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Decomposition of Invariant States and Nonseparable C^*-Algebras

By

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Introduction

The main purpose of this paper is to give a proof of the following

Theorem. *Let $\mathscr{A}$ be a C^*-algebra of operators on a Hilbert space with cyclic vector ξ, let $\mathscr{A}$ contain the identity operator and $S(\mathscr{A})$ be the state space of $\mathscr{A}$. Then there exist a positive Radon measure μ on $S(\mathscr{A})$ and for each $\varphi \in S(\mathscr{A})$ an irreducible representation π^φ of $\mathscr{A}$ such that*

$$\mathscr{A} \cong \int^{\oplus} \pi^\varphi(\mathscr{A}) d\mu(\varphi)$$

i.e. $\mathscr{A}$ is isomorphic to a direct integral of irreducible C^*-algebras $\pi^\varphi(\mathscr{A})$ with respect to μ.

Actually, one can get μ to be the orthogonal measure corresponding to the vector state ω defined by ξ and a maximal abelian subalgebra of the commutant $\mathscr{A}'$ of $\mathscr{A}$. (Such a measure is maximal with respect to the Choquet ordering also in the nonseparable case, see [7], for a short proof).

It may be surprising that no separability condition on $\mathscr{A}$ is assumed. The example given by J. L. Taylor in [15] shows that every state φ in the support of such a measure may fail to be a pure state, contradicting an assertion in an earlier paper of M. Tomita. Therefore in our theorem π^φ will *not* be the GNS representation π_φ corresponding to φ, in general.

The counterexample given by the author in [7], Theorem 2, tells, moreover, that being interested in a decomposition into factor representations with respect to a Radon measure on a locally compact space one cannot get a topological

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direct integral, in general (in the sense of R. Godement [3], with respect to continuous fields of Hilbert spaces and operators). Therefore we use the more general definition of direct integrals as introduced by W. Wils in [17].

Then the proof of the Theorem depends heavily on Tomita's generalization of known facts on decomposition of states to nonseparable C^*-algebras in his later 1959 paper [14].

Definition. For a state $\varphi \in S(\mathscr{A})$ let $N_\varphi = \{T \in \mathscr{A};\ \varphi(T^*T)=0\}$ be the corresponding left ideal. A state φ is said to be pure relative to $T \in \mathscr{A}$ if $\rho(T) = \varphi(T)$ for all states ρ such that $N_\rho \supset N_\varphi$.

By Kadison's result, a state is pure if and only if it is pure relative to all $T \in \mathscr{A}$.

Theorem (Tomita). *Let $\mathscr{A}$ be a C^*-algebra of operators on a Hilbert space, $I \in \mathscr{A}$, μ be the orthogonal measure corresponding to a cyclic vector ξ and a maximal abelian subalgebra of $\mathscr{A}'$. Then for $T \in \mathscr{A}$ μ-almost all $\varphi \in S(\mathscr{A})$ are pure relative to T.*

The set of measure zero depends on the given operator $T \in \mathscr{A}$, hence uncountable many of them can occur in the nonseparable case. We shall show in Section 2 that this doesn't matter in the proof of our first theorem.

Tomita's theorem seems not to be well known, in the proof fields of Hilbert spaces and operators are used. In the appendix we shall give a shorter and more straightforward proof of this theorem without using direct integral theory. The main ideas, however, are the same as in Tomita's original proof and "shorter proof" also means that we use without proof such methods and results which are now well known and can be found also in recent monographs on C^*-algebras. For instance, the polar decomposition of functionals is introduced in that paper, the noncommutative Lusin's theorem has been proved and also the concept of regular projections has been used to prove the theorem.

In Sections 3 and 4 we deal with the central decomposition of states and decompositions of invariant states into ergodic states. We obtain

Theorem. *Let $\mathscr{A}$ be a C^*-algebra with identity, $\omega \in S(\mathscr{A})$ and μ be the central measure associated with ω. Then there is a map $\varphi \to \psi_\varphi$ from $\mathscr{A}$ into the set of factorial states of $\mathscr{A}$ such that*

$$\omega(T) = \int_{S(\mathscr{A})} \psi_\varphi(T)\,d\mu(\varphi) \qquad \textit{for all} \quad T \in \mathscr{A}$$

Moreover, $\pi_\omega(\mathscr{A})\cong\int^{\oplus}\pi_{\psi_\varphi}(\mathscr{A})d\mu(\varphi)$.

A similar result holds concerning decompositions of invariant states into ergodic ones. We finish by discussing some uniqueness problems which arise also in the case of the central decomposition and in the simplex case.

In Section 1 we show some aspects of the counterexample given in [7] in the framework of C^*-algebras in order to see to what extent we can expect results for nonseparable algebras.

Theorem. *There is a C^*-Algebra $\mathscr{A}$ and a state ω of $\mathscr{A}$ such that*

1) *The weak-$*$-closure of the smallest face containing ω is disjoint from the set $P(\mathscr{A})$ of pure states.*

2) *There does not exist a regular Borel measure m on the topological space $P(\mathscr{A})$ such that*

$$\omega(T)=\int\varphi(T)dm(\varphi)\qquad \text{for all}\quad T\in\mathscr{A}.$$

Throughout this paper we use the following

Notation. For a C^*-algebra $\mathscr{A}$ let be

$S(\mathscr{A})$ the state space of $\mathscr{A}$

$P(\mathscr{A})$ the set of pure states of $\mathscr{A}$

$(\pi_\varphi, H_\varphi, \xi_\varphi)$ the GNS-representation for $\varphi\in S(\mathscr{A})$

$M(\varphi)=\{\psi\in S(\mathscr{A});\ \psi\leqq r\varphi$ for some $r\geqq 0\}$

the face generated by $\varphi\in S(\mathscr{A})$

$N_\varphi=\{T\in\mathscr{A};\ \varphi(T^*T)=0\}$

the left ideal corresponding to φ and

$K_\varphi=\{\psi\in S(\mathscr{A});\ N_\psi\supset N_\varphi\}$

For $T\in\mathscr{A}$ $\hat{T}$ is defined as $\hat{T}(\varphi)=\varphi(T)$, $\varphi\in S(\mathscr{A})$

$[M]$ the closed subspace of a Hilbert space H generated by $M\subset H$

$\mathscr{A}'$ the commutant of a set $\mathscr{A}$ of operators on H.

For a set X

$\mathscr{C}(X)$ the continuous functions on X

$f|Y$ the restriction of a function f onto a subset $Y\subset X$.

supp μ the support of a Radon measure μ

§ 1. The Counterexample

Let $D=\boldsymbol{R}^{\boldsymbol{R}}$ denote the group of all functions $f: \boldsymbol{R} \to \boldsymbol{R}$ and let G be the *discrete* group of all triangular matrices

$$x=\begin{pmatrix} 1 & f & h \\ 0 & 1 & g \\ 0 & 0 & 1 \end{pmatrix}, \quad f, g, h \in D, \quad f-g \in \boldsymbol{Z}1.$$

Let L and R, resp., denote the left and right regular representation of G on $l^2(G)$, resp. $\mathscr{A}=C^*(L, R)$ is the C^*-algebra generated by all operators L_x, R_x, $x \in G$ where for $\xi \in l^2(G)$, $y \in G$

$$L_x\xi(y)=\xi(x^{-1}y), \quad R_x\xi(y)=\xi(yx).$$

There is a cyclic vector $\xi_e \in l^2(G)$ for $\mathscr{A}$,

$$\xi_e(y)=\begin{cases} 1 & y=e \\ 0 & y \neq e \end{cases}$$

where $e \in G$ is the identity element. Let ω be the corresponding vector state on $\mathscr{A}$. Then we have

Theorem 1.1. a) *The weak-*-closure* $\overline{M(\omega)}$ *in* $S(\mathscr{A})$ *of the smallest face* $M(\omega)$ *containing* ω *is disjoint from the set* $P(\mathscr{A})$ *of pure states of* $\mathscr{A}$ *(even from the set* $F(\mathscr{A})$ *of factorial states).*

b) *There is no regular Borel measure* m *on the topological space* $P(\mathscr{A})$ *such that*

$$\omega(T)=\int_{P(\mathscr{A})} \varphi(T)dm(\varphi) \qquad \text{for all} \quad T \in \mathscr{A}.$$

Proof. a) ([7], proof of Theorem 2.) For $h \in D$ the element

$$z(h)=\begin{pmatrix} 1 & 0 & h \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

belongs to the centre of G hence $L_{z(h)}$ is in the centre of $\mathscr{A}$ as $L_xR_y=R_yL_x$ for all $x, y \in G$. If π is a factorial representation of $\mathscr{A}$, there is a character χ of D such that

$$\pi(L_{z(h)})=\chi(h)I, \quad h \in D.$$

Because of the cardinality of D there exists $k \in D$ such that $\chi(k)=1$, $k \neq 0$. Let

$$x=\begin{pmatrix}1 & k & 0\\ 0 & 1 & k\\ 0 & 0 & 1\end{pmatrix}.$$

If

$$y=\begin{pmatrix}1 & f & h\\ 0 & 1 & g\\ 0 & 0 & 1\end{pmatrix}\in G,\ f-g=nl\,,$$

then

(1.1) $$yxy^{-1}=\begin{pmatrix}1 & 0 & nk\\ 0 & 1 & 0\\ 0 & 0 & 1\end{pmatrix}x$$

hence $\pi(L_yL_x)=\chi(k)^n\pi(L_xL_y)=\pi(L_xL_y)$. Thus $\pi(L_x)$ is a unitary operator in the centre of $\pi(\mathscr{A})$, say $\pi(L_x)=\lambda I$, $|\lambda|=1$. Therefore we have shown that for every factorial state φ of $\mathscr{A}$ there is an element $x=\begin{pmatrix}1 & k & 0\\ 0 & 1 & k\\ 0 & 0 & 1\end{pmatrix}$, $k\neq 0$ such that $|\varphi(L_x)|=1$. We shall see that $\rho(L_x)=0$ for all states ρ in the closure of the smallest face $M(\omega)$ generated by ω.

