy\int |u(x+y)-u(x)|^2|y|^{-2s}dx=\int \|(u(\cdot+y)-u(\cdot))\,|y|^{-s}\|^2_{L_2(\boldsymbol{R}^n)}|y|^{-n}\,dy\,,$$

the norm $\|u\|_{r+s}$ is equivalent to the norm

$$\|u\|_r+\Big(\int_{|y|\le 1}\|(u(\cdot+y)-u(\cdot))\,|y|^{-s}\|_r^2|y|^{-n}dy\Big)^{1/2}.$$

Lemma 2.9. *For any integer $m\geqq[n/2]+1$, $u_j\in H^m(\boldsymbol{R}^n)$, $j=1,\dots,l$, and multi-indices ν_j, $j=1,\dots,l$, $|\nu_1|+\cdots+|\nu_l|\leqq m$, the estimate*

$$\|(D^{\nu_1}u_1)\cdots(D^{\nu_l}u_l)\|\leqq C\|u_1\|_m\cdots\|u_l\|_m$$

holds where $C>0$ is a constant depending only on n, m, l. $H^m(\boldsymbol{R}^n)$ is an algebra, i.e., if u, $v\in H^m(\boldsymbol{R}^n)$ then $uv\in H^m(\boldsymbol{R}^n)$ and $\|uv\|_m\leqq C\|u\|_m\|v\|_m$.

Remark 2.10. By Remark 2.8 and simple calculations, we have the estimate

$$\|(D^{\nu_1}u_1)\cdots(D^{\nu_l}u_l)\|_r\leqq C\|u_1\|_{m+r}\cdots\|u_l\|_{m+r}$$

for $u_j\in H^{m+r}(\boldsymbol{R}^n)$, $0<r<1$. Therefore $H^s(\boldsymbol{R}^n)$ is an algebra for any real $s\geqq[n/2]+1$. Let u, $v\in\mathscr{S}(\boldsymbol{R}^n)$, $s>n/2$. From

$$(1+|\xi|)^s\widehat{uv}(\xi)=(2\pi)^{-n}\int(1+|\xi|)^s\hat{u}(\xi-\eta)\hat{v}(\eta)d\eta$$

and $(1+|\xi|)^s\leqq(1+|\xi-\eta|+|\eta|)^s\leqq 2^s(1+|\xi-\eta|)^s+2^s(1+|\eta|)^s$ we obtain by Lemma 2.1

$$\|uv\|_s\leqq C\|u\|_s\int(1+|\eta|)^s|\hat{v}(\eta)|(1+|\eta|)^{-s}d\eta$$
$$+C\|v\|_s\int(1+|\xi|)^s|\hat{u}(\xi)|(1+|\xi|)^{-s}d\xi\leqq C\|u\|_s\|v\|_s.$$

Consequently $H^s(\boldsymbol{R}^n)$ is an algebra for $s>n/2$.

Lemma 2.11. *For u, $v_j\in H^m(\boldsymbol{R}^n)$, $j=1,\dots,l$, the estimate*

$$\|F(\cdot,v)u\|_m\leqq C\|F\|_{\mathscr{B}^m(\Omega)}(1+\|v\|_m)^m\|u\|_m$$

holds where m is an integer $\geqq[n/2]+1$, Ω is an open set containing $\{(x,v(x))\in\boldsymbol{R}^n\times\boldsymbol{C}^l\,|\,x\in\boldsymbol{R}^n\}$, F belongs to $\mathscr{B}^m(\Omega)$ and $C=C(m,n,l)>0$.

Remark 2.12. Using Remark 2.8 we obtain

$$\|F(\cdot,v)u\|_{m+r}\leqq C\|F\|_{\mathscr{B}^{m+1}(\Omega)}(1+\|v\|_m)^m(1+\|v\|_{m+r})\,\|u\|_{m+r}$$

where $0<r<1$ and Ω is an open set containing $\{(x,z)\,|\,x\in\boldsymbol{R}^n,\ z\in\boldsymbol{C}^l,\ |z|\leqq\sup|v|\}$.

2.4. Estimates for Commutators. Here we deal with the case $n=1$, so we omit $\boldsymbol{R}^1$ in the notations. It is known that

$$(2.13)\quad \widehat{\text{v.p.}\frac{1}{x}}=-\pi i\operatorname{sgn}\xi,\quad \widehat{\frac{c}{x^2+c^2}}=\pi e^{-c|\xi|},\quad \widehat{\frac{x}{x^2+c^2}}=-\pi i e^{-c|\xi|}\operatorname{sgn}\xi$$

where $\operatorname{sgn}\xi=1$ for $\xi>0$, $\operatorname{sgn}\xi=-1$ for $\xi<0$ and $c>0$.

Lemma 2.14. *Let $r\geqq 0$, $s>1/2$ and m be an integer $\geqq 2$. For $a, u\in\mathscr{S}$ we have*

1) $\|[\operatorname{sgn} D, a]u\|_r\leqq C\|a\|_{r+t}\|u\|_{s-t},\quad t\geqq 0$
2) $\|[D^m, a]u\|\leqq C\|a\|_m\|u\|_{m-1}$
3) $\|[(1+|D|)^t, a]u\|\leqq C\|a\|_t\|u\|_{t-1},\quad t>3/2$
4) $\|[|D|^t, a]u\|\leqq C\|a\|_{1+s}\|u\|_{t-1},\quad 0<t\leqq 1$

where $[A, B]=AB-BA$ and C is a constant independent of a and u.

Proof. 1) Put $v=[\operatorname{sgn} D, a]u$. Then we have

$$(1+|\xi|)^r\hat{v}(\xi)=(2\pi)^{-1}\int(1+|\xi|)^r(\operatorname{sgn}\xi-\operatorname{sgn}\eta)\hat{a}(\xi-\eta)\hat{u}(\eta)d\eta\,.$$

If $\operatorname{sgn}\xi-\operatorname{sgn}\eta\neq 0$ then $\operatorname{sgn}\xi=-\operatorname{sgn}\eta$ and $|\xi|+|\eta|=\xi\operatorname{sgn}\xi+\eta\operatorname{sgn}\eta=(\xi-\eta)\cdot\operatorname{sgn}\xi\leqq|\xi-\eta|$. Since t, $r\geqq 0$ we have

$$(1+|\xi|)^r|\hat{v}(\xi)|\leqq\frac{1}{\pi}\int(1+|\xi-\eta|)^r(1+|\xi-\eta|)^t|\hat{a}(\xi-\eta)|(1+|\eta|)^{-t}|\hat{u}(\eta)|d\eta\,.$$

Taking L_2-norm with respect to ξ and using Hausdorff-Young's inequality, we obtain

$$\begin{aligned}\|v\|_r&\leqq\frac{1}{\pi}\|a\|_{r+t}\int(1+|\eta|)^{-t}|\hat{u}(\eta)|d\eta\\&\leqq\frac{1}{\pi}\|a\|_{r+t}\Big(\int(1+|\eta|)^{-2s}d\eta\Big)^{1/2}\Big(\int(1+|\eta|)^{2s-2t}|\hat{u}(\eta)|^2d\eta\Big)^{1/2}\\&=C\|a\|_{r+t}\|u\|_{s-t}\,.\end{aligned}$$

2) Put $v=[D^m, a]u$. From the estimate $|\xi^m-\eta^m|\leqq C|\xi-\eta|\{(1+|\xi-\eta|)^{m-1}+(1+|\eta|)^{m-1}\}$ we obtain

$$\begin{aligned}|\hat{v}(\xi)|\leqq C&\int(1+|\xi-\eta|)^m|\hat{a}(\xi-\eta)\hat{u}(\eta)|d\eta\\&+C\int(1+|\xi-\eta|)|\hat{a}(\xi-\eta)|(1+|\eta|)^{m-1}|\hat{u}(\eta)|d\eta\,.\end{aligned}$$

Since $m\geqq 2$, we can choose q such that $1/2<q\leqq m-1$. Therefore

$$\|v\| \leqq C\|a\|_m \int (1+|\eta|)^q |\hat{u}(\eta)| (1+|\eta|)^{-q} d\eta$$

$$+ C\|u\|_{m-1} \int (1+|\xi|)^{1+q} |\hat{a}(\xi)| (1+|\xi|)^{-q} d\xi$$

$$\leqq C\|a\|_m \|u\|_q + C\|u\|_{m-1} \|a\|_{1+q} \leqq C\|a\|_m \|u\|_{m-1}.$$

3) From $|(1+|\xi|)^t - (1+|\eta|)^t| \leqq C|\xi-\eta| \{(1+|\xi-\eta|)^{t-1} + (1+|\eta|)^{t-1}\}$ and $1/2<q<t-1$, we obtain 3) in the same way as in 2).

4) Put $v=[|D|^t, a]u$. Then we have

$$\hat{v}(\xi) = (2\pi)^{-1} \int (|\xi|^t - |\eta|^t)(1+|\eta|)^{1-t} \hat{a}(\xi-\eta)(1+|\eta|)^{t-1} \hat{u}(\eta) d\eta.$$

Noting that $0<t\leqq 1$, we estimate $f = ||\xi|^t - |\eta|^t|(1+|\eta|)^{1-t}$. For $|\eta| \leqq 1$, $f \leqq C(1+|\xi-\eta|)$. For $|\eta| \geqq 1$ and $|\xi| \geqq |\eta|$,

$$f \leqq |\xi|^t \left(1 - \left(\frac{|\eta|}{|\xi|}\right)^t\right)(2|\eta|)^{1-t} \leqq |\xi|^t \left(1 - \frac{|\eta|}{|\xi|}\right)(2|\eta|)^{1-t}$$

$$\leqq (|\xi| - |\eta|) \left(\frac{2|\eta|}{|\xi|}\right)^{1-t} \leqq 2|\xi-\eta|.$$

For $|\eta| \geqq 1$ and $|\xi| \leqq |\eta|$,

$$f \leqq \left(1 - \left(\frac{|\xi|}{|\eta|}\right)^t\right)|\eta|^t (2|\eta|)^{1-t} \leqq \left(1 - \frac{|\xi|}{|\eta|}\right) 2^{1-t} |\eta| \leqq 2|\xi-\eta|.$$

Hence we have $f \leqq C(1+|\xi-\eta|)$ and

$$|\hat{v}(\xi)| \leqq C \int (1+|\xi-\eta|) |\hat{a}(\xi-\eta)| (1+|\eta|)^{t-1} |\hat{u}(\eta)| d\eta.$$

In the same way as in 1), we obtain $\|v\| \leqq C\|a\|_{1+s} \|u\|_{t-1}$. The proof is complete.

Lemma 2.15. *Let $h>0$, $s\geqq 0$. For $u \in H^0$, the estimates*

$$\|(\operatorname{sgn} D - \tanh(hD))u\|_s \leqq C\|u\|, \quad \|(1-\tanh^2(hD))u\|_s \leqq C\|u\|$$

hold where $C=C(h, s)>0$.

Proof. Since $\tanh(h\xi) = (e^{h\xi} - e^{-h\xi})(e^{h\xi} + e^{-h\xi})^{-1} = (\operatorname{sgn} \xi)\{1 - 2e^{-h|\xi|} \cdot (e^{h|\xi|} + e^{-h|\xi|})^{-1}\}$, we have

$$|\operatorname{sgn} \xi - \tanh(h\xi)| + |1 - \tanh^2(h\xi)| \leqq Ce^{-h|\xi|}.$$

From this we obtain the required estimates.

Lemma 2.16. *For $0<s<1$ and an integer $m\geqq 0$, there exists $A=A(s, m)>0$ such that for any $u \in \mathscr{S}$*

$$(2\pi)^{-1}\int|\hat{u}(\xi)|^2(1+|\xi|^{2m+2s})d\xi$$
$$=\int|u(x)|^2dx+A\iint\left|D_y^m\frac{u(x+y)-u(x)}{y}\right|^2|y|^{1-2s}dxdy.$$

Moreover

$$2^{-2m-2s}\|u\|_{m+s}^2\leqq(2\pi)^{-1}\int|\hat{u}(\xi)|^2(1+|\xi|^{2m+2s})d\xi\leqq 2\|u\|_{m+s}^2.$$

Proof. Using the Parseval's formula,

$$\iint\left|D_y^m\frac{u(x+y)-u(x)}{y}\right|^2|y|^{1-2s}dxdy$$
$$=(2\pi)^{-1}\int|\hat{u}(\xi)|^2d\xi\int\left|D_y^m\frac{e^{iy\xi}-1}{y}\right|^2|y|^{1-2s}dy$$

by the transformation $y\to\xi^{-1}z$,

$$=(2\pi)^{-1}\left(\int|\hat{u}(\xi)|^2|\xi|^{2m+2s}d\xi\right)\left(\int\left|D_z^m\frac{e^{iz}-1}{z}\right|^2|z|^{1-2s}\,dz\right).$$

Since $|D_z^m(e^{iz}-1)/z|^2|z|^{1-2s}\leqq C(1+|z|)^{-2}|z|^{1-2s}$ and $0<s<1$, the integral

$$\int\left|D_z^m\frac{e^{iz}-1}{z}\right|^2|z|^{1-2s}dz$$

converges. It is obvious that if we put this integral equal to A^{-1} then we obtain the equality in question. For $a>0$, $b>0$, we have $1+b^a\leqq 1+(1+b)^a\leqq 2(1+b)^a$. By a substitution $a\to a^{-1}$, we have $1+b^{1/a}\leqq 2(1+b)^{1/a}$, and by $b\to b^a$, $1+b\leqq 2(1+b^a)^{1/a}$. Hence $2^{-a}(1+b)^a\leqq 1+b^a\leqq 2(1+b)^a$ is valid. Putting $a=2m+2s$, $b=|\xi|$, we obtain $2^{-2m-2s}(1+|\xi|)^{2m+2s}\leqq 1+|\xi|^{2m+2s}\leqq 2(1+|\xi|)^{2m+2s}$. If we multiply these by $(2\pi)^{-1}|\hat{u}(\xi)|^2$ and integrate, then we obtain the required inequality.

§3. Representation of the Operator *K*

In this section we give the operator K the representation which is adequate when we investigate the dependence of K on functions defining the bottom and the free surface. We can not assert the validity of the following calculation if we do not indicate which space the functions under consideration belong to, but we proceed with calculations under the ambiguous assumption that all occurring functions are smooth, small and tend to zero when variables tend to the infinity.

Let Ω be the domain in the y_1, y_2-space which is identified with the $z=y_1+iy_2$ plane. Assume that the boundary of Ω consists of Γ_s and Γ_b which are given by

$$(3.1)\quad \begin{cases} \Gamma_s\colon\ (x+X_1(x),\ X_2(x)) \quad\text{or}\quad z(x)=x+X_1(x)+iX_2(x), \\ \Gamma_b\colon\ (x,\ -h+b(x)) \quad\text{or}\quad w(x)=x+i(-h+b(x)), \qquad -\infty<x<+\infty . \end{cases}$$

Let v_1, v_2 be defined in Ω and satisfy the equations

$$\frac{\partial}{\partial y_1}v_1+\frac{\partial}{\partial y_2}v_2=\frac{\partial}{\partial y_2}v_1-\frac{\partial}{\partial y_1}v_2=0 \quad\text{in}\quad \Omega$$
$$v\cdot N=0 \quad\text{on}\quad \Gamma_b .$$

Then $F=v_1-iv_2$ is holomorphic in Ω. Put

$$(3.2)\qquad \begin{cases} f(x)=f_1(x)+if_2(x)=F(z(x)) \\ g(x)=g_1(x)+ig_2(x)=F(w(x)) \\ b_1(x)=\dfrac{db}{dx}(x) . \end{cases}$$

From $v\cdot N=0$ we have $g_2(x)=-b_1(x)g_1(x)$. Taking $z_0\in\Gamma_s$ and the closed path γ in Ω and letting $\gamma\to\Gamma_s\cup\Gamma_b$, we obtain

$$0=\frac{1}{2\pi i}\int_\gamma \frac{F(z)}{z-z_0}dz=\frac{-\pi i}{2\pi i}F(z_0)-\frac{1}{2\pi i}\ \text{v.p.}\int_{\Gamma_s}\frac{F(z)}{z-z_0}dz+\frac{1}{2\pi i}\int_{\Gamma_b}\frac{F(z)}{z-z_0}dz .$$

Using (3.1) and (3.2) we have

$$(3.3)\quad f(x)+\frac{1}{\pi i}\ \text{v.p.}\int\frac{f(y)}{z(y)-z(x)}\frac{dz(y)}{dy}dy=\frac{1}{\pi i}\int\frac{g(y)}{w(y)-z(x)}\frac{dw(y)}{dy}dy .$$

By

$$\begin{aligned}\frac{1}{z(y)-z(x)}\frac{dz(y)}{dy}&=\frac{\partial}{\partial y}\log(z(y)-z(x))=\frac{\partial}{\partial y}\left(\log(y-x)+\log\frac{z(y)-z(x)}{y-x}\right)\\ &=\frac{1}{y-x}+\frac{\partial}{\partial y}\log\left(1+\frac{X_1(y)-X_1(x)}{y-x}+i\frac{X_2(y)-X_2(x)}{y-x}\right),\\ \frac{1}{w(y)-z(x)}\frac{dw(y)}{dy}&=\frac{\partial}{\partial y}\log(w(y)-z(x))\\ &=\frac{\partial}{\partial y}\left(\log(y-x-ih)+\log\frac{w(y)-z(x)}{y-x-ih}\right)\\ &=\frac{y-x+ih}{(y-x)^2+h^2}+\frac{\partial}{\partial y}\log\left(1+\frac{-X_1(x)+ib(y)-iX_2(x)}{y-x-ih}\right)\end{aligned}$$

and (2.13), the real part of (3.3) becomes after the partial integration in the integrals containing log, (3.4) $f_1+i\,\mathrm{sgn}\,Df_2+A_1f_1+A_2f_2=e^{-h|D|}g_1+i\,\mathrm{sgn}\,De^{-h|D|}g_2+A_3g_1+A_4g_2$

where

$$(3.5)\quad \begin{cases} A_ju=\int a_j(x,y)\dfrac{du}{dy}(y)dy, \quad j=1,2,3,4, \\ a_1=-\dfrac{1}{\pi}\,\mathrm{Im}\,\log\left(1+\dfrac{X_1(x)-X_1(y)}{x-y}+i\dfrac{X_2(x)-X_2(y)}{x-y}\right) \\ a_2=-\dfrac{1}{\pi}\,\mathrm{Re}\,\log\left(1+\dfrac{X_1(x)-X_1(y)}{x-y}+i\dfrac{X_2(x)-X_2(y)}{x-y}\right) \\ a_3=-\dfrac{1}{\pi}\,\mathrm{Im}\,\log\left(1+\dfrac{X_1(x)+iX_2(x)-ib(y)}{x-y+ih}\right) \\ a_4=-\dfrac{1}{\pi}\,\mathrm{Re}\,\log\left(1+\dfrac{X_1(x)+iX_2(x)-ib(y)}{x-y+ih}\right). \end{cases}$$

