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Isomorphism Theorems for Cohomology Groups of Weakly 1-Complete Manifolds

By

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Introduction

In the theory of complex manifolds, there are two different extreme objects: compact manifolds and holomorphically complete ones. We have a lot of good knowledge about the fundamental properties of both classes of manifolds, contributions to which have been made by many celebrated authors in this century.

In 1970, S. Nakano [18] succeeded in solving a problem on the inverse of monoidal transformation by proving the vanishing of cohomology groups for line bundles over a class of complex manifolds. This class includes the above extremes and was called by him weakly 1-complete manifolds. The definition is as follows; a complex manifold is said to be weakly 1-complete if it carries a

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C^∞ plurisubharmonic exhaustion function. It is trivial that a compact complex manifold is weakly 1-complete. It follows immediately from the Remmert's proper embedding theorem that holomorphically complete manifolds are weakly 1-complete.

From the definition, it is quite natural to expect that a weakly 1-complete manifold is a nice intermediate object between compact complex manifolds and holomorphically complete ones.

In the last decade, more or less inspired by this philosophy, several authors have studied cohomological properties of weakly 1-complete manifolds: [1], [12], [19], [20], [21], [22], [23], [24], [25], [26], [27], [28], [29]. The following theorem is due to S. Nakano [21].

Theorem 1. *Let X be a weakly 1-complete manifold and $\pi: B \to X$ a holomorphic line bundle. Assume that B has a metric along the fibers whose curvature form is positive. Then,*

$$H^q(X, \Omega^p(B)) = 0, \quad p+q > \dim X.$$

Here we denote by $H^q(X, \Omega^p(B))$ the q-th cohomology group of X with coefficients in the sheaf of the germs of B-valued holomorphic p-forms.

Let X_c be the sublevel set $\{x; \varphi(x) < c\}$ of the exhaustion function φ. In [24], the author has extended the above theorem as follows:

Theorem 2. *Let X be a weakly 1-complete manifold and $\pi: B \to X$ a holomorphic line bundle. Assume that B has a metric along the fibers whose curvature form is positive outside a compact subset K of X. Then, the natural restriction maps*

$$\rho_c: H^q(X, \Omega^p(B)) \longrightarrow H^q(X_c, \Omega^p(B)), \quad p+q > \dim X,$$

are bijective if $X_c \supset K$.

The purpose of the present article is to extend the methods employed in [24] and show more explicitly how they unite each other to yield a fundamental theorem on cohomology groups of weakly 1-complete manifolds; the results including all the known ones will be deduced from the isomorphism theorem in Chapter 2.

As a by-product we obtain simple proofs of results obtained by Andreotti-Grauert [3], Andreotti-Vesentini [4], and Hörmander [10]. Our viewpoint is that of [4] and the argument is essentially included in [10]; the use of complete metrics renders the derivation of the basic estimates very easy, and the argument borrowed from [10] enables us to avoid the use of so called 'bumping lemma'. Of course the advantage of our method lies in that we can prove the isomorphism theorems on weakly 1-complete manifolds in the same way.

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Chapter 1. Preliminaries

§1. Hermitian Geometry

Let X be a paracompact complex manifold of dimension n and $\pi\colon E\to X$ a complex vector bundle. We denote by E^*, $\bar{E}$, $\overset{m}{\wedge} E$, and $E^{(m)}$, the dual, the conjugate, m-fold exterior power and m-fold symmetric power of E.

Definition 1.1. A section h of $\mathrm{Hom}(E, \bar{E}^*)$ is called a hermitian metric along the fibers of E if, for any point $x\in X$ and any two vectors $v, w\in E_x := \pi^{-1}(x)$,

$$(1)\qquad \begin{cases} \overline{(h(x)(w))(\bar{v})}=(h(x)(v))(\bar{w}) \\ (h(x)(v))(\bar{v})>0, \quad v\neq 0. \end{cases}$$

Hermitian metrics are assumed to be C^∞ unless otherwise stated. Let h be a hermitian metric along the fibers of E. For two sections f and g of E, we set

$$(2)\qquad \langle f, g\rangle = h(f)(\bar{g}).$$

$\langle f, g\rangle$ is called *the pointwise inner product* of f and g. Canonically, h induces metrics along the fibers of E^*, $\bar{E}$, $E^{(m)}$, $\overset{m}{\wedge} E$, and $\overset{p}{\wedge} E\otimes \overset{q}{\wedge} \bar{E}$. We also denote by $\langle\ ,\ \rangle$ the pointwise inner product with respect to the induced metrics.

Let T_X be the tangent bundle of X and $T'_X\oplus T''_X$ the splitting of $T^*_X\otimes_{\mathbf{R}}\mathbf{C}$ into types $(1, 0)$ and $(0, 1)$ with respect to the complex structure of T_X. As a complex vector bundle we always identify T_X (resp. T^*_X) with T'^*_X (resp. T'_X).

Definition 1.2. A section of $E\otimes \overset{p}{\wedge} T'_X\otimes \overset{q}{\wedge} T''_X$ is called an *E-valued* (p, q)*-form*. In particular, a section of $\overset{p}{\wedge} T'_X\otimes \overset{q}{\wedge} T''_X$ is called a (p, q)*-form*.

We can naturally identify $\overset{p}{\wedge} T'_X \otimes \overset{q}{\wedge} T''_X$ with a subbundle of $\overset{p+q}{\wedge}(T_X \otimes_{\mathbf{R}} \mathbf{C})$. For simplicity we denote $\overset{r}{\wedge}(T_X \otimes_{\mathbf{R}} \mathbf{C})$ by T^r_X and the subbundle of T^r_X corresponding to $\overset{p}{\wedge} T'_X \otimes \overset{q}{\wedge} T''_X$ by $T^{p,q}_X$. A section u of T^r_X is called an *r-form*. We set $\deg u = r$. We express $v \in (E \otimes T^{p,q}_X)_x$ as

(3) $$v = \sum_{\substack{i_1 < \cdots < i_p \\ j_1 < \cdots < j_q}} v_{i_1 \cdots i_p \bar{j}_1 \cdots \bar{j}_q} \tau_{i_1} \wedge \cdots \wedge \tau_{i_p} \wedge \bar{\tau}_{j_1} \wedge \cdots \wedge \bar{\tau}_{j_q},$$

where $v_{i_1 \cdots i_p \bar{j}_1 \cdots \bar{j}_q} \in E_x$ and $(\tau_1, \dots, \tau_n)$ is a basis of $T'_{X,x}$.

Let ds^2 be a hermitian metric along the fibers of T_X. We denote also by $\langle f, g \rangle$ the pointwise inner product of E-valued (p, q)-forms f and g with respect to the metric induced by h and ds^2. The length of f is defined by $|f| = \sqrt{\langle f, f \rangle}$. ds^2 is pointwise expressed as

(4) $$ds^2 = \sum_{i=1}^{n} \tau_i \otimes \bar{\tau}_i,$$

for a suitable choice of the basis $(\tau_1, \dots, \tau_n)$. $(\tau_1, \dots, \tau_n)$ is called an *orthonormal basis* with respect to ds^2. We set

(5) $$\omega = \sqrt{-1} \sum_{i=1}^{n} \tau_i \wedge \bar{\tau}_i.$$

ω is called *the fundamental form* associated to ds^2.

Let E_1 and E_2 be two complex vector bundles over X provided with hermitian metrics along the fibers. Let $\xi: E_1 \to E_2$ be a morphism or the conjugate of a morphism. *The adjoint* ξ^* of ξ is defined by the following formula:

(6) $$\langle \xi f, g \rangle_2 = \langle f, \xi^* g \rangle_1,$$

where f and g run the sections of E_1 and E_2, respectively, and $\langle\ ,\ \rangle_i$ denotes the pointwise inner product of E_i.

The conjugate star operator $\bar{*}_E: E \otimes T^{p,q}_X \to E^* \otimes T^{n-p,n-q}_X$ is defined to be a conjugate linear operator satisfying

(7) $$\begin{cases} f \wedge \bar{*}_E f = \langle f, f \rangle dv \\ (\bar{*}_E)^* \bar{*}_E f = (-1)^{p+q} f \end{cases}$$

for any E-valued (p, q)-form f, where

(8) $$dv = \frac{1}{n!} \overbrace{\omega \wedge \cdots \wedge \omega}^{n}$$

and

$$(e_1 \otimes g_1) \wedge (e_2^* \otimes g_2) = e_2^*(e_1) g_1 \wedge g_2, \tag{9}$$
$$e_1 \in E_x,\ e_2^* \in E_X^*,\ g_1 \in T_{X,x}^{p,q},\ g_2 \in T_{X,x}^{p',q'}.$$

dv is called *the volume form* with respect to ds^2. $\bar{*}_E$ is unique. In particular, for the trivial bundle $\mathbb{1}$ with fiber $\mathbb{C}$ and with trivial metric along the fibers, we denote $\bar{*}_1$ by $\bar{*}$. $\bar{*}$ operates naturally on $E \otimes T_X^{p,q}$ so that $\bar{*}(e \otimes f) = \bar{e} \otimes \bar{*} f$ for $e \otimes f \in (E \otimes T_X^{p,q})_x$. We have $\bar{*}_E = \bar{h} \otimes \bar{*}$. Let F be another complex vector bundle over X. For a $\mathrm{Hom}(E, F)$-valued (p, q)-form f, we denote by

$$e(f) \colon E \otimes T_X^{s,t} \longrightarrow F \otimes T_X^{s+p,t+q} \tag{10}$$

the left multiplication by f. We set

$$\begin{cases} L = e(\omega) \\ \Lambda f = (-1)^{\deg f} \bar{*} L \bar{*} f, \quad \text{for} \quad f \in T_{X,x}^r \end{cases} \tag{11}$$

L is called *the Lefschetz' operator* with respect to ds^2. Λ is the adjoint of L with respect to the metrics induced by ds^2. We set $\bar{C} f = \sum (-1)^{q-p} \pi_{p,q} \bar{f}$, for $f \in T_X^r$, where we denote by $\pi_{p,q}$ the projection from T_X^{p+q} to $T_X^{p,q}$. By abuse of notation we denote $\mathrm{id}_E \otimes L$ by L, and so on.

In what follows let E be a holomorphic vector bundle. We denote by $C^{p,q}(X, E)$ the set of E-valued (p, q)-forms of class C^∞. We set $C^{\cdot}(X, E) = \sum_{p,q} C^{p,q}(X, E)$. The complex exterior differentiations ∂ and $\bar{\partial}$ operate naturally on $C^{\cdot}(X, \bar{E})$ and on $C^{\cdot}(X, E)$, respectively. We set

$$\begin{aligned} D_E &= \bar{\partial} + h^{-1} \partial h \\ D'_E &= h^{-1} \partial h \\ \vartheta_E &= -\bar{*}_{E^*} \bar{\partial} \bar{*}_E (= -\bar{*} \bar{h}^{-1} \bar{\partial} \bar{h} \bar{*}) \\ \bar{\vartheta} &= -\bar{*} \partial \bar{*}, \end{aligned} \tag{12}$$

where by abuse of notation we denote $h \otimes \mathrm{id}$ by h, and so on.