Therefore let $\rho\in M(\omega)$, $\rho\leqq r\omega$, $0<r$, and for $y\in G$ let U_y denote the unitary operator L_yR_y on $l^2(G)$. Then $\rho((U_y-I)^*(U_y-I))\leqq r\omega((U_y-I)^*(U_y-I))=r\|(U_y-I)\xi_e\|^2=0$ for $y\in G$. Since $U_y^*=U_{y^{-1}}$, $y\in G$, we get by the Cauchy-Schwarz inequality for all $T\in\mathscr{A}$

$$\rho(U_yT)=\rho(T)=\rho(TU_y)$$

in particular, for x, $y\in G$

(1.2) $$\rho(L_x)=\rho(U_yL_xU_{y^{-1}})=\rho(L_{yxy^{-1}})$$

(thus $x\to\rho(L_x)$ is a positive definite class function on G). Finally, if x is as in (1.1) and if for $n\in\mathbf{N}$

$$y_n=\begin{pmatrix}1 & nl & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{pmatrix}\in G,$$

then by (1.1) the elements $y_nxy_n^{-1}=\begin{pmatrix}1 & 0 & nk\\ 0 & 1 & 0\\ 0 & 0 & 1\end{pmatrix}x$ are mutually distinct (the conjugacy class of x is infinite). It is known that then $\rho(L_x)=0$. For completeness we give a short proof by using the Cauchy-Schwarz inequality again. Since $\{L_{y_n x y_n^{-1}}\xi_e;\ n \in \boldsymbol{N}\}$ is an orthonormal set in $l^2(G)$ for $m \in \boldsymbol{N}$

$$\|B_m\xi_e\|^2 = \frac{1}{m} \qquad \text{where} \quad B_m = \frac{1}{m}\sum_{n=1}^{m} L_{y_n x y_n^{-1}}\,.$$

By (1.2), $\rho(L_x) = \rho(B_m)$, hence

$$|\rho(L_x)|^2 \leqq \rho(I)\rho(B_m^* B_m) \leqq r\omega(B_m^* B_m) = r\|B_m\xi_e\|^2 = \frac{r}{m}$$

for all $m \in \boldsymbol{N}$, thus $\rho(L_x) = 0$. Consequently, no state $\rho \in \overline{M(\omega)}$ is pure or factorial.

b) Assume that m is a Borel measure on the space $X = P(\mathscr{A})$ (or $X = F(\mathscr{A})$) such that

1) m is inner regular, i.e. for every Borel subset $Y \subset X$ $m(Y) = \sup\{m(K);\ K \subset Y,\ K \text{ compact}\}$,

2) $\omega(T) = \int_X \varphi(T)dm(\varphi) \qquad$ for all $T \in \mathscr{A}$.

Then m is a finite Radon measure on X in the sense of Definition R_3 in [10], p. 13. Let μ be the positive Radon measure on the state space $S(\mathscr{A})$ defined by

$$\mu(f) = \int_X f(\varphi)dm(\varphi),\ f \in \mathscr{C}(S(\mathscr{A}))\,.$$

Then by 2), $\mu(\hat{T}) = \omega(T)$, i.e. μ has resultant ω. It is well known that the support of μ is contained in $\overline{M(\omega)}$ which by a) does not contain any pure state (factorial state). Hence X is contained in the open set $\mathscr{Q} = S(\mathscr{A}) \backslash \overline{M(\omega)}$. Let $1_{\mathscr{Q}}$ denote the characteristic function of $\mathscr{Q}$ and let $\{f_i\}_{i\in J} \subset \mathscr{C}(S(\mathscr{A}))$ be an increasing directed family of non-negative functions such that $1_{\mathscr{Q}} = \sup_{i\in J} f_i$. Then for all $i \in J$

$$0 = \mu(1_{\mathscr{Q}}) = \mu(f_i) = \int_X f_i(\varphi)dm(\varphi)\,.$$

Since $\sup_{i\in J} f_i(\varphi) = 1$ for $\varphi \in X$ we get by [10], Proposition 5, p. 42,

$$m(X) = \sup_{i\in J} \int_X f_i(\varphi)dm(\varphi) = 0,$$

a contradiction.

§2. Orthogonal Measures and Direct Integrals

Let $\mathscr{A}$ be a C^*-algebra of operators on a Hilbert space H such that $I \in \mathscr{A}$ and $\xi \in H$ is a cyclic vector for $\mathscr{A}$. Let $\mathscr{C}$ be an abelian von Neumann subalgebra of $\mathscr{A}'$ and let $P\in\mathscr{C}'$ be the projection of H onto $[\mathscr{C}\xi]$. Since ξ is a separating vector for $\mathscr{A}'$ $\mathscr{C}$ is isomorphic to $\mathscr{C}_P$. Hence $\mathscr{C}_P$ is an abelian von Neumann algebra with cyclic vector ξ therefore

$$\mathscr{C}_P=(\mathscr{C}_P)'=\mathscr{C}'_P\,.$$

Thus for every $T\in\mathscr{C}'$ there is a unique operator $\Phi(T)$ in $\mathscr{C}$ such that

$$PTP=\Phi(T)P. \tag{2.1}$$

Then $T\to\Phi(T)$ is a positive linear map from $\mathscr{C}'$ onto $\mathscr{C}$ satisfying

$$\begin{aligned}&\Phi(P)=\Phi(I)=I\\&\Phi(ST)=S\Phi(T),\ S\in\mathscr{C},\ T\in\mathscr{C}'\end{aligned} \tag{2.2}$$

Let Ω be the spectrum of $\mathscr{C}$ and for $\gamma\in\Omega$ define a state φ_γ of $\mathscr{A}$ by $\varphi_\gamma(T)=\gamma(\Phi(T))$. Then the map $\gamma\to\varphi_\gamma$ is weak-$*$ continuous and its transpose θ is a $*$-homomorphism from $\mathscr{C}(S(\mathscr{A}))$ into $\mathscr{C}(\Omega)=\mathscr{C}$ such that

$$\theta(h\hat{T})=\theta(h)\Phi(T),\ h\in\mathscr{C}(S(\mathscr{A})),\ T\in\mathscr{A}.$$

In particular, $\theta(\hat{T})=\Phi(T)$ for $T\in\mathscr{A}$.

Definition 2.1. The measure μ on $S(\mathscr{A})$ defined as

$$\mu(h)=(\theta(h)\xi,\ \xi),\ h\in\mathscr{C}(S(\mathscr{A}))$$

is called the orthogonal measure corresponding to $\mathscr{C}$ and ω, where ω is the vector state on $\mathscr{A}$ defined by ξ ([12]). Then for $h\in\mathscr{C}(S(\mathscr{A}))$, $T\in\mathscr{A}$

$$\begin{aligned}\mu(h\hat{T})&=(\theta(h)\Phi(T)\xi,\ \xi)=(\theta(h)P\Phi(T)P\xi,\ \xi)\\&=(\theta(h)T\xi,\ \xi)\,.\end{aligned} \tag{2.3}$$

In particular, μ has resultant $r(\mu)=\omega$.

Remarks 2.2. 1) By definition, $h\circ\varphi_\gamma=\gamma(\theta(h))$, $\gamma\in\Omega$, $h\in\mathscr{C}(S(\mathscr{A}))$, hence μ is the image of the spectral measure defined by ξ on Ω under the map $\gamma\to\varphi_\gamma$.

2) The map θ can be extended to a $*$-isomorphism θ_μ of $L^\infty(\mu)$ onto $\mathscr{C}$ such that

$$(\theta_\mu(h)T\xi,\ \xi)=\int h(\varphi)\varphi(T)d\mu(\varphi),\ h\in L^\infty(\mu) \tag{2.4}$$

(see [13], Proposition 6.23).

3) That μ is a maximal measure with respect to the Choquet ordering in case $\mathscr{C}$ is a maximal abelian subalgebra of $\mathscr{A}'$ is proved in [7], Theorem 1.