Taking $w_0\in\Gamma_b$ and proceeding in the same way as the above, we obtain

(3.6) $g_1-i\,\mathrm{sgn}\,Dg_2+A_5g_1+A_6g_2=e^{-h|D|}f_1-i\,\mathrm{sgn}\,De^{-h|D|}f_2+A_7f_1+A_8f_2$

where

$$(3.7)\quad \begin{cases} A_ju=\int a_j(x,y)\dfrac{du}{dy}(y)dy, \quad j=5,6,7,8, \\ a_5=\dfrac{1}{\pi}\,\mathrm{Im}\,\log\left(1+i\dfrac{b(x)-b(y)}{x-y}\right) \\ a_6=\dfrac{1}{\pi}\,\mathrm{Re}\,\log\left(1+i\dfrac{b(x)-b(y)}{x-y}\right) \\ a_7=\dfrac{1}{\pi}\,\mathrm{Im}\,\log\left(1+\dfrac{-X_1(y)+ib(x)-iX_2(y)}{x-y-ih}\right) \\ a_8=\dfrac{1}{\pi}\,\mathrm{Re}\,\log\left(1+\dfrac{-X_1(y)+ib(x)-iX_2(y)}{x-y-ih}\right). \end{cases}$$

Eliminating g_1, g_2 from (3.4), (3.6) and $g_2=-b_1g_1$, we obtain

$$\{1-e^{-2h|D|}-i\,\mathrm{sgn}\,D(1+e^{-2h|D|})B_2\}f_1=-i\,\mathrm{sgn}\,D(1+e^{-2h|D|})(1+B_1)f_2\,.$$

Since $f_1=v_1$, $f_2=-v_2$, $K=K(X)=K(X,b,h)$ can be written in the form

$$(3.8)\quad \begin{aligned} K&=-(1+B_1)^{-1}(i\tanh(hD)+B_2) \\ &=-i\tanh(hD)-B_2+B_1(1+B_1)^{-1}(i\tanh(hD)+B_2) \\ &=-i\tanh(hD)+K_1 \end{aligned}$$

where

$$(3.9)\quad \begin{cases} B_1=i\,\mathrm{sgn}\,D\,(1+e^{-2h|D|})^{-1}\{-A_2+e^{-h|D|}A_8+B_3(-i\,\mathrm{sgn}\,De^{-h|D|} \\ \qquad +A_8)-(e^{-h|D|}+B_3)B_4(1+B_4)^{-1}(-i\,\mathrm{sgn}\,De^{-h|D|}+A_8)\}\,, \\ B_2=i\,\mathrm{sgn}\,D(1+e^{-2h|D|})^{-1}\{A_1-e^{-h|D|}A_7-B_3(e^{-h|D|}+A_7) \\ \qquad +(e^{-h|D|}+B_3)B_4(1+B_4)^{-1}(e^{-h|D|}+A_7)\}\,, \end{cases}$$

$$\begin{cases} B_3 = -i \operatorname{sgn} De^{-h|D|}b_1 + A_3 - A_4 b_1\,, \\ B_4 = i \operatorname{sgn} Db_1 + A_5 - A_6 b_1\,, \\ A_j,\ j=1,\dots,8, \text{ are defined by (3.5), (3.7)}\,, \\ b_1 = \dfrac{db}{dx}\,. \end{cases}$$

§4. Estimates for Integral Operators

We shall show that, roughly speaking, K_1 and $[\partial_t^k \partial_x^j, K]$ are operators of order -1. To this end, first of all, we consider the integral operator of the form

$$Au(x) = \int \Big(\prod_1^M \frac{a_j(x) - a_j(y)}{x-y} \Big) F\Big(\frac{b(x) - b(y)}{x-y} \Big) u(y)\,dy \tag{4.1}$$

where $b = (b_1, \dots, b_N)$, a_j, b_j are real-valued and F is smooth in a neighbourhood of $0 \in \boldsymbol{R}^N$, or

$$Au(x) = \int \Big(\prod_1^M \frac{a_j(x) - b_j(y)}{x - y \pm ih} \Big) F\Big(\frac{f(x) - g(y)}{x - y \pm ih} \Big) u(y)\,dy \tag{4.2}$$

where $f = (f_1, \dots, f_N)$, $g = (g_1, \dots, g_N)$ and F is smooth in a neighbourhood of $0 \in \boldsymbol{C}^N$.

In the following two articles we shall show that if functions $a, b, \dots$, occurring in the kernel are in H^s, then Au is in H^s and Lipschitz continuous with respect to $a, b, \dots$. Since $\mathscr{S}$ is dense in H^s, to simplify the statements we shall assume that, unless the contrary is stated, all functions $a, b, \dots$, are in $\mathscr{S}$.

4.1. Operators of the Form (4.1).

Lemma 4.3. *Let k be an integer $\geqq 0$, $r \geqq 0$ and $s > 1/2$. For A defined by*

$$Au(x) = \int \frac{a(x) - a(y)}{x-y} D^k u(y)\,dy = \int \frac{a(x) - a(x-y)}{y} D^k u(x-y)\,dy$$
$$= \int \Big(D_y^k \frac{a(x) - a(x-y)}{y} \Big) u(x-y)\,dy$$

we have $\|Au\|_r \leqq C \|a\|_{k+r} \|u\|_s$ where $C = C(k, r, s) > 0$.

Proof. Since

$$Au = a\Big(\text{v.p.}\ \frac{1}{x} * D^k u \Big) - \text{v.p.}\ \frac{1}{x} * (aD^k u)$$
$$= -\pi i [a, \operatorname{sgn} D] D^k u$$

the lemma follows from Lemma 2.14.

Lemma 4.4. *Let j be an integer $\geqq 0$ and*

$$I(x)=\int\left|D_y^j \frac{a(x)-a(x-y)}{y}\right||u(x-y)|dy .$$

Then $\|I\|\leqq C\|a\|_{j+(1/2)}\|u\|$ *where* $C=C(j)>0$.

Proof. Since

$$\begin{aligned}\int I(x)^2\,dx&\leqq\int dx\int\left|D_y^j\frac{a(x)-a(x-y)}{y}\right|^2 dy\int|u(x-y)|^2dy\\&=\int\left|D_y^j\frac{a(-x-y)-a(-x)}{y}\right|^2 dxdy\,\|u\|^2,\end{aligned}$$

we obtain the required estimate if we put $m=j$, $s=1/2$ in Lemma 2.16.

Lemma 4.5. *Let j, k be integers $\geqq 0$, $s>1/2$ and*

$$I(x)=\int\left|D_y^j\frac{a(x)-a(x-y)}{y}\right|\left|D_y^k\frac{b(x)-b(x-y)}{y}\right|dy .$$

Then $\|I\|\leqq C\|a\|_{j+s}\|b\|_{k+1}$ *where* $C=C(j,k,s)>0$.

Proof. Let $0<r<1$. Then

$$\begin{aligned}\left|\frac{a(x)-a(x-y)}{y}\right|&=|a(x)-a(x-y)|\,|y|^{-r}|y|^{r-1}\\&\leqq C\|a\|_{\mathscr{B}^r}(1+|y|)^{-r}|y|^{r-1}.\end{aligned}\tag{4.6}$$

If $m\geqq 1$ then

$$\begin{aligned}\partial_y^m\frac{a(x)-a(x-y)}{y}&=\partial_y^m\int_{-1}^0 a^{(1)}(x+ty)\,dt=\int_{-1}^0 t^m a^{(m+1)}(x+ty)\,dt\\&=\int_{-1}^0 t^m\frac{1}{y}\frac{\partial}{\partial t}a^{(m)}(x+ty)\,dt\\&=\frac{1}{y}\left\{-(-1)^m a^{(m)}(x-y)-\int_{-1}^0 mt^{m-1}a^{(m)}(x+ty)\,dt\right\}\\&=\frac{m}{y}\int_{-1}^0 t^{m-1}\{a^{(m)}(x-y)-a^{(m)}(x+ty)\}\,dt\\&=-\frac{1}{y}\left\{(-1)^m a^{(m)}(x-y)+m\partial_y^{m-1}\frac{a(x)-a(x-y)}{y}\right\}\end{aligned}\tag{4.7}$$

where $a^{(j)}(x)=\partial_x^j a(x)$. Therefore for $s=0$, r we have

$$\begin{aligned}\left|\partial_y^m\frac{a(x)-a(x-y)}{y}\right|\leqq m|y|^{-1}\int_{-1}^0 &|t|^{m-1}|a^{(m)}(x-y)\\&-a^{(m)}(x+ty)|\,|y+ty|^{-s}|1+t|^s|y|^s\,dt\end{aligned}$$

$$\leqq 2m|y|^{s-1}\left\{\int_{-1}^{0}|t|^{m-1}|t+1|^{s}dt\right\}\|D^m a\|_{\mathscr{B}^s}.$$

Since $0<r<1$, $\int_{-1}^{0}|t|^{m-1}|t+1|^{r}dt<\infty$. These inequalities and (4.6) show that

$$\left|D_y^m\frac{a(x)-a(x-y)}{y}\right|\leqq C\|D^m a\|_{\mathscr{B}^r}(1+|y|)^{-r}|y|^{r-1}$$

holds where $m\geqq 0$ and $C=C(m, r)>0$. From this it follows that

$$\int I(x)^2dx=\int dx\left\{\int\left|D_y^j\frac{a(x)-a(x-y)}{y}\right|\left|D_y^k\frac{b(x)-b(x-y)}{y}\right|dy\right\}^2$$

$$\leqq C\|D^j a\|_{\mathscr{B}^r}^2\left\{\int dx\int(1+|y|)^{-r}|y|^{(1/2)(r-1)}\right.$$

$$\left.\times\left|D_y^k\frac{b(x)-b(x-y)}{y}\right||y|^{(1/2)-(1-(r/2))}dy\right\}^2$$

$$\leqq C\|D^j a\|_{\mathscr{B}^r}^2\int(1+|y|)^{-2r}|y|^{r-1}dy$$

$$\times\int\left|D_y^k\frac{b(x)-b(x-y)}{y}\right|^2|y|^{1-2(1-(r/2))}dxdy.$$

Let r be so small that $0<r<s-(1/2)$. Then from Lemma 2.5 we obtain

$$\|D^j a\|_{\mathscr{B}^r}\leqq C\|D^j a\|_s\leqq C\|a\|_{j+s}.$$

If we put $m=k$, $s=1-(r/2)$ in Lemma 2.16, then the integral containing b is smaller than $C\|b\|_{k+(1-(r/2))}^2$, which proves the lemma.

Lemma 4.8. *Let k be an integer $\geqq 0$, $s>1/2$, $b=(b_1,\dots, b_N)$, b_j be real-valued and*

$$Au(x)=\int\left(D_y^k\frac{a(x)-a(x-y)}{y}\right)F\left(\frac{b(x)-b(x-y)}{y}\right)u(x-y)dy.$$

Then we have $\|Au\|\leqq C\|F\|_{\mathscr{B}^1(\Omega)}\|a\|_k(1+\|b\|_2)\|u\|_s$ where $C=C(k, s, N)>0$, Ω is an open set containing the convex hull of

$$\left\{\frac{b(x)-b(x-y)}{y}\;\middle|\;-\infty<x,\ y<+\infty\right\}\quad \textit{and}\quad F\in\mathscr{B}^1(\Omega).$$

Proof. For a function $f(x)$, we put

$$\tilde{f}(x,y)=\frac{f(x)-f(x-y)}{y},\quad f'(x)=\frac{df}{dx}(x).$$

It is easily seen that $\partial\tilde{f}(x, y)/\partial y+\tilde{f}'(x, y)=-(1/y)(\tilde{f}(x, y)-f'(x))$. Using the formula

$$F(\tilde{b})-F(b')=\sum_1^N\left\{\int_0^1\frac{\partial F}{\partial z_j}(t\tilde{b}+(1-t)b')dt\right\}(\tilde{b}_j-b'_j)$$

(we write this in the form $F(\tilde{b})-F(b')=\sum_1^N F_j(\tilde{b}_j-b'_j)$), we can write Au in the form

$$Au=F(b')\int\Big(D_y^k\frac{a(x)-a(x-y)}{y}\Big)u(x-y)\,dy+\sum_1^N A_ju=Bu+\sum_1^N A_ju$$

$$A_ju=\int\Big(D_y^k\frac{a(x)-a(x-y)}{y}\Big)F_j(\tilde{b}_j-b'_j)u(x-y)\,dy\,.$$

First we assume that $k=0$. Then we have

$$A_ju=\int a(x)F_j\frac{\tilde{b}_j-b'_j}{y}u(x-y)\,dy+\int a(x-y)F_j\Big(\frac{\partial}{\partial y}\tilde{b}_j+\tilde{b}'_j\Big)u(x-y)\,dy\,.$$

For $0<r<1$,

$$\begin{aligned}\Big|\frac{\tilde{b}_j-b'_j}{y}\Big|&\leqq\Big|\frac{1}{y}\int_{-1}^0(b'_j(x+ty)-b'_j(x))\,dt\Big|\\&\leqq\int_{-1}^0|b'_j(x+ty)-b'_j(x)|\,|ty|^{-r}|t|^r|y|^{r-1}\,dt\\&\leqq C\|Db_j\|_{\mathscr{B}^r}(1+|y|)^{-r}|y|^{r-1}\,.\end{aligned}$$

By choosing r, p and q such that $0<r<1/2$, $2<p<\infty$, $(1-r)q<1$ and $(1/p)+(1/q)=1$, we obtain from Corollary 2.6

$$\int(1+|y|)^{-r}|y|^{r-1}|u(x-y)|\,dy\leqq\|(1+|y|)^{-r}|y|^{r-1}\|_{L_q}\|u\|_{L_p}\leqq C\|u\|_s$$

and from Lemma 2.5

$$\|Db_j\|_{\mathscr{B}^r}\leqq C\|Db_j\|_1\leqq C\|b_j\|_2,\quad \sup|u(x)|\leqq C\|u\|_s\,.$$

Therefore we have

$$\begin{aligned}|A_ju(x)|\leqq{}&C\sup|F_j|\,\|b_j\|_2\|u\|_s|a(x)|\\&+C\sup|F_j|\,\|u\|_s\int\Big\{\Big|D_y\frac{b_j(x)-b_j(x-y)}{y}\Big|\,|a(x-y)|\\&+\Big|\frac{b'_j(x)-b'_j(x-y)}{y}\Big|\,|a(x-y)|\Big\}\,dy\,.\end{aligned}$$

Using Lemma 4.4, we have $\|A_ju\|\leqq C\sup|F_j|\,\|a\|\,\|b_j\|_2\|u\|_s$. Next we assume that $k>0$. From (4.7) we see that

$$\partial_y^k\frac{a(x)-a(x-y)}{y}=-\frac{1}{y}\Big\{(-1)^ka^{(k)}(x-y)+k\partial_y^{k-1}\frac{a(x)-a(x-y)}{y}\Big\}\,.$$

This and $\tilde{b}_j-b'_j=-y(\partial_y\tilde{b}_j+\tilde{b}'_j)$ show that

$$A_j u = (-i)^k$$
$$\times \int \left\{ (-1)^k a^{(k)}(x-y) + k\partial_y^{k-1} \frac{a(x)-a(x-y)}{y} \right\} F_j(\partial_y \tilde{b}_j + \tilde{b}'_j) u(x-y) dy .$$

By Lemmas 4.4 and 4.5, we have

$$\begin{aligned} \|A_j u\| &\leqq C \sup |F_j| (\|b_j\|_{1+1/2} + \|b'_j\|_{1/2}) \|a^{(k)}\| \sup |u| \\ &\quad + C \sup |F_j| \|a\|_{k-1+1} (\|b_j\|_{1+1} + \|b'_j\|_{0+1}) \sup |u| \\ &\leqq C \sup |F_j| \|a\|_k \|b_j\|_2 \|u\|_s . \end{aligned}$$

Applying Lemma 4.3 to Bu we have for any $k \geqq 0$

$$\|Au\| \leqq C \sup |F(b')| \|a\|_k \|u\|_s + C \sum_1^N \sup |F_j| \|a\|_k \|b_j\|_2 \|u\|_s , \tag{4.9}$$

which proves the lemma.

If F has the form $F(z) = z_1 \cdots z_M G(z_{M+1}, \dots, z_N)$, then

$$\sup |F(b')| \leqq (\sup |b'_1|) \cdots (\sup |b'_M|) \sup |G|$$

and

$$\sup |F_j| \leqq \begin{cases} (\prod_1^M \sup |b'_i|) \sup \left| \dfrac{\partial G}{\partial z_j} \right| & \text{for } M+1 \leqq j \leqq N \\ (\prod_{i \neq j} \sup |b'_i|) \sup |G| & \text{for } 1 \leqq j \leqq M . \end{cases}$$

Using (4.9) and $\sup |b'_j| \leqq C \|b_j\|_2$, we have

Lemma 4.10. *Let k be an integer $\geqq 0$, $d = (d_1, \dots, d_N)$, d_j be real-valued and*

$$Au(x) = \int \left(D_y^k \frac{a(x)-a(x-y)}{y} \right)$$
$$\times \prod_1^M \frac{b_j(x)-b_j(x-y)}{y} G\left(\frac{d(x)-d(x-y)}{y} \right) u(x-y) dy.$$

Then we have

$$\|Au\| \leqq C \|G\|_{\mathscr{B}^1(\Omega)} \|a\|_k \|b_1\|_2 \cdots \|b_M\|_2 (1 + \|d\|_2) \|u\|_s ,$$

where $C = C(k, s, M, N) > 0$, Ω is an open set containing the convex hull of $\{(d(x)-d(x-y))/y \mid -\infty < x, y < +\infty\}$ and $G \in \mathscr{B}^1(\Omega)$.