Theorem 1.3. *Let the notations be as above. Then there is a linear operator T_1 (resp. T_2) belonging to the algebra of operators generated by $e(\bar{\partial}\omega)$ (resp. $e(\partial\omega)$), L, Λ, $\bar{*}$, and $\bar{C}^{-1}$ over the field of rational numbers, satisfying*

$$\begin{aligned} &D'_E \Lambda - \Lambda D'_E = \sqrt{-1}(\vartheta_E + T_1) \\ &(\textit{resp. } \bar{\partial} \Lambda - \Lambda \bar{\partial} = -\sqrt{-1}(\bar{\vartheta} + T_2)), \end{aligned} \tag{13}$$

for any vector bundle E and h. Furthermore T_i are linear with respect to $d\omega$ and $T_1=T_2=0$ if $d\omega=0$.

Proof. See Appendix.

Theorem 1.4 (*Extended Bochner-Calabi-Nakano formula*).

$$\begin{aligned}(14)\qquad & -\sqrt{-1}(D_E^2\Lambda-\Lambda D_E^2)\\ &=\bar\partial\vartheta_E+\vartheta_E\bar\partial-(\bar\vartheta D'_E+D'_E\bar\vartheta)+\bar\partial T_1+T_1\bar\partial\\ &\quad -T_2D'_E-D'_ET_2.\end{aligned}$$

Proof. Since $\bar\partial^2=0$ and $(D'_E)^2=0$,

$$(15)\qquad D_E^2=(\bar\partial+D'_E)(\bar\partial+D'_E)=\bar\partial D'_E+D'_E\bar\partial\,.$$

Hence,

$$\begin{aligned}(16)\qquad & -\sqrt{-1}(D_E^2\Lambda-\Lambda D_E^2)\\ &=-\sqrt{-1}(\bar\partial D'_E\Lambda+D'_E\bar\partial\Lambda-\Lambda\bar\partial D'_E-\Lambda D'_E\bar\partial)\\ &=-\sqrt{-1}(\bar\partial(D'_E\Lambda-\Lambda D'_E)+(D'_E\Lambda-\Lambda D'_E)\bar\partial\\ &\quad +(\bar\partial\Lambda-\Lambda\bar\partial)D'_E+D'_E(\bar\partial\Lambda-\Lambda\bar\partial))\\ &=\bar\partial\vartheta_E+\vartheta_E\bar\partial-(\bar\vartheta D'_E+D'_E\bar\vartheta)+\bar\partial T_1+T_1\bar\partial\\ &\quad -T_2D'_E-D'_ET_2\,.\end{aligned}$$

q.e.d.

We set

$$(17)\qquad (f,g)=\int_X\langle f,g\rangle dv\,,$$

for $f, g\in C^{p,q}(X, E)$ such that $\operatorname{supp} f\cap\operatorname{supp} g\Subset X$, where $\operatorname{supp} f$ denotes the support of f. Then, by Stokes' theorem,

$$(18)\qquad \begin{aligned}(\bar\partial f, g)&=(f,\vartheta_E g)\,,\\ (D'_E f, g)&=(f,\bar\vartheta g)\,,\end{aligned}$$

provided that $\operatorname{supp} f\cap\operatorname{supp} g$ is compact. We put

$$(19)\qquad C_0^{p,q}(X, E)=\{f\in C^{p,q}(X, E);\ \operatorname{supp} f \text{ is compact}\}\,.$$

$C_0^{p,q}(X, E)$ is provided with the structure of a pre-Hilbert space with a norm $\|f\|=\sqrt{(f,f)}$. When we need to be more preciese, we denote $\|f\|$ by $\|f\|_{h,ds^2}$.

Combining Theorem 1.4 with arithmetic-geometric inequality we obtain

Theorem 1.5. *Under the above notations,*

$$(20)\quad \frac{3}{2}(\|\vartheta_E f\|^2+\|\bar{\partial} f\|^2)$$
$$\geqq(-\sqrt{-1}(D_E^2\Lambda-\Lambda D_E^2)f, f)$$
$$-\frac{1}{2}(\|T_1 f\|^2+\|T_1^* f\|^2+\|T_2 f\|^2+\|T_2^* f\|^2),\quad f\in C_0^{p,q}(X, E).$$

Here T_i^* *denote the adjoints of* T_i.

D_E^2 is a multiplication of a Hom(E, E)-valued $(1, 1)$-form. We set

$$(21)\qquad -D_E^2=e(\Theta_h),\qquad \Theta_h\in C^{1,1}(X, \mathrm{Hom}(E, E)).$$

Θ_h is called *the curvature form* of E with respect to h. Then (20) becomes

$$(22)\qquad \frac{3}{2}(\|\vartheta_E f\|^2+\|\bar{\partial} f\|^2)\geqq\sqrt{-1}((e(\Theta_h)\Lambda-\Lambda e(\Theta_h))f, f)$$
$$-\frac{1}{2}(\|T_1 f\|^2+\|T_1^* f\|^2+\|T_2 f\|^2+\|T_2^* f\|^2).$$

T_i and T_i^* are called *the torsions* of ds^2. In what follows we call ds^2 a *hermitian metric on* X, and if $d\omega=0$ we say that ds^2 *is* (or simply X *is*) *Kählerian.*

§2. L^2 Estimates of $\bar{\partial}$

Let H_1 and H_2 be two Hilbert spaces with norms $\|\ \|_i$, $i=1, 2$ and T: $H_1\to H_2$ a linear operator with a domain of definition D_T. T is called a *closed operator* if $G_T:=\{(u, Tu); u\in D_T\}$ is a closed subspace of $H_1\times H_2$. In what follows every linear operator is assumed to be closed and with a dense domain. T^*: $H_2\to H_1$, *the adjoint of* T, is defined as follows;

$$(1)\qquad G_{T^*}=\{(v, w); (u, w)_1=(Tu, v)_2\quad \text{for any}\quad u\in D_T\},$$

where $(\ ,\ )_i$ denote the inner products of H_i. We denote by R_T (resp. N_T) the range (resp. the kernel) of T. The closure of R_T is denoted by $[R_T]$.

Assume that we are given another Hilbert space H_3 with a norm $\|\ \|_3$ and a linear operator S: $H_2\to H_3$ such that

$$(2)\qquad S\circ T=0.$$

Then $N_S\supset R_T$ and $N_{T^*}\supset R_{S^*}$. (2) implies that R_T and R_{S^*} are orthogonal, and the intersection of the orthogonal complements of these spaces is $\mathscr{H}:=N_S\cap N_{T^*}$. Hence we have $H_2=\mathscr{H}\oplus[R_T]\oplus[R_{S^*}]$.

Theorem 2.1 (cf. Theorem 1.1.3 in [10]). *Assume that from every sequence* $g_k\in D_{T^*}\cap D_S$ *with* $\|g_k\|_2$ *bounded and* $T^*g_k\to 0$ *in* H_1, $Sg_k\to 0$ *in* H_3, *one can* *select a strongly convergent subsequence. Then both R_T and R_S are closed, and $\mathscr{H}$ is finite dimensional.*

Elements of $\mathscr{H}$ are said to be harmonic.

Theorem 2.2 (cf. Theorem 1.1.4 in [10]). *Let F be a closed subspace of H_2 containing R_T. Assume that*

(3) $$\|f\|_2^2 \leqq \|T^*f\|_1^2 + \|Sf\|_3^2, \quad f \in D_{T^*} \cap D_S \cap F,$$

then we have

(4) *If $g \in N_S \cap F$, we can find $u \in D_T$ so that $Tu = g$ and $\|u\|_1 \leqq \|h\|_2$.*

(5) *If $v \in R_{T^*}$, we can choose $f \in D_{T^*}$ so that $T^*f = v$ and $\|f\|_2 \leqq \|v\|_1$.*

Let the notations be as in Section 1. We denote by $L^{p,q}(X, E)$ the space of square integrable E-valued (p, q)-forms on X with respect to the metrics h and ds^2. By Riesz-Fischer's theorem $L^{p,q}(X, E)$ is naturally identified with the completion of $C_0^{p,q}(X, E)$ with respect to the norm $\|f\|$. When we need to be more precise we denote $L^{p,q}(X, E)$ by $L^{p,q}(X, E, h, ds^2)$. $\bar\partial$ and other differential operators introduced in Section 1 are naturally extended to closed linear operators on $L^{p,q}(X, E)$ with dense domains

$$D_{\bar\partial} = \{f \in L^{p,q}(X, E); \text{ there is a } g \in L^{p,q+1}(X, E) \text{ such that } (f, \vartheta_E \varphi) = (g, \varphi) \text{ for any } \varphi \in C_0^{p,q+1}(X, E)\}.$$

and so on. ϑ_E, so extended, is called *the formal adjoint* of $\bar\partial$. When we need to be more precise we denote $D_{\bar\partial}$, $R_{\bar\partial}$, and $N_{\bar\partial}$ by $D_{\bar\partial}^{p,q}$ or $D_{\bar\partial}^{p,q}(h, ds^2)$, and so on.

In general $D_{\bar\partial^*} \subsetneqq D_{\vartheta_E}$ because of the presence of the boundary of X. To avoid this technical difficulty we provide with X a complete hermitian metric. This viewpoint is due to Andreotti-Vesentini [4].

Definition 2.3. We say a hermitian metric ds^2 is complete if the distance function on X with respect to ds^2 provides X with a structure of a complete metric space.

Theorem 2.4 (cf. Theorem 1.1 in [30] p. 22). *If the hermitian metric ds^2 is complete, then*

(6) $C_0^{p,q}(X, E)$ is dense in $D_{\bar\partial}^{p,q}$ with respect to the norm $(\|f\|^2 + \|\bar\partial f\|^2)^{1/2}$.

(7) $C_0^{p,q}(X, E)$ is dense in $D_{\vartheta_E}^{p,q}$ with respect to the norm $(\|f\|^2 + \|\vartheta_E f\|^2)^{1/2}$

(8) $C_0^{p,q}(X, E)$ is dense in $D_{\bar\partial}^{p,q} \cap D_{\vartheta_E}^{p,q}$ with respect to the norm
$$(\|f\|^2+\|\bar\partial f\|^2+\|\vartheta_E f\|^2)^{1/2}.$$

Corollary 2.5. *If the hermitian metric ds^2 is complete, then $\bar\partial^*=\vartheta_E$ and $D_E'^*=\bar\vartheta$.*

Proof. Clearly $D_{\bar\partial^*}\subset D_{\vartheta_E}$. Let $f\in D_{\vartheta_E}$. Then, by Theorem 2.4, there is a sequence f_k in $C_0^{p,q}(X, E)$ such that $\|f_k-f\|+\|\vartheta_E f_k-\vartheta_E f\|\to 0$. Hence, for any $\varphi\in D_{\bar\partial}$, we have

$$(9)\qquad (\vartheta_E f, \varphi)=\lim_{k\to\infty}(\vartheta_E f_k, \varphi)=\lim_{k\to\infty}(f_k, \bar\partial\varphi)=(f, \bar\partial\varphi).$$

Therefore, $f\in D_{\bar\partial^*}$ and $\vartheta_E f=\bar\partial^* f$. The proof of $D_E'^*=\bar\vartheta$ is similar. q.e.d.