4) For $T\in\mathscr{A}$ let $\xi_T\in\prod_{\varphi\in S(\mathscr{A})}H_\varphi$ be the vector field defined by $\xi_T(\varphi)$

$=\pi_\varphi(T)\xi_\varphi$. Then $\Lambda=\{\xi_T;\ T\in\mathscr{A}\}$ is a fundamental family of continuous vector fields in the sense of [3]. By (2.3), we have for $T_i\in\mathscr{A}$, $h_i\in\mathscr{C}(S(\mathscr{A}))$, $1\leqq i\leqq n$,

$$\|\sum_{i=1}^n h_i\cdot\xi_{T_i}\|^2=\int\sum_{i,j}h_i(\varphi)\overline{h_j(\varphi)}\varphi(T_j^*T_i)d\mu(\varphi)$$
$$=\|\sum_{i=1}^n \theta(h_i)T_i\xi\|^2.$$

Hence H is isomorphic to the topological integral $\int^\oplus H_\varphi d\mu(\varphi)$.

Moreover, $\mathscr{A}$ is isomorphic to the topological direct integral of C^*-algebras $\pi_\varphi(\mathscr{A})$ in the sense of Godement and Tomita. If $\mathscr{A}$ is separable and $\mathscr{C}\subset\mathscr{A}'$ is maximal abelian, it is well known that μ-almost all $\varphi\in S(\mathscr{A})$ are pure states and the so-called Godement-Mautner-Segal decomposition of $\mathscr{A}$ is obtained ([3], [8], [11]).

The counterexamples given by J. L. Taylor in [15] and by the author (see § 1) show that in the nonseparable case each state φ_γ may fail to be pure, even if $\mathscr{C}$ is maximal abelian. Moreover, it can be shown as in [7], proof of Theorem 2, that the C^*-algebra $\mathscr{A}$ in Section 1 cannot be decomposed as a topological direct integral of irreducible C^*-algebras.

In [14], M. Tomita has extended the Godement-Mautner-Segal Theorem to nonseparable C^*-algebras in the following way: For a state φ of $\mathscr{A}$ denote by $N_\varphi=\{T\in\mathscr{A};\ \varphi(T^*T)=0\}$ the corresponding left ideal and let $K_\varphi=\{\psi\in S(\mathscr{A});\ N_\psi\supset N_\varphi\}$.

By Kadison's theorem, $K_\varphi=\{\varphi\}$ if and only if φ is pure.

Theorem 2.3 (Tomita [14], Theorem 7). *Let μ be the orthogonal measure on $S(\mathscr{A})$ corresponding to a maximal abelian subalgebra of $\mathscr{A}'$. Then for $T\in\mathscr{A}$, μ-almost everywhere*

$$\psi(T)=\varphi(T)\qquad \textit{for all}\quad \psi\in K_\varphi.$$

A proof of Tomita's theorem is given in the appendix.

In what follows we shall use the more general concept of direct integrals of Hilbert spaces as introduced by W. Wils in [17]. In our applications, however, the basic measure space X will be the state space of $\mathscr{A}$ and the measure will be a positive Radon measure, actually, the orthogonal measure corresponding to abelian von Neumann subalgebras $\mathscr{C}\subset\mathscr{A}'$. Let us recall the definition.

Definition 2.4 ([17]) Let $\{H^\varphi;\ \varphi\in X\}$ be a field of Hilbert spaces on X and let $\Gamma\subset\prod_{\varphi\in X}H^\varphi$ be a subspace of vector fields such that

1) $\varphi \to \|\eta(\varphi)\|^2$ is μ-integrable for every $\eta \in \Gamma$

2) If $\eta \in \Gamma$ and f is a bounded measurable function on X ($f \in M^\infty(X, \mu)$), the vector field $f \cdot \eta$, $\varphi \to f(\varphi)\eta(\varphi)$, belongs to Γ

3) If for a vector field η there is $\eta' \in \Gamma$ such that

$$\eta(\varphi) = \eta'(\varphi) \qquad \text{for almost all} \quad \varphi\,,$$

then $\eta \in \Gamma$.

4) The seminormed space $(\Gamma, \|\ \|)$ where $\|\eta\| = (\int \|\eta(\varphi)\|^2 d\mu(\varphi))^{1/2}$ is complete.

The corresponding Hilbert space is called the *direct integral* of the spaces H^φ, denoted by $\int^\Gamma H^\varphi d\mu(\varphi)$ or $\int^\oplus H^\varphi d\mu(\varphi)$.

Remarks 2.5. 1) In our applications, for every $\varphi \in X$ the set $\Gamma(\varphi) = \{\eta(\varphi);\ \eta \in \Gamma\}$ will be dense in H^φ.

2) If $\Gamma_0 \subset \prod_{\varphi \in X} H^\varphi$ satisfies only 1) of Definition 2.4, there is a unique smallest subspace Γ such that $\Gamma_0 \subset \Gamma$ and Γ satisfies 1)–4). In fact, let Γ_1 be the vector space generated by all vector fields $f \cdot \eta$, $f \in M^\infty(X, \mu)$, $\eta \in \Gamma_0$, and let Γ be the space of vector fields which are limits of Cauchy-sequences in Γ_1 with respect to $\|\ \|$ ([17], Corollary 2.3).

We can prove now

Theorem 2.6. *Let μ be the orthogonal measure on $S(\mathscr{A})$ corresponding to a maximal abelian subalgebra of $\mathscr{A}'$. For $\varphi \in S(\mathscr{A})$ let ρ_φ be a pure state of $\mathscr{A}$ such that $N_{\rho_\varphi} \supset N_\varphi$, let $H^\varphi = H_{\rho_\varphi}$ and $\pi^\varphi = \pi_{\rho_\varphi}$. Then $\mathscr{A}$ is a direct integral of irreducible algebras $\pi^\varphi(\mathscr{A})$ with respect to μ.*

Proof. For every $\varphi \in S(\mathscr{A})$ the set K_φ is a closed face in $S(\mathscr{A})$, therefore every extremal point of K_φ is a pure state. If pure states ρ_φ in K_φ, $\varphi \in S(\mathscr{A})$, have been chosen, for every $T \in \mathscr{A}$

$$\rho_\varphi(T) = \varphi(T) \qquad \text{for almost all} \quad \varphi$$

by Tomita's theorem. For $T \in \mathscr{A}$ let $\eta_T \in \prod H^\varphi$ be the vector field defined as $\eta_T(\varphi) = \pi_{\rho_\varphi}(T)\xi_{\rho_\varphi}$, $\varphi \in S(\mathscr{A})$, and let $\Gamma_0 = \{\eta_T;\ T \in \mathscr{A}\}$. Then $\varphi \to \|\eta_T(\varphi)\|^2 = \rho_\varphi(T^*T)$ is equal to the continuous function $\varphi \to \varphi(T^*T)$ almost everywhere, hence it is μ-integrable. Moreover, for

$$\eta = \sum_{i=1}^n f_i \cdot \eta_{T_i} \in \Gamma_1, \quad f_i \in M^\infty(S(\mathscr{A}), \mu), \quad \eta_{T_i} \in \Gamma_0$$

it follows from formula (2.4) (Remark 2.2) that

$$\begin{aligned}\|\eta\|^2 &= \int_{S(\mathscr{A})} \sum_{i,j=1}^{n} f_i(\varphi)\overline{f_j(\varphi)}\rho_\varphi(T_j^* T_i)d\mu(\varphi) \\ &= \int_{S(\mathscr{A})} \sum_{i,j=1}^{n} f_i(\varphi)\overline{f_j(\varphi)}\varphi(T_j^* T_i)d\mu(\varphi) \\ &= \|\sum_{i=1}^{n} \theta_\mu(f_i)T_i\xi\|^2.\end{aligned}$$

Since Γ_1 is dense in Γ by definition (Remark 2.5, 2)), the map $T\xi\to\eta_T$ from $\mathscr{A}\xi$ into Γ_1 can be extended to an isometric linear map U from H onto the Hilbert space $\int^\Gamma H^\varphi d\mu(\varphi)$.

Finally, for arbitrary operators $R\in\mathscr{A}$ we see

$$\pi^\varphi(R)\eta_T(\varphi)=\pi_{\rho_\varphi}(RT)\xi_{\rho_\varphi}=\eta_{RT}(\varphi),\ T\in\mathscr{A}.$$

Hence $\varphi\to\pi^\varphi(R)$ is a measurable operator field in the sense of [16], Definition 1.1, such that $\|\pi^\varphi(R)\|\leqq\|R\|$. Therefore it defines a bounded operator $\int^\Gamma \pi^\varphi(R)d\mu(\varphi)$ on $\int^\Gamma H^\varphi d\mu(\varphi)$ such that

$$U(R(T\xi))=\eta_{RT}=\int^\Gamma \pi^\varphi(R)d\mu(\varphi)\ (U(T\xi)),$$

hence $URU^{-1}=\int^\Gamma \pi^\varphi(R)d\mu(\varphi)$ for every $R\in\mathscr{A}$ as ξ is a cyclic vector.