Lemma 4.11. *Let m be an integer $\geqq 2$, $s > 1/2$, $b = (b_1, \dots, b_N)$, b_j be real-valued and*

$$Au(x) = \int \frac{a(x)-a(y)}{x-y} F\left(\frac{b(x)-b(y)}{x-y} \right) u(y) dy .$$

Then we have

$$\|Au\|_m \leqq C\|F\|_{\mathscr{B}^m(\Omega)}\|a\|_m(1+\|b\|_m)^m\|u\|_s$$

where $C=C(m, s, N)>0$, Ω *is an open set containing the convex hull of* $\{(b(x)-b(y))/(x-y) \mid -\infty<x, y<+\infty\}$ *and* $F\in\mathscr{B}^m(\Omega)$.

Proof. Note that $\|Au\|_m \leqq C(\|Au\|+\|D^m Au\|)$. Putting $k=0$ in Lemma 4.8 and noting that $m\geqq 2$, we have

$$\|Au\| \leqq C\|F\|_{\mathscr{B}^m(\Omega)}\|a\|_m(1+\|b\|_m)^m\|u\|_s .$$

After the replacement of y by $x-y$, m-times differentiation with respect to x under the integral sign and the partial integration, $D^m Au(x)$ can be written in the form

$$D^m Au(x)=\sum_0^m \binom{m}{k}\int \left\{D_y^k D_x^{m-k}(\tilde{a}F(\tilde{b}))\right\}u(x-y)\,dy=\sum_0^m \binom{m}{k}A_k u(x)\,,$$

$$\begin{aligned} A_k u(x)=&\int (D_y^k D_x^{m-k}\tilde{a})F(\tilde{b})u(x-y)\,dy \\ &+\sum_1^N \int \tilde{a}\frac{\partial F}{\partial z_j}(\tilde{b})(D_y^k D_x^{m-k}\tilde{b}_j)u(x-y)\,dy \\ &+\sum_{J,n,p,q}\int (D_y^{p_0}D_x^{q_0}\tilde{a})F^{J,n,p,q}(\tilde{b})\prod_{j=1}^J (D_y^{p_j}D_x^{q_j}\tilde{b}_{n_j})u(x-y)\,dy \end{aligned}$$

where J, n, p and q move in the set such that

$$\text{(4.12)}\qquad \begin{cases} 1\leqq J\leqq m\,, \\ p_j\leqq k,\ q_j\leqq m-k,\ p_j+q_j\leqq m-1, \quad (0\leqq j\leqq J), \\ 1\leqq n_j\leqq N,\ 1\leqq p_j+q_j, \quad (1\leqq j\leqq J), \\ p_0+q_0+\cdots+p_J+q_J=m \end{cases}$$

and $F^{J,p,q,n}(z)$ is the linear combination of $(\partial/\partial z)^\alpha F(z)$, $|\alpha|\leqq J$. We put $A_k u=A_{k1}u+A_{k2}u+A_{k3}u$. Since $D_x^{m-k}\tilde{a}=\widetilde{D^{m-k}a}$, applying Lemma 4.8 to $A_{k1}u$, we have

$$\begin{aligned} \|A_{k1}u\| &\leqq C\|F\|_{\mathscr{B}^1(\Omega)}\|D^{m-k}a\|_k(1+\|b\|_2)\,\|u\|_s \\ &\leqq C\|F\|_{\mathscr{B}^m(\Omega)}\|a\|_m(1+\|b\|_m)^m\|u\|_s . \end{aligned}$$

From Lemma 4.10 we have

$$\begin{aligned} \|A_{k2}u\| &\leqq \sum_1^N C\left\|\frac{\partial F}{\partial z_j}\right\|_{\mathscr{B}^1(\Omega)}\|D^{m-k}b_j\|_k\|a\|_2(1+\|b\|_2)\,\|u\|_s \\ &\leqq C\|F\|_{\mathscr{B}^m(\Omega)}\|a\|_m(1+\|b\|_m)^m\|u\|_s . \end{aligned}$$

We may carry out the estimate for $A_{k3}u$ under the assumption that

(4.13) $$p_1+q_1\geqq p_j+q_j, \quad 1\leqq j\leqq J.$$

When $p_0+q_0+p_1+q_1\geqq 2$, we see from (4.12) that $p_j+q_j\leqq m-2$ for $2\leqq j\leqq J$. Using (4.7), we have for $2\leqq j\leqq J$

$$|D_y^{p_j}D_x^{q_j}\tilde{b}_{n_j}|\leqq \sup |D^{1+p_j+q_j}b_{n_j}|\leqq C\|b_{n_j}\|_{1+p_j+q_j+1}\leqq C\|b_{n_j}\|_m .$$

Hence

$$\begin{aligned} I&=|(D_y^{p_0}D_x^{q_0}\tilde{a})\prod_{j=1}^{J}(D_y^{p_j}D_x^{q_j}\tilde{b}_{n_j})F^{J,n,p,q}(\tilde{b})| \\ &\leqq C|D_y^{p_0}D_x^{q_0}\tilde{a}|\,|D_y^{p_1}D_x^{q_1}\tilde{b}_{n_1}|\,\|b_{n_2}\|_m\cdots\|b_{n_J}\|_m \sup|F^{J,n,p,q}(\tilde{b})| . \end{aligned}$$

When $p_0+q_0+p_1+q_1<2$, by (4.13) and $p_j+q_j\geqq 1$ for $j\geqq 1$ we have $p_0+q_0=0$, $p_j+q_j=1$ for $j\geqq 1$. Moreover from (4.12) it follows that $J=m$. If $m\geqq 3$ then

$$|D_y^{p_j}D_x^{q_j}\tilde{b}_{n_j}|\leqq C\|b_{n_j}\|_3\leqq C\|b_{n_j}\|_m \qquad \text{for} \quad j\geqq 2 .$$

If $m=2$ then $J=m=2$ and

$$\begin{aligned} I&\leqq C(\sup|\tilde{a}|)|D_y^{p_1}D_x^{q_1}\tilde{b}_{n_1}|\,|D_y^{p_2}D_x^{q_2}\tilde{b}_{n_2}| \sup|F^{J,n,p,q}(\tilde{b})| \\ &\leqq C|D_y^{p_1}D_x^{q_1}\tilde{b}_{n_1}|\,|D_y^{p_2}D_x^{q_2}\tilde{b}_{n_2}|\,\|a\|_2 \sup|F^{J,n,p,q}(\tilde{b})| . \end{aligned}$$

By Lemma 4.5 we have

$$\|A_{k3}u\|\leqq C\|F\|_{\mathscr{B}^m(\Omega)}\|a\|_m(1+\|b\|_m)^m\|u\|_s .$$

This completes the proof.

The same consideration as in the derivation of Lemma 4.10 from (4.9) leads to the following lemma.

Lemma 4.14. *Let m be an integer $\geqq 2$, $s>1/2$, $b=(b_1,\dots, b_N)$, b_j be real-valued and*

$$Au(x)=\int\Big(\prod_{j=1}^{M}\frac{a_j(x)-a_j(y)}{x-y}\Big)F\Big(\frac{b(x)-b(y)}{x-y}\Big)u(y)\,dy .$$

Then we have

$$\|Au\|_m\leqq C\|F\|_{\mathscr{B}^m(\Omega)}\Big(\prod_1^M\|a_j\|_m\Big)(1+\|b\|_m)^m\|u\|_s$$

where $C=C(m, s, M, N)>0$, Ω is an open set containing the convex hull of $\{(b(x)-b(y))/(x-y)\mid -\infty<x, y<+\infty\}$ and $F\in\mathscr{B}^m(\Omega)$.

Lemma 4.15. *Let m be an integer $\geqq 2$, $s>1/2$ and $A=A(a, b)$ be the operator defined in the above lemma. Then we have*

$$\|A(a^1, b^1)u - A(a^2, b^2)u\|_m \leqq C\|F\|_{\mathscr{B}^{m+1}(\Omega)}(1+\|a^1\|_m+\|a^2\|_m)^M(1+\|b^1\|_m + \|b^2\|_m)^m(\|a^1-a^2\|_m+\|b^1-b^2\|_m)\|u\|_s$$

where $C=C(m, s, M, N)>0$, Ω *is an open set containing the convex hull of* $\{(b^k(x)-b^k(y))/(x-y) \mid -\infty<x, y<+\infty, k=1, 2\}$ *and* $F\in\mathscr{B}^{m+1}(\Omega)$.

Proof. Note that

$$(4.16)\quad A(a^1, b^1)u - A(a^2, b^2)u = \int_0^1 \frac{\partial}{\partial t} A(ta^1+(1-t)a^2, tb^1+(1-t)b^2)u\,dt = \int_0^1 dt\int Gu\,dy$$

where

$$G=\sum_{k=1}^{M}(\check{a}_k^1-\check{a}_k^2)\prod_{j\neq k}\{t\check{a}_j^1+(1-t)\check{a}_j^2\}F(t\check{b}^1+(1-t)\check{b}^2) + \prod_{j=1}^{M}\{t\check{a}_j^1+(1-t)\check{a}_j^2\}\sum_{k=1}^{N}\frac{\partial F}{\partial z_k}(t\check{b}^1+(1-t)\check{b}^2)(\check{b}_k^1-\check{b}_k^2),$$

$$\check{a}=\frac{a(x)-a(y)}{x-y},\quad \check{b}=\frac{b(x)-b(y)}{x-y}.$$

Since

$$\|A(a^1, b^1)u-A(a^2, b^2)u\|_m = \left\|\int_0^1 dt\int Gu\,dy\right\|_m \leqq \int_0^1 dt\left\|\int Gu\,dy\right\|_m$$

the required estimate is obtained from Lemma 4.14.

Remark 4.17. The above two lemmas hold also for $m+r$, $m\geqq 2$, $0<r<1$. If we define the translation operator T_z, $z\in\mathbb{R}^1$, by $T_zf(x)=f(x+z)$ then it is clear that $T_zA(a, b)u=A(T_za, T_zb)T_zu$. Since $T_zA(a, b)u-A(a, b)u = A(T_za, T_zb)T_zu-A(a, b)T_zu+A(a, b)(T_zu-u)$, by Remark 2.8, Lemma 4.14 and (4.16) we have

$$\|Au\|_{m+r}\leqq C\|F\|_{\mathscr{B}^{m+1}(\Omega)}\prod_1^M\|a_j\|_{m+r}(1+\|b\|_{m+r})^{m+1}\|u\|_{s+r}.$$

This combined with (4.16) leads to the estimate

$$\|A(a^1, b^1)u-A(a^2, b^2)u\|_{m+r} \leqq C\|F\|_{\mathscr{B}^{m+2}(\Omega)}(1+\|a^1\|_{m+r}+\|a^2\|_{m+r})^M(1+\|b^1\|_{m+r}+\|b^2\|_{m+r})^{m+1} \times(\|a^1-a^2\|_{m+r}+\|b^1-b^2\|_{m+r})\|u\|_{s+r}.$$

4.2. Operators of the Form (4.2).

Lemma 4.18. *Let m be an integer* $\geqq 1$, $h>0$ *and*

$$Au(x)=\int\Big(\prod_1^M \frac{a_j(x)-b_j(y)}{x-y\pm ih}\Big)F\Big(\frac{f(x)-g(y)}{x-y\pm ih}\Big)u(y)dy\,.$$

Then

$$\|Au\|_m \leqq C\|F\|_{\mathscr{B}^m(\Omega)}\prod_1^M(\|a_j\|_m+\|b_j\|_1)(1+\|f\|_m+\|g\|_1)^m\|u\|$$

where $C=C(m, h, M, N)>0$, Ω *is an open set containing*

$$\Big\{\frac{f(x)-g(y)}{x-y\pm ih}\Big| -\infty<x,\ y<+\infty\Big\} \quad \textit{and} \quad F\in\mathscr{B}^m(\Omega)\,.$$

Proof. Put

$$I(x)=\int\Big|\frac{u(y)v(y)}{x-y\pm ih}\Big|dy\,.$$

If $v=1$ then $I(x)\leqq \|(x\pm ih)^{-1}\|\,\|u\|\leqq C\|u\|$. Since

$$I(x)^2\leqq\|u\|^2\int\Big|\frac{v(y)}{x-y\pm ih}\Big|^2dy,$$

by Hausdorff-Young's inequality, we have

$$\|I\|^2\leqq\|u\|^2\Big\|\frac{1}{|x\pm ih|^2}\Big\|_{L_1}\||v|^2\|_{L_1},\ \text{i.e.,}\quad \|I\|\leqq C\|u\|\,\|v\|\,.$$

From

$$|Au(x)|\leqq\Big(|a_1(x)|\int\Big|\frac{u(y)}{x-y\pm ih}\Big|dy+\int\Big|\frac{b_1(y)u(y)}{x-y\pm ih}\Big|dy\Big)\sup_{x,y}\Big|\prod_2^M\frac{a_j(x)-b_j(y)}{x-y\pm ih}F\Big|$$

we obtain

$$\|Au\|\leqq C(\|a_1\|\,\|u\|+\|b_1\|\,\|u\|)\Big\{\prod_2^M\frac{1}{h}(\|a_j\|_1+\|b_j\|_1)\Big\}\,\|F\|_{\mathscr{B}^0(\Omega)}\,.$$

By the differentiation under the integral sign we can divide D^mAu into two parts: The first contains D^ma, D^mf and the second D^ka, D^kf, $k<m$. The above method is available for the first part and also for the second if we note that $|D_x^j(x-y\pm ih)^{-1}|\leqq C|x-y\pm ih|^{-1}$. Since $\|Au\|_m\leqq C\|Au\|+C\|D^mAu\|$, we obtain the required inequality.

Lemma 4.19. *Let m be an integer* $\geqq 1$, $h>0$ *and* $A=A(a, b, f, g)$ *be the operator defined in the above lemma. Then we have*

$$
\begin{aligned}
&\|A(a^1, b^1, f^1, g^1)u - A(a^2, b^2, f^2, g^2)u\|_m \\
&\quad \leqq C\|F\|_{\mathscr{B}^{m+1}(\Omega)}(1+\|a^1\|_m+\|a^2\|_m+\|b^1\|_1+\|b^2\|_1)^M \\
&\qquad \times (1+\|f^1\|_m+\|f^2\|_m+\|g^1\|_1+\|g^2\|_1)^m \\
&\qquad \times (\|a^1-a^2\|_m+\|b^1-b^2\|_1+\|f^1-f^2\|_m+\|g^1-g^2\|_1)\|u\|
\end{aligned}
$$

where $C=C(m, h, M, N)>0$, Ω *is an open set containing the convex hull of*

$$
\left\{\frac{f^k(x)-g^k(y)}{x-y\pm ih}\,\Big|\, -\infty<x,\ y<+\infty,\ k=1,\ 2\right\} \quad \textit{and} \quad F\in\mathscr{B}^{m+1}(\Omega)\,.
$$

The lemma is proved by the method used for Lemma 4.15.

Remark 4.20. By the same consideration as in Remark 4.17, we have for $m\geqq 1$, $0<r<1$,

$$
\|Au\|_{m+r}\leqq C\|F\|_{\mathscr{B}^{m+1}(\Omega)}\prod_1^M(\|a_j\|_{m+r}+\|b_j\|_{1+r})(1+\|f\|_{m+r}+\|g\|_{1+r})^{m+1}\|u\|_r,
$$

$$
\begin{aligned}
&\|A(a^1, b^1, f^1, g^1)u - A(a^2, b^2, f^2, g^2)u\|_{m+r} \\
&\quad \leqq C\|F\|_{\mathscr{B}^{m+2}(\Omega)}(1+\|a^1\|_{m+r}+\|a^2\|_{m+r}+\|b^1\|_{1+r}+\|b^2\|_{1+r})^M \\
&\qquad \times(1+\|f^1\|_{m+r}+\|f^2\|_{m+r}+\|g^1\|_{1+r}+\|g^2\|_{1+r})^{m+1}(\|a^1-a^2\|_{m+r} \\
&\qquad +\|b^1-b^2\|_{1+r}+\|f^1-f^2\|_{m+r}+\|g^1-g^2\|_{1+r})\|u\|_r.
\end{aligned}
$$

4.3. The Operator *K*. Let A be an operator of the form (4.1) or (4.2) which we write in the form $Au(x)=\int A(x, y)u(y)dy$. Since

$$
\left[\frac{\partial}{\partial x}, A\right]u=\int\left\{\left(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}\right)A(x, y)\right\}u(y)dy
$$

$$
\left(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}\right)\frac{a(x)-a(y)}{x-y}=\frac{a'(x)-a'(y)}{x-y}, \quad a'(x)=\partial_x a(x)
$$

$$
\left(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}\right)\frac{a(x)-b(y)}{x-y\pm ih}=\frac{a'(x)-b'(y)}{x-y\pm ih},
$$

we see that $[\partial/\partial x, A]$ is the sum of operators of the form (4.1) or (4.2). If $A(x, y)$ and u depend on t, then $[\partial/\partial t, A]u=\int(\partial A(x, y)/\partial t)u(y)dy$. Hence $[\partial/\partial t, A]$ is the linear combination of operators of the form (4.1) or (4.2).

Note that $\log(1+z)=zf(z)$ where $f(z)$ is holomorphic in z, $\operatorname{Re} z>-1$ and

$$
\left|\frac{a(x)-a(y)}{x-y}\right|\leqq \sup|a'(x)|\leqq C\|a\|_{1+s},
$$

$$
\left|\frac{a(x)-b(y)}{x-y\pm ih}\right|\leqq\frac{1}{h}(|a(x)|+|b(y)|)\leqq C(\|a\|_s+\|b\|_s), \quad s>\frac{1}{2}.
$$

By (3.5) and (3.7) we see that, if X and b are small in H^s, then A_j can be written in the form (4.1) or (4.2) where u is replaced by du/dy. This situation leads us to the following

Definition 4.21. Let $0 \leqq r \leqq s$, $0 \leqq t \leqq s$. $L(r, s; t)$ is the totality of M satisfying the following conditions: 1) $M = M(P; P(J))$ is the linear operator depending on $P = (P_1, \dots, P_k)$ where P_j is the real-valued function, J is the subset of $\{1, \dots, k\}$, $P(J) = (P_{j_1}, \dots, P_{j_l})$ if $J = \{j_1, \dots, j_l\}$ and we write $M = M(P; 0)$ if J is empty. 2) There exists $d = d(M, t) > 0$ such that if $d_0 > 0$, P, $P^0 \in H^s$, $\|P\|_s$, $\|P^0\|_s \leqq d_0$, $\|P(J)\|_t$, $\|P^0(J)\|_t \leqq d$ then $\|M(P; P(J))u\|_s \leqq C\|u\|_r$, $\|M(P; P(J))u - M(P^0; P^0(J))u\|_s \leqq C\|P - P^0\|_s \|u\|_r$ for $u \in H^r$ where $C = C(r, s, t, d, d_0) > 0$. $L_0(r, s; t)$ consists of $M \in L(r, s; t)$ such that $\|M(P; P(J))u\|_s \leqq C\|P\|_s \|u\|_r$.

Lemma 4.22.

1) *$L(r, s; t)$, $L_0(r, s; t)$ are algebras.*
2) *$L_0(r, s; t)$ is a two-sided $L(r, s; t)$-module.*
3) *If f is smooth in a neighbourhood of $0 \in \mathbf{R}^k$ then the operator M defined by $M(P; P)u = f(P)u$ belongs to $L(s, s; t)$ for $1/2 < t \leqq s$, $1 \leqq s$.*
4) *$M(P; P) = (1 + P_1 + P_2\, i \operatorname{sgn} D)^{-1} \in L(s, s; t)$ for $1/2 < t \leqq s$.*

Proof. 1) and 2) are trivial. 3) follows from Lemma 2.11 and Remark 2.12. It remains to show 4). Let $P_1, P_2 \in H^t$. Then by Remark 2.10, $\|(P_1 + P_2\, i \operatorname{sgn} D)u\|_t \leqq C\|P\|_t \|u\|_t$. Hence if P_1 and P_2 are small in H^t, then we have (a) $\|M(P; P)u\|_t \leqq C\|u\|_t$. Note that $T_z u(x) = u(x+z) = \exp(izD)u(x)$ and $[T_z, P_j] = (P_j(\cdot + z) - P_j)T_z$. Since $T_z Mu - Mu = [T_z, M]u + M(T_z u - u) = -M[T_z, M^{-1}]Mu + M(T_z u - u)$ and

$$[T_z, 1 + P_1 + P_2\, i \operatorname{sgn} D] = (P_1(\cdot + z) - P_1)T_z + (P_2(\cdot + z) - P_2)T_z\, i \operatorname{sgn} D,$$

we have (b) $\|T_z Mu - Mu\|_t \leqq C\|T_z P - P\|_t \|u\|_t + C\|T_z u - u\|_t$. From Remark 2.8 it follows that (c) $\|Mu\|_{t+r} \leqq C\|u\|_{t+r}$ if $0 < r < 1$, P_1, $P_2 \in H^{t+r}$. Using (c) in place of (a) we have (b) with $t + r$, which leads to (c) with $t + 2r$. Repeating this procedure, we have $\|M(P; P)u\|_s \leqq C\|u\|_s$. This, combined with $M - M_0 = -M(M^{-1} - M_0^{-1})M_0$, $M = M(P; P)$ and $M_0 = M(P^0; P^0)$, shows that $\|M(P; P)u - M(P^0; P^0)u\|_s \leqq C\|P - P^0\|_s \|u\|_s$. The proof is complete.

By the facts stated in the beginning of this article and Lemmas 4.14, 4.15, 4.17–4.20 we have

Lemma 4.23. *Let m be an integer $\geqq 2$, $0 \leqq r < 1$ and $1/2 < s$, $t \leqq 1$. Then for the operators A_j defined by* (3.5) *and* (3.7),

$$A_j(X;\, X_1)\in L_0(1+s+r,\, m+r;\, 1+t),\quad j=1,\, 2$$

$$A_j(X,\, b;\, X,\, b)\in L_0(1+r,\, m-1+r;\, t),\quad j=3,\, 4,\, 7,\, 8$$

$$A_j(b;\, 0)\in L_0(1+s+r,\, m+r;\, 0),\quad j=5,\, 6.$$

Lemma 4.24. *Let m be an integer $\geqq 3$ and $0\leqq r<1$. Then*

$$K_1=K_1(X,\, b;\, X,\, b)\in L_0(2+r,\, m+r;\, 3).$$

Proof. By the above lemma we see that

$$B_3(X,\, b;\, X,\, b)=-i\,\mathrm{sgn}\, De^{-h|D|}b_x+A_3-A_4b_x\in L_0(1+r,\, m+r;\, 1),$$

$$B_4(b,\, b_x;\, 0)=i\,\mathrm{sgn}\, Db_x+A_5-A_6b_x\in L_0(2,\, 2;\, 0).$$

Therefore

$$M_1(b,\, b_x;\, b,\, b_x)=(1+B_4)^{-1}\in L(2,\, 2;\, 2),$$

$$M_2(b,\, b_x;\, b,\, b_x)=B_4(1+B_4)^{-1}\in L_0(2,\, 2;\, 2).$$

From (3.9) it follows that

$$B_j(X,\, b;\, X,\, b)\in L_0(2+r,\, m+r;\, 3),\quad j=1,\, 2,$$

$$B_1(X,\, b,\, b_x;\, X,\, b,\, b_x)\in L_0(2,\, 2;\, 2).$$

Hence $M_3(X,\, b,\, b_x;\, X,\, b,\, b_x)=(1+B_1)^{-1}\in L(2,\, 2;\, 2)$. In the same way as in the proof of Lemma 4.22 we see that $M_3\in L(2+r,\, 2+r;\, 2)$. Consequently we see from (3.8) that $K_1\in L_0(2+r,\, m+r;\, 3)$. The proof is complete.

Remark 4.25. It is easily seen from the above proof that

$$K_1(X,\, b,\, b_x;\, X,\, b,\, b_x)\in L_0(2+r,\, 2+r;\, 2),\quad 0\leqq r<1.$$

Assuming that X depends also on t, we define

$$A_{j,k,l}(X,\ldots,\, \partial_t^k\partial_x^l X,\, b,\ldots,\, \partial_x^l b;\, X,\, b),\quad 1\leqq j\leqq 8,\ 0\leqq k,\, l,$$

(by $X,\ldots,\, \partial_t^k\partial_x^l X$ we denote the derivatives $\partial_t^p\partial_x^q X$, $p\leqq k$, $q\leqq l$), inductively by

$$A_{j,0,0}=A_j,\ A_{j,0,l}=\left[\frac{\partial}{\partial x},\, A_{j,0,l-1}\right],\ l\geqq 1,\quad A_{j,k,l}=\left[\frac{\partial}{\partial t},\, A_{j,k-1,l}\right],\ k\geqq 1,\ l\geqq 0.$$

We have seen that $A_{j,k,l}$ can be written in the form of the linear combination of operators of the type (4.1) or (4.2). After this is done, there is no necessity to regard $\partial_t^p\partial_x^q X$ in $A_{j,k,l}$ as the derivatives of X. Hence we replace $\partial_t^p\partial_x^q X$ by X^{pq}. By the results of the preceding two articles we have

Lemma 4.26. *Let m be an integer* $\geqq 2$ *and* $0 \leqq r < 1$. *Then*

$$A_{j,k,l}(X^{00},\dots, X^{kl}, b,\dots, \partial_x^l b; X^{00}, b) \in L_0(2+r, m+r; 2).$$

Since K_1 is rational in A_j and $[R, ST]=[R, S]T+S[R, T]$, $[R, (1+T)^{-1}] =(1+T)^{-1}[T, R](1+T)^{-1}$ hold for operators R, S and T, we define

$$K_{1,k,l}(X,\dots, \partial_t^k \partial_x^l X, b,\dots, \partial_x^l b; X, b)$$

by the formula used for the definition of $A_{j,k,l}$ and replace $\partial_t^p \partial_x^q X$ in K_1 by X^{pq}. By the same consideration as in the proof of Lemma 4.24 we have

Lemma 4.27. *Let m be an integer* $\geqq 3$ *and* $0 \leqq r < 1$. *Then*

$$K_{1,k,l}(X^{00},\dots, X^{kl}, b,\dots, \partial_x^l b; X^{00}, b) \in L_0(2+r, m+r; 3).$$

Corollary 4.28. *For any integer* $m \geqq 3$,

$$M(X, Z, b; X, Z, b)=\{1+Z_1+Z_2K(X, b; X, b)\}^{-1} \in L(m, m; 3).$$

Proof. Since $K(X, b; X, b) = -i\tanh(hD) + K_1(X, b; X, b) \in L(3, 3; 3)$ we have $M_1(X, Z, b; X, b)=Z_1+Z_2K(X, b; X, b) \in L_0(3, 3; 3)$. Hence $M =(1+M_1)^{-1} \in L(3, 3; 3)$. Since

$$\left[\frac{\partial}{\partial x}, M\right] = -M\left[\frac{\partial}{\partial x}, M^{-1}\right]M = -M\left\{Z_{1x}+Z_{2x}K+Z_2\left[\frac{\partial}{\partial x}, K_1\right]\right\}M,$$

we have by the above lemma

$$\begin{aligned}\|Mu\|_4 &\leqq C\|Mu\|_3 + C\left\|\frac{\partial}{\partial x}Mu\right\|_3 \\ &\leqq C\|Mu\|_3 + C\left\|\left[\frac{\partial}{\partial x}, M\right]u + M\frac{\partial}{\partial x}u\right\|_3 \leqq C\|u\|_4\end{aligned}$$

if X, Z, b are in H^4 and small in H^3. In the same way as in the proof of Lemma 4.22 we see that $M \in L(m, m; 3)$.

§5. Reduction to the Quasilinear System

We shall reduce the system (1.6) to the quasilinear system such that the unique solvability of the initial value problem for this system assures one for that system and the successive approximation is available for this system. For the usual procedure for the reduction to the quasilinear system, see [9], Chapter I, Section 7.2, Chapter V, Section 1.7 and [10].

5.1. Quasilinearization. In quasilinearizing the system

$$(5.1)\qquad (1+X_{1x})X_{1tt}+X_{2x}(1+X_{2tt})=0, \quad X_{2t}=KX_{1t},$$

we use the commutators in order to single out the principal parts of operators. Put $F_{jk}=[\partial_t^j\partial_x^k, K]X_{1t}$. Since $P^nQ=QP^n+\sum_1^n\binom{n}{j}[P, Q]_jP^{n-j}$ holds for operators P and Q where $[P, Q]_1=[P, Q]$ and $[P, Q]_{k+1}=[P, [P, Q]_k]$, $k\geqq 1$, we have

$$\partial_t^j\partial_x^kK=K\partial_t^j\partial_x^k+\sum_1^j\binom{j}{p}K_{1,p,0}\partial_t^{j-p}\partial_x^k+\sum_1^k\binom{k}{q}K_{1,0,q}\partial_t^j\partial_x^{k-q}$$
$$+\sum_1^k\binom{k}{q}\sum_1^j\binom{j}{p}K_{1,p,q}\partial_t^{j-p}\partial_x^{k-q}.$$

In virtue of Lemma 4.27 the operator $K_{1,p,q}\partial_x^{k-q}$ is of order 0 if all functions contained in the kernels are in H^m for the sufficiently large m, therefore we use the notations

$$F_{j0}=F_{j0}(X,\dots, \partial_t^jX), \quad F_{jk}=F_{jk}(X,\dots, \partial_t^j\partial_x^kX, \partial_t^{j+1}X_1), \quad j\geqq 0,\ k\geqq 1.$$

In the precise form,

$$(5.2)\qquad \begin{cases} F_{j0}(X^{00},\dots, X^{j0})=\sum_1^j\binom{j}{p}K_{1,p,0}(X^{00},\dots, X^{p0})X_1^{j-p+1,0}, \\ F_{jk}(X^{00},\dots, X^{jk}, X_1^{j+1,0})=\sum_1^j\binom{j}{p}K_{1,p,0}(X^{00},\dots, X^{p0})\partial_x^kX_1^{j-p+1,0} \\ \qquad +\sum_1^k\binom{k}{q}K_{1,0,q}(X^{00},\dots, X^{0q})\partial_x^{k-q}X_1^{j+1,0} \\ \qquad +\sum_1^k\binom{k}{q}\sum_1^j\binom{j}{p}K_{1,p,q}(X^{00},\dots, X^{pq})\partial_x^{k-q}X_1^{j-p+1,0} \end{cases}$$

where in $K_{1,k,l}$ we omit $b,\dots, \partial_x^lb$; X^{00}, b. From (5.1) we obtain

$$(5.3)\qquad \partial_t^j\partial_x^kX_{2t}=K\partial_t^j\partial_x^kX_{1t}+F_{jk}.$$

Put

$$(5.4)\qquad Y=X_{tt},\ Z=X_x,\ W=(X, Y, Z),\ W'=(X, Y_1).$$

From (5.1) it follows that

$$(5.5)\qquad \frac{\partial}{\partial t}\{(1+Z_1)Y_1+Z_2(1+Y_2)\}=Y_1Z_{1t}+(1+Y_2)Z_{2t}+(1+Z_1)Y_{1t}+Z_2Y_{2t}=0.$$

From (5.3) and (3.8) we obtain

$$X_{2tx}=KX_{1tx}+F_{01}(X, X_x, X_{1t})$$
$$=-i\operatorname{sgn} DX_{1tx}+i(\operatorname{sgn} D-\tanh(hD))\frac{\partial}{\partial x}X_{1t}+K_1\frac{\partial}{\partial x}X_{1t}+F_{01}.$$

This yields the equation

$$Z_{2t} = -i \operatorname{sgn} D Z_{1t} + F_{010}(X, Z, X_{1t}). \tag{5.6}$$

By the elimination of Z_{2t} from this and (5.5), we have

$$Z_{1t} = -\{(1+Y_2)(-i \operatorname{sgn} D) + Y_1\}^{-1}\{(1+Y_2)F_{010} + (1+Z_1)Y_{1t} + Z_2 Y_{2t}\}. \tag{5.7}$$

In virtue of (5.3) with $j=2$, $k=0$ we have $Y_{2t} = K(X)Y_{1t} + F_{20}(X, X_t, Y) = f_2(W, W_t')$, therefore the substitution of Y_{2t} in (5.7) by f_2 and Z_{1t} in (5.6) by the right-hand side of (5.7) lead to the equations

$$Y_{2t} = f_2(W, W_t'), \quad Z_{1t} = f_3(W, W_t'), \quad Z_{2t} = f_4(W, W_t'). \tag{5.8}$$

Remark 5.9. If functions X, Y, Z satisfy the equations (5.8), then

$$\begin{cases} Z_{2t} = -i \operatorname{sgn} D Z_{1t} + F_{010}(X, Z, X_{1t}), \\ \dfrac{\partial}{\partial t}\{(1+Z_1)Y_1 + Z_2(1+Y_2)\} = 0. \end{cases} \tag{5.10}$$

Let us now proceed to the equation for Y_1. From (5.1) we have $(\partial/\partial t)^2 \cdot \{(1+X_{1x})Y_1 + X_{2x}(1+Y_2)\} = 0$. Replacing X_{xtt} by Y_x and X_{xt} by Z_t, we obtain $(1+Z_1)Y_{1tt} + Z_2 Y_{2tt} + Y_1 Y_{1x} + (1+Y_2)Y_{2x} + 2Y_t \cdot Z_t = 0$. By (5.3) with $j=3$, $k=0$ we have $Y_{2tt} = K(X)Y_{1tt} + F_{30}(X, X_t, Y, Y_t)$. Eliminating Y_{2tt} from these equations, we obtain

$$Y_{1tt} = -(1+Z_1+Z_2K)^{-1}(Y_1 Y_{1x} + (1+Y_2)Y_{2x} + Z_2 F_{30} + 2Y_t \cdot Z_t).$$

By (5.3) with $j=k=1$, we have $Y_{2x} = K(X)Y_{1x} + F_{11}(X, X_t, Z, Z_t, Y_1)$. Hence the above two equations yield

$$\begin{aligned} Y_{1tt} = &-(1+Z_1+Z_2K)^{-1}\{Y_1 + (1+Y_2)K\}Y_{1x} \\ &-(1+Z_1+Z_2K)^{-1}\{(1+Y_2)F_{11} + Z_2F_{30} + 2Y_t \cdot Z_t\}. \end{aligned} \tag{5.11}$$

Using the identity

$$\begin{aligned} &(1+Z_1-Z_2K)(1+Z_1+Z_2K) \\ &\quad = (1+Z_1)^2 + Z_2^2 - Z_2\{[K, Z_1] + [K, Z_2]K + Z_2(1+K^2)\}, \end{aligned}$$

we obtain

$$(1+Z_1+Z_2K)^{-1} = \{(1+Z_1)^2 + Z_2^2\}^{-1}(1+Z_1-Z_2K) + Q(X, Z) \tag{5.12}$$

where

$$\begin{aligned} &Q(X, Z) \\ &\quad = \{(1+Z_1)^2 + Z_2^2\}^{-1} Z_2\{[K, Z_1] + [K, Z_2]K + Z_2(1+K^2)\}(1+Z_1+Z_2K)^{-1}. \end{aligned}$$

The identity

$$\begin{aligned}(1+Z_1-Z_2K)\{Y_1+(1+Y_2)K\}\\ =(1+Z_1)Y_1+Z_2(1+Y_2)+\{(1+Z_1)(1+Y_2)-Z_2Y_1\}K\\ -Z_2\{[K,\ Y_1]+(1+Y_2)(K^2+1)+[K,\ Y_2]K\},\end{aligned}$$

(5.11) and (5.12) lead to the equation

$$\begin{aligned}Y_{1tt}=-\{(1+Z_1)^2+Z_2^2\}^{-1}\{(1+Z_1)Y_1+Z_2(1+Y_2)\\ +((1+Z_1)(1+Y_2)-Z_2Y_1)K\}Y_{1x}+\cdots.\end{aligned}$$

Noting that $(1+Z_1)Y_1+Z_2(1+Y_2)=0$ and $K=-i\operatorname{sgn}D+i(\operatorname{sgn}D-\tanh(hD))+K_1$, and replacing Y_{2t}, Z_{1t} and Z_{2t} in the above equation by f_2, f_3 and f_4, respectively, we can write the equation for Y_1 in the form

$$Y_{1tt}=-a(-i\operatorname{sgn}D)Y_{1x}+f_1(W,\ W_t')=-a|D|Y_1+f_1(W,\ W_t') \tag{5.13}$$

where $a=\{(1+Z_1)(1+Y_2)-Z_2Y_1\}\{(1+Z_1)^2+Z_2^2\}^{-1}$.