Let

$$(10)\qquad \varphi=\varphi_{i_1\cdots i_p}\tau_1\wedge\cdots\wedge\tau_n\wedge\bar\tau_{i_1}\wedge\cdots\wedge\bar\tau_{i_p},$$

where $(\tau_1,\dots,\tau_n)$ is an orthonormal basis of $T'_{X,x}$ with respect to ds^2. Then we have

$$(11)\qquad \langle\sqrt{-1}e(\tau_i\wedge\bar\tau_i)\Lambda\varphi, \varphi\rangle=0 \qquad \text{if}\quad i\notin\{i_1,\dots,i_p\}$$

and

$$(12)\qquad \langle\sqrt{-1}e(\tau_{i_\alpha}\wedge\bar\tau_{i_\alpha})\Lambda\varphi, \varphi\rangle=\langle\varphi, \varphi\rangle \qquad \text{for}\quad 1\leqq\alpha\leqq p.$$

Hence, from (22) in Section 1, we obtain

Proposition 2.6. *Suppose that the sums of q eigenvalues of $\partial\bar\partial\Phi$ with respect to ds^2 are bigger than a nonnegative function λ. Then,*

$$(13)\qquad \frac{3}{2}(\|\bar\partial^* f\|^2+\|\bar\partial f\|^2)\geqq\sqrt{-1}(e(\Theta_h)\Lambda f, f)+(\lambda f, f)$$
$$-\frac{1}{2}(\|T_1 f\|^2+\|T_1^* f\|^2+\|T_2 f\|^2+\|T_2^* f\|^2),\quad f\in C_0^{n,p}(X, E),\ p\geqq q.$$

Similarly we have

Proposition 2.7. *Suppose that the sums of q eigenvalues of $\partial\bar\partial\Phi$ with respect to ds^2 are less than a nonpositive function λ. Then,*

$$(14)\qquad \frac{3}{2}(\|\bar\partial^* f\|^2+\|\bar\partial f\|^2)\geqq-\sqrt{-1}(\Lambda e(\Theta_h)f, f)-(\lambda f, f)$$
$$-\frac{1}{2}(\|T_1 f\|^2+\|T_1^* f\|^2+\|T_2 f\|^2+\|T_2^* f\|^2),$$
$$f\in C_0^{0,p}(X, E),\ p\leqq n-q.$$

In view of Theorem 2.2 and Theorem 2.4, the meaning of these propositions will be clear.

Chapter 2. Isomorphism Theorems for Pseudo-Runge Pairs

§1. Basic Estimates

Let X be an n-dimensional complex manifold with a *complete* hermitian metric ds^2, and let E be a holomorphic vector bundle over X with a hermitian metric h along the fibers.

Definition 1.1. We say that the basic estimate holds at bi-degree (p, q) if there exist a compact subset $K \subset X$ and a constant C_0, satisfying

$$(1) \qquad \|f\|^2 \leqq C_0\{\|\bar{\partial}^* f\|^2 + \|\bar{\partial} f\|^2 + \int_K \langle f, f\rangle dv\}, \qquad f \in D_{\bar{\partial}^*}^{p,q} \cap D_{\bar{\partial}}^{p,q}.$$

K is called an exceptional set and C_0 is called a subellipticity constant.

Proposition 1.2. *Assume that the basic estimate holds at bi-degree (p, q). Then both $R_{\bar{\partial}}^{p,q}$ and $R_{\bar{\partial}^*}^{p,q-1}$ are closed, and* $\dim N_{\bar{\partial}}^{p,q}/R_{\bar{\partial}}^{p,q} < \infty$.

Proof. In view of Theorem 2.1 in Chapter 1, we have only to show that from any sequence $g_k \in D_{\bar{\partial}^*}^{p,q} \cap D_{\bar{\partial}}^{p,q}$ with $\|g_k\|$ bounded and $\|\bar{\partial}^* g_k\| \to 0$ in $L^{p,q-1}(X, E)$, $\|\bar{\partial} g_k\| \to 0$ in $L^{p,q+1}(X, E)$, one can select a strongly convergent subsequence.

By the completeness of the metric ds^2, we can take a sequence $f_k \in C_0^{p,q}(X, E)$ so that

$$(2) \qquad \begin{aligned} &\|f_k - g_k\| < 1/k, \\ &\bar{\partial} f_k \to 0 \quad \text{in} \quad L^{p,q+1}(X, E), \\ &\bar{\partial}^* f_k \to 0 \quad \text{in} \quad L^{p,q-1}(X, E) \end{aligned}$$

(cf. Theorem 2.4 in Chapter 1).

Let K_1 be any compact subset of X and K_2 another compact subset of X containing K_1 in the interior. Let $\chi: X \to \mathbf{R}$ be a C^∞ function satisfying $\chi = 1$ on K_1 and $\chi = 0$ on $X - K_2$. Since

$$((\bar{\partial}\vartheta_E + \vartheta_E \bar{\partial}) f_k, f_k) + (f_k, f_k)$$

is bounded,

$$((\bar{\partial}\vartheta_E + \vartheta_E \bar{\partial})(\chi f_k), \chi f_k) + (\chi f_k, \chi f_k)$$

is also bounded. Since $\bar\partial\vartheta_E+\vartheta_E\bar\partial$ is a strongly elliptic operator, in virtue of Gårding's inequality (cf. Theorem 6.5.1 in [16]) and Rellich's lemma (cf. Theorem 3.4.4 in [16]), χf_k has a strongly convergent subsequence.

Hence f_k has a subsequence f_{i_k} converging strongly on K. By the basic estimate, f_{i_k} converges strongly on X, hence so does g_{i_k}. q.e.d.

§ 2. Pseudo-Runge Pairs and an Approximation Theorem

We shall present here an abstract form of Proposition 6 in [24] (cf. also Approximation theorem in [29]).

Let X be a complex manifold and E a holomorphic vector bundle over X. Let X_1 and X_2 be two open subsets of X.

Definition 2.1. The pair (X_1, X_2) is called a pseudo-Runge pair at bi-degree (p, q) with respect to E, if $X_1\subset X_2$ and there exist a complete hermitian metric ds_0^2 on X_1, a hermitian metric h_0 along the fibers of $E|_{X_1}$, a sequence of complete hermitian metrics ds_k^2 $(k=1, 2,\dots)$ on X_2, and a sequence of hermitian metrics h_k along the fibers of $E|_{X_2}$, satisfying the following properties;

(*) ds_k^2, h_k, and their derivatives converge on every compact subset of X_1 uniformly to ds_0^2, h_0, and to their derivatives, respectively.

(**) The basic estimates hold with respect to ds_k^2 and h_k at bi-degree $(p, q+1)$ with a common subellipticity constant and a common exceptional set contained in X_1.

(***) $L^{p,q}(X_2, E, h_k, ds_k^2)\subset L^{p,q}(X_2, E, h_{k+1}, ds_{k+1}^2)$ and there is a constant C_2 independent of k such that

$$\|\varphi|_{X_1}\|_{h_0,ds_0^2}\leqq C_2\|\varphi\|_{h_k,ds_k^2},$$
$$\varphi\in C_0^{p,q+i}(X_2, E|_{X_2}),\quad i=0, 1.$$

For simplicity we set

$$C_0^{p,q}(X_1, E)=C_0^{p,q}(X_1, E|_{X_1}), \tag{1}$$

and so on.

Note that under the above conditions the basic estimate holds with respect to h_0 and ds_0^2, too.

The following lemma is essentially the same as Proposition 3.4.5 in [10].

Lemma 2.2. *Let (X_1, X_2) be a pseudo-Runge pair at bi-degree (p, q) with* *respect to E. Let h_0, ds_0^2, h_k, and ds_k^2 be chosen as above. Then there is an integer k_0 and a constant C_4 such that for any $k \geqq k_0$,*

$$C_4(\|\bar{\partial}^* f\|^2 + \|\bar{\partial} f\|^2) \geqq \|f\|^2, \tag{2}$$

for any $f \in L^{p,q+1}(X_2, E, h_k, ds_k^2)$ satisfying $f|_{X_1} \perp N_{\bar{\partial}} \cap N_{\bar{\partial}^*}$.

Proof. Assume that the assertion is false. Then there is a sequence f_k satisfying

$$f_k \in L^{p,q+1}(X_2, E, h_k, ds_k^2) \tag{3}$$

$$\|f_k\| = 1 \tag{4}$$

$$\lim_{m \to \infty} \inf_{k \geqq m} \|\bar{\partial}^* f_k\| = 0 \tag{5}$$

$$\lim_{m \to \infty} \inf_{k \geqq m} \|\bar{\partial} f_k\| = 0 \tag{6}$$

and

$$f_k|_{X_1} \perp N_{\bar{\partial}} \cap N_{\bar{\partial}^*}. \tag{7}$$

Choosing a subsequence if necessary, we may assume that

$$\|\bar{\partial}^* f_k\| < \frac{1}{k} \tag{8}$$

and

$$\|\bar{\partial} f_k\| < \frac{1}{k}. \tag{9}$$

Then there is a subsequence of $f_k|_{X_1}$ weakly convergent in $L^{p,q+1}(X_1, E)$ and strongly convergent on a common exceptional set K of the basic estimates. Let the weak limit be f. Then f must be zero. In fact, by (7) we have $f \perp N_{\bar{\partial}} \cap N_{\bar{\partial}^*}$, and the completeness of the metric ds_0^2 implies that we have both $\bar{\partial} f = 0$ and $\bar{\partial}^* f = 0$.

On the other hand, combining (8) and (9) with the basic estimates,

$$\int_K \langle f_k, f_k \rangle dv_k > \frac{1}{C_5} - \frac{2}{k^2}, \tag{10}$$

where C_5 is a positive number and dv_k denotes the volume form with respect to ds_k^2. From the strong convergence of f_k on K, we obtain

$$\int_K \langle f, f \rangle dv_0 \geqq \frac{1}{C_5}, \tag{11}$$

where dv_0 denotes the volume form for ds_0^2. Therefore $f \neq 0$. A contradiction!

q. e. d.

Theorem 2.3 (*Approximation theorem*). *Let (X_1, X_2) be a pseudo-Runge pair at bi-degree (p, q) with respect to E. Let h_0, ds_0^2, h_k, and ds_k^2 be chosen as above, and let $f \in N_{\bar\partial}^{p,q}(h_0, ds_0^2)$. Then, for any positive number ε, there exist an integer k_0 and an $\tilde{f} \in N_{\bar\partial}^{p,q}(X_2, E, h_{k_0}, ds_{k_0}^2)$ satisfying*

$$\|\tilde{f}\,|_{X_1} - f\| < \varepsilon. \tag{12}$$

Proof. In virtue of Hahn-Banach's theorem, we have only to prove the following assertion:

Let $u \in L^{p,q}(X_1, E)$ and

$$(u, g\,|_{X_1}) = 0, \quad g \in N_{\bar\partial}^{p,q}(h_k, ds_k^2), \quad k = 1, 2, \dots, \tag{13}$$

then we have

$$(u, f) = 0, \quad f \in N_{\bar\partial}^{p,q}(h_0, ds_0^2). \tag{14}$$

To prove the assertion we observe first that by the assumption (***),

$$|(u, v\,|_{X_1})| \leqq C_2 \|u\| \cdot \|v\|. \tag{15}$$

Hence $(u, \cdot\,|_{X_1})$ is continuous on $L^{p,q}(X_2, E, h_k, ds_k^2)$ and its norm does not exceed $C_2\|u\|$. From the Riesz representation theorem there is a $u_k \in L^{p,q}(X_2, E, h_k, ds_k^2)$ such that

$$\begin{cases} (u_k, \cdot) = (u, \cdot\,|_{X_1}) \\ \|u_k\| \leqq C_2 \|u\|. \end{cases} \tag{16}$$

Clearly $u_k = 0$ on $X_2 - X_1$, so that u_k are orthogonal to $N_{\bar\partial}^{p,q}(h_k, ds_k^2)$. Furthermore

$$\|u_k\,|_{X_1}\| \leqq C_2^2 \|u\|. \tag{17}$$

In fact we have

$$|(u_k\,|_{X_1}, w)| \leqq C_2 \|u_k\| \cdot \|w\| \leqq C_2^2 \|u\| \cdot \|w\|. \tag{18}$$

On the other hand, from (*) combined with (16), we have

$$(u_k\,|_{X_1}, \varphi) \longrightarrow (u, \varphi), \tag{19}$$

for any $\varphi \in C_0^{p,q}(X_1, E)$. (19), combined with (16), implies that

$$(u_k\,|_{X_1}, f) \longrightarrow (u, f), \qquad \text{for any} \quad f \in L^{p,q}(X_1, E). \tag{20}$$

Since u_k are orthogonal to $N_{\bar\partial}^{p,q}(h_k, ds_k^2)$,

$$u_k \in [R_{\bar\partial^*}^{p,q}(h_k, ds_k^2)]. \tag{21}$$

Hence, in virtue of Lemma 2.2 and Theorem 2.2 in Chapter 1, there exist a constant C_3 and $w_k \in L^{p,q+1}(X_2, E, h_k, ds_k^2)$ satisfying

$$\text{(22)} \qquad \begin{cases} \bar{\partial}^* w_k = u_k \\ \|w_k\| \leqq C_3 \|u_k\| . \end{cases}$$

Since

$$\text{(23)} \qquad \|w_k|_{X_1}\| \,(\leqq C_2 \|w_k\| \leqq C_2 C_3 \|u_k\|) \leqq C_2^2 C_3 \|u\| ,$$

there is a subsequence of $w_k|_{X_1}$ converging weakly in $L^{p,q+1}(X_1, E)$. Let the weak limit be w. Then

$$\text{(24)} \qquad (w, \bar{\partial}\varphi) = (u, \varphi), \qquad \varphi \in C_0^{p,q}(X_1, E).$$

Therefore, in virtue of Corollary 2.5 in Chapter 1, we obtain

$$\text{(25)} \qquad \bar{\partial}^* w = u.$$

Consequently,

$$\text{(26)} \qquad (u, f) = 0, \qquad \text{for any} \quad f \in N_{\bar{c}}^{p,q}(h_0, ds_0^2). \qquad \text{q.e.d.}$$

§ 3. Isomorphism Theorems

In this paragraph we shall prove isomorphism theorems for pseudo-Runge pairs. First we recall a fundamental fact about cohomology groups. Notations are as in Chapter 1.

Let $H^q(X, \Omega^p(E))$ be the q-th cohomology group of X with coefficients in the sheaf of holomorphic sections of $E \otimes \overset{p}{\wedge} T'_X$. Let $L_{\mathrm{loc}}^{p,q}(X, E)$ be the space of locally square integrable E-valued (p, q)-forms on X. $L_{\mathrm{loc}}^{p,q}(X, E)$ is naturally identified with the completion of $C^{p,q}(X, E)$ with respect to the semi-norms

$$p_K(\psi) = \left(\int_K \langle \psi, \psi \rangle dv \right)^{1/2}, \quad \psi \in C^{p,q}(X, E),$$

where K runs through the compact subsets of X.

Proposition 3.1.

(1) $\quad H^q(X, \Omega^p(E))$
$\cong \{f \in L_{\mathrm{loc}}^{p,q}(X, E);\ \bar{\partial} f = 0\} / \{g \in L_{\mathrm{loc}}^{p,q}(X, E);$ *there is an* $h \in L_{\mathrm{loc}}^{p,q-1}(X, E)$ *satisfying* $\bar{\partial} h = g\}$.

Here $\bar{\partial} u = v$ *should read*

(2) $\quad (u, \vartheta_E \varphi) = (v, \varphi)$ *for any* $\varphi \in C_0^{p,q}(X, E)$, $p \geqq 0$, $q \geqq 0$.

Proof. Let $\mathscr{U}=\{U_i\}$ be a locally finite open cover of X such that U_i are biholomorphic to a polydisc. By solving the $\bar{\partial}$-equation for the domains of holomorphy in $\mathbf{C}^n$ (cf. [10] or [11]), we have

$$\begin{aligned}(3)\qquad &\{f\in L^{0,q}_{\mathrm{loc}}(U_{i_1\cdots i_p}, E);\ \bar{\partial}f=0\}\\ &=\{g\in L^{0,q}_{\mathrm{loc}}(U_{i_1\cdots i_p}, E);\ \text{there is an } h\in L^{0,q-1}_{\mathrm{loc}}(U_{i_1\cdots i_p}, E)\\ &\qquad \text{satisfying } \bar{\partial}h=g\},\ p\geqq 1,\ q\geqq 1\,.\end{aligned}$$

Here we put $U_{i_1\cdots i_p}=U_{i_1}\cap\cdots\cap U_{i_p}$. It is easy to see that the equality (3) implies the equivalence (2).

Theorem 3.2 (*Weak isomorphism theorem*). *Let (X_1, X_2) be a pseudo-Runge pair with respect to E, at bi-degrees (p, q) and $(p, q+1)$. Then there is an integer k_0 such that the natural restriction maps*

$$\begin{aligned}(4)\qquad \rho_k\colon &N^{p,q+1}_{\bar{\partial}}(h_k, ds_k^2)/R^{p,q+1}_{\bar{\partial}}(h_k, ds_k^2)\longrightarrow\\ &N^{p,q+1}_{\bar{\partial}}(h_0, ds_0^2)/R^{p,q+1}_{\bar{\partial}}(h_0, ds_0^2),\quad k\geqq k_0\end{aligned}$$

are bijective.

Proof. Since the basic estimate holds at bi-degree $(p, q+1)$, we have

$$(5)\qquad \dim N^{p,q+1}_{\bar{\partial}}(h_0, ds_0^2)/R^{p,q+1}_{\bar{\partial}}(h_0, ds_0^2)<\infty\,.$$

Hence we have only to show that, for sufficiently large k, ρ_k is injective and the image of ρ_k is dense. The density of the images follows from Theorem 2.3. On the other hand, the injectivity follows at once from Lemma 2.2. q.e.d.

Given a pseudo-Runge pair (X_1, X_2), we set for simplicity

$$(6)\qquad \begin{cases} N^{p,q}(X_1)=N^{p,q}_{\bar{\partial}}(h_0, ds_0^2)\\ R^{p,q}(X_1)=R^{p,q}_{\bar{\partial}}(h_0, ds_0^2)\,.\end{cases}$$

Theorem 3.3 (*Isomorphism theorem*). *Let X be a complex manifold, let E be a holomorphic vector bundle over X, and let D be an open subset of X. Assume that there exist a family $X_k (k=1, 2,\dots)$ of open subsets of X and a family D_k of open subsets of D such that*

$$(7)\qquad X=\bigcup_{k\geqq 1} X_k,\qquad X_k\Subset X_{k+1}$$

$$(8)\qquad D=\bigcup_{k\geqq 1} D_k,\qquad D_k\Subset D_{k+1}$$

$$(9)\qquad X_1=D_1$$

$$(10)\qquad X_2=D$$

(11) *(X_k, X_{k+2}) is a pseudo-Runge pair at bi-degrees (p, q) and $(p, q+1)$ with respect to E.*

(12) *(D_k, D_{k+1}) is a pseudo-Runge pair at bi-degrees (p, q) and $(p, q+1)$ with respect to E.*

Then the natural restriction map

$$\rho: H^{q+1}(X, \Omega^p(E)) \longrightarrow H^{q+1}(D, \Omega^p(E))$$

is bijective. Moreover, $H^{q+1}(X, \Omega^p(E))$ and $H^{q+1}(D, \Omega^p(E))$ are finite dimensional.

Proof. Injectivity: Let $f \in L^{p,q+1}_{\mathrm{loc}}(X, E)$, $\bar\partial f=0$, and $f|_D=\bar\partial g$ for some $g \in L^{p,q}_{\mathrm{loc}}(D, E)$. By the pseudo-Rungeness of (X_1, X_3) and Theorem 3.2, there is a $g_1 \in L^{p,q}_{\mathrm{loc}}(X_3, E)$ satisfying $\bar\partial g_1 = f|_{X_3}$. Thus, inductively we obtain $g_k \in L^{p,q}_{\mathrm{loc}}(X_{k+2}, E)$ satisfying $\bar\partial g_k = f|_{X_{k+2}}$. By Theorem 2.3, there is a sequence $f_k \in L^{p,q}_{\mathrm{loc}}(X_{k+3}, E)$ satisfying

$$\text{(13)} \qquad \begin{cases} \bar\partial f_k = 0 \\ \|g_{k+1}|_{X_k} - g_k|_{X_k} + f_{k-1}|_{X_k} - f_k|_{X_k}\| < \dfrac{1}{2^k}, \end{cases}$$

where we put $f_0=0$, and the norm is taken with respect to a fixed metrics on X and E. Hence we can define $\tilde g \in L^{p,q}_{\mathrm{loc}}(X, E)$ by putting

$$\text{(14)} \qquad \tilde g = g_k - f_{k-1} + \sum_{m=1}^{\infty} (g_{k+m} - g_{k+m-1} + f_{k+m-2} - f_{k+m-1})$$

on X_{k+2}. Clearly, $\bar\partial \tilde g = f$.

Note that by the same argument we can prove that, in the following triangle of the natural restriction maps,

$$\text{(15)} \qquad \begin{tikzcd} H^{q+1}(X, \Omega^p(E)) \arrow[d, "\rho"] \arrow[dr, "\rho^X"] & \\ H^{q+1}(D, \Omega^p(E)) \arrow[r, "\rho^D"'] & N^{p,q+1}(X_1)/R^{p,q+1}(X_1) \end{tikzcd}$$

ρ^X and ρ^D are injective, too. Hence by Proposition 1.2 $\dim H^{q+1}(X, \Omega^p(E)) < \infty$ and $\dim H^{q+1}(D, \Omega^p(E)) < \infty$.

Surjectivity: Let $f \in L^{p,q+1}_{\mathrm{loc}}(D, E)$ and $\bar\partial f=0$. By Theorem 2.2, for any $\varepsilon>0$, there is an $f_1 \in L^{p,q+1}_{\mathrm{loc}}(X_3, E)$ satisfying

$$\text{(16)} \qquad \begin{cases} \bar\partial f_1 = 0 \\ \|f_1|_{X_1} - f|_{X_1}\| < \dfrac{\varepsilon}{2}. \end{cases}$$

Thus, inductively we can choose $f_k \in L^{p,q+1}_{\text{loc}}(X_{k+2}, E)$ satisfying

$$(17)\qquad \begin{cases} \bar\partial f_k = 0 \\ \| f_k | X_k - f_{k-1} | X_k \| < \dfrac{\varepsilon}{2^k}, \end{cases}$$

where we put $f_0 = f$. Hence we can define $\tilde f \in L^{p,q+1}_{\text{loc}}(X, E)$ by putting

$$(18)\qquad \tilde f = f_k + \sum_{m \geqq k} (f_{m+1} - f_m) \qquad \text{on} \quad X_{k+2}.$$

Clearly, $\tilde f$ satisfies

$$(19)\qquad \begin{cases} \bar\partial \tilde f = 0 \\ \| \tilde f | X_1 - f | X_1 \| < \sum_{k=1} \dfrac{\varepsilon}{2^k} = \varepsilon. \end{cases}$$

Therefore the image of ρ^X is dense in the image of ρ^D. Combining this with the injectivity of ρ^D and the finite dimensionality of $H^{q+1}(D, \Omega^p(E))$, we obtain the surjectivity of ρ. q.e.d.

§4. Examples of Pseudo-Runge Pairs

In accordance with Andreotti-Grauert [3], we adopt the following

Definition 4.1. Let X be a complex manifold of dimension n, and let q be a positive integer. X is said to be strongly q-pseudoconvex (resp. strongly q-pseudoconcave) if there is a real-valued C^∞ function Φ on X satisfying

(1) $X_c := \{x; \Phi(x) < c\} \Subset X$ or $= X$, for any c,

(2) the Levi form of Φ has at least $n-q+1$ positive (resp. $n-q+1$ negative) eigenvalues outside a compact subset K of X.

We call Φ an exhaustion function and K an exceptional set.

Note that X_c is also strongly q-pseudoconvex (resp. strongly q-pseudoconcave) if $X_c \supset K$.

Theorem 4.2. *Let X be a strongly q-pseudoconvex (resp. strongly q-pseudoconcave) manifold of dimension n with an exhaustion function Φ and an exceptional set K, and let E be a holomorphic vector bundle over X. Let $c < d$, $X_c \supset K$, and $X_d \Subset X$. Then, the pair (X_c, X_d) is a pseudo-Runge pair with respect to E at bi-degrees (n, p), $p \geqq q-1$ (resp. at bi-degrees $(0, p)$, $p \leqq n-q-2$).*

For the proof we need the following lemmas.

Lemma 4.3 (cf. Lemma 4.1 in [30] § 12). *Let X be a paracompact complex manifold of dimension n, and let $\mathscr{L}$ be a real C^∞ (1, 1)-form having at least $n-q+1$ positive eigenvalues. Then, for any positive integer N, there is a hermitian metric ds^2 on X such that with respect to ds^2,*

(3) *at least $n-q+1$ eigenvalues of $\mathscr{L}$ are bigger than N*

and

(4) *negative eigenvalues of $\mathscr{L}$ are bigger than $-1/N$.*

Proof. By the continuity of the eigenvalues it is clear that there is a continuous hermitian metric on X satisfying (3) and (4). Hence there is a C^∞ hermitian metric satisfying (3) and (4). q.e.d.

Lemma 4.4. *There is a sequence $\lambda_k(t)\,(k=1, 2,\ldots)$ of C^∞ functions on $(-\infty, d)$, satisfying the following conditions.*

$$\lambda_k(t)=\frac{1}{\left(c+\frac{1}{k}-t\right)^2}+11 \quad \text{on} \quad (-\infty, c)\,, \tag{5}$$

$$\{\lambda_k'(t)\}^2<11\{\lambda_k(t)\}^3\,, \tag{6}$$

$$\int_{d-1}^{d}\sqrt{\lambda_k(t)}\,dt=\infty\,, \tag{7}$$

$$\lambda_k(t)>11,\ \lambda_k'(t)\geqq 0\,. \tag{8}$$

Proof. We put

$$\eta_k(t)=\begin{cases}\dfrac{1}{\left(c+\frac{1}{k}-t\right)^2}+11 & \text{on} \quad (-\infty, c] \\[2ex] 2k^3(t-c)+k^2+11 & \text{on} \quad \left(c, \dfrac{c+d}{2}\right] \\[2ex] \dfrac{(d-c)^3k^3}{8(d-t)^2}+\dfrac{(d-c)k^3}{2}+k^2+11 & \text{on} \quad \left(\dfrac{c+d}{2}, d\right).\end{cases} \tag{9}$$

It is clear that $\eta_k(t)$ are differentiable and satisfy (5) to (8). Hence, approximating $\eta_k(t)$ by C^∞ functions, we obtain a sequence $\lambda_k(t)$ of C^∞ functions satisfying (5) to (8). q.e.d.

Lemma 4.5.

$$(c-t)^{-m}e^{-\frac{1}{c-t}}<2^m m!\left(c+\frac{1}{k}-t\right)^{-m}\cdot e^{-\frac{1}{c+\frac{1}{k}-t}}\,, \tag{10}$$

$$k=1, 2,\ldots,\quad m=0, 1,\ldots,\quad t<c\,,$$

where $e=\lim_{k\to\infty}(1+k^{-1})^k$.

Proof. Left to the reader.

Proof of Theorem 4.2: Let ds^2 be a hermitian metric on X and h a hermitian metric along the fibers of E.

Pseudoconvex case: Let X be strongly q-pseudoconvex. We may assume that outside $X_{c'}$, such that $c'<c$ and $X_{c'}\supset K$, at least $n-q+1$ eigenvalues of $\partial\bar\partial\Phi$ are bigger than q and other eigenvalues are bigger than -1. We put

$$\text{(11)}\qquad \begin{cases} ds_k^2=\lambda_k(\Phi)ds^2 \\ h_k=he^{-A\int_{\inf\Phi}^{\Phi}\lambda_k(t)dt}\,. \end{cases}$$

Here $\lambda_k(t)$ is as in Lemma 4.4, and A is a positive number. By (7), ds_k^2 is a complete hermitian metric on X_d (cf. Proposition 1 in [18] or Proposition 3.1 in [25]). Let ω_k be the fundamental form associated to ds_k^2 and let ω be associated to ds^2. Then

$$\text{(12)}\qquad d\omega_k=\lambda_k'(\Phi)d\Phi\wedge\omega+\lambda_k(\Phi)d\omega\,.$$

In virtue of (6), $e(d\omega_k)$ is a bounded operator with respect to the pointwise norm with respect to ds_k^2. Hence by Theorem 1.3 in Chapter 1,

$$\text{(13)}\qquad \begin{cases} \|T_i^k\varphi\|_{h_k,ds_k^2}^2\leqq C_6\|\varphi\|_{h_k,ds_k^2}^2 \\ \|T_i^{k*}\varphi\|_{h_k,ds_k^2}^2\leqq C_6\|\varphi\|_{h_k,ds_k^2}^2 \end{cases}$$
$$i=1,2,\quad k=1,2,\dots,\ \varphi\in C_0^{n,p}(X_d,E),\ p\geqq 0\,.$$

Here, T_i^k and T_i^{k*} are the torsions of ds_k^2, and C_6 is a constant which is independent of A. As for the curvature form of E with respect to h_k, we have

$$\text{(14)}\qquad \Theta_{h_k}=\Theta_h+A\{\lambda_k(\Phi)\partial\bar\partial\Phi+\lambda_k'(\Phi)\partial\Phi\wedge\bar\partial\Phi\}\,.$$

Since the eigenvalues of $\lambda_k'(\Phi)\partial\Phi\wedge\bar\partial\Phi$ are nonnegative and by the choice of ds^2 the sums of q eigenvalues of $\lambda_k(\Phi)\partial\bar\partial\Phi$ with respect to ds_k^2 are bigger than 1 outside $X_{c'}$, we deduce from Proposition 2.6 in Chapter 1 that

$$\text{(15)}\qquad \begin{aligned} &3\{\|\bar\partial^*\varphi\|^2+\|\bar\partial\varphi\|^2\} \\ &\quad\geqq 2\{\sqrt{-1}(e(\Theta_h)\Lambda_k\varphi,\varphi)+A(\varphi,\varphi)\} \\ &\qquad -\{\|T_1^k\varphi\|^2+\|T_1^{k*}\varphi\|^2+\|T_2^k\varphi\|^2+\|T_2^{k*}\varphi\|^2\} \end{aligned}$$

and whence, for sufficiently large A,

$$\text{(16)}\qquad \begin{aligned} &\|\bar\partial^*\varphi\|^2+\|\bar\partial\varphi\|^2 \\ &\quad\geqq\|\varphi\|^2,\quad\text{for any}\quad \varphi\in C_0^{n,p}(X_d-\overline{X_{c'}},E),\ p\geqq q\,. \end{aligned}$$

Here, the inner products and the norms as well as the torsions T_i^k and the adjoints Λ_k of the Lefschetz' operators are defined with respect to ds_k^2 and h_k. We fix such an A.

Let δ_1 be a positive number satisfying $c'<c'+\delta_1<c-\delta_1<c$, and σ a C^∞ function on X_d satisfying

$$\sigma=\begin{cases} 0 & \text{on} \quad X_{c'+\delta_1} \\ 1 & \text{on} \quad X_{c-\delta_1}\,. \end{cases} \tag{17}$$

Applying (16) to $\sigma\varphi$, where $\varphi\in C_0^{n,p}(X_d, E)$, we obtain the following estimate:

$$\begin{aligned} &\|\bar{\partial}^*\varphi\|^2+\|\bar{\partial}\varphi\|^2+C_7\int_{X_{c-\delta_1}}\langle\varphi,\varphi\rangle dv_k \\ &\geqq\|\varphi\|^2, \quad \varphi\in C_0^{n,p}(X_d, E), \quad p\geqq q\,. \end{aligned} \tag{18}$$

Here C_7 is a constant and the norms are with respect to h_k and ds_k^2.

Hence we obtain

$$\begin{aligned} &C_8\left\{\|\bar{\partial}^*f\|^2+\|\bar{\partial}f\|^2+\int_{X_{c-\delta_1}}\langle f,f\rangle dv_k\right\} \\ &\geqq\|f\|^2, \quad f\in D_{\bar{\partial}}^{n,p}(h_k, ds_k^2)\cap D_{\bar{\partial}^*}^{n,p}(h_k, ds_k^2), \quad p\geqq q\,, \end{aligned} \tag{19}$$

for some constant C_8. Therefore the basic estimates hold for ds_k^2 and h_k with common exceptional set $\overline{X_{c-\delta_1}}\subset X_c$ and common subellipticity constant C_8.

By the definition of ds_k^2 and h_k, they converge with their derivatives to hermitian metrics

$$ds_0^2=\frac{ds^2}{(c-\Phi)^2}+11ds^2$$

and

$$h_0=h\exp\left(-\frac{A}{c-\Phi}+\frac{A}{c-\inf\Phi}-11A(\Phi-\inf\Phi)\right),$$

respectively. The completeness of ds_0^2 is clear. So it remains to show that (***) is true. That $L^{n,p}(X_2, E, h_k, ds_k^2)\subset L^{n,p}(X_2, E, h_{k+1}, ds_{k+1}^2)$ is clear. we note that

$$\begin{aligned} &\langle\varphi,\varphi\rangle_{h,ds_0^2}\leqq\langle\varphi,\varphi\rangle_{h,ds_k^2} \qquad \text{on} \quad X_c, \\ &\varphi\in C^{s,t}(X_d, E), \quad s\geqq 0,\ t\geqq 0, \end{aligned} \tag{20}$$

since

$$11+\frac{1}{(c-t)^2}>\lambda_k(t) \quad \text{on} \quad (-\infty, c)\,. \tag{21}$$

Hence, by Lemma 4.5,

$$\langle\varphi, \varphi\rangle_{h_0, ds_0^2} dv_0 \leqq 2^{2n}(2n)!\, \langle\varphi, \varphi\rangle_{h_k, ds_k^2} dv_k \tag{22}$$
$$\text{on} \quad X_c, \qquad \text{for} \quad \varphi\in C^{s,t}(X_d, E), \quad s\geqq 0,\ t\geqq 0\,.$$

Thus (*), (**), and (***) have been verified for the corresponding bi-degrees.

Pseudoconcave case: Assume that X is strongly q-pseudoconcave. We set

$$\begin{cases} ds_k^2 = ds^2 + \lambda_k(\Phi)\partial\Phi\otimes\bar\partial\Phi \\ h_k = h\cdot e^{-B\int_{\inf\Phi}^{\Phi}\lambda_k(t)dt}\,, \end{cases} \tag{23}$$

where B is a positive number.

By (7), ds_k^2 are complete on X_d. Let ω_k and ω be as above. Then

$$d\omega_k = d\omega + \lambda_k(\Phi)\bar\partial\partial\Phi\wedge(\partial\Phi+\bar\partial\Phi)\,. \tag{24}$$

Hence the pointwise norms of T_i^k and T_i^{k*}, $i=1, 2$, with respect to ds_k^2, are bounded by $C_{10}\sqrt{\lambda_k(\Phi)}$ for some constant C_{10}. We may assume that outside some $X_{c'}$ with $c'<c$ and $X_{c'}\supset K$, at least $n-q+1$ eigenvalues of $\partial\bar\partial\Phi$ with respect to ds^2 are less than $-q-3$ and other eigenvalues are less than 1. Let $x\in X_d - X_{c'}$ be any point, let $\gamma_1\geqq\cdots\geqq\gamma_n$ be the eigenvalues of $\partial\bar\partial\Phi$ at x with respect to ds^2, and let $\gamma_1^k\geqq\cdots\geqq\gamma_n^k$ be the eigenvalues of $\partial\bar\partial\Phi$ with respect to ds_k^2 at x. Since the rank of $\lambda_k(\Phi)\partial\Phi\wedge\bar\partial\Phi$ is $\leqq 1$, by the minimum-maximum principle[*] we have

$$\begin{cases} \gamma_n \leqq \gamma_n^k \leqq \gamma_{n-1} \leqq \cdots \leqq \gamma_q^k < 0 \\ \gamma_i^k \leqq \max\{\gamma_1, 0\} < 1, \quad i<q\,. \end{cases} \tag{25}$$

Hence

$$\gamma_n^k \leqq \cdots \leqq \gamma_{q+1}^k < -q-3\,. \tag{26}$$

As for the curvature form, we have

$$\Theta_{h_k} = \Theta_h + B(\lambda_k(\Phi)\partial\bar\partial\Phi + \lambda_k'(\Phi)\partial\Phi\wedge\bar\partial\Phi)\,. \tag{27}$$

By (6) and (8),

$$\frac{\lambda_k'(t)}{\lambda_k(t)} < \sqrt{11\lambda_k(t)} < \lambda_k(t)\,. \tag{28}$$

---

[*] The reader is referred to Courant-Hilbert's book 'Metoden der Mathematischen Physik I, Springer-Verlag, Berlin-Heidelberg-New York', Erstes Kapitel, Paragraph 4.

Hence, if $\Gamma_1 \geqq \cdots \geqq \Gamma_n$ are the eigenvalues of $\lambda_k(\Phi)\partial\bar{\partial}\Phi + \lambda'_k(\Phi)\partial\Phi \wedge \bar{\partial}\Phi$ at x with respect to ds_k^2, we have

$$(29)\qquad \begin{cases} \Gamma_n \leqq \Gamma_{n-1} \leqq \cdots \leqq \Gamma_{q+1} < -(q+2)\lambda_k(\Phi(x)) \\ \Gamma_q \leqq \cdots \leqq \Gamma_2 < \lambda_k(\Phi(x)) \\ \Gamma_1 < 2\lambda_k(\Phi(x)). \end{cases}$$

Therefore, for sufficiently large B, the sums of $q+1$ eigenvalues of $B(\lambda_k(\Phi)\partial\bar{\partial}\Phi + \lambda'_k(\Phi)\partial\Phi \wedge \bar{\partial}\Phi)$ are less than

$$-B\lambda_k(\Phi(x)),$$

at any point $x \in X_d - X_{c'}$. Hence, for sufficiently large B, we deduce from Proposition 2.7 in Chapter 1 that, for any $\varphi \in C_0^{0,p}(X_d - \bar{X}_{c'}, E)$ $(P \leqq n-q-1)$,

$$\begin{aligned}(30)\qquad & 3\{\|\vartheta_k\varphi\|_k^2 + \|\bar{\partial}\varphi\|_k^2\} \\ & \geqq 2\{-\sqrt{-1}(\Lambda_k e(\Theta_h)\varphi, \varphi)_k + B(\lambda_k(\Phi)\varphi, \varphi)_k\} - 4C_{10}^2(\lambda_k(\Phi)\varphi, \varphi)_k \\ & \geqq B(\lambda_k(\Phi)\varphi, \varphi)_k \\ & \geqq \|\varphi\|_k^2 .\end{aligned}$$

Here we denote by ϑ_k, $\| \;\|_k$, Λ_k, and $(\;,\;)_k$ the formal adjoint of $\bar{\partial}$, the norm, the adjoint of the Lefschetz' operator, and the inner product with respect to ds_k^2 and h_k. Hence similarly as in pseudoconvex case, we obtain

$$\begin{aligned}(31)\qquad & C_{11}\left(\|\bar{\partial}^* f\|^2 + \|\bar{\partial} f\|^2 + \int_{X_{c-\delta_2}} \langle f, f\rangle dv_k\right) \\ & \geqq \|f\|^2, \quad f \in D_{\bar{\partial}}^{0,p}(h_k, ds_k^2) \cap D_{\bar{\partial}^*}^{0,p}(h_k, ds_k^2), \quad p \leqq n-q-1,\end{aligned}$$

with a common exceptional set $\overline{X_{c-\delta_2}}$ contained in X_c and a common subellipticity constant C_{11}.

Now ds_k^2 and h_k converge to

$$ds_0^2 = ds^2 + \frac{\partial\Phi \otimes \bar{\partial}\Phi}{(c-\Phi)^2} + 11\partial\Phi \otimes \bar{\partial}\Phi$$

and

$$h_0 = h\exp\left(-\frac{B}{c-\Phi} + \frac{B}{c-\inf\Phi} - 11B(\Phi - \inf\Phi)\right),$$

respectively. Clearly ds_0^2 is complete. Thus (*) and (**) have verified. The verification of (***) is the same as in the pseudoconvex case. q.e.d.

Theorem 4.6. *Let the situations be as in Theorem* 4.2, *and let ds^2 and h be a hermitian metric on X and a hermitian metric along the fibers of E, respectively.*

(32) *If X is strongly q-pseudoconvex and $X_c \supset K$, then the natural restriction maps*

$$\rho_1 \colon H^p(X, \Omega^n(E)) \longrightarrow H^p(X_c, \Omega^n(E)), \quad p \geqq q,$$

are bijective. Furthermore, if $p \geqq q$, $H^p(X, \Omega^n(E))$ are finite dimensional and every cohomology class of $H^p(X_c, \Omega^n(E))$ is represented uniquely by a harmonic form with respect to the following metrics:

$$\begin{cases} ds_1^2 = \dfrac{ds^2}{(c-\Phi)^2} \\ h_1 = h \cdot e^{-\frac{\alpha}{c-\Phi}}, \quad \alpha \gg 0 . \end{cases}$$

(33) *If X is strongly q-pseudoconcave and $X_c \supset K$, then the natural restriction maps*

$$\rho_2 \colon H^p(X, E) \longrightarrow H^p(X_c, E), \quad p \leqq n-q-1$$

are bijective. Furthermore, if $p \leqq n-q-1$, $H^p(X, E)$ are finite dimensional and every cohomology class of $H^p(X_c, E)$ is represented uniquely by a harmonic form with resepect to the following metrics:

$$\begin{cases} ds_2^2 = ds^2 + \dfrac{\partial\Phi \otimes \bar{\partial}\Phi}{(c-\Phi)^2} \\ h_2 = h \cdot e^{-\frac{\alpha}{c-\Phi}}, \quad \alpha \gg 0 . \end{cases}$$

Proof. Combining Theorem 4.2 with Theorem 3.3, we obtain the former parts of (32) and (33). The latter parts follow from the proof of Theorem 4.2, since the two metrics $\lambda_k(\Phi)ds^2$ (resp. $ds^2 + \lambda_k(\Phi)\partial\Phi \otimes \bar{\partial}\Phi$) and $\alpha\lambda_k(\Phi)ds^2$ (resp. $ds^2 + \alpha\lambda_k(\Phi)\partial\Phi \otimes \bar{\partial}\Phi$) are equivalent if $\alpha > 0$, so the closedness of the range of $\bar{\partial}$ is also valid for the above metrics. q.e.d.

Letting $K = \emptyset$ and $c = \inf \Phi - 1$ in the above theorem, we have

Corollary 4.7. *Let X be a strongly q-pseudoconvex manifold with empty exceptional set. Then,*

$$H^p(X, \Omega^n(E)) = 0, \quad p \geqq q,$$

for any holomorphic vector bundle E over X. Moreover, if $X_c \Subset X$, then for any $f \in L^{n,p}(X_c, E, ds_1^2, h_1)$, $p \geqq q$, with $\bar{\partial} f = 0$, there is a $g \in L^{n,p-1}(X_c, E, ds_1^2, h_1)$ with $\bar{\partial} g = f$ and $\|g\| \leqq C\|f\|$, where C is a constant independent of f.

A new feature of Theorem 4.6 is that the harmonic forms representing the cohomology classes need not satisfy any kind of boundary conditions, which was not the case in Hörmander's work [10]. In fact, in virtue of Stampacchia Inequality (cf. Theorem 1.2 in [30]), the completeness of the hermitian metric implies that f is harmonic if and only if $(\bar{\partial}\vartheta_E+\vartheta_E\bar{\partial})f=0$ in the sense described in (2), Section 3.

The advantage of using extended Bochner-Calabi-Nakano formula will be shown in the next chapter.

Chapter 3. Isomorphism Theorems on Weakly 1-Complete Manifolds

In this chapter we shall present several extensions and variations of so called 'vanishing theorems' on weakly 1-complete manifolds. Contrary to Theorem 4.6 in Chapter 2, the working hypothesis is the positivity of the curvature forms of the metrics along the fibers of holomorphic vector bundles. Two different notions of positivity are well known; one is due to S. Nakano [17] and the other is due to P. A. Griffiths [8], [9]. Both of these shall be examined here.

§1. Coarse Isomorphism Theorems

Let X be a complex manifold of dimension n. For a vector space V, $S\lhd V$ shall mean that S is a subspace of V.

Definition 1.1. A holomorphic vector bundle $E\to X$ with a hermitian metric h along the fibers, in short *a hermitian vector bundle* (E, h), is said to be *q-positive* (resp. *q-negative*) if, for any point $x\in X$, there is a subspace $S_x\lhd T_{X,x}$ of dimension $n-q+1$ such that $(h\otimes id)(\Theta_h|_{S_x})$ is a positive definite (resp. negative definite) hermitian form on $E_x\otimes S_x$.

Note that if E is a line bundle, i.e. if the rank of E is 1, then (E, h) is q-positive (resp. q-negative) if and only if the curvature form Θ_h has everywhere at least $n-q+1$ positive eigenvalues (resp. negative eigenvalues). It follows from the definition of strongly q-pseudoconvex manifolds (resp. strongly q-pseudoconcave manifolds) that every holomorphic vector bundle over them is q-positive (resp. q-negative) outside a compact subset.

The following definition is due to Nakano [19].

Definition 1.2. X is said to be *weakly 1-complete* if there is a C^∞ plurisubharmonic function Φ: $X\to\mathbb{R}$ such that X_c: $=\{x;\ \Phi(x)<c\}\Subset X$ for any c. Φ is called an *exhaustion function.*

In what follows, let X be a weakly 1-complete manifold of dimension n with an exhaustion function Φ and a hermitian metric ds^2.

Theorem 1.3. *Let X be a weakly 1-complete manifold and let (B, a) be a hermitian line bundle over X. Assume that (B, a) is q-positive outside a compact subset $K\subset X_c$. Then, for any holomorphic vector bundle E over X, there is an integer m_0 such that $H^p(X_c, E\otimes B^m)$, $m>m_0$, are finite dimensional and the natural restriction maps*

$$H^p(X_c, E\otimes B^m) \longrightarrow H^p(X_d, E\otimes B^m), \qquad m>m_0,\ c>d,$$

are bijective, if $X_d\supset K$ and $p\geqq q$.

As a special case, we have

Corollary 1.4. *Let the situations be as above. If $K=\emptyset$, then for any holomorphic vector bundle $E\to X$, there is an integer m_0 such that*

$$H^p(X_c, E\otimes B^m)=0, \qquad \text{for} \quad p\geqq q \quad \text{and} \quad m>m_0\,.$$

For the proof we need the following

Lemma 1.5. *Let H_1 and H_2 be two hermitian matrices of rank n, and let $\gamma_1\geqq\cdots\geqq\gamma_n$ be the eigenvalues of H_1. Assume that $\gamma_1\geqq\cdots\geqq\gamma_{n-q+1}\geqq\varepsilon>0$ and H_2 is positive semi-definite. Let $v_1,\dots, v_{n-q+1}$ be the eigenvectors of H_1 with $H_1v_i=\gamma_iv_i$, and set*

$$(1) \qquad V=\{v\in\mathbf{C}^n;\ v=\sum_{i=1}^{n-q+1} c_iv_i,\ c_i\in\mathbf{C}\}\,.$$

Then,

$$(2) \qquad \frac{{}^tvH_1v+\varepsilon{}^tvH_2v}{\sum_{\alpha=1}^{n}|v^\alpha|^2+{}^tvH_2v}\geqq\varepsilon, \qquad \text{for} \quad v\in V-\{0\}\,,$$

where we put $v={}^t(v^1,\dots, v^n)$.

Proof. Trivial.

Proof of Theorem 1.3: We have to verify that for sufficiently large m, (X_d, X_c) is a pseudo-Runge pair at bi-degrees (n, p), $p\geqq q-1$, with respect to $K_X^*\otimes E\otimes B^m$, where K_X denotes the canonical bundle of X. Let ds^2 be so chosen that at any point $x\in X-X_{d-\delta}$ the eigenvalues $\gamma_1\geqq\cdots\geqq\gamma_n$ of Θ_a satisfy

$$(3) \qquad \begin{cases} \gamma_i>q, & 1\leqq i\leqq n-q+1 \\ \gamma_i>-1, & n-q+1<i\leqq n\,, \end{cases}$$

where δ is fixed so that $K\subset X_{d-\delta}\subset X_d$, and let h be a hermitian metric along the fibers of $K_X^*\otimes E$.

We set

$$
\begin{cases} ds_k^2=ds^2+\partial\bar\partial\chi_k(\Phi)\,, \\ h_{k,m}=h\cdot a^m\cdot e^{-mq\chi_k(\Phi)}\,, \end{cases} \tag{4}
$$

where we put

$$
\chi_k(t)=\int_{\inf\Phi}^t \lambda_k(u)du\,, \tag{5}
$$

letting λ_k be as in Lemma 4.4 in Chapter 2 except that c and d are interchanged, and where we regard $\partial\bar\partial\chi_k(\Phi)$ naturally as a section of $T'_X\otimes T''_X$.

We have

$$
\Theta_{h_{k,m}}=m\Theta_a+\Theta_h+mq\partial\bar\partial\chi_k(\Phi)\,. \tag{6}
$$

Since $\chi_k(t)$ is a convex increasing function, $\partial\bar\partial\chi_k(\Phi)$ is a positive semi-definite form. Hence by Lemma 1.5, at any point $x\in X_c-X_{d-\delta}$, the eigenvalues $\Gamma_1^k\geqq\cdots\geqq\Gamma_n^k$ of $\Theta_a+q\partial\bar\partial\chi_k(\Phi)$ with respect to ds_k^2 satisfy

$$
\Gamma_i^k>q,\quad 1\leqq i\leqq n-q+1\,. \tag{7}
$$

Clearly, we have

$$
\Gamma_i^k>-1,\quad n-q+1<i\leqq n\,. \tag{8}
$$

On the other hand, letting ω and ω_k be the fundamental forms associated to ds^2 and ds_k^2, respectively, we have

$$
d\omega_k=d\omega\,, \tag{9}
$$

so $e(d\omega_k)$ is bounded. Hence similarly as in the pseudoconvex case of Theorem 4.2 in Chapter 2, we conclude that, for sufficiently large m, (X_d, X_c) is a pseudo-Runge pair with respect to $K_X^*\otimes E\otimes B^m$ at bi-degrees (n, p), $p\geqq q-1$, whence follows the theorem. q.e.d.

The following definition is essentially due to Griffiths [8].

Definition 1.6. A hermitian vector bundle (E, h) over a complex manifold X of dimension n is said to be *weakly q-positive* if, for any point $x\in X$ and for any $v\in E_x-\{0\}$, $h(\Theta_h v)(\bar v)$ is a hermitian form on $T_{X,x}$ having at least $n-q+1$ positive eigenvalues. Here we put $h(\Theta_h v)(\bar v)=e(h(e(\Theta_h)v))\bar v$ for simplicity.

Let $\pi\colon P(E)\to X$ be the bundle of projective spaces associated to $E\to X$. Over $P(E)$ there is a tautological line bundle $L(E)$ whose fiber $L(E)_\xi$ ($\xi\in P(E)$)

is the line $[\xi]\to E_{\pi(\xi)}$. By Leray's theorem, we have a canonical isomorphism

$$H^p(X, \mathscr{S}\otimes E^{(m)})\cong H^p(P(E^*), \pi^*\mathscr{S}\otimes L^m), \quad m\geqq 0, \tag{10}$$

where $L=(L(E^*))^*$ and $\mathscr{S}$ is any locally free coherent analytic sheaf over X.

Proposition 1.7. *If (E, h) is weakly q-positive, then $L\to P(E^*)$, with the induced metric, is q-positive.*

Proof. See (2.36) in [9].

From Proposition 1.7, combined with the canonical isomorphism (10), we obtain the following corollaries to Theorem 1.3.

Corollary 1.8. *Let (E, h) be a hermitian vector bundle over a weakly 1-complete manifold X. Assume that (E, h) is weakly q-positive outside a compact subset $K\subset X_c$. Then, for any holomorphic vector bundle F over X, there is an integer m_0 such that $H^p(X_c, F\otimes E^{(m)})$, $m>m_0$, are finite dimensional and the natural restriction map*

$$H^p(X_c, F\otimes E^{(m)})\longrightarrow H^p(X_d, F\otimes E^{(m)}), \quad c>d,\ m>m_0$$

is bijective, if $p\geqq q$ and $X_d\supset K$.

Corollary 1.9. *Let the situations be as above. If moreover $K=\emptyset$, then for any holomorphic vector bundle F over X, there is an integer m_0 such that*

$$H^p(X_c, F\otimes E^{(m)})=0, \qquad \text{for} \quad p\geqq q,\ m>m_0.$$

Remark. Corollary 1.9, which is along the line of Nakano-Hironaka (cf. [19]), is a generalization of the coarse vanishing theorems for compact complex manifolds obtained by Andreotti-Grauert [3] and Griffiths [8], whose original form is found in Kodaira [13].

As a generalization of the dual of Corollary 1.9 for compact manifolds, we obtain the following

Theorem 1.10. *Let (E, a) be a hermitian vector bundle of rank s over a weakly 1-complete manifold X. Assume that (E, a) is q-negative outside a compact subset $K\subset X_d$ and that the rank of $\partial\bar\partial(e^{\Phi})$ is everywhere $\leqq r$. Then, for any $c\in\mathbf{R}$ with $X_c\supset X_d$, and for any vector bundle F over X, there is an integer m_0 such that the natural restriction maps*

$$H^p(X_c, F\otimes E^{(m)})\longrightarrow H^p(X_d, F\otimes E^{(m)}),$$

are bijective and $H^p(X_c, F\otimes E^{(m)})$ are finite dimensional, if $X_d\to K$, $m>m_0$, and $p\leqq n-r-q-s+1$.

Proof. By (2.36) in [9], $L=(L(E^*))^*$ is $q+s-1$ negative outside K. Hence by the isomorphism (10), we may assume that E is a line bundle.

Let χ_k be as before. By hypothesis, the rank of $\partial\bar{\partial}\chi_k(\Phi)$ is $\leqq r$. We put

$$(11) \qquad \begin{cases} ds_k^2 = ds^2 + \partial\bar{\partial}\chi_k(\Phi) \\ h_{k,m} = h\ a^m e^{-m\chi_k(\Phi)}, \end{cases}$$

where ds^2 is so chosen that at least $n-q+1$ eigenvalues of Θ_a are less than $-2(q+r)$ and other eigenvalues are less than 1, both outside a fixed $X_{d-\delta}$ such that $K\subset X_{d-\delta}\subset X_d$, and h is a hermitian metric along the fibers of F. Then, at any point $x\in X_c-X_{d-\delta}$, the eigenvalues $\Gamma_1^k\geqq\cdots\geqq\Gamma_n^k$ of $\Theta_a+\partial\bar{\partial}\chi_k(\Phi)$ with respect to ds_k^2 satisfy

$$(12) \qquad \begin{cases} \Gamma_i^k\leqq 2, & 1\leqq i\leqq q+r-1 \\ \Gamma_i^k\leqq -2(q+r), & q+r-1<i\leqq n. \end{cases}$$

The rest of the proof is the same as in the proof of Theorem 1.3. q.e.d.

In particular, if $K=\emptyset$, then we have a vanishing theorem as Corollary 1.9.

Remark 1. By the same argument we can prove the corresponding coarse vanishing theorems (see Corollary 1.4) for semi-positive bundles (resp. semi-negative bundles) of type q (for the definition see the next chapter) over strongly q-pseudoconvex (resp. strongly q-pseudoconcave) manifolds.

Remark 2. It will be interesting to know whether

$$\dim H^p(X_c, F\otimes E^{(m)}),\ p\geqq q,$$

are at most of polynomial growth of degree n with respect to m, where E is as in Theorem 1.3. The corresponding result for strongly q-pseudoconvex manifolds and strongly q-pseudoconcave manifolds has already been obtained by D. Leistner [14].

§ 2. Precise Isomorphism Theorems

Let (E, h) be a hermitian vector bundle over a complex manifold X of dimension n. (E, h) is said to be *semi-positive* (resp. *semi-negative*) if, for any point $x\in X$, $(h\otimes id)\Theta_h$ is a positive semi-definite (resp. negative semi-definite) hermitian form on $(E\otimes T_X)_x$. (E, h) is said to be *semi-positive of type q* (resp. *semi-negative of type q*), if (E, h) is both semi-positive (resp. semi-negative) and q-positive (resp. q-negative).

Proposition 2.1. *Let (E, h) be a semi-positive hermitian vector bundle of type q over a complex manifold X with a hermitian metric ds^2. Then, at any point $x \in X$,*

$$\sqrt{-1}\langle e(\Theta_h)\Lambda f, f\rangle \geqq \gamma(x)\langle f, f\rangle \qquad \text{for } f \in C^{n,p}(X, E), \quad p \geqq q, \tag{1}$$

where

$$\gamma(x) := \max_{\substack{S_x \triangleleft T_{X,x} \\ \dim S_x = n-q+1}} \min_{\substack{f \in E_x \otimes S_x \\ f \neq 0}} \frac{((h \otimes \mathrm{id})(\Theta_h f))(f)}{\langle f, f\rangle}. \tag{2}$$

Proof. Let $(\tau_1^*, \dots, \tau_n^*)$ be an orthonormal basis of $T_{X,x}$ with respect to ds^2 such that, if S_x is the linear subspace of $T_{X,x}$ spanned by $\tau_1^*, \dots, \tau_{n-q+1}^*$, we have

$$\gamma(x) = \min_{\substack{f \in E_x \otimes S_x \\ f \neq 0}} \frac{((h \otimes \mathrm{id})(\Theta_h f))(f)}{\langle f, f\rangle}. \tag{3}$$

By direct computation, we have

$$\begin{aligned} &\sqrt{-1}\langle e(\tau_{j_\alpha} \wedge \bar{\tau}_\beta)\Lambda \tau_1 \wedge \cdots \wedge \tau_n \wedge \bar{\tau}_{j_1} \wedge \cdots \wedge \bar{\tau}_{j_p} \\ &\tau_1 \wedge \cdots \wedge \tau_n \wedge \bar{\tau}_{j_1} \wedge \cdots \wedge \bar{\tau}_{j_{\alpha-1}} \wedge \bar{\tau}_\beta \wedge \bar{\tau}_{j_{\alpha+1}} \wedge \cdots \wedge \bar{\tau}_{j_p}\rangle \\ &= 1, \quad p \geqq 1, \end{aligned} \tag{4}$$

where $(\tau_1, \dots, \tau_n)$ denotes the dual basis of $(\tau_1^*, \dots, \tau_n^*)$. We set

$$\sqrt{-1}\langle e(\Theta_h)\Lambda\varphi, \varphi\rangle = \sum_{1 \leqq \alpha, \beta \leqq n} \Theta_{\alpha\beta}(\varphi_\alpha, \varphi_\beta), \tag{5}$$

where

$$\varphi = \sum_{1 \leqq \alpha \leqq n} \varphi_\alpha \tau_1 \wedge \cdots \wedge \tau_n \wedge \bar{\tau}_\alpha, \qquad \varphi_\alpha \in E_x, \tag{6}$$

and $\Theta_{\alpha\beta}$ is the coefficient of $\tau_\alpha \wedge \bar{\tau}_\beta$ in $\sqrt{-1}\Theta_h$. For any multi-index $J = (j_1, \dots, j_p)$ with $j_1 < \cdots < j_p$ and for any $J' = (j_1, \dots, \check{j}_\alpha, \dots, j_p)$, we put

$$\begin{aligned} &f_{J'\beta}\tau_1 \wedge \cdots \wedge \tau_n \wedge \bar{\tau}_{j_1} \wedge \cdots \wedge \check{\bar{\tau}}_{j_\alpha} \wedge \cdots \wedge \bar{\tau}_{j_p} \wedge \bar{\tau}_\beta \\ &= f_{j_1' \cdots j_p'}\tau_1 \wedge \cdots \wedge \tau_n \wedge \bar{\tau}_{j_1'} \wedge \cdots \wedge \bar{\tau}_{j_p'}, \quad j_1' < \cdots < j_p', \end{aligned} \tag{7}$$

where $\{j_1', \dots, j_p'\} = \{j_1, \dots, \check{j}_\alpha, \dots, j_p, \beta\}$ and

$$f = \sum_{j_1 < \cdots < j_p} f_{j_1 \cdots j_p}\tau_1 \wedge \cdots \wedge \tau_n \wedge \bar{\tau}_{j_1} \wedge \cdots \wedge \bar{\tau}_{j_p}, \quad f_{j_1 \cdots j_p} \in E_x. \tag{8}$$

Then, by (4),

$$\begin{aligned} &\sqrt{-1}\langle e(\Theta_h)\Lambda f, f\rangle \\ &= \sum_{|J'| = p-1} \sum_{1 \leqq \alpha, \beta \leqq n} \Theta_{\alpha\beta}(f_{J'\alpha}, f_{J'\beta}) \\ &= \sum_{|J'| = p-1} \sum_{1 \leqq \alpha, \beta \leqq n-p+1} \Theta_{i_\alpha i_\beta}(f_{J' i_\alpha}, f_{J' i_\beta}) \end{aligned} \tag{9}$$

$$\geqq \sum_{|J'|=p-1} \sum_{1 \leqq \alpha,\beta \leqq n-p+1} \Theta_{\alpha\beta}(f_{J'\alpha}, f_{J'\beta})$$

$$\geqq \gamma(x) \sum_{|J'|=p-1} \left\langle \sum_{\alpha=1}^{n-p+1} f_{J'\alpha} \otimes \tau_\alpha^*, \sum_{\alpha=1}^{n-p+1} f_{J'\alpha} \otimes \tau_\alpha^* \right\rangle .$$

Hence,

$$\begin{aligned} &\sqrt{-1}\langle e(\Theta_h)\Lambda f, f\rangle \\ &\geqq \gamma(x) \sum_{\alpha=1}^{n-p+1} \left(\sum_{J \ni \alpha} \langle f_J, f_J\rangle\right) \\ &\geqq \gamma(x)\langle f, f\rangle . \end{aligned} \tag{10}$$

q.e.d.

The following theorem has been proved by Ohsawa [23] and by Nakano-Rhai [22] for $q=1$ (rank $E=1$ is assumed in [23]). The original form where $K=\emptyset$ and $q=1$ was proved by Nakano [17], [19], [20], [21] and Kazama [12], and reproved by Suzuki [27].

Theorem 2.2. *Let X be a weakly 1-complete manifold, and let (E, h) be a hermitian vector bundle over X. Assume that (E, h) is semi-positive of type q outside a compact subset $K \subset X_d$ and ds^2 is Kählerian outside K. Then $H^p(X, \Omega^n(E))$ are finite dimensional and the natural restriction maps*

$$H^p(X, \Omega^n(E)) \longrightarrow H^p(X_d, \Omega^n(E))$$

are bijective, if $p \geqq q$. In particular, if $K=\emptyset$,

$$H^p(X, \Omega^n(E))=0, \qquad \text{for} \quad p \geqq q .$$

Proof. For any $c>d$, (X_d, X_c) is a pseudo-Runge pair with respect to E at bi-degrees (n, p), $p \geqq q-1$. In fact we have only to put

$$\begin{cases} ds_k^2 = ds^2 + \partial\bar{\partial}\chi_k(\Phi) \\ h_k = he^{-\chi_k(\Phi)} . \end{cases} \tag{11}$$

Then, in virtue of Proposition 2.1, we can prove the basic estimates. The rest of the proof is the same as in the proof of Theorem 1.3.

Proposition 2.3. *Let (E, h) be a semi-negative vector bundle of type q over a complex manifold X of dimension n with a hermitian metric ds^2. Then, at any point $x \in X$,*

$$\begin{aligned} &-\sqrt{-1}\langle \Lambda e(\Theta_h) f, f\rangle \geqq \delta(x)\langle f, f\rangle , \\ &f \in C^{0,p}(X, E), \quad p \leqq n-q . \end{aligned} \tag{12}$$

Here,

$$(13) \qquad \delta(x) := -\min_{\substack{S_x \triangleleft T_{X,x} \\ \dim S_x = n-q+1}} \max_{\substack{f \in E_x \otimes S_x \\ f \neq 0}} \frac{((h \otimes \mathrm{id})(\Theta_h f))(f)}{\langle f, f \rangle} .$$

Proof. Similar to Proposition 2.1.

As a dual of Theorem 2.2, we obtain

Theorem 2.4. *Let X be a weakly 1-complete manifold and let (E, h) be a hermitian vector bundle over X. Assume that (E, h) is semi-negative of type q outside a compact subset $K \subset X_d$, X is Kählerian outside K, and that $\partial\bar{\partial}(e^{\Phi})$ is of rank $\leqq r$. Then $H^p(X, E)$ are finite dimensional and the natural restriction maps*

$$H^p(X, E) \longrightarrow H^p(X_d, E)$$

are bijective, