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On the G-Homotopy Types of G-ANR's

By

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§0. Introduction

J. Milnor [3] pointed out that spaces have the homotopy types of separable ANR's iff they have the homotopy types of countable CW-complexes. We study parallel properties of this for G-ANR's (defined below).

Let G be a *finite* group throughout this paper. Let $\mathscr{W}^G$ denote the category of G-spaces having the G-homotopy types of G-CW complexes and G-maps. Let $\mathscr{W}_c^G$ denote the full subcategory of $\mathscr{W}^G$ whose objects have the G-homotopy types of countable G-CW complexes.

The main results of this paper are the following theorems.

Theorem 1. *The following restrictions on the G-space X are equivalent*:

a) *X belongs to $\mathscr{W}^G$*,

b) *X is G-dominated by a G-CW complex*,

c) *X has the G-homotopy type of a G-ANR.*

Theorem 2. (*An equivariant version of Milnor* [3], *Theorem* 1.) *The following restrictions on the G-space X are equivalent*:

a) *X belongs to $\mathscr{W}_c^G$*,

b) *X is G-dominated by a countable G-CW complex*,

c) *X has the G-homotopy type of a separable G-ANR.*

§1. G-ANR's

Definition 1. A metrizable G-space X is called a *G-ANR* (a *G-absolute neighbourhood retract*) iff X has the G-neighbourhood extension property for all metrizable G-spaces, i.e., any G-map $f: A \to X$ of every closed G-subspace A

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of every metrizable G-space Y can be extended equivariantly to an open G-neighbourhood U of A in Y.

Definition 2. A Banach space B is called a *Banach G-space* iff G acts on B linearly and the norm $\| \ \|$ is G-invariant, i.e., $\|gb\| = \|b\|$ for $g \in G$ and $b \in B$.

From now on a metric d of a metrizable G-space X is assumed to be *G-invariant*, i.e., $d(gx, gy) = d(x, y)$ for $g \in G$ and $x, y \in X$, since we can choose a G-invariant metric by averaging any metric over G.

Proposition 1.1. *For any metrizable G-space X there exists a Banach G-space B(X) with a G-embedding*

$$i: X \longrightarrow B(X)$$

such that i(X) is closed in the convex hull C(X) of i(X) in B(X). Then C(X) becomes G-invariant.

Proof. Let $B(X)$ be the set of all bounded continuous real-valued functions on X. Define

$$(f+f')(x) = f(x) + f'(x), \qquad (rf)(x) = r \cdot f(x),$$
$$\|f\| = \sup_{x \in X} |f(x)| \quad \text{and} \quad (gf)(x) = f(g^{-1}x),$$

for $f, f' \in B(X)$, $x \in X$, $r \in R$ and $g \in G$. Then we see easily that $B(X)$ is a Banach G-space. (Cf. [2], pp. 63–64.)

We choose a bounded metric d of X. (We can do it, for we can define a bounded metric d from any metric d' by $d(x, y) = d'(x, y)/(1 + d'(x, y))$ for $x, y \in X$.) For $x \in X$ we define $i(x) \in B(X)$ by

$$i(x)(y) = d(x, y) \qquad \text{for} \quad y \in X.$$

Then i is an embedding by [2], Chapter II, Lemma 16.2, and $i(X)$ is closed in $C(X)$ by [2], Chapter III, Theorem 2.1. Since d is G-invariant, i is a G-map:

$$i(gx)(y) = d(gx, y) = d(x, g^{-1}y) = i(x)(g^{-1}y) = (gi(x))(y).$$

$C(X)$ consists of the points of the form

$$t_0x_0 + \cdots + t_nx_n \qquad \text{for} \quad x_0, \ldots, x_n \in i(X), \ \sum_{j=0}^{n} t_j = 1 \quad \text{and} \quad t_j \geqq 0.$$

For $t_0x_0 + \cdots + t_nx_n \in C(X)$

$$g(t_0x_0 + \cdots + t_nx_n) = t_0gx_0 + \cdots + t_ngx_n$$

is contained in $C(X)$, for $i(X)$ is G-invariant. This shows that $C(X)$ is G-invariant. q. e. d.

Proposition 1.2. *A convex G-set C in a Banach G-space is a G-ANR.*

Proof. Let A be a closed G-subspace of a metrizable G-space Y. Let $f: A \to C$ be a G-map. By [2], Chapter II, Corollary 14.2 there exists an extension f' of f to Y. Define a G-extension F of f by

$$F(y) = \frac{1}{|G|} \sum_{g \in G} gf'(g^{-1}y) \qquad \text{for} \quad y \in Y.$$

Then $F(y) \in C$, since $gf'(g^{-1}y) \in C$ and $\sum_{g \in G} 1/|G| = 1$. q. e. d.

Definition 3. A G-subspace X of a G-space Y is said a *G-neighbourhood retract* of Y iff X is a G-retract of an open G-subspace U of Y (i.e., there is a G-retraction $r: U \to X$).

Proposition 1.3. *Every G-neighbourhood retract X of a G-ANR Y is a G-ANR.*

Proof. Let A be a closed G-subspace of a metrizable G-space Z and f: $A \to X$ a G-map. Let $r: U \to X$ be a G-neighbourhood retraction. We regard f as a G-map to Y. Then there is a G-extension $f': V' \to Y$ of f to a G-neighbourhood V' of A in Z, for Y is a G-ANR. Let $V = f'^{-1}(U)$. Define a G-map $F: V \to X$ by

$$F = r \circ f' |_V : V \xrightarrow{f'|} U \xrightarrow{r} X .$$

Then V is a G-neighbourhood of A in Z and F is a G-extension of f. q. e. d.

Proposition 1.4. *A metrizable G-space X is a G-ANR iff every G-homeomorphic image of X as a closed G-subspace in any metrizable G-space Y is a G-neighbourhood retract.*

Proof. Let X be a G-ANR G-embedded as a closed G-subspace in a metrizable G-space Y. Consider the identity map of X. Then the map is a G-map and has a G-extension to a G-neighbourhood of X. This shows the "only if" part.

Putting $Y = C(X)$, the converse follows from Propositions 1.1, 1.2 and 1.3. q. e. d.

§ 2. Simplicial G-Complexes

A *simplicial G-complex K* is a simplicial complex endowed with a group G of automorphisms of its simplicial structure. Then its geometric realization K_m (resp. K_w) with the metric (resp. weak) topology is a G-space. (Cf., [1], p. 206.)

Proposition 2.1. *Every G-subcomplex L of any simplicial G-complex K with the metric topology is a G-neighbourhood retract.*

Proof. The regular neighbourhood of L is G-invariant and the retraction is a G-map. q.e.d.

A simplicial G-complex K is said to be *full* iff every finite set of its vertices spans a simplex of K. Any simplicial G-complex K can be G-embedded in a full simplicial G-complex $F(K)$ with the same vertices.

Proposition 2.2. *Every simplicial G-complex with the metric topology is a G-ANR.*

Proof. Let $\{v_\lambda \mid \lambda \in \Lambda\}$ be the set of all vertices of a simplicial G-complex K with the metric topology. We define a G-action on Λ by $v_{g\lambda} = gv_\lambda$. Consider the Banach G-space S which consists of all real-valued functions s: $\Lambda \to R$ such that

$$\sum_{\lambda \in \Lambda} |s(\lambda)|$$

is convergent. The norm of $s \in S$ is defined by

$$\|s\| = \sum_{\lambda \in \Lambda} |s(\lambda)| .$$

The G-action on S is defined by $(gs)(\lambda) = s(g^{-1}\lambda)$. Define a G-map h: $F(K) \to S$ as follows: Let $x \in F(K)$. Let $\{x_\lambda \mid \lambda \in \Lambda\}$ denote the barycentric coordinates of x. Then $h(x)$ is given by

$$h(x)(\lambda) = x_\lambda \qquad \text{for} \quad \lambda \in \Lambda .$$

This h is isometric and one can easily see that h is a G-embedding. $h(F(K))$ is a convex G-set in the Banach G-space S, for $F(K)$ is full. The proposition follows from Propositions 1.2, 1.3 and 2.1. q.e.d.

§ 3. G-Domination

As to the definitions of a G-covering and a G-partition of unity we refer to [1], p. 208.

Proposition 3.1. *Let X be a G-ANR. Then X is G-dominated by a G-CW complex K.*

Proof. X is a G-neighbourhood retract of $C(X)$ with a G-retraction r: $U \to X$ by Propositions 1.1 and 1.4. Since $C(X)$ is convex and $B(X)$ is locally convex, $C(X)$ is locally convex and we can find a G-covering $\mathscr{V}' = \{V'_\lambda \mid \lambda \in \Lambda\}$ of X by open convex sets V'_λ in U. Put

$$\mathscr{V} = \{V_\lambda = V'_\lambda \cap X \mid \lambda \in \Lambda\}.$$

Since X is metrizable, X is paracompact and fully normal.

Assertion. For any open G-covering $\mathscr{V}$ of X there exists a locally finite open G-covering $\mathscr{U} = \{U_\delta \mid \delta \in \Delta\}$ with points $\{x_\delta\}$ in X which satisfies

i) $gx_\delta = x_{g\delta}$ for any $\delta \in \Delta$ and

ii) for any point $x \in X$ both the star $S(x, \mathscr{U}) = \cup \{U_\delta \mid x \in U_\delta \in \mathscr{U}\}$ of x with respect to $\mathscr{U}$ and the points x_δ with $x \in U_\delta$ are contained in a certain $V_\lambda \in \mathscr{V}$.

Proof. Since X is fully normal, there is a G-covering $\mathscr{S} = \{S_x = \text{a slice at } x \mid x \in X\}$ which is an open star-refinement of $\mathscr{V}$. (Slices are open, for G is finite.) Choose a locally finite open G-covering $\mathscr{U} = \{U_\delta \mid \delta \in \Delta\}$ which is a refinement of $\mathscr{S}$. For each $\delta \in \Delta$ we choose $x_\delta \in X$ such that $U_\delta \subset S_{x_\delta} \in \mathscr{U}$ and $gx_\delta = x_{g\delta}$. These $\mathscr{U}$ and $\{x_\delta\}$ satisfy i) and ii). (For detail, see [1], Theorem 2.3.) q.e.d.

Proof of Proposition 3.1. We choose a G-partition of unity $\{p_\delta \mid \delta \in \Delta\}$ subordinate to $\mathscr{U}$. Let K denote the geometric nerve with the weak topology. The barycentric subdivision of K is a G-CW complex. Define

$$P: X \longrightarrow K$$

by letting $P(x)$ be the point in K with barycentric coordinates $\{p_\delta(x)\}$ for $x \in X$. Then P is a well-defined G-map.

Define a map q: $K \to B(X)$ by

$$q(y) = \sum_{\delta \in \Delta} y_\delta x_\delta ,$$

where y_δ denotes the δ-th barycentric coordinate of $y \in K$. Then q is a well-defined G-map. Let $y_{\delta_0},\dots, y_{\delta_n}$ be the non-zero barycentric coordinates of y. Then x_{δ_i}'s are contained in some $V_\lambda \subset V'_\lambda$. Since V'_λ is convex, $q(y) = \sum_{i=0}^n y_{\delta_i} x_{\delta_i}$ is contained in $V'_\lambda \subset U$. Thus $q(K) \subset U$ and we regard q as a G-map

$$q: K \longrightarrow U$$

to U. Put

$$s = r \circ q: K \longrightarrow X.$$

Define a G-homotopy h_t: $1_X \underset{G}{\simeq} s \circ P$ by

$$h_t(x) = r((1-t)x + t \cdot q \circ P(x)) \qquad \text{for} \quad x \in X.$$

Note that $(1-t)x + t \cdot q \circ P(x)$ is contained in U; Because, if $S(x, \mathscr{U}) = \cup_{i=0}^n U_{\delta_i}$ and the points x_{δ_i} $(i = 0, 1,\dots, n)$ are contained in $V_\lambda \subset V'_\lambda$, then both x and $q \circ P(x) = \sum_{i=0}^n p_{\delta_i}(x) x_{\delta_i}$ are contained in the convex set $V'_\lambda \subset U$, and so is $(1-t)x + t \cdot q \circ P(x)$. Thus h_t is a well-defined G-map. With G-maps P, s and a G-homotopy h_t, X is G-dominated by the G-CW complex K. q.e.d.

Corollary 3.2. *Every separable G-ANR is G-dominated by a countable G-CW complex.*

Proof. Since a separable metrizable space has the Lindelöf property, we can choose $\mathscr{U}$ to be countable. Then the nerve K is countable. q.e.d.

§4. Proof of Theorems

Theorem 1 follows from Propositions 2.2, 3.1 and [1], Theorem 2.1.

Proof of Theorem 2. The implication c)⇒b) follows from Corollary 3.2, and a)⇒c) follows from Proposition 2.2 and [2], Chapter III, Lemma 11.4.

We show b)⇒a) similar to [4], Theorem 24. Let X be G-dominated by a countable G-CW complex K with G-maps $f: X \to K$ and $f': K \to X$ such that $f' \circ f \underset{G}{\simeq} 1_X$. By the same argument as [3], p. 275, there is a G-map $k: K \to |S(X)|$ such that $k' = k \circ f$ is a G-homotopy inverse to j: $|S(X)| \to X$. (Cf. [1], Propositions 1.5–1.7 and Theorem 2.1.)

Since G is finite and closed G-cells Ge are compact, $k(Ge)$ are contained in finite G-subcomplexes. Thus $k(K)$ is contained in a countable G-subcomplex L_0 of $|S(X)|$, for K is countable. Let h_t: $|S(X)| \to |S(X)|$ be a G-homotopy of $h_0 = k' \circ j$ into $h_1 = 1_{|S(X)|}$. Then there is a countable G-subcomplex L_1 of $|S(X)|$

such that $h_t(L_0)\subset L_1$, for the same reason that $k(K)\subset L_0$. By repeating this argument, we have a sequence of countable G-subcomplexes

$$L_0\subset L_1\subset L_2\subset\cdots\subset L_n\subset\cdots$$

of $|S(X)|$ such that $h_t(L_n)\subset L_{n+1}$. The union $L=\bigcup_n L_n$ is a countable G-subcomplex such that $k'(X)\subset L$ and $h_t(L)\subset L$. Therefore $j'=j|_L$ is a G-homotopy equivalence of L to X, for $h_t|_L$: $k'\circ j'\underset{G}{\simeq}1_L$ and $j'\circ k'=j\circ k'\underset{G}{\simeq}1_X$.

q.e.d.

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