Remark 5.14. Since $Y=X_{tt}$ and $Z=X_x$, we observe that

$$\begin{aligned}&\{(1+Z_1)^2+Z_2^2\}^{1/2}a\\ &=\{(1+X_{1x})^2+X_{2x}^2\}^{-1/2}(-X_{2x},\ 1+X_{1x})\cdot(X_{1tt},\ 1+X_{2tt})\\ &=N\cdot(X_{tt}+(0,\ 1))=-N\cdot\operatorname{grad}p.\end{aligned}$$

Therefore $\{(1+Z_1)^2+Z_2^2\}^{1/2}a$ is the gradient of the pressure in the inner normal direction on the free surface.

Remark 5.15. If X, Y, Z satisfy the equations (5.8), (5.13) and $(1+Z_1)Y_1+Z_2(1+Y_2)=0$, then the equation (5.11) holds.

The required quasilinear system has the form

$$\begin{cases}X_{tt}=Y,\\ Y_{1tt}+a(W)|D|Y_1=f_1(W,\ W_t'),\\ Y_{2t}=f_2(W,\ W_t'),\ Z_{1t}=f_3(W,\ W_t'),\ Z_{2t}=f_4(W,\ W_t')\end{cases} \tag{5.16}$$

where

$$\begin{cases}a(W)=a(Y,\ Z)=\{(1+Z_1)(1+Y_2)-Z_2Y_1\}\{(1+Z_1)^2+Z_2^2\}^{-1},\\ f_2=KY_{1t}+F_{20}(X,\ X_t,\ Y),\\ f_3=-\{(1+Y_2)(-i\operatorname{sgn}D)+Y_1\}^{-1}\{(1+Y_2)F_{010}(X,\ Z,\ X_{1t})\\ \qquad +(1+Z_1)Y_{1t}+Z_2f_2(W,\ W_t')\},\end{cases}$$

$$
(5.17)\quad \begin{cases} F_{010}=i(\operatorname{sgn} D-\tanh(hD))\dfrac{\partial}{\partial x}X_{1t}+K_1\dfrac{\partial}{\partial x}X_{1t}+F_{01}(X,\ Z,\ X_{1t}), \\ f_4=-i\operatorname{sgn} Df_3(W,\ W_t')+F_{010}(X,\ Z,\ X_{1t}), \\ f_1=-a(W)(i\operatorname{sgn} D-i\tanh(hD)+K_1)\dfrac{\partial}{\partial x}Y_1 \\ \qquad +Z_2\{(1+Z_1)^2+Z_2^2\}^{-1}\{[K,\ Y_1]+[K,\ Y_2]K+(1+Y_2)(K^2+1)\} \\ \qquad \dfrac{\partial}{\partial x}Y_1 \\ \qquad -Z_2\{(1+Z_1)^2+Z_2^2\}^{-1}\{[K,\ Z_1]+[K,\ Z_2]K+Z_2(K^2+1)\} \\ \qquad \times(1+Z_1+Z_2K)^{-1}\{Y_1+(1+Y_2)K\}\dfrac{\partial}{\partial x}Y_1 \\ \qquad -(1+Z_1+Z_2K)^{-1}\{(1+Y_2)F_{11}(X,\ X_t,\ Z,\ Z_t,\ Y_1) \\ \qquad +Z_2F_{30}(X,\ X_t,\ Y,\ Y_t)+2Y_t\cdot Z_t\}, \\ \text{(in the last term } Y_{2t},\ Z_{1t} \text{ and } Z_{2t} \text{ are replaced by } f_2, f_3 \text{ and } f_4, \\ \text{respectively)}, \\ K=K(X)=-i\tanh(hD)+K_1(X). \end{cases}
$$

Though f depends on W, W_t', b and h, we omit b and h in the notations.

5.2. Properties of $a(W)$ and $f(W, W_t')$. In solving the initial value problem for the system (5.16) we need only the properties of a and f which will be shown in this article. The explicit form (5.17) of a and f will play an important role in dealing with the original problem (1.6) and (1.7).

Lemma 5.18. *There exists $c_0>0$ such that if $s\geqq 2$, $d_0>0$ and*

$$
(5.19)\qquad W=(0,\ Y,\ Z)\in H^s,\ \|W\|_2\leqq c_0,\ \|W\|_s\leqq d_0,
$$

then for $a=a(W)=a(Y, Z)$

1) $a'\equiv a-1\in H^s$
2) $|a'|\leqq e_3<1$, *i.e.*, $0<e_1\leqq a\leqq e_2$
3) $\|[a,\ |D|^{1/2}]u\|\leqq C_1\|u\|_{-1/2},\quad u\in H^{-1/2}$
4) $\|[a,\ |D|]u\|\leqq C_2\|u\|,\quad u\in H^0$
5) $\|[a,\ (1+|D|)^t]u\|\leqq C_3\|u\|_{t-1},\quad u\in H^{t-1},\ 2\leqq t\leqq s$
6) *for $W^0=(0,\ Y^0,\ Z^0)$ satisfying the condition* (5.19)

$$
\|a(W)-a(W^0)\|_s\leqq C\|W-W^0\|_s
$$

where $e_j=e_j(c_0)>0$, $j=1, 2, 3$, $C_j=C_j(c_0)>0$, $j=1, 2$, $C_3=C_3(c_0,\ d_0,\ s)>0$ and $C=C(c_0,\ d_0,\ s)>0$.

Proof. From Lemma 2.5 it follows that

$$|W(x)| \leqq C\|W\|_1 \leqq Cc_0 .$$

Since $a(0)=1$, we can choose c_0 so small that $a(W)$ is everywhere defined, i.e., $(1+Z_1)^2+Z_2^2>0$ and 2) holds for some e_j. Let $\Omega \subset \boldsymbol{R}^4$ be the closed ball of radius Cc_0 with the center at the origin. Since

$$a(W)-a(W^0)=\sum_3^6 (W_j-W_j^0)\int_0^1 \frac{\partial a}{\partial w_j}(tW+(1-t)W^0)\,dt ,$$

it follows from Lemma 2.11 and Remark 2.12 that

$$\|a(W)-a(W^0)\|_r \leqq C\|a\|_{\mathscr{B}^{m+1}(\Omega)}(1+\|W\|_r+\|W^0\|_r)^{m+1}\|W-W^0\|_r, \quad 2\leqq r\leqq s\leqq m .$$

This proves 6). Putting $W^0=0$ we see that $a'=a-1\in H^s$ in virtue of $a(0)=1$. Noting that $[a, P(D)]=[a', P(D)]$ and $\|a'\|_2\leqq C$, (C depends only on c_0), we see by Lemma 2.14 that 3), 4) and 5) are valid. The proof is complete.

Lemma 5.20. *Let c_0 be the constant chosen in Lemma 5.18, $0<T<\infty$, $s\geqq 2$ and*

(5.21) $W=(0, Y, Z)\in C^1([0, T], H^s)$,

$$\|W(t)\|_2\leqq c_0, \quad \|W(t)\|_s\leqq d_0, \quad \|W_t(t)\|_2\leqq d, \quad 0\leqq t\leqq T .$$

Then

1) $a-1\in C^1([0, T], H^s)$.
2) $|a_t|\leqq C_4 d$.
3) $\|[a_t, |D|^{1/2}]u\|\leqq C_5 d\|u\|_{-1/2}, \quad u\in H^{-1/2}$,

where $a_t=(\partial/\partial t)a(W)$, $C_j=C_j(c_0)>0$, $j=4, 5$.

Proof. It follows from Lemma 5.18, 6) that $\|a(W(t))-a(W(t_0))\|_s\leqq C\|W(t)-W(t_0)\|_s$, which proves that $a-1\in C^0([0, T], H^s)$. Note that $a_t(W(t))=\sum_3^6 \partial a(W(t))/\partial W_j\cdot\partial W_j(t)/\partial t$. This gives

$$|a_t(W(t))|\leqq\left(\sup\left|\frac{\partial a}{\partial W_j}\right|\right)|W_t|\leqq C\|W_t\|_1\leqq C_4 d$$

and

$$\|a_t\|_r\leqq C\|a\|_{\mathscr{B}^{m+2}(\Omega)}(1+\|W(t)\|_r)^{m+1}\|W_t(t)\|_r, \qquad 2\leqq r\leqq s\leqq m ,$$

in virtue of Lemma 2.11 and Remark 2.12. Using Lemma 2.14, 4) we have

$$\|[a_t, |D|^{1/2}]u\|\leqq C\|a_t\|_2\|u\|_{-1/2}\leqq C_5 d\|u\|_{-1/2} .$$

Since

$$a_t(W(t))-a_t(W(t_0))=\sum_3^6 \frac{\partial a}{\partial W_j}(W(t))\left\{\frac{\partial W_j}{\partial t}(t)-\frac{\partial W_j}{\partial t}(t_0)\right\}$$
$$+\sum_3^6 \left\{\frac{\partial a}{\partial W_j}(W(t))-\frac{\partial a}{\partial W_j}(W(t_0))\right\}\frac{\partial W_j}{\partial t}(t_0),$$

we have $\|a_t(W(t))-a_t(W(t_0))\|_s \leqq C\|W(t)-W(t_0)\|_s$, which proves that $a_t \in C^0([0, T], H^s)$. The proof is complete.

Lemma 5.22. *There exists a small positive constant* c_0 *such that if* m *is an integer* $\geqq 3$, $0<T<\infty$, $d_0>0$ *and* $b\in H^{m+1}$, $\|b\|_3\leqq c_0$, $\|b\|_{m+1}\leqq d_0$,

(5.23) $$W,\ W_t' \in C^0([0, T], H^m),\quad \|W(t)\|_3\leqq c_0,$$
$$\|W(t)\|_m\leqq d_0,\ \|W_t'(t)\|_m\leqq d_0,\qquad 0\leqq t\leqq T,$$

then

1) $f=f(W, W_t')=f(W, W_t', b)\in C^0([0, T], H^m)$,
$\|f(W, W_t')\|_m\leqq k(\|W\|_m^2+\|W_t'\|_m^2)^{1/2}$,
2) *for* W^0, $W_t^{0\prime}$ *satisfying* (5.23),
$\|f(W, W_t')-f(W^0, W_t^{0\prime})\|_m\leqq C(\|W-W^0\|_m+\|W_t'-W_t^{0\prime}\|_m)$

where $k=k(c_0, d_0, m)>0$ *and* $C=C(c_0, d_0, m)>0$.

Proof. We see from (5.2) and (5.17) that

$$f_2=K(X)Y_{1t}+F_{20}(X, X_t, Y)$$
$$=K(X, b; X, b)Y_{1t}+2K_{1,1,0}(X, X_t, b; X, b)Y_1+K_{1,2,0}(X, X_t, Y, b; X, b)X_{1t}.$$

Hence Lemma 4.27 shows that 1) and 2) are valid for f_2. As to

$$f_3=-\{(1+Y_1)(-i\,\mathrm{sgn}\,D)+Y_2\}^{-1}$$
$$\times\{(1+Y_2)F_{010}(X, Z, X_{1t})+(1+Y_1)Y_{1t}+Z_2f_2\},$$

first of all, note that $\{(1+Y_1)(-i\,\mathrm{sgn}\,D)+Y_2\}^{-1}=i\,\mathrm{sgn}\,D(1+Y_1+Y_2\ i\,\mathrm{sgn}\,D)^{-1}$ $\in L(m, m; 1)$ by Lemma 4.22. Since

$$F_{010}=\{i\,\mathrm{sgn}\,D-i\tanh(hD)+K_1(X, b; X, b)\}\frac{\partial}{\partial x}X_{1t}$$
$$+K_{1,0,1}(X, Z, b, b_x; X, b)X_{1t},$$

we see by Lemmas 2.15 and 4.27 that F_{010} has the properties 1) and 2). Since H^m is an algebra, we see in virtue of Definition 4.21 that 1) and 2) hold for f_3. Similarly 1) and 2) hold for $f_4=-i\,\mathrm{sgn}\,Df_3+F_{010}$. For the operators occurring in the definition of f_1 we see that

$a(Y, Z)$, $Z_2\{(1+Z_1)^2+Z_2^2\}^{-1}\in L(m, m; 1)$ by 4.22, 3);

$\{i \operatorname{sgn} D - i \tanh(hD) + K_1(X, b; X, b)\}\frac{\partial}{\partial x} \in L(3, m; 3)$ by 2.15 and 4.27;

$[K, Y_1] = -i[\operatorname{sgn} D, Y_1] + [i \operatorname{sgn} D - i \tanh(hD) + K_1(X, b; X, b), Y_1]$
$\in L_0(2, m; 3)$ by 2.14, 2.15 and 4.27;

$1 + K(X, b; X, b)^2 = 1 + \{-i \tanh(hD) + K_1\}^2$
$= \{1 - \tanh^2(hD)\} - i \tanh(hD)K_1 - iK_1 \tanh(hD) + K_1^2$
$\in L(2, m; 3)$ by 2.15 and 4.27;

$\{1 + Z_1 + Z_2K(X, b, b_x; X, b, b_x)\}^{-1} \in L(2, 2; 2)$ by 4.25;

$\{1 + Z_1 + Z_2K(X, b; X, b)\}^{-1} \in L(m, m; 3)$ by 4.28.

These combined with the already proved properties of f_2, f_3 and f_4 show that 1) and 2) hold for f_1. The proof is finished.

5.3. Transformation of Initial Values. Suppose that X is a solution of the initial value problem:

(5.24) $\qquad X_{2t} = K(X)X_{1t},\ (1 + X_{1x})X_{1tt} + X_{2x}(1 + X_{2tt}) = 0, \qquad t \geqq 0,$

(5.25) $\qquad X = U,\ X_{1t} = V,\ t = 0.$

We shall determine the initial values of $W = (X, Y, Z)$ and $W_t' = (X_t, Y_{1t})$ at $t = 0$ for the system (5.16) from U and V by means of (5.4) and (5.24). Since $X_{2t} = K(X)X_{1t}$ and $Z = X_x$, we put $X_{2t} = K(U)V$ and $Z = U_x$, $t = 0$. From (5.3) with $j = 1$ and $k = 0$ we obtain

(5.26) $$X_{2tt} = K(X)X_{1tt} + F_{10}(X, X_t).$$

This combined with the second equation in (5.24) shows that

$$(1 + X_{1x} + X_{2x}K(X))X_{1tt} = -X_{2x} - X_{2x}F_{10}(X, X_t).$$

Therefore we put

$$Y_1 = -\{1 + Z_1 + Z_2K(X)\}^{-1}Z_2\{1 + F_{10}(X, X_t)\}, \quad t = 0.$$

In view of (5.26) we put $Y_2 = K(X)Y_1 + F_{10}(X, X_t)$, $t = 0$. In view of $Y = X_{tt}$ and $Y_{2t} = K(X)Y_{1t} + F_{20}(X, X_t, Y)$, we have

$$0 = \frac{\partial}{\partial t}\{(1 + X_{1x})Y_1 + X_{2x}(1 + Y_2)\}$$
$$= (1 + X_{1x})Y_{1t} + X_{2x}(KY_{1t} + F_{20}) + Y_1X_{1tx} + (1 + Y_2)X_{2tx}.$$

Hence we put

$$Y_1 = -\{1+Z_1+Z_2K\}^{-1}\Big\{Z_2F_{20}(X, X_t, Y) + Y_1\frac{\partial}{\partial x}X_{1t} + (1+Y_2)\frac{\partial}{\partial x}X_{2t}\Big\}, \quad t=0.$$

Thus the transformation of U and V into W and W_t', $t=0$, is as follows:

$$(5.27)\quad \begin{cases} X=U, \quad X_{1t}=V, \quad X_{2t}=K(U)V, \quad Z=U_x, \\ Y_1=-\{1+Z_1+Z_2K\}^{-1}Z_2\{1+F_{10}(X, X_t)\}, \\ Y_2=KY_1+F_{10}(X, X_t), \\ Y_{1t}=-\{1+Z_1+Z_2K\}^{-1}\Big\{Z_2F_{20}(X, X_t, Y) \\ \qquad + Y_1\frac{\partial}{\partial x}X_{1t} + (1+Y_2)\frac{\partial}{\partial x}X_{2t}\Big\}. \end{cases}$$

Remark 5.28. For values of W and W_t' at $t=0$ defined by (5.27) we have

$$(1+Z_1)Y_1+Z_2(1+Y_2)=0,$$
$$(1+Z_1)Y_{1t}+Z_2\{KY_{1t}+F_{20}(X, X_t, Y)\}+Y_1\frac{\partial}{\partial x}X_{1t}+(1+Y_2)\frac{\partial}{\partial x}X_{2t}=0.$$

In Section 6 it will be shown that the initial value problem for the system (5.16) is uniquely solvable if the initial values are small. Hence in solving the problem (1.6) and (1.7) we need the following lemma.

Lemma 5.29. *There exists $c_0>0$ such that if m is an integer $\geqq 3$, $d_0>0$ and*

$$b\in H^m, \quad U\in H^{m+(1/2)}, \quad V\in H^m, \quad \|b\|_3\leqq c_0, \quad \|U\|_3\leqq c_0,$$
$$\|b\|_m, \quad \|U\|_{m+(1/2)}, \quad \|V\|_m\leqq d_0$$

then by (5.27) *U, V are transformed into W, W_t', $t=0$, such that*

$$X\in H^{m+(1/2)}, \quad X_t\in H^m, \quad Y, Z\in H^{m-(1/2)}, \quad Y_{1t}\in H^{m-1}$$

and

$$\|X\|_{m+(1/2)}+\|X_t\|_m+\|Y\|_{m-(1/2)}+\|Z\|_{m-(1/2)}+\|Y_{1t}\|_{m-1} \leqq C(\|U\|_{m+(1/2)}+\|V\|_m)$$

where $C=C(c_0, d_0, m)>0$.

Proof. Using Remark 2.10, Lemma 4.27 and the definition of F_{j0} we obtain the lemma by the same consideration as in the proof of Lemma 5.22.

§ 6. Unique Existence Theorems

In this section we shall show the unique solvability of the initial value problems for the quasilinear system (5.16) and the original (1.6). Throughout this section we assume that every function is real-valued.

6.1. Preliminaries. We use the notations:

$$W=(X, Y, Z), \quad W'=(X, Y_1), \quad \Lambda=\Lambda(D)=1+|D|, \quad D=\frac{1}{i}\frac{\partial}{\partial x}$$

$$a=a(W)=a(Y, Z)=\{(1+Z_1)(1+Y_2)-Z_2Y_1\}\{(1+Z_1)^2+Z_2^2\}^{-1}, \quad a_t=\frac{\partial a}{\partial t}.$$

In view of the identity

$$a|D|=\frac{1}{2}\Lambda^{-m}(a|D|+|D|a)\Lambda^m+\lambda+\frac{1}{2}\Lambda^{-m}(a|D|-|D|a)\Lambda^m \\ -\lambda-\Lambda^{-m}[a, \Lambda^m]|D|$$

we introduce the operators

$$(6.1)\quad \begin{cases} G=G(W)=\frac{1}{2}(a|D|+|D|a)+\lambda \\ G_m=G_m(W)=\Lambda^{-m}G(W)\Lambda^m \\ G'_m=G'_m(W)=\lambda-\frac{1}{2}\Lambda^{-m}[a, |D|]\Lambda^m+\Lambda^{-m}[a, \Lambda^m]|D| \\ G_t=G_t(W)=\left[\frac{\partial}{\partial t}, G(W)\right]=\frac{1}{2}(a_t|D|+|D|a_t). \end{cases}$$

Assumption 6.2. Let $\lambda=1+C_1$, m be an integer $\geqq 2$, d_0, $d>0$,

$$W=(0, Y, Z)\in C^1([0, T], H^2)\cap C^0([0, T], H^m),$$

$$\|W(t)\|_2\leqq c_0, \quad \|W(t)\|_m\leqq d_0, \quad \|W_t(t)\|_2\leqq d, \qquad 0\leqq t\leqq T$$

where c_0, $C_1=C_1(c_0)$ are constants occurring in Lemma 5.18.

Lemma 6.3. *Under Assumption* 6.2,

1) $a|D|=G_m-G'_m$
2) $(Gu, v)=(u, Gv)$, $(G_mu, v)_m=(u, G_mv)_m$, $\quad u, v\in H^{m+1}$
3) $(G_mu, u_t)_m=\frac{1}{2}\frac{d}{dt}(G^m\Lambda u, \Lambda^m u)-\frac{1}{2}(G_t\Lambda^m u, \Lambda^m u)$, $\quad u\in C^1([0, T], H^{m+1})$
4) $e_1\| |D|^{1/2}u\|^2+\|u\|^2\leqq(Gu, u)\leqq e_4(e_1\| |D|^{1/2}u\|^2+\|u\|^2)$, $\quad u\in\mathscr{S}$
5) $|(G_tu, u)|\leqq d(C_4e_1^{-1}+C_5)(Gu, u)$, $\quad u\in\mathscr{S}$

6) *for* $W^0=(0, Y^0, Z^0)$ *satisfying conditions in* 6.2 *and* $-m+1\leqq s\leqq m$,

$$\|G'_m u\|_m\leqq\left(1+C_1+\frac{1}{2}C_2+C_3\right)\|u\|_m,$$
$$\|\{G'_m(W)-G'_m(W^0)\}u\|_m\leqq C(\|Y-Y^0\|_m+\|Z-Z^0\|_m)\|u\|_m,$$
$$\|G(W)u\|_{s-1}\leqq C\|u\|_s,$$
$$\|\{G(W)-G(W^0)\}u\|_{s-1}\leqq C(\|Y-Y^0\|_m+\|Z-Z^0\|_m)\|u\|_s$$

where e_j, C_j, $j=1, 2, 3$ *are constants occurring in Lemma* 5.18, C_4, C_5 *in* 5.20, $C=C(c_0, d_0, m)>0$ *and* $e_4=\max\{e_2e_1^{-1}, 1+2C_1\}$.

Proof. Since $a-1\in C^1([0, T], H^1)$ by Lemma 5.20, H^1 is an algebra and $(u, v)_m=(\Lambda^m u, \Lambda^m v)$, we have 1), 2) and 3) in virtue of (6.1). Note that

$$\begin{aligned}(Gu, u)&=\frac{1}{2}(a|D|u, u)+\frac{1}{2}(|D|au, u)+\lambda(u, u)\\&=(a|D|u, u)+\lambda(u, u)\\&=(a|D|^{1/2}u, |D|^{1/2}u)+([a, |D|^{1/2}]|D|^{1/2}u, u)+\lambda(u, u).\end{aligned}$$

Therefore from Lemma 5.18 we obtain

$$\begin{aligned}(Gu, u)&\geqq e_1\||D|^{1/2}u\|^2+(\lambda-C_1)\|u\|^2,\\(Gu, u)&\leqq e_2\||D|^{1/2}u\|^2+(\lambda+C_1)\|u\|^2.\end{aligned}$$

These prove 4). Similarly we have by Lemma 5.20

$$\begin{aligned}|(G_t u, u)|&=|(a_t|D|^{1/2}u, |D|^{1/2}u)+([a_t, |D|^{1/2}]|D|^{1/2}u, u)|\\&\leqq C_4 d\||D|^{1/2}u\|^2+C_5 d\||D|^{1/2}u\|_{-1/2}\|u\|\\&\leqq d(C_4e_1^{-1}+C_5)(e_1\||D|^{1/2}u\|^2+\|u\|^2).\end{aligned}$$

Hence we have 5) in virtue of 4). Since $\|u\|_m=\|\Lambda^m u\|$, by Lemma 5.18 we have

$$\begin{aligned}\|G'_m u\|_m&\leqq\lambda\|u\|_m+\frac{1}{2}\|[a, |D|]\Lambda^m u\|+\|[a, \Lambda^m]|D|u\|\\&\leqq\lambda\|u\|_m+\frac{1}{2}C_2\|\Lambda^m u\|+C_3\||D|u\|_{m-1}\\&\leqq\left(\lambda+\frac{1}{2}C_2+C_3\right)\|u\|_m,\end{aligned}$$

which proves the first inequality of 6). Using Lemma 2.14, 3) and 4) we have

$$\begin{aligned}\|\{G'_m(W)-G'_m(W^0)\}u\|_m&\leqq\frac{1}{2}\|[a(W)-a(W^0), |D|]\Lambda^m u\|\\&+\|[a(W)-a(W^0), \Lambda^m]|D|u\|\leqq C\|a(W)-a(W^0)\|_m\|u\|_m.\end{aligned}$$

Since $\|a(W)-a(W^0)\|_m\leqq C(\|Y-Y^0\|_m+\|Z-Z^0\|_m)$ in virtue of Lemma 5.18, 6), we have the second inequality of 6). We obtain the inequalities for $G=$

$2^{-1}(a|D|+|D|a)+\lambda$ if we show that

$$\|au\|_s \leqq C\|u\|_s, \quad \|\{a(W)-a(W^0)\}u\|_s \leqq C\|a(W)-a(W^0)\|_m\|u\|_s, \quad -m\leqq s\leqq m.$$

Note that $a'=a-1\in H^m$, (Lemma 5.18) and H^s, $s>1/2$, is an algebra, (Remark 2.10). For $1/2<s\leqq m$ we have

$$\|au\|_s \leqq C(1+\|a'\|_s)\|u\|_s \leqq C(1+\|a'\|_m)\|u\|_s,$$
$$\|\{a(W)-a(W^0)\}u\|_s \leqq C\|a(W)-a(W^0)\|_s\|u\|_s.$$

For $0\leqq s\leqq 1/2$, from Lemma 2.14 it follows that

$$\begin{aligned}\|au\|_s &\leqq C\|(1+|D|^s)au\| \leqq C\|au\|+C\|[|D|^s, a]u\|+C\|a|D|^s u\| \\ &\leqq C(\sup|a|)\|u\|+C\|a'\|_2\|u\|_{s-1}+C(\sup|a|)\|u\|_s \\ &\leqq C(1+\|a'\|_m)\|u\|_s\end{aligned}$$

and in the same way

$$\|\{a(W)-a(W^0)\}u\|_s \leqq C\|a(W)-a(W^0)\|_m\|u\|_s.$$

For $-m\leqq s<0$, by the above results we have

$$|(au, v)|=|(u, av)|\leqq \|u\|_s\|av\|_{-s}\leqq C\|u\|_s(1+\|a'\|_m)\|v\|_{-s}.$$

Hence the duality between H^s and H^{-s} shows that

$$\|au\|_s \leqq C(1+\|a'\|_m)\|u\|_s.$$

It is easily seen that

$$\|\{a(W)-a(W^0)\}u\|_s \leqq C\|a(W)-a(W^0)\|_m\|u\|_s.$$

Thus the proof is finished.

6.2. Linear Equations. Consider the initial value problem:

$$u_{tt}-\varepsilon^2 u_{xx}+A(t)u=f, \quad 0\leqq t\leqq T \tag{6.4}$$

$$u=u_0, \; u_t=u_1, \quad t=0. \tag{6.5}$$

Let B, C be Banach spaces. We denote by $\mathscr{L}(B, C)$ the Banach space consisting of linear continuous operators from B to C.

Lemma 6.6. *Let* $0<\varepsilon$, $0<T<\infty$, $-\infty<s<+\infty$ *and* $A\in C^0([0, T], \mathscr{L}(H^{s+1}, H^s))$. *If* $u_0\in H^{s+1}$, $u_1\in H^s$ *and* $f\in C^0([0, T], H^s)$, *then there exists the unique solution* u *of* (6.4), (6.5) *such that*

$$u\in C^j([0, T], H^{s+1-j}), \quad j=0, 1, 2.$$

Proof. We shall obtain u as the limit of the sequence u^j, $j \geqq 0$, such that $u^0 = 0$ and u^j, $j \geqq 1$, is a solution of

$$u^j_{tt} - \varepsilon^2 u^j_{xx} = -A(t)u^{j-1} + f, \quad 0 \leqq t \leqq T,$$
$$u^j = u_0, \ u^j_t = u_1, \quad t = 0.$$

Since the solution v of $v_{tt} - \varepsilon^2 v_{xx} = g$ can be written in the form

$$\hat{v}(t) = (\cos \varepsilon|\xi|t)\hat{v}(0) + \frac{\sin \varepsilon|\xi|t}{\varepsilon|\xi|}\hat{v}_t(0) + \int_0^t \frac{\sin \varepsilon|\xi|(t-s)}{\varepsilon|\xi|}\hat{g}(s)ds,$$

it is clear that

$$\|v(t)\|_{s+1} + \|v_t(t)\|_s \leqq C\left\{\|v(0)\|_{s+1} + \|v_t(0)\|_s + \int_0^t \|g(t)\|_s dt\right\}, \quad 0 \leqq t \leqq T.$$

These combined with the assumption on $A(t)$ show that u^j is defined and converges to u, which is the required solution. The uniqueness of u is easily proved. The proof is complete.

Next consider the initial value problem:

$$u_{tt} + G_m(W)u = f, \quad 0 \leqq t \leqq T, \tag{6.7}$$

$$u = u_0, \ u_t = u_1, \quad t = 0. \tag{6.8}$$

Lemma 6.9. *Let m be an integer $\geqq 2$ and the assumption 6.2 hold. If $u_0 \in H^{m+(1/2)}$, $u_1 \in H^m$, $f \in C^0([0, T], H^m)$ then there exists the unique solution u of (6.7), (6.8) such that*

$$u \in C^j([0, T], H^{m+(1/2)-(j/2)}), \quad j = 0, 1, 2.$$

Moreover u satisfies the estimate

$$|u(t)|_m \leqq e^{Ct}\sqrt{e_4}\,|u(0)|_m + \int_0^t e^{C(t-s)}\|f(s)\|_m ds \tag{6.10}$$

where $C = 2^{-1}d(C_4 e_1^{-1} + C_5)$ and

$$|u(t)|_m^2 = \|u_t\|_m^2 + e_1\| \, |D|^{1/2}u\|_m^2 + \|u\|_m^2.$$

Proof. The proof is divided into three steps.

Step 1: Let $u \in C^2([0, T], H^{m+1})$ satisfy the equation (6.7) and put

$$E_m(u(t))^2 = \|u_t\|_m^2 + (G\Lambda^m u, \Lambda^m u).$$

Then

$$\frac{1}{2}\frac{d}{dt}E_m(u(t))^2 = (u_{tt}, u_t)_m + (G\Lambda^m u, \Lambda^m u_t) + \frac{1}{2}(G_t\Lambda^m u, \Lambda^m u)$$

$$=(f,\ u_t)_m+\frac{1}{2}(G_t\Lambda^m u,\ \Lambda^m u)$$
$$\leqq \|f\|_m\|u_t\|_m+\frac{1}{2}d(C_4 e_1^{-1}+C_5)(G\Lambda^m u,\ \Lambda^m u)$$

in virtue of Lemma 6.3. Therefore for t such that $E_m(u(t))>0$,

$$\frac{d}{dt}E_m(u(t))\leqq \|f\|_m+CE_m(u(t)).$$

Since $E_m(u(t))$ is continuous in t we have

$$E_m(u(t))\leqq e^{Ct}E_m(u(0))+\int_0^t e^{C(t-s)}\|f(s)\|_m ds.$$

Using $|u(t)|_m\leqq E_m(u(t))\leqq\sqrt{e_4}\,|u(t)|_m$ (see Lemma 6.3, 4)), we obtain (6.10). Let u be the solution stated in this lemma. Since $\varphi_\delta * u\in C^2([0,\ T],\ H^{m+1})$ and

(6.11) $$(\varphi_\delta * u)_{tt}+G_m(\varphi_\delta * u)=\varphi_\delta * f-\varphi_\delta * G_m u+G_m(\varphi_\delta * u),$$

$\varphi_\delta * u$ satisfies the estimate obtained in the above if we replace f by the right-hand side of (6.11). By Lemma 2.3

$$\|\varphi_\delta * u-u\|_{m+(1/2)}+\|(\varphi_\delta * u)_t-u_t\|_m\to 0,\quad \delta\to +0.$$

Hence

$$|\varphi_\delta * u(t)|_m\to |u(t)|_m,\quad \delta\to +0,$$
$$\|\varphi_\delta * f\|_m\leqq \|f\|_m,\quad \|\varphi_\delta * f-f\|_m\to 0,\quad \delta\to +0.$$

By the simple calculation we have

$$\|\varphi_\delta * G_m u-G_m(\varphi_\delta * u)\|_m=\|\Lambda^m(\varphi_\delta * \Lambda^{-m}G\Lambda^m u)-G\Lambda^m(\varphi_\delta * u)\|$$
$$=\left\|[\varphi_\delta *,\ a]D\operatorname{sgn}D\Lambda^m u+\frac{1}{2}\varphi_\delta * [|D|,\ a]\Lambda^m u-\frac{1}{2}[|D|,\ a](\varphi_\delta * \Lambda^m u)\right\|.$$

Lemmas 2.3, 2.4 and 5.18, 4) show that this is bounded when $0\leqq t\leqq T$ and $0<\delta<1$, and tends to zero when $\delta\to +0$. Consequently we have the estimate (6.10), which assures the uniqueness of u.

Step 2: Let $0<\varepsilon<1$ and consider the initial value problem

(6.12) $$v_{tt}-\varepsilon^2 v_{xx}+G_m(W)v=g,\quad 0\leqq t\leqq T$$

(6.13) $$v=v_0,\ v_t=v_1,\quad t=0$$

where $v_0\in H^{m+2}$, $v_1\in H^{m+1}$ and $g\in C^0([0,\ T],\ H^{m+1})$. It follows from Lemma 6.3, 6) that

(6.14) $$G_m\in C^0([0,\ T],\ \mathscr{L}(H^s,\ H^{s-1})),\quad 1\leqq s\leqq 2m.$$

Since $1\leqq m+2\leqq 2m$ in virtue of $m\geqq 2$, putting $s=m+1$ in Lemma 6.6 we see that there exists the unique solution $v=v^{\varepsilon}$ of (6.12) and (6.13) such that

$$v\in C^{j}([0, T], H^{m+2-j}), \quad j=0, 1, 2.$$

Put $E_{m+1,\varepsilon}(v(t))^2=\|v_t\|_{m+1}^2+\varepsilon^2\|v_x\|_{m+1}^2+(G\Lambda^{m+1}v, \Lambda^{m+1}v)$. Since $\Lambda^{m+1}(G_{m+1}-G_m)=2^{-1}[a, |D|^2]\Lambda^m=2^{-1}[a, (1+|D|)^2-2|D|]\Lambda^m$, in virtue of Lemma 5.18 we have

$$\begin{aligned}
&|(\Lambda^{m+1}(G_{m+1}-G_m)v, \Lambda^{m+1}v_t)|\\
&\leqq\left(\frac{1}{2}\|[(1+|D|)^2, a]\Lambda^m v\|+\|[|D|, a]\Lambda^m v\|\right)\|\Lambda^{m+1}v_t\|\\
&\leqq\left(\frac{1}{2}C_3+C_2\right)\|v\|_{m+1}\|v_t\|_{m+1}.
\end{aligned}$$

We have

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}E_{m+1,\varepsilon}(v(t))^2\\
&=(v_{tt}, v_t)_{m+1}+\varepsilon^2(v_x, v_{xt})_{m+1}+(G\Lambda^{m+1}v, \Lambda^{m+1}v_t)+\frac{1}{2}(G_t\Lambda^{m+1}v, \Lambda^{m+1}v)\\
&=(v_{tt}-\varepsilon^2 v_{xx}+G_{m+1}v, v_t)_{m+1}+\frac{1}{2}(G_t\Lambda^{m+1}v, \Lambda^{m+1}v)\\
&=(g+G_{m+1}v-G_m v, v_t)_{m+1}+\frac{1}{2}(G_t\Lambda^{m+1}v, \Lambda^{m+1}v)\\
&\leqq\|g\|_{m+1}E_{m+1,\varepsilon}(v(t))+CE_{m+1,\varepsilon}(v(t))^2.
\end{aligned}$$

Therefore

$$E_{m+1,\varepsilon}(v(t))\leqq e^{Ct}E_{m+1,\varepsilon}(v(0))+\int_0^t e^{C(t-s)}\|g(s)\|_{m+1}ds. \tag{6.15}$$

Since

$$\begin{aligned}
E_{m+1,\varepsilon}(v(0))^2&=\|v_1\|_{m+1}^2+\varepsilon^2\|v_{0x}\|_{m+1}^2+(G\Lambda^{m+1}v_0, \Lambda^{m+1}v_0)\\
&\leqq\|v_1\|_{m+1}^2+\|v_0\|_{m+2}^2+e_4(e_1\||D|^{1/2}\Lambda^{m+1}v_0\|^2+\|\Lambda^{m+1}v_0\|^2),
\end{aligned}$$

there exists a constant C independent of $0<\varepsilon<1$, $0\leqq t\leqq T$ such that

$$\|\varepsilon v_x^{\varepsilon}(t)\|_{m+1}\leqq C, \quad 0<\varepsilon<1, \ 0\leqq t\leqq T. \tag{6.16}$$

Let $0<\varepsilon$, $\delta<1$ and put $v=v^{\varepsilon}-v^{\delta}$. Then

$$\begin{cases} v_{tt}+G_m v=\varepsilon^2 v_{xx}^{\varepsilon}-\delta^2 v_{xx}^{\delta}, & 0\leqq t\leqq T\\ v=0, \quad v_t=0, \quad t=0. \end{cases}$$

Applying the estimate (6.10) to v, we have

$$|v(t)|_m\leqq e^{CT}\int_0^t\|\varepsilon^2 v_{xx}^{\varepsilon}-\delta^2 v_{xx}^{\delta}\|_m ds\leqq e^{CT}CT(\varepsilon+\delta).$$

Hence there exists u such that, when $\varepsilon \to +0$,

$$v^{\varepsilon}(t) \to u(t) \quad \text{in} \quad H^{m+(1/2)}, \quad v_t^{\varepsilon}(t) \to u_t(t) \quad \text{in} \quad H^m$$

uniformly in $0 \leqq t \leqq T$. These, combined with (6.14) and (6.16), show that

$$\varepsilon^2 v_{xx}^{\varepsilon} \to 0 \quad \text{in} \quad H^m, \quad G_m v^{\varepsilon} \to G_m u \quad \text{in} \quad H^{m-(1/2)}$$

uniformly in $0 \leqq t \leqq T$. By the equation $v_{tt}^{\varepsilon} = \varepsilon^2 v_{xx}^{\varepsilon} + G_m v^{\varepsilon} + f$, we see that

$$v_{tt}^{\varepsilon} \to u_{tt} \quad \text{in} \quad H^{m-(1/2)} \text{ uniformly in } 0 \leqq t \leqq T.$$

Consequently $u \in C^j([0, T], H^{m+(1/2)-(j/2)})$, $j = 0, 1, 2$, and u is the solution of

$$\text{(6.17)} \qquad \begin{cases} u_{tt} + G_m u = g, & 0 \leqq t \leqq T \\ u = v_0, \quad u_t = v_1, & t = 0. \end{cases}$$

Step 3: Let $0 < \delta < 1$ and put $v_0 = \varphi_\delta * u_0$, $v_1 = \varphi_\delta * u_1$, $g = \varphi_\delta * f$. Then by the step 2, there exists the unique solution u^δ of (6.17). Putting $w = u^{\varepsilon} - u^\delta$ and applying the estimate (6.10) to w, we have

$$|w(t)|_m \leqq e^{CT}\Big\{\sqrt{e_4}\,(\|\varphi_\delta * u_1 - \varphi_\varepsilon * u_1\|_m^2 + (e_1 + 1)\|\varphi_\delta * u_0 - \varphi_\varepsilon * u_0\|_{m+(1/2)}^2)^{1/2}$$
$$+ \int_0^T \|\varphi_\delta * f - \varphi_\varepsilon * f\|_m ds\Big\}, \quad 0 \leqq t \leqq T.$$

Lemma 2.3 and the argument as in the step 2 show that when $\delta \to +0$, the limit u of u^δ exists and u is the required solution, which completes the proof.

Now consider the initial value problem:

$$\text{(6.18)} \qquad u_{tt} + a(W)|D|u = f, \quad 0 \leqq t \leqq T,$$

$$\text{(6.19)} \qquad u = u_0, \quad u_t = u_1, \quad t = 0.$$

Theorem 6.20. *Under Assumption* 6.2, *if* $u_0 \in H^{m+(1/2)}$, $u_1 \in H^m$ *and* $f \in C^0([0, T], H^m)$, *then there exists the unique solution* u *of* (6.18) *and* (6.19) *such that* $u \in C^j([0, T], H^{m+(1/2)-(j/2)})$, $j = 0, 1, 2$. *Moreover* u *satisfies the estimate*

$$\text{(6.21)} \qquad |u(t)|_m \leqq \sqrt{e_4}\, e^{Ct}|u(0)|_m + \int_0^t e^{C(t-s)}\|f(s)\|_m ds$$

where $|u(t)|_m^2 = \|u_t\|_m^2 + e_1\| \, |D|^{1/2}u\|_m^2 + \|u\|_m^2$ *and* $C = 1 + C_1 + 2^{-1}C_2 + C_3 + (d/2) \times (C_4 e_1^{-1} + C_5)$.

Proof. Using $a|D| = G_m - G_m'$ we write (6.18) in the form

$$\text{(6.22)} \qquad u_{tt} + G_m u = f + G_m' u.$$

The solution u will be obtained as the limit of the sequence u^j, $j\geqq 0$, such that $u^0=0$ and u^j, $j\geqq 1$, is a solution of

$$\begin{cases} u^j_{tt}+G_m u^j=f+G'_m u^{j-1} \\ u^j=u_0, \quad u^j_t=u_1, \quad t=0\,. \end{cases}$$

Since $G'_m\in C^0([0, T], \mathscr{L}(H^m, H^m))$ by Lemma 6.3, it follows from Lemma 6.9 that u^j, $j\geqq 1$, is uniquely defined and

$$u^j\in C^k([0, T], H^{m+(1/2)-(k/2)}), \quad k=0, 1, 2\,.$$

By the estimate (6.10) and Lemma 6.3 we have

$$|u^{j+1}(t)-u^j(t)|_m\leqq C\int_0^t \|u^j(s)-u^{j-1}(s)\|_m ds\leqq C\int_0^t |u^j(s)-u^{j-1}(s)|_m ds,$$

which assures the existence of the required solution u. Applying the estimate (6.10) to the solution u of (6.22) we have in virtue of Lemma 6.3

$$\begin{aligned} e^{-Ct}|u(t)|_m\leqq \sqrt{e_4}\,|u(0)|_m+\int_0^t e^{-Cs}\|f(s)\|_m ds \\ +\left(1+C_1+\frac{1}{2}C_2+C_3\right)\int_0^t e^{-Cs}|u(s)|_m ds \end{aligned}$$

where $C=(d/2)(C_4 e_1^{-1}+C_5)$. Therefore we have

$$\begin{aligned} e^{-Ct}|u(t)|_m\leqq \sqrt{e_4}\,|u(0)|_m \exp\left(1+C_1+\frac{1}{2}C_2+C_3\right)t \\ +\int_0^t \|f(s)\|_m e^{-Cs}\exp\left\{\left(1+C_1+\frac{1}{2}C_2+C_3\right)(t-s)\right\}ds, \end{aligned}$$

which gives (6.21). The proof is complete.

6.3. Quasilinear System. We use the notations:

$$\begin{aligned} &W=(X, Y, Z), \ W'=(X, Y_1), \quad |Y_1(t)|_m^2=\|Y_1\|_m^2+e_1\|\,|D|^{1/2}Y_1\|_m^2+\|Y_{1t}\|_m^2, \\ &|W|_m^2=|W(t)|_m^2=\|W(t)\|_m^2+\|W'_t(t)\|_m^2+e_1\|\,|D|^{1/2}Y_1\|_m^2 \\ &\quad=\|X\|_m^2+\|X_t\|_m^2+\|Y_1\|_m^2+e_1\|\,|D|^{1/2}Y_1\|_m^2+\|Y_{1t}\|_m^2+\|Y_2\|_m^2+\|Z\|_m^2\,. \end{aligned}$$

We shall consider the initial value problem for the quasilinear system (see (5.16)):

$$\text{(6.23)} \qquad \begin{cases} X_{tt}=Y \\ Y_{1tt}+a(W)|D|Y_1=f_1 \\ Y_{2t}=f_2, \quad Z_{1t}=f_3, \quad Z_{2t}=f_4\,. \end{cases}$$

To simplify the notations we write the initial condition at $t=0$ in the form

$$\text{(6.24)} \qquad W(0)=\tilde{W}=(\tilde{X}, \tilde{Y}, \tilde{Z}), \quad W'_t(0)=\widetilde{W'_t}=(\widetilde{X_t}, \widetilde{Y_{1t}})\,.$$

Theorem 6.25. *Let c_0 be so small that Lemmas 5.18 and 5.22 hold, m be an integer $\geqq 4$ and $b \in H^{m+1}$, $\|b\|_3 \leqq c_0$. If $\tilde{W}$, $\widetilde{W'_t}$ satisfy the conditions;*

$$\tilde{W},\ \widetilde{W'_t},\ |D|^{1/2}\widetilde{Y_1} \in H^m \quad \text{and} \quad \|\tilde{W}\|_3 < c_0,$$

then we can choose $T>0$ such that the initial value problem (6.23) and (6.24) has the unique solution W such that

$$X \in C^2([0, T], H^m),\ Y_2,\ Z \in C^1([0, T], H^m),$$
$$Y_1 \in C^j([0, T], H^{m+(1/2)-(j/2)}),\quad j=0, 1, 2, \qquad \|W(t)\|_3 \leqq c_0, \quad 0 \leqq t \leqq T.$$

Proof. The proof is decomposed into several steps.

Step 1: Take d, d_0, d_1, d_3 and d_m such that

$$(6.26) \qquad \begin{cases} \|b\|_{m+1} \leqq d_0, \quad \|\tilde{W}\|_m \leqq d_0, \quad \|\widetilde{W'_t}\|_m \leqq d_0, \quad \|\widetilde{W'_t}\|_3 \leqq d_1 \\ d_j = (2+\sqrt{e_4})\{\|\tilde{W}\|_j^2 + \|\widetilde{W'_t}\|_j^2 + e_1 \| |D|^{1/2}\widetilde{Y_1}\|_j^2\}^{1/2}, \quad j=3, m, \\ d_3 < d_1, \quad d_m < d_0, \quad d = \{d_1^2 + k_0^2(c_0^2 + d_1^2)\}^{1/2} \end{cases}$$

where $k_0 = k(c_0, \max(c_0, d_1), 3)$ is the constant occurring in Lemma 5.22. Now we shall estimate the solution W satisfying the conditions,

$$\|W(t)\|_3 \leqq c_0, \quad \|W'_t(t)\|_3 \leqq d_1, \quad \|W(t)\|_m \leqq d_0, \quad \|W'_t(t)\|_m \leqq d_0, \quad 0 \leqq t \leqq T.$$

We see that $\|Y(t)\|_2^2 + \|Z(t)\|_2^2 \leqq \|W(t)\|_3^2 \leqq c_0^2$, $\|Y(t)\|_m^2 + \|Z(t)\|_m^2 \leqq \|W(t)\|_m^2 \leqq d_0^2$ and in virtue of Lemma 5.22

$$\begin{aligned} \|Y_t(t)\|_2^2 + \|Z_t(t)\|_2^2 &\leqq \|Y_{1t}(t)\|_3^2 + \|(Y_{2t}, Z_t)\|_3^2 \\ &\leqq d_1^2 + \|f(W, W'_t)\|_3^2 \\ &\leqq d_1^2 + k_0^2(c_0^2 + d_1^2) = d^2. \end{aligned}$$

Therefore we can use (6.21), which gives

$$\begin{aligned} |Y_1(t)|_m &\leqq \sqrt{e_4}\,|Y_1(0)|_m e^{Ct} + \int_0^t e^{C(t-s)} \|f_1(W, W'_t)\|_m ds \\ &\leqq \sqrt{e_4}\,|W(0)|_m e^{Ct} + k(c_0, d_0, m)\int_0^t e^{C(t-s)} |W(s)|_m ds \end{aligned}$$

in virtue of Lemma 5.22. For Y_2, Z we have

$$\begin{aligned} \|(Y_2(t), Z(t))\|_m &= \|(Y_2(0), Z(0)) + \int_0^t (f_2, f_3, f_4) ds\|_m \\ &\leqq |W(0)|_m + k\int_0^t |W(s)|_m ds. \end{aligned}$$

Since

$$\frac{1}{2}\frac{d}{dt}\{(X_t, X_t)_m + (X, X)_m\} = (X_{tt} + X, X_t)_m = (X + Y, X_t)_m$$

$$\leqq \|X+Y\|_m \|X_t\|_m \leqq 2|W(t)|_m \|(X, X_t)\|_m,$$

we have

$$\|(X, X_t)\|_m \leqq |W(0)|_m + 2\int_0^t |W(s)|_m ds.$$

Hence

$$\begin{aligned}|W(t)|_m &\leqq \|(X, X_t)\|_m + \|(Y_2, Z)\|_m + |Y_1|_m \\ &\leqq (2+\sqrt{e_4})|W(0)|_m e^{Ct} + (2+2k)\int_0^t e^{C(t-s)}|W(s)|_m ds,\end{aligned}$$

which gives the estimate

$$|W(t)|_m \leqq (2+\sqrt{e_4})|W(0)|_m \exp(k_m t) = d_m \exp(k_m t)$$

where $k_m = 2 + 2k(c_0, d_0, m) + 1 + C_1 + 2^{-1}C_2 + C_3 + 2^{-1}d(C_4 e_1^{-1} + C_5)$, $C_3 = C_3(c_0, d_0, m)$. Replacing m by 3 we have

$$|W(t)|_3 \leqq (2+\sqrt{e_4})|W(0)|_3 \exp(k_3 t) = d_3 \exp(k_3 t)$$

where $k_3 = 2 + 2k_0 + 1 + C_1 + 2^{-1}C_2 + C_3 + 2^{-1}d(C_4 e_1^{-1} + C_5)$, $C_3 = C_3(c_0, c_0, 3)$.

Step 2: Put

$$T = \min\left\{\frac{1}{k_3}\log\frac{d_1}{d_3}, \frac{1}{k_m}\log\frac{d_0}{d_m}, \frac{c_0 - \|\tilde{W}\|_3}{(1+k_0)d_1}\right\}.$$

Since $0 \leqq d_3 < d_1$, $0 \leqq d_m < d_0$, $0 < c_0 - \|\tilde{W}\|_3$, $0 < k, k_3, k_m$ we see that $0 < T < \infty$. By S we denote the totality of W satisfying the following conditions:

$$W, W_t, |D|^{1/2}Y_1 \in C^0([0, T], H^m), \tag{6.27}$$

$$\begin{cases} \|W(t) - \tilde{W}\|_3 \leqq c_0 - \|\tilde{W}\|_3, \quad \|W_t'(t)\|_3 \leqq d_1, \quad \|W(t)\|_m \leqq d_0, \\ \|W_t'(t)\|_m \leqq d_0 \\ \|(Y_t(t), Z_t(t))\|_3 \leqq d, \quad |W(t)|_j \leqq d_j \exp(k_j t), \quad j = 3, m \\ \qquad\qquad\qquad\qquad\qquad\qquad \text{for} \quad 0 \leqq t \leqq T, \ W(0) = \tilde{W}. \end{cases} \tag{6.28}$$

We denote by $M(W^0)$ the solution W of the initial value problem,

$$\begin{cases} X_{tt} + X = X^0 + Y^0 \\ Y_{1tt} + a(W^0)|D|Y_1 = f_1(W^0, W_t^{0\prime}) \\ Y_{2t} = f_2(W^0, W_t^{0\prime}), \quad Z_{1t} = f_3(W^0, W_t^{0\prime}), \quad Z_{2t} = f_4(W^0, W_t^{0\prime}) \end{cases} \tag{6.29}$$

$$W(0) = \tilde{W}, \quad W_t'(0) = \widetilde{W_t'}. \tag{6.30}$$

We shall show that if $W^0 \in S$ then $W = M(W^0) \in S$. Since

$$\|W^0\|_3 \leqq \|W^0 - \tilde{W}\|_3 + \|\tilde{W}\|_3 \leqq c_0,$$

it follows from (6.28) that Y^0, Z^0 satisfy Assumption 6.2 and in virtue of Lemma 5.22 the right-hand sides of (6.29) belong to $C^0([0, T], H^m)$. Therefore by the integration and Theorem 6.20 we see that the initial value problem (6.29) and (6.30) has the unique solution W satisfying the condition (6.27). In the same way as in the step 1 we obtain

$$|W(t)|_j \leqq d_j \exp(k_j t), \quad j=3,\ m.$$

By the definition of T we have for $0 \leqq t \leqq T$

$$\|W(t)\|_j, \quad \|W'_t(t)\|_j \leqq |W(t)|_j \leqq d_j \exp(k_j t) \begin{cases} \leqq d_0, & j=m \\ \leqq d_1, & j=3. \end{cases}$$

Since $W^0 \in S$, we have

$$\begin{aligned} \|Y_{1t}\|_3^2 + \|(Y_{2t}, Z_t)\|_3^2 &\leqq d_1^2 + \|f(W^0, W_t^{0\prime})\|_3^2 \\ &\leqq d_1^2 + k_0^2 |W^0(t)|_3^2 \leqq d_1^2 + k_0^2(c_0^2 + d_1^2) = d^2, \end{aligned}$$

$$\begin{aligned} \|W(t) - \tilde{W}\|_3 &\leqq \int_0^t \left\| \frac{d}{dt} W(t) \right\|_3 dt \leqq \int_0^t \{\|W'_t(t)\|_3 + \|(Y_{2t}, Z_t)\|_3\} dt \\ &\leqq \int_0^t \{\|W'_t(t)\|_3 + k_0 |W^0(t)|_3\} dt \leqq T(d_1 + k_0 d_1) \leqq c_0 - \|\tilde{W}\|_3. \end{aligned}$$

Consequently $W \in S$, which means that M is the mapping from S to itself.

Step 3: Put $W^0(t) = \tilde{W}$, $0 \leqq t \leqq T$. Then $X_t^0 = Y_t^0 = 0$, $\|W^0(t)\|_m = \|\tilde{W}\|_m \leqq d_0$,

$$\begin{aligned} |W^0(t)|_j &= (\|\tilde{W}\|_j^2 + e_1 \||D|^{1/2} \tilde{Y}_1\|_j^2)^{1/2} \\ &\leqq (2 + \sqrt{e_4})(\|\tilde{W}\|_j^2 + \|\tilde{W}'_t\|_j^2 + e_1 \||D|^{1/2} \tilde{Y}_1\|_j^2)^{1/2} = d_j \leqq d_j \exp(k_j t), \quad j=3, m. \end{aligned}$$

We see that W^0 satisfies (6.27) and (6.28), i.e., $W^0 \in S$. The result of the step 2 shows that $W^{j+1} = M(W^j)$, $j \geqq 0$, are defined and $W^j \in S$. Note that $W = W^{j+1} - W^j$, $j \geqq 1$, is the solution of

$$(6.31) \qquad \begin{cases} X_{tt} + X = X^j - X^{j-1} + Y^j - Y^{j-1} \\ Y_{1tt} + a(W^j)|D| Y_1 = f_1(W^j, W_t^{j\prime}) - f_1(W^{j-1}, W_t^{j-1\prime}) \\ \qquad\qquad - (a(W^j) - a(W^{j-1}))|D| Y_1^j \\ Y_{2t} = f_2(W^j, W_t^{j\prime}) - f_2(W^{j-1}, W_t^{j-1\prime}) \\ Z_{kt} = f_{2+k}(W^j, W_t^{j\prime}) - f_{2+k}(W^{j-1}, W_t^{j-1\prime}), \quad k=1, 2, \\ W(0) = 0, \quad W'_t(0) = 0. \end{cases}$$

Since $m-1 \geqq 3$ and $W^j \in S$, we have in virtue of Lemma 5.18 with $s = m-1$,

$$\begin{aligned} &\|\{a(W^j) - a(W^{j-1})\} |D| Y_1^j\|_{m-1} \\ &\quad \leqq C\|a(W^j) - a(W^{j-1})\|_{m-1} \|Y_1^j\|_m \leqq C\|W^j - W^{j-1}\|_{m-1} \|W^j\|_m. \end{aligned}$$

Therefore, in virtue of Lemma 5.22, we see that H^{m-1}-norms of the right-hand sides of (6.31) are smaller than $C|W^j(t)-W^{j-1}(t)|_{m-1}$ where C is independent of j. In the same way as in the step 1, we have

$$\|(Y_2(t), Z(t))\|_{m-1}+\|(X(t), X_t(t))\|_{m-1}+|Y_1(t)|_{m-1} \leqq C\int_0^t |W^j-W^{j-1}|_{m-1}ds\,.$$

Therefore we obtain

$$|W^{j+1}(t)-W^j(t)|_{m-1}\leqq C\int_0^t |W^j(s)-W^{j-1}(s)|_{m-1}ds\,.$$

This means that there exists W such that

$$W,\ W'_t,\ |D|^{1/2}Y_1\in C^0([0, T], H^{m-1}),\quad \sup_{0\leqq t\leqq T}|W^j(t)-W(t)|_{m-1}\to 0,\quad j\to\infty\,.$$

Noting that $W^{j+1}=M(W^j)\in S$ and letting $j\to\infty$ we see that $(X^j_{tt}, Y^j_{2t}, Z^j_t)$ converges in H^{m-1} and Y^j_{1tt} converges in $H^{m-1-(1/2)}$. Hence

$$\begin{cases} X_{tt}=Y, \quad Y_{1tt}+a(W)|D|Y_1=f_1(W, W'_t), \\ Y_{2t}=f_2(W, W'_t), \quad Z_{jt}=f_{2+j}(W, W'_t), \quad j=1, 2, \end{cases}$$

$$(6.32)\qquad \begin{cases} X\in C^2([0, T], H^{m-1}), \quad Y_2, Z\in C^1([0, T], H^{m-1}), \\ Y_1\in C^j([0, T], H^{m-1+(1/2)-(j/2)}), \quad j=0, 1, 2, \\ \|W(t)\|_3\leqq c_0, \quad \|(Y_t(t), Z_t(t))\|_3\leqq d\,. \end{cases}$$

Step 4: We shall show that W is the solution required in this theorem, i.e., in (6.32) we can replace $m-1$ by m. Noting that $W^{j+1}=M(W^j)\in S$ and using Lemma 5.22 we have

$$(6.33)\qquad \begin{cases} \|W^j(t)\|_m\leqq d_0, \quad \|W^{j\prime}_t(t)\|_m\leqq d_0, \quad |W^j(t)|_m\leqq d_m\exp(k_mt)\,, \\ \|X^j_{tt}(t)\|_m+\|Y^j_{1tt}(t)\|_{m-(1/2)}+\|Y^j_{2t}(t)\|_m+\|Z^j_t(t)\|_m\leqq C \end{cases}$$

where $C>0$ is independent of t and j. Since any bounded sequence in a Hilbert space is weakly precompact, each sequence occurring in (6.33) has a weak-limit. By the result of the step 3, they have the strong-limits if m is replaced by $m-1$. Hence for any fixed t,

$$W(t),\quad W'_t(t),\quad Y_{2t}(t),\quad |D|^{1/2}Y_1(t),\quad X_{tt}(t),\quad Z_t(t)\in H^m,\quad Y_{1tt}(t)\in H^{m-(1/2)}.$$

Taking the inferior limits of sequences in (6.33) we see that

$$(6.34)\qquad \begin{cases} \|W(t)\|_m\leqq d_0, \quad \|W'_t(t)\|_m\leqq d_0, \quad |W(t)|_m\leqq d_m\exp(k_mt)\,, \\ \|X_{tt}(t)\|_m+\|Y_{1tt}(t)\|_{m-(1/2)}+\|Y_{2t}(t)\|_m+\|Z_t(t)\|_m\leqq C\,. \end{cases}$$

For $0 \leqq t_0 < t \leqq T$, we have

$$\| W^j(t) - W^j(t_0) \|_m \leqq \int_{t_0}^t \| W_t^j(t) \|_m dt \leqq (t - t_0)(d_0 + 2C).$$

Taking the inferior limit we have $\| W(t) - W(t_0) \|_m \leqq (t - t_0)(d_0 + 2C)$. Hence $W \in C^0([0, T], H^m)$. Similarly we see that $X_t \in C^0([0, T], H^m)$. By $X_{tt} = Y$, we have $X \in C^2([0, T], H^m)$. Since $m - 1 \geqq 2$, (Y, Z) satisfies Assumption 6.2 in virtue of (6.32) and (6.34). Hence by Theorem 6.20 there exists u^δ, $\delta > 0$, such that

$$\begin{aligned} &u_{tt}^\delta + a(W)|D|u^\delta = f_1(W, \varphi_\delta * W_t'), \quad 0 \leqq t \leqq T, \\ &u^\delta(0) = \widetilde{Y_1}, \quad u_t^\delta(0) = \widetilde{Y_{1t}}, \\ &u^\delta \in C^j([0, T], \quad H^{m+(1/2)-(j/2)}), \quad j = 0, 1, 2. \end{aligned}$$

Using (6.21) we have

$$\begin{aligned} |u^\varepsilon(t) - u^\delta(t)|_m &\leqq C \int_0^t \| f_1(W, \varphi_\varepsilon * W_t') - f_1(W, \varphi_\delta * W_t') \|_m dt \\ &\leqq C \int_0^t \| \varphi_\varepsilon * W_t' - \varphi_\delta * W_t' \|_m dt \to 0, \quad \varepsilon, \delta \to +0. \end{aligned}$$

On the other hand,

$$\begin{aligned} |u^\delta(t) - Y_1(t)|_{m-1} &\leqq C \int_0^t \| f_1(W, \varphi_\delta * W_t') - f_1(W, W_t') \|_{m-1} dt \\ &\leqq C \int_0^t \| \varphi_\delta * W_t' - W_t' \|_{m-1} dt \to 0, \quad \delta \to +0. \end{aligned}$$

Therefore $Y_1 \in C^j([0, T], H^{m+(1/2)-(j/2)})$, $j = 0, 1$. Thus we proved that W, $W_t' \in C^0([0, T], H^m)$. Consequently, in virtue of Lemma 5.22, $f(W, W_t') \in C^0([0, T], H^m)$, by means of which we see that $Y_{1tt} \in C^0([0, T], H^{m-(1/2)})$ and Y_{2t}, $Z_t \in C^0([0, T], H^m)$. The proof is complete.

Remark 6.35. In the step 2 we defined T by

$$T = \min \left\{ \frac{1}{k_3} \log \frac{d_1}{d_3}, \ \frac{1}{k_m} \log \frac{d_0}{d_m}, \ \frac{c_0 - \| \tilde{W} \|_3}{(1 + k_0) d_1} \right\}.$$

(For d, d_0, d_1, d_3 and d_m see (6.26) and for k_3, k_m see the end of the step 1.) Putting $d_1 = \sqrt{d_3}$ we see that $T \to \infty$ if $d_3, d_m \to 0$, i.e. the initial values, $\tilde{W}$, $\widetilde{W_t'}$ tend to zero.

6.4. Nonlinear Equations. Consider the initial value problem:

(6.36) $\quad (1 + X_{1x}) X_{1tt} + X_{2x}(1 + X_{2tt}) = 0, \ X_{2t} = K X_{1x}, \quad 0 \leqq t \leqq T,$

(6.37) $\quad X = U, \quad X_{1t} = V, \quad t = 0.$

Theorem 6.38. *Take c_0 be so small that Lemmas* 5.18, 5.22 *and* 5.29 *hold. Let m be an integer $\geqq 5$ and $b \in H^m$, $\|b\|_3 \leqq c_0$. There exists $\delta > 0$ such that if*

$$U \in H^{m+(1/2)}, \quad V \in H^m, \quad \|U\|_{4+(1/2)} \leqq \delta, \quad \|V\|_4 \leqq \delta$$

then there exists $T > 0$ such that the initial value problem (6.36) *and* (6.37) *has the unique solution*

$$X \in C^1([0, T], H^m) \cap C^2([0, T], H^{m-(1/2)}).$$

Proof. Existence. Define W, W'_t, $t=0$, by (5.27). Then it follows from Lemma 5.29 that we can take $\delta > 0$ so small that W, W'_t, $t=0$, satisfy the conditions of Theorem 6.25 if m is replaced by $m-1$. Therefore by Theorem 6.25 we have the solution W of the system (6.23) such that

$$X \in C^2([0, T], H^{m-1}), \quad Y_2, Z \in C^1([0, T], H^{m-1}),$$

$$Y_1 \in C^j([0, T], H^{m-1+(1/2)-(j/2)}), \quad j=0, 1, 2,$$

for some $T > 0$. It is clear that X satisfies (6.37). We have $(1+Z_1)Y_1 + Z_2(1+Y_2)=0$, $t=0$, by Remark 5.28 and $\partial/\partial t\{(1+Z_1)Y_1+Z_2(1+Y_2)\}=0$, $0 \leqq t \leqq T$, by Remark 5.9. Hence $(1+Z_1)Y_1+Z_2(1+Y_2)=0$, $0 \leqq t \leqq T$. Since $Y = X_{tt}$, it remains to show that $Z = X_x$, $X_{2t} = KX_{1t}$, $0 \leqq t \leqq T$, and X has the required differentiability. Since $Y = X_{tt}$ and

$$Y_{2t} = f_2 = KY_{1t} + F_{20}(X, X_t, Y), \tag{6.39}$$

we have $(X_{2t} - KX_{1t})_{tt} = 0$, $0 \leqq t \leqq T$. On the other hand, it follows from (5.27) that $X_{2t} - KX_{1t} = 0$, $(X_{2t} - KX_{1t})_t = Y_2 - KY_1 - F_{10}(X, X_t) = 0$, $t=0$. Thus $X_{2t} = KX_{1t}$, $0 \leqq t \leqq T$. Differentiating this, we have

$$\begin{cases} Y_2 = KY_1 + F_{10}(X, X_t) \\ Y_{2tt} = KY_{1tt} + F_{30}(X, X_t, Y, Y_t) \\ Y_{2x} = KY_{1x} + F_{11}(X, X_t, X_x, X_{xt}, X_{1tt}) \\ X_{2tx} = KX_{1tx} + F_{01}(X, X_x, X_{1t}). \end{cases} \tag{6.40}$$

By Remark 5.9,

$$\begin{aligned} Z_{2t} &= -i \operatorname{sgn} DZ_{1t} + F_{010}(X, Z, X_{1t}) \\ &= -i \operatorname{sgn} DZ_{1t} + (i \operatorname{sgn} D - i \tanh(hD) + K_1)\frac{\partial}{\partial x} X_{1t} + F_{01}(X, Z, X_{1t}) \\ &= -i \operatorname{sgn} D(Z_1 - X_{1x})_t + K\frac{\partial}{\partial x} X_{1t} + F_{01}(X, Z, X_{1t}). \end{aligned}$$

From this we obtain

(6.41) $$Z_{2tt}=-i\,\mathrm{sgn}\,D(Z_1-X_{1x})_{tt}+KY_{1x}+F_{11}(X, X_t, Z, Z_t, Y_1).$$

Using (6.40) we have

(6.42) $$(Z_2-X_{2x})_t=-i\,\mathrm{sgn}\,D(Z_1-X_{1x})_t+F_{01}(X, Z, X_{1t})-F_{01}(X, X_x, X_{1t}).$$

By Remark 5.15,

$$\begin{aligned}0&=(1+Z_1)Y_{1tt}+Z_2KY_{1tt}+Z_2F_{30}+Y_1Y_{1x}+(1+Y_2)KY_{1x}+(1+Y_2)F_{11}+2Y_tZ_t\\&=(1+Z_1)Y_{1tt}+Z_2Y_{2tt}+Y_1Y_{1x}+(1+Y_2)(KY_{1x}+F_{11})+2Y_tZ_t.\end{aligned}$$

Since

$$\begin{aligned}0&=\{(1+Z_1)Y_1+Z_2(1+Y_2)\}_{tt}\\&=(1+Z_1)Y_{1tt}+Z_2Y_{2tt}+Y_1Z_{1tt}+(1+Y_2)Z_{2tt}+2Y_tZ_t,\end{aligned}$$

we have

$$\begin{aligned}0&=Y_1(Z_{1tt}-Y_{1x})+(1+Y_2)(Z_{2tt}-KY_{1x}-F_{11})\\&=Y_1(Z_1-X_{1x})_{tt}+(1+Y_2)(-i\,\mathrm{sgn}\,D)(Z_1-X_{1x})_{tt}\\&=\{Y_1+(1+Y_2)(-i\,\mathrm{sgn}\,D)\}(Z_1-X_{1x})_{tt}\end{aligned}$$

where we used (6.41). Thus $(Z_1-X_{1x})_{tt}=0$, $0\leqq t\leqq T$. In virtue of (5.27), $Z_1-X_{1x}=0$, $t=0$. By Remark 5.28 and (6.39),

$$(1+Z_1)Y_{1t}+Z_2Y_{2t}+Y_1X_{1tx}+(1+Y_2)X_{2tx}=0,\quad t=0.$$

On the other hand,

$$\{(1+Z_1)Y_1+Z_2(1+Y_2)\}_t=(1+Z_1)Y_{1t}+Z_2Y_{2t}+Y_1Z_{1t}+(1+Y_2)Z_{2t}=0.$$

Therefore, $Y_1(Z_1-X_{1x})_t+(1+Y_2)(Z_2-X_{2x})_t=0$, $t=0$. Putting $t=0$ in (6.42), we have $(Z_2-X_{2x})_t=-i\,\mathrm{sgn}\,D(Z_1-X_{1x})_t$, $t=0$. Thus $(Z_1-X_{1x})_t=0$, $t=0$. Consequently, $Z_1-X_{1x}=0$, $0\leqq t\leqq T$. Since $Z_2-X_{2x}=0$, $t=0$, (6.42) gives

$$Z_2-X_{2x}=\int_0^t\{F_{01}(X, Z, X_{1t})-F_{01}(X, X_x, X_{1t})\}dt.$$

We have

$$\|Z_2-X_{2x}\|_{m-2}\leqq C\int_0^t\|Z_2-X_{2x}\|_{m-2}dt,$$

which shows that $Z_2-X_{2x}=0$, $0\leqq t\leqq T$. Thus we have proved that X satisfies (6.36), (6.37). Since X, $X_x=Z\in C^1([0, T], H^{m-1})$, we see that $X\in C^1([0, T], H^m)$. Since $X_{1tt}=Y_1\in C^0([0, T], H^{m-(1/2)})$ and

$$\begin{aligned}X_{2tt}&=Y_2=K(X)Y_1+F_{10}(X, X_t)\\&=K(X, b; X, b)Y_1+K_{1,1,0}(X, X_t, b; X, b)X_{1t}\end{aligned}$$

we see by Lemma 4.27 that $X_{2tt} \in C^0([0, T], H^{m-(1/2)})$.

Uniqueness. Put $W=(X, X_{tt}, X_x)$. By the estimate as in the end of the step 2 of the proof of Theorem 6.25 we see that

$$\|W(t)-W(0)\|_3 \leqq \int_0^t \left\|\frac{d}{dt}W(t)\right\|_3 dt$$
$$\leqq t(d_1+k_0 d_1) < c_0 - \|W(0)\|_3, \quad 0 \leqq t < T,$$

in virtue of the definition of T. Let X^0 be a solution of (6.36), (6.37) and put $W^0=(X^0, X^0_{tt}, X^0_x)$. It is easily seen that if $\|W^0(t)\|_3 \leqq c_0$ then W^0 is a solution of (6.23) having the properties stated in Theorem 6.25 where m is replaced by $m-1$. Since $W^0(0)=W(0)$, $\|W(0)\|_3 < c_0$ we see that $\|W(t)\|_3 \leqq c_0$, $0 \leqq t \leqq t_0$ for sufficiently small $t_0>0$. By Theorem 6.25 we have $W^0(t)=W(t)$, $0 \leqq t \leqq t_0$. Since $\|W(t_0)\|_3 < c_0$, we see that $\|W^0(t)\|_3 \leqq c_0$, $t_0 \leqq t \leqq t_0+t_1$ for small $t_1>0$. Hence $W^0(t)=W(t)$, $0 \leqq t \leqq t_1$. Repeating this procedure we see that $W^0(t) = W(t)$, $0 \leqq t < T$, i.e., $X^0(t)=X(t)$, $0 \leqq t \leqq T$. The proof is complete.

Remark 6.43. By Remark 6.35 we see that $T \to \infty$ if $U \to 0$ in $H^{m+(1/2)}$, $V \to 0$ in H^m.

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