Remarks 2.7. 1) In the construction of Theorem 2.6 we even have $\Gamma_0(\varphi)$ to be all of H^φ as ρ_φ is a pure state for all $\varphi\in S(\mathscr{A})$.

2) In [14], Theorem 7′, such a map $\varphi\to\rho_\varphi(\in K_\varphi)$ from $S(\mathscr{A})$ into the pure states $P(\mathscr{A})$ has been used to define a measure ν on $P(\mathscr{A})$. From Theorem 1.1 in Section 1 we know that such a measure cannot be a regular Borel measure on $P(\mathscr{A})$, in general.

3) Using Theorem 2.6, one can show that every unitary representation of an arbitrary locally compact group can be decomposed into irreducible representations as a direct integral with respect to a Radon measure on a locally compact space.

For applications it would be useful to answer the following question.

Problem 2.8. Let π be a cyclic representation of a C^*-algebra $\mathscr{A}$ and let

$$\pi(\mathscr{A})=\int_{S(\mathscr{A})}^{\oplus} \pi^\varphi(\mathscr{A})d\mu(\varphi)$$

be an integral decomposition as in Theorem 2.6 such that all π^φ are equivalent to a single irreducible representation $\pi_0 \in \mathscr{A}$. Is π of type I?

§ 3. Central Decompositions

Let $\mathscr{A}$ be a C^*-algebra with identity and for $\varphi \in S(\mathscr{A})$ let $\mathfrak{Z}(\varphi)$ be the centre of $\pi_\varphi(\mathscr{A})''$. If $\mu_{\mathfrak{Z}(\varphi)}$ denotes the orthogonal measure on $S(\pi_\varphi(\mathscr{A}))$ corresponding to $\mathscr{C} = \mathfrak{Z}(\varphi)$ and ξ_φ, the image $\mu = \mu_\varphi$ of $\mu_{\mathfrak{Z}(\varphi)}$ under the map $\psi \to \psi \circ \pi_\varphi$, $S(\pi_\varphi(\mathscr{A})) \to S(\mathscr{A})$, is called the central measure of φ. Let $Z(\varphi)$ be the set of all states ψ of $\mathscr{A}$ such that for some $0 \leqq T \in \mathfrak{Z}(\varphi)$

$$\psi(A) = (\pi_\varphi(A)\xi_\varphi, T\xi_\varphi), \ A \in \mathscr{A}$$

Definition 3.1. Let $\mathscr{A}$ be a C^*-algebra of operators on a Hilbert space H with $I \in \mathscr{A}$, $\mathscr{B} = C^*(\mathscr{A}, \mathscr{A}')$ be the C^*-algebra generated by $\mathscr{A}$ and $\mathscr{A}'$. For a state φ of $\mathscr{A}$ define

$$S(\varphi) = \bigcap_{f \in S(\mathscr{B}),\, f|\mathscr{A} = \varphi} K_f|\mathscr{A}.$$

Remark 3.2. 1) $S(\varphi)$ is a compact, convex subset of $S(\mathscr{A})$ containing φ. Moreover, $\psi \in S(\varphi)$ if and only if for every extension $f \in S(\mathscr{B})$ of φ there is an extension $g \in S(\mathscr{B})$ of ψ such that $N_g \supset N_f$.

2) $S(\psi) \subset S(\varphi)$ for all $\psi \in S(\varphi)$; for, let $\psi \in S(\varphi)$, $\rho \in S(\psi)$ and f be an extension of φ. Then there exist an extension g of ψ with $N_g \supset N_f$ and an extension h of ρ with $N_h \supset N_g$ hence $N_h \supset N_f$.

Lemma 3.3. *Let $\mathscr{A}$ be a C^*-algebra of operators on a Hilbert space H, $I \in \mathscr{A}$. Then for $\varphi \in S(\mathscr{A})$*

$$Z(\varphi) \subset S(\varphi)$$

holds.

Proof. Let $\psi \in Z(\varphi)$ and $f \in S(\mathscr{B})$ be an extension of φ. Since $\varphi(A) = (\pi_f(A)\xi_f, \xi_f)$, $A \in \mathscr{A}$, we may assume that $\xi_\varphi = \xi_f$ and π_φ is the restriction of $\pi_f|\mathscr{A}$ onto the invariant subspace $H_\varphi = [\pi_f(\mathscr{A})\xi_f]$. Let $F \in \pi_f(\mathscr{A})'$ be the projection of H_f onto H_φ. Then

$$\pi_\varphi(\mathscr{A})'' = \pi_f(\mathscr{A})''_F \quad \text{and} \quad \mathfrak{Z}(\varphi) = (\pi_f(\mathscr{A})'' \cap \pi_f(\mathscr{A})')_F .$$

Since $\psi \in Z(\varphi)$, there is an operator $0 \leqq T \in \pi_f(\mathscr{A})'' \cap \pi_f(\mathscr{A})'$ such that $\psi(A) = (\pi_f(A)\xi_f, T_F\xi_f) = (\pi_f(A)\xi_f, T\xi_f)$, $A \in \mathscr{A}$. Define $g(B) = (\pi_f(B)\xi_f, T\xi_f)$, $B \in \mathscr{B}$. Since

$$T\in\pi_f(\mathscr{A})''\cap\pi_f(\mathscr{A})'\subset\pi_f(\mathscr{A}')'\cap\pi_f(\mathscr{A})'$$
$$=\pi_f(C^*(\mathscr{A},\mathscr{A}'))'=\pi_f(\mathscr{B})'$$

$g\in M(f)$, in particular, $N_g\supset N_f$ and $g\mid\mathscr{A}=\psi$, thus $\psi\in S(\varphi)$.

As a corollary we get

Theorem 3.4. *Let $\mathscr{A}$ be a C^*-algebra of operators on a Hilbert space with $I\in\mathscr{A}$. Then for $\varphi\in S(\mathscr{A})$*

1) $S(\varphi)$ is a compact convex set such that the extremal points of $S(\varphi)$ are factorial states of $\mathscr{A}$.

2) The support of the central measure μ_φ is contained in $S(\varphi)$.

Proof. 1) Let ψ be an extremal point of $S(\varphi)$. By Lemma 3.3 and Remark 3.2,

$$Z(\psi)\subset S(\psi)\subset S(\varphi),$$

hence $Z(\psi)=\{\psi\}$ as ψ is an extremal point of $S(\varphi)$, thus ψ is factorial.

2) Let $\mathscr{B}(\varphi)=C^*(\pi_\varphi(\mathscr{A}),\pi_\varphi(\mathscr{A})')$ and $f\in S(\mathscr{B}(\varphi))$ be the state defined by ξ_φ. Let ν be the orthogonal measure on $S(\mathscr{B}(\varphi))$ corresponding to ξ_φ and $\mathscr{B}(\varphi)'=\pi_\varphi(\mathscr{A})'\cap\pi_\varphi(\mathscr{A})''=\mathfrak{Z}(\varphi)$. Then $\operatorname{supp}\nu\subset\overline{M(f)}$ and $\operatorname{supp}\mu_\varphi\subset\overline{\{g\circ\pi_\varphi;\, g\in M(f)\}}$ because μ_φ is the image of ν under the map $g\to g\circ\pi_\varphi$. If $g\in M(f)$, $g(B)=(B\xi_\varphi, T\xi_\varphi)$, $B\in\mathscr{B}(\varphi)$, for some $0\leqq T\in\mathscr{B}(\varphi)'=\mathfrak{Z}(\varphi)$ hence

$$(g\circ\pi_\varphi)(A)=(\pi_\varphi(A)\xi_\varphi, T\xi_\varphi),\ A\in\mathscr{A}$$

thus $g\circ\pi_\varphi\in Z(\varphi)\subset S(\varphi)$ by Lemma 3.3.

Theorem 3.5. *Let $\mathscr{A}$ be a C^*-algebra of operators on a Hilbert space H with cyclic vector ξ, $\|\xi\|=1$, and identity $I\in\mathscr{A}$. Let μ be the central measure of the vector state $\omega\in S(\mathscr{A})$ defined by ξ. If $T\in\mathscr{A}$, then for μ-almost all $\varphi\in S(\mathscr{A})$*

$$\varphi(T)=\psi(T)\qquad \textit{for all}\quad \psi\in S(\varphi).$$

Proof. Let $\mathscr{B}=C^*(\mathscr{A},\mathscr{A}')$, then $\mathscr{B}'=\mathscr{A}'\cap\mathscr{A}''$ is the centre of $\mathscr{A}'$. Let λ be the orthogonal measure on $S(\mathscr{B})$ corresponding to $\mathscr{B}'$ and ξ. Then μ is the image of λ under the restriction map $f\to f\mid\mathscr{A}$, $S(\mathscr{B})\to S(\mathscr{A})$. If $T\in\mathscr{A}$, $\varepsilon>0$, there is a compact $K\subset\operatorname{supp}\lambda$ such that $\lambda(K)\geqq 1-\varepsilon$ and f is pure relative to T for all $f\in K$, by Tomita's theorem. If $\varphi=f\mid\mathscr{A}$ for some $f\in K$ and $\psi\in S(\varphi)$, there is an extension $g\in S(\mathscr{B})$ of ψ such that $N_g\supset N_f$, hence

$$g(T)=f(T)$$

thus $\psi(T)=\varphi(T)$ as $T\in\mathscr{A}$. Since

$$\mu\{f|\mathscr{A}; f\in K\}\geqq 1-\varepsilon$$

the assertion follows.

Corollary 3.6. *Let $\mathscr{A}$ be as in Theorem 3.5. If in addition $\mathscr{A}$ is separable, the central measure is supported by factorial states.*

Proof. If $\mathscr{A}$ is separable, for almost all $\varphi\in S(\mathscr{A})$

$$S(\varphi)=\{\varphi\}$$

hence by Theorem 3.4, φ is factorial almost everywhere.

Remark 3.7. 1) In this proof of the well known fact stated in Corollary 3.6 $\mathscr{B}=C^*(\mathscr{A}, \mathscr{A}')$ may be nonseparable.

2) If in Theorem 3.5 $\mathscr{A}$ is a von Neumann algebra, $\mathscr{C}=\mathscr{B}'$ is contained in $\mathscr{A}$. Hence $\gamma\to\varphi_\gamma$ is a homeomorphism from the spectrum Ω of $\mathscr{C}$ onto supp μ. Moreover, if $\varphi\in\operatorname{supp}\mu$, $\varphi=\varphi_\gamma$, there is a unique extension $\tilde{\varphi}\in\operatorname{supp}\lambda$ of φ, namely

$$\tilde{\varphi}(B)=\gamma(\Phi(B)),\ B\in\mathscr{B}\,.$$

Hence the assertion in Theorem 3.5 can be sharpened in the following way: If $T\in\mathscr{A}$, then for μ-almost all $\varphi\in\operatorname{supp}\mu$

$$\psi(T)=\varphi(T)\qquad\text{for all}\quad \psi\in S(\varphi)'=K_{\tilde{\varphi}}\,|\,\mathscr{A}.$$

Every extremal point of $S(\varphi)'\supset S(\varphi)$ is factorial because it is the restriction of an extremal point of $K_{\tilde{\varphi}}$ which is a pure state of $\mathscr{B}=C^*(\mathscr{A}, \mathscr{A}')$ (see proof of Lemma 3.3).

Theorem 3.8. *Let $\mathscr{A}$ be a C^*-algebra with identity, $\omega\in S(\mathscr{A})$ and μ be the central measure of ω. Then there is a map $\varphi\to\psi_\varphi$ from $S(\mathscr{A})$ into $F(\mathscr{A})$ (the set of factorial states of $\mathscr{A}$) such that for every $T\in\mathscr{A}$ $\varphi\to\psi_\varphi(T)$ is μ-measurable and*

$$\omega(T)=\int_{S(\mathscr{A})}\psi_\varphi(T)d\mu(\varphi)\,.$$

Proof. We may assume that $\mathscr{A}$ is a C^*-algebra of operators on a Hilbert space and ω is the vector state defined by a cyclic vector ξ. If for every $\varphi\in S(\mathscr{A})$ an extremal point ψ_φ of $S(\varphi)$ has been chosen, the assertion follows from Theorems 3.4 and 3.5.

Remark 3.9. Such a map $\varphi \to \psi_\varphi$ may fail to be Lusinmeasurable (see [7], Theorem 2).

As in Section 2, proof of Theorem 2.6, we can show

Theorem 3.10. *Let ω be a state of a C^*-algebra $\mathscr{A}$ with identity and μ the central measure. Then the cyclic representation π_ω of $\mathscr{A}$ can be written as a direct integral of factor representations π_{ψ_φ} with respect to μ. In particular, H_ω is isometric isomorphic to $\int^\oplus H_{\psi_\varphi} d\mu(\varphi)$.*

§4. Ergodic Decompositions of Invariant States

Let $I \in \mathscr{A}$ be a C^*-algebra of operators on a Hilbert space H with cyclic vector ξ, $\|\xi\| = 1$, and $\mathfrak{U}$ be a group of unitary operators on H such that

$$U \mathscr{A} U^{-1} \subset \mathscr{A} \quad \text{and} \quad U\xi = \xi \qquad \text{for all} \quad U \in \mathfrak{U}.$$

Let

$$S(\mathscr{A}, \mathfrak{U}) = \{\varphi \in S(\mathscr{A});\ \varphi(UAU^{-1}) = \varphi(A),\ A \in \mathscr{A},\ U \in \mathfrak{U}\}$$

denote the set of G-invariant states, the extremal points of $S(\mathscr{A}, \mathfrak{U})$ are called G-ergodic states where G is the group of automorphisms of $\mathscr{A}$ induced by $\mathfrak{U}$.

Let $\mathscr{B} = C^*(\mathscr{A}, \mathfrak{U})$ be the C^*-algebra generated by $\mathscr{A}$ and $\mathfrak{U}$ and let

$$S_{\mathfrak{U}}(\mathscr{B}) = \{\rho \in S(\mathscr{B});\ \rho(U) = 1,\ U \in \mathfrak{U}\}.$$

$S_{\mathfrak{U}}(\mathscr{B})$ is a closed, convex subset of $S(\mathscr{B})$ such that

$$K_\rho \subset S_{\mathfrak{U}}(\mathscr{B}) \quad \text{if} \quad \rho \in S_{\mathfrak{U}}(\mathscr{B}).$$

For, a state ρ is in $S_{\mathfrak{U}}(\mathscr{B})$ if and only if $\rho((U-I)^*(U-I)) = 0$ for all $U \in \mathfrak{U}$. Hence

Remark 4.1. $S_{\mathfrak{U}}(\mathscr{B})$ is a closed face in $S(\mathscr{B})$ and $\rho \to \rho \mid \mathscr{A}$ is a homeomorphism of $S_{\mathfrak{U}}(\mathscr{B})$ onto $S(\mathscr{A}, \mathfrak{U})$, it is also an order isomorphism. Hence $\rho \in S_{\mathfrak{U}}(\mathscr{B})$ is a pure state iff $\rho \mid \mathscr{A}$ is G-ergodic.

Definition 4.2. For $\varphi \in S(\mathscr{A}, \mathfrak{U})$ let $S_{\mathfrak{U}}(\varphi) = \{\rho \mid \mathscr{A};\ N_\rho \supset N_{\check{\varphi}}\}$ where $\check{\varphi}$ is the unique extension of φ in $S_{\mathfrak{U}}(\mathscr{B})$.

Remark 4.3. a) $S_{\mathfrak{U}}(\varphi)$ is a compact convex subset of $\{\psi \in S(\mathscr{A}, \mathfrak{U}):\ N_\psi \supset N_\varphi\}$.

b) Clearly, $N_\psi \supset N_\varphi$ if $N_{\check{\psi}} \supset N_{\check{\varphi}}$ for $\varphi, \psi \in S(\mathscr{A}, \mathfrak{U})$, but the converse does not hold, in general. For example, let $\mathfrak{U}$ be the group of all unitary elements of $\mathscr{A}$ and φ, ψ two ergodic states (factorial traces) such that $N_\varphi = N_\psi$ (see [4], §3, proof of Proposition 2). Then $\check{\varphi}$ and $\check{\psi}$ are different pure states of $\mathscr{B}$ hence neither $N_{\check{\varphi}} \supset N_{\check{\psi}}$ nor $N_{\check{\psi}} \supset N_{\check{\varphi}}$.

Therefore the inclusion in a) may be proper. Nevertheless, we have always

c) The extremal points of $S_\mathfrak{U}(\varphi)$ are G-ergodic. For, if ψ is an extremal point of $S_\mathfrak{U}(\varphi) = K_{\check{\varphi}} | \mathscr{A}$, there is an extremal point ρ of $K_{\check{\varphi}}$ such that $\rho | \mathscr{A} = \psi$. Hence $\rho = \check{\psi}$ is a pure state of $\mathscr{B}$ thus ψ is ergodic.

Theorem 4.4. *Let $\omega \in S(\mathscr{A}, \mathfrak{U})$ be the vector state defined by ξ. Then there is a maximal measure μ on $S(\mathscr{A}, \mathfrak{U})$ with resultant ω such that for $T \in \mathscr{A}$, for almost all $\varphi \in S(\mathscr{A}, \mathfrak{U})$*

$$\psi(T) = \varphi(T), \quad \psi \in S_\mathfrak{U}(\varphi).$$

Proof. Let ν be a maximal orthogonal measure on $S(\mathscr{B})$ corresponding to $\check{\omega}$, then ν is maximal ([7], Theorem 1) and $\operatorname{supp} \nu \subset S_\mathfrak{U}(\mathscr{B})$. Hence ν can be considered as a maximal measure on $S_\mathfrak{U}(\mathscr{B})$. The image μ of ν under the restriction map $\rho \to \rho | \mathscr{A}$ is a maximal measure on $S(\mathscr{A}, \mathfrak{U})$ (Actually, the orthogonal measure corresponding to a maximal abelian subalgebra $\mathscr{C}$ of $\mathscr{B}' = \mathscr{A}' \cap \mathfrak{U}'$). For $T \in \mathscr{A}$, and ν-almost all $f \in S_\mathfrak{U}(\mathscr{B})$ we have by Tomita's theorem,

$$g(T) = f(T) \qquad \text{for} \quad g \in K_f$$

hence for μ-almost all $\varphi \in S(\mathscr{A}, \mathfrak{U})$

$$\psi(T) = \varphi(T) \qquad \text{for all} \quad \psi \in S_\mathfrak{U}(\varphi) = K_{\check{\varphi}} | \mathscr{A}.$$

By Remark 4.3, c), we can choose ergodic states $\psi_\varphi \in S_\mathfrak{U}(\varphi)$ for every φ thus using the GNS-construction for states ω of arbitrary C^*-algebras we get

Corollary 4.5. *Let $\mathscr{A}$ be a C^*-algebra with identity, G be a group of $*$-automorphisms of $\mathscr{A}$ and let $S(\mathscr{A}, G)$ be the set of G-invariant states. Then for $\omega \in S(\mathscr{A}, G)$ there exist a maximal measure μ on $S(\mathscr{A}, G)$ with resultant ω and a map $\varphi \to \rho_\varphi$ of $S(\mathscr{A}, G)$ into the set of G-ergodic states such that*

$$\omega(T) = \int_{S(\mathscr{A}, G)} \rho_\varphi(T) d\mu(\varphi) \qquad \textit{for all} \quad T \in \mathscr{A}.$$

Remark 4.6. Let $\omega \in S(\mathscr{A}, \mathfrak{U})$ be as in Theorem 4.4. Then ω is the resultant of a unique normalized maximal measure μ on $S(\mathscr{A}, \mathfrak{U})$ if and only if $\mathscr{A}' \cap \mathfrak{U}'$ is abelian ([1], Proposition 4.3.3, [12], see also [6], Satz 1). But, even if $S(\mathscr{A}, \mathfrak{U})$ is a simplex, each $\varphi \in \operatorname{supp} \mu$ may fail to be ergodic.

Example ([5]). Let G be the discrete group in Section 1, $\mathscr{A}_0 = C^*(L)$ the C^*-algebra generated by the left regular representation of G on $l^2(G)$ (the group C^*-aglebra as G is amenable). Let $\mathfrak{U} = \{U_x = L_x R_x;\ x \in G\}$, then $S(\mathscr{A}_0, \mathfrak{U})$ is the set of all normalized traces on $\mathscr{A}_0$. Moreover, $\mathscr{B} = C^*(\mathscr{A}_0, \mathfrak{U})$ is the C^*-algebra which is denoted by $\mathscr{A}$ in Section 1. Then $\omega \in S_{\mathfrak{U}}(\mathscr{B})$, let $\omega_e = \omega \mid \mathscr{A}_0 \in S(\mathscr{A}_0, \mathfrak{U})$. Since the restriction map defines an order isomorphism of $S_{\mathfrak{U}}(\mathscr{B})$ onto $S(\mathscr{A}_0, \mathfrak{U})$ no state $\varphi \in \overline{M(\omega_e)}$ is ergodic (factorial) by Theorem 1.1. The unique maximal measure μ on $S(\mathscr{A}_0, \mathfrak{U})$ with resultant ω_e is also the central measure of ω_e on $S(\mathscr{A}_0)$. Therefore for every $\varphi \in \operatorname{supp} \mu$ the convex, compact sets

$$S_{\mathfrak{U}}(\varphi) \quad \text{and} \quad S(\varphi), \text{ resp.},$$

do not consist of a single point $\{\varphi\}$, by Remark 4.3, c) and Theorem 3.4, resp.

Therefore, different selections of ergodic states $\rho_\varphi, \rho'_\varphi \in S_{\mathfrak{U}}(\varphi)$, i.e. $\rho_\varphi \neq \rho'_\varphi$ for all $\varphi \in \operatorname{supp} \mu$, give the same integral

$$\omega_e(T) = \int \rho_\varphi(T) d\mu(\varphi) = \int \rho'_\varphi(T) d\mu(\varphi) \qquad \text{for all} \quad T \in \mathscr{A}_0 .$$

Therefore one may ask the following questions.

Problem 4.7. a) Is there a canonical way to distinguish certain ergodic states $\rho_\varphi \in S_{\mathfrak{U}}(\varphi)$, at least in the case that the measure is unique?

b) Can non-uniqueness described in 4.6 be given any physical interpretation? For instance, let observables be selfadjoint operators in a possibly non separable C^*-algebra $\mathscr{A}$ which are measured at a state ω. Does failure of uniqueness mean that one cannot get information about ergodic states (factorial states) decomposing ω, but only about the sets $S_{\mathfrak{U}}(\varphi)$ $(S(\varphi))$?

§5. Appendix

All results presented in this appendix are due to M. Tomita. The main purpose is to give a proof of Theorem 2.3 without using fields of Hilbert spaces.

Lemma 5.1 ([14], p. 88, 2.1). *Let $\mathscr{A}$ be a C^*-algebra of operators on a Hilbert space H and $\mathscr{M}$ be its weak closure. For every $T \in \mathscr{M}$, projection $E \in \mathscr{M}$, $\varepsilon > 0$ and every set $\{\xi_1, \ldots, \xi_n\}$ in H there exist a projection $F \leqq E$ in $\mathscr{M}$ and $A \in \mathscr{A}$ such that*

$$\|(T-A)F\| \leqq \varepsilon,\ \|A\| \leqq \|TE\| \quad and \quad \|(E-F)\xi_k\| \leqq \varepsilon,\ 1 \leqq k \leqq n.$$

Remark. For a short proof of this Lemma see [9], Lemma 2.7.2. In [14], the Lemma has been stated in a slightly stronger version, namely one can even have $\|A\| \leqq \|TF\|$. This can be easily seen by defining $A = A_0\|TF\|/\|TE\|$ if A_0 is as in Lemma 5.1. But this is not needed for the proof of the following Noncommutative Lusin's Theorem as given in [13], Theorem 4.15.

Theorem 5.2 ([14], Theorem 6). *Let $\mathscr{A}$ be a C^*-algebra of operators on a Hilbert space H and $\mathscr{M}$ be its weak closure. For every $T \in \mathscr{M}$, projection $E \in \mathscr{M}$ and every strong neighbourhood U of E in $\mathscr{M}$ there exist a projection $F \leqq E$ in U and $A \in \mathscr{A}$ such that*

$$TF = AF \quad and \quad \|A\| \leqq \|TF\|(1+\varepsilon).$$

Remark. Since $\|TF\| \leqq \|TE\|$ one can always have $\|A\| \leqq \|TF\| + \varepsilon$, but the weaker estimate $\|A\| \leqq \|TE\| + \varepsilon$ as in [9], Theorem 2.7.3 is not sufficient for the proof of the next Theorem 5.4.

Definition 5.3. Let $\mathscr{A}$ be a C^*-algebra of operators on a Hilbert space H and $\mathscr{B}$ a C^*-subalgebra of $\mathscr{A}$. A projection E in the strong closure $\mathscr{M}$ of $\mathscr{A}$ is said to be *regular relative to* $\mathscr{B}$ if

$$\|BE\| = \inf_{A \in N(E)} \|B+A\| \qquad \text{for every} \quad B \in \mathscr{B}$$

where $N(E)$ is the left ideal $N(E) = \{A \in \mathscr{A};\ AE = 0\}$.

Theorem 5.4 ([14], Theorem 5). *Let $\mathscr{B}$ be a separable C^*-subalgebra of $\mathscr{A}$ and E a projection in the weak closure $\mathscr{M}$ of $\mathscr{A}$. Then every strong neighbourhood of E in $\mathscr{M}$ contains a projection $F \leqq E$ which is regular relative to $\mathscr{B}$.*

Proof. Let $\{T_j\}_{j \in \boldsymbol{N}} \subset \mathscr{B}$ be a sequence of operators which is uniformly dense in $\mathscr{B}$, let $\xi_1, \ldots, \xi_n$ be in H and $\delta > 0$.

a) It is sufficient to show: There is a projection $F \leqq E$ in $\mathscr{M}$ such that $\|(E-F)\xi_i\| \leqq \delta$, $1 \leqq i \leqq n$, and

$$\|T_jF\| = \inf\{\|A\|;\ A \in \mathscr{A},\ T_jF = AF\} \qquad \text{for all} \quad j.$$

For, if $B \in \mathscr{B}$ and $\varepsilon > 0$, there exist T_j and $A \in \mathscr{A}$ such that

$$\|T_j - B\| \leqq \varepsilon,\ T_jF = AF,\ \|A\| \leqq \|T_jF\| + \varepsilon$$

hence $(B - T_j + A)F = BF$ and $\|B + (A - T_j)\| \leqq \varepsilon + \|T_jF\| + \varepsilon \leqq 3\varepsilon + \|BF\|$ thus F is regular relative to $\mathscr{B}$.

b) Let $0<\varepsilon<\delta$ and $F_0=E$. By Theorem 5.2, for $k\in\boldsymbol{N}$ we can find projections $F_k\in\mathscr{M}$, $A_k\in\mathscr{A}$ and vectors $\eta_k\in H$, $\|\eta_k\|=1$, such that

1) $\|T_kF_{k-1}\eta_k\|\geqq\|T_kF_{k-1}\|-\varepsilon$,
2) $F_k\leqq F_{k-1}$, $\|(F_{k-1}-F_k)\xi_i\|\leqq 2^{-k}\varepsilon$ for $1\leqq i\leqq n$,
$\|(F_{k-1}-F_k)\eta_j\|\leqq 2^{-k}\varepsilon(1+\|T_jF_{j-1}\|)^{-1}$ for $1\leqq j\leqq k$

and

$$T_kF_k=A_kF_k,\ \|T_kF_k\|\geqq\|A_k\|-\varepsilon.$$

Let F be the limit of the F'_{ks}, then

$$\|(E-F)\xi_i\|\leqq\sum_{k=1}^{\infty}\|(F_{k-1}-F_k)\xi_i\|\leqq\varepsilon<\delta,\ 1\leqq i\leqq n,$$

$$\|F_{j-1}\eta_j-F\eta_j\|\leqq\sum_{k=j}^{\infty}\|F_k\eta_j-F_{k-1}\eta_j\|\leqq\varepsilon(1+\|T_jF_{j-1}\|)^{-1},$$

$$A_jF=A_jF_jF=T_jF_jF=T_jF,$$

and

$$\begin{aligned}\|T_jF\|&\geqq\|T_jF\eta_j\|\geqq\|T_jF_{j-1}\eta_j\|-\|T_jF_{j-1}(F_{j-1}\eta_j-F\eta_j)\|\\&\geqq\|T_jF_{j-1}\|-2\varepsilon\geqq\|T_jF_{j-1}F_j\|-2\varepsilon=\|T_jF_j\|-2\varepsilon\geqq\|A_j\|-3\varepsilon.\end{aligned}$$

Corollary 5.5. ([14], p. 92). *Let $\mathscr{A}$ be a C^*-algebra of operators on a Hilbert space H with cyclic vector ξ such that $I\in\mathscr{A}$ and $\mathscr{C}=\mathscr{A}'$ is abelian. Let $P\in\mathscr{A}''=\mathscr{M}$ be the projection onto $[\mathscr{A}'\xi]$ and $\Phi\colon\mathscr{M}=\mathscr{C}'\to\mathscr{C}$ the projection map onto $\mathscr{C}$ (as introduced in* §2) *such that*

$$PTP=\Phi(T)P,\ T\in\mathscr{M}\ \text{(see 2.1)}.$$

Then for every $T\in\mathscr{A}$, $\varepsilon>0$ there exist a projection $E\in\mathscr{M}$, $E\leqq P$ and $B\in\mathscr{A}$ such that

$$\|(P-E)\xi\|\leqq\varepsilon,\ ETE=BE$$

and E is regular relative to the C^*-algebra $\mathscr{B}=C^*(T,B,I)\subset\mathscr{A}$ generated by T, B, I.

Proof. By Theorem 5.2, there exist a projection $F\leqq P$ in $\mathscr{M}$ and $B\in\mathscr{A}$ such that $\Phi(T)F=BF$ and $\|(P-F)\xi\|\leqq\varepsilon/2$. Then $\mathscr{B}=C^*(T,B,I)\subset\mathscr{A}$ and by Theorem 5.4, there is a projection $E\leqq F$ in $\mathscr{M}$ regular relative to $\mathscr{B}$ with

$$\|(F-E)\xi\|\leqq\varepsilon/2.$$

Then

$$ETE = EPTPE = \Phi(T)E = \Phi(T)FE = BE$$

and $\|(P-E)\xi\| \leqq \varepsilon$.

Remark 5.6. In the following we use the notation introduced in Section 2. For $\gamma \in \Omega$ let f_γ be the state of $\mathscr{C}'$ defined as

$$f_\gamma(T) = \gamma(\Phi(T)),\ T \in \mathscr{C}'$$

Then f_γ is pure. For, if $q \in S(\mathscr{C}')$ and $N_q \supset N_{f_\gamma}$, then $q \mid \mathscr{C} = f_\gamma \mid \mathscr{C} = \gamma$ since $N_{q|\mathscr{C}} \supset N_\gamma$. Moreover, $q(I-P)=0$ as $f_\gamma(I-P)=0$ hence by the Cauchy-Schwarz inequality

$$0 = q(T(I-P) = q((I-P)T) \qquad \text{for all} \quad T \in \mathscr{C}',$$

thus

$$q(T) = q(PTP) = q(\Phi(T)P) = q(\Phi(T)) = \gamma(\Phi(T)) = f_\gamma(T).$$

Assume, for a moment, that $\mathscr{A}'$ is abelian and $\mathscr{C} = \mathscr{A}' \subset \mathscr{A}$ (the C^*-algebra $\mathscr{R} = C^*(\mathscr{A}, \mathscr{C})$ has these properties, if $\mathscr{C}$ is an arbitrary maximal abelian subalgebra of $\mathscr{A}'$). Then $\mathscr{A}$ is weakly dense in $\mathscr{C}' = \mathscr{A}''$. If the restriction φ of f_γ onto $\mathscr{A}$ is not a pure state, there is a state $\psi \in S(\mathscr{A})$ such that $\psi \neq \varphi$, $\psi \leqq r\varphi$ for some $r > 0$, which has no extension $q \in S(\mathscr{A}'')$ such that $q(P)=1$ ($N_q \supset N_{f_\gamma}$).

Conversely, if for every $\psi \in M(\varphi)$, $\varphi = f_\gamma \mid \mathscr{A}$, there is a continuous linear functional f on $\mathscr{A}''$ extending ψ such that $f(TP) = f(T)$ for all $T \in \mathscr{A}''$ (but not necessarily $f(PT) = f(T)$ for all $T \in \mathscr{A}''$), one has to use a polar decomposition of f to find an extension q of ψ such that $q(TP) = q(PT) = q(T)$, $T \in \mathscr{A}''$. Then it follows

$$\psi(T) = q(PTP) = q(\Phi(T)P) = \psi(\Phi(T)) = \gamma(\Phi(T)) = \varphi(T)$$

hence φ is a pure state of $\mathscr{A}$.

Therefore the next lemma will be the key for the proof of Tomita's theorem.

Lemma 5.7 ([14], Lemma 2.2, p. 90). *Let $\mathscr{A}$ be a C^*-algebra of operators on a Hilbert space H, $\mathscr{B}$ be a C^*-subalgebra of $\mathscr{A}$ with $I \in \mathscr{B}$ and $E \in \mathscr{M} = \mathscr{A}''$ a projection regular relative to $\mathscr{B}$. Then for every state φ of $\mathscr{A}$ with $\varphi(A)=0$ for $A \in N(E)$, i.e. $AE=0$, there is a state q of $\mathscr{M}$ such that*

$$q(E) = 1 \quad \textit{and} \quad q \mid \mathscr{B} = \varphi \mid \mathscr{B}.$$

Proof. Let $\mathfrak{F} = \{f \in \mathscr{M}^*;\ \|f\| \leqq 1, f(T(I-E)) = 0 \ \text{ for all } \ T \in \mathscr{M}\}$, then $\mathfrak{F}$ is a convex, $\sigma(\mathscr{M}^*, \mathscr{M})$-compact and balanced set. For $B \in \mathscr{B}$, $A \in N(E)$

$$|\varphi(B)| = |\varphi(B+A)| \leqq \|B+A\|$$

hence, because E is regular relative to $\mathscr{B}$

$$\begin{aligned} |\varphi(B)| &\leqq \inf_{A\in N(E)} \|B+A\| \\ &= \|BE\| = \sup_{\|\xi\|=1, \|\eta\|=1} |(BE\xi, \eta)| \leqq \sup_{f\in\mathfrak{F}} |f(B)| \end{aligned}$$

But then there is $f\in\mathfrak{F}$ such that $f|\mathscr{B}=\varphi|\mathscr{B}$. Otherwise, by using the geometric form of the Hahn-Banach theorem, $\varphi|\mathscr{B}$ could be strictly separated from the convex, $\sigma(\mathscr{B}^*, \mathscr{B})$-compact set $\mathfrak{F}_{\mathscr{B}}=\{f|\mathscr{B}; f\in\mathfrak{F}\}$ by the real part of a $\sigma(\mathscr{B}^*, \mathscr{B})$-continuous linear functional. Since $\mathfrak{F}_{\mathscr{B}}$ is balanced, multiplying by an appropriate $z\in\mathbf{C}$, $|z|\leqq 1$, we could find an element $B\in\mathscr{B}$ such that

$$|\varphi(B)| > \sup_{f\in\mathfrak{F}} |f(B)|$$

which is impossible.

Let $q=|f/\|f\||$ be the absolute value of $f/\|f\|$. Since f is not necessarily ultraweakly continuous, we use the enveloping polar decomposition of f, $\mathscr{M}$ considered as C^*-algebra. As $\|f\|\leqq 1$ we get for $B\in\mathscr{B}$

$$|\varphi(B)|^2 = |f(B)|^2 \leqq \left|\frac{f(B)}{\|f\|}\right|^2 \leqq q(B^*B)$$

hence $\varphi|\mathscr{B}=q|\mathscr{B}$ since φ is a state ([2], Proposition 12.2.9). Finally, there is an element v in the universal enveloping von Neumann algebra $\tilde{\mathscr{M}}$ of $\mathscr{M}$ such that $\tilde{q}=v\cdot\tilde{f}/\|f\|$ hence $q(I-E)=(\tilde{f}/\|f\|)(v(I-E))=0$ as $f(\mathscr{M}(I-E))=\{0\}$ $\tilde{f}$ being the ultraweakly continuous extension of f onto $\tilde{\mathscr{M}}$.

Theorem 5.8 (Tomita). *Let $\mathscr{A}$ be a C^*-algebra of operators on a Hilbert space H with $I\in\mathscr{A}$ and cyclic vector ξ, $\|\xi\|=1$. Let ω be the corresponding vector state and μ a maximal orthogonal measure on $S(\mathscr{A})$ with resultant ω. Then if $T\in\mathscr{A}$, almost all $\varphi\in S(\mathscr{A})$ are pure relative to T.*

Proof. 1) Assume first that $\mathscr{A}'=\mathscr{C}$ is abelian, let $T\in\mathscr{A}$ and $\varepsilon>0$. Then $P\in\mathscr{A}''=\mathscr{M}$ and by Corollary 5.5 there exist a projection $E\leqq P$ in $\mathscr{M}$ and $B\in\mathscr{A}$ such that E is regular relative to $\mathscr{B}=C^*(T, B, I)$, $\|(P-E)\xi\|^2\leqq\varepsilon$ and

$$ETE = BE = \Phi(T)E\,.$$

Thus $E=PEP=\Phi(E)P$ where $Q=\Phi(E)$ is a projection in $\mathscr{C}=\mathscr{A}'$. Let ν be the spectral measure on Ω defined by ξ. Since $1-\nu(Q)=((I-Q)P\xi, \xi)=\|(P-E)\xi\|^2 \leqq\varepsilon$ and μ is the image of ν under the continuous map $\gamma\to\varphi_\gamma$, the assertion will follow in this case, if we have shown that φ_γ is pure relative to T for all $\gamma\in\Omega$ with $\gamma(Q)=1$.

If $A\in\mathscr{A}$, $AE=0$, then

$$0=(PEA^*AEP\xi, \xi)=(QPA^*AP\xi, \xi)=(Q\Phi(A^*A)\xi, \xi)$$
$$=\int_\Omega \gamma(Q)\varphi_\gamma(A^*A)d\nu(\gamma),$$

hence $\varphi_\gamma(A^*A)=0$ if $\gamma(Q)=1$. Thus

(5.1) $$N(E)\subset N_{\varphi_\gamma} \quad \text{if} \quad \gamma(Q)=1.$$

Let $D=B-\varphi_\gamma(T)I$, then

(5.2) $$D\in N_{\varphi_\gamma} \quad \text{if} \quad \gamma(Q)=1.$$

For, if $\gamma(Q)=1, f_\gamma(E)=\gamma(\Phi(E))=\gamma(Q)=1$ hence

$$\begin{aligned}\varphi_\gamma(D^*D)&=f_\gamma(D^*D)=f_\gamma(ED^*DE)\\&=f_\gamma((BE-\varphi_\gamma(T)E)^*(BE-\varphi_\gamma(T)E))\\&=f_\gamma((\Phi(T)E-f_\gamma(T)E)^*(\Phi(T)E-f_\gamma(T)E))\\&=f_\gamma((\Phi(T)-f_\gamma(T)I)^*(\Phi(T)-f_\gamma(T)I))\\&=|\gamma(\Phi(T)-f_\gamma(T)I)|^2=|f_\gamma(T)-f_\gamma(T)|^2=0.\end{aligned}$$

Consequently, let $\gamma(Q)=1$ and let $\psi\in S(\mathscr{A})$ such that $N_\psi\supset N_{\varphi_\gamma}$. From (5.2) we see

(5.3) $$0=\psi(D^*D)=\psi(D)=\psi(B)-\varphi_\gamma(T).$$

By (5.1), $N(E)\subset N_\psi$ hence by Lemma 5.7, there is a state q of $\mathscr{M}$ such that $q\,|\,\mathscr{B}=\psi\,|\,\mathscr{B}$ and $q(E)=1$. Therefore

$$\psi(T)=q(T)=q(ETE)=q(BE)=q(B)=\psi(B),$$

thus $\psi(T)=\varphi_\gamma(T)$ by (5.3).

2) Let $\mathscr{C}$ be a maximal abelian subalgebra of $\mathscr{A}'$ and let $\mathscr{R}=C^*(\mathscr{A}, \mathscr{C})$ be the C^*-algebra generated by $\mathscr{A}$ and $\mathscr{C}$. Then $\mathscr{R}'=\mathscr{A}'\cap\mathscr{C}'=\mathscr{C}$ is abelian, let λ be the orthogonal measure on $S(\mathscr{R})$ corresponding to $\mathscr{C}$ and ξ. Then μ is the image of λ under the restriction map $g\to g\,|\,\mathscr{A}$. For $g\in\operatorname{supp}\lambda$ there is a $\gamma\in\Omega$ such that for $T\in\mathscr{R}$

$$g(T)=\gamma(\Phi(T)) \qquad \text{(see Remark 2.2)}.$$

Thus for $A\in\mathscr{A}$, $T\in\mathscr{C}$

$$g(AT)=\gamma(\Phi(AT))=\gamma(\Phi(A)T)=g(A)\gamma(T)$$

therefore

(5.4) $$\pi_g(\mathscr{R})=\pi_g(\mathscr{A}) \qquad \text{for} \quad g\in\operatorname{supp}\lambda$$

Let $\varphi \in \operatorname{supp} \mu$, and $g \in \operatorname{supp} \lambda$ with $g \mid \mathscr{A} = \varphi$.
If ψ is a state of $\mathscr{A}$ with $N_\psi \supset N_\varphi$, we define a state $\tilde{\psi}$ of $\mathscr{R}$ (depending on the extension g of φ) as follows. For $T \in \mathscr{R}$ put $\tilde{\psi}(T) = \psi(A)$, if $\pi_g(T) = \pi_g(A)$ for $A \in \mathscr{A}$ (see 5.4). If $\pi_g(A) = 0$, then $g(A^*A) = 0 = \varphi(A^*A)$ hence

$$\psi(A^*A) = 0 = \psi(A) \quad \text{as} \quad N_\psi \supset N_\varphi .$$

It is easily seen that $\tilde{\psi}$ is a state of $\mathscr{R}$, moreover,

$$N_{\tilde{\psi}} \supset N_g$$

holds. For, if $g(T^*T) = 0$ and

$$\pi_g(T) = \pi_g(A),$$

then

$$\tilde{\psi}(T^*T) = \psi(A^*A) = 0$$

because

$$\begin{aligned} \varphi(A^*A) &= g(A^*A) = (\pi_g(A^*A)\xi_g, \xi_g) = (\pi_g(T^*T)\xi_g, \xi_g) \\ &= g(T^*T) = 0. \end{aligned}$$

3) Finally, let $T \in \mathscr{A}$ and $\varepsilon > 0$. By the proof in I, there is a compact set $K \subset \operatorname{supp} \lambda$ such that $\lambda(K) \geqq 1 - \varepsilon$ and all $g \in K$ are pure relative to T, then $\mu(S_\varepsilon) \geqq 1 - \varepsilon$ where $S_\varepsilon = \{g \mid \mathscr{A};\ g \in K\}$ is compact. Let $\varphi \in S_\varepsilon$, say $\varphi = g \mid \mathscr{A}$, $g \in K$, and let $\psi \in S(\mathscr{A})$ such that $N_\psi \supset N_\varphi$. Then by 2),

$$N_{\tilde{\psi}} \supset N_g$$

therefore

$$\psi(T) = \tilde{\psi}(T) = g(T) = \varphi(T)$$

hence all $\varphi \in S_\varepsilon$ are pure relative to T.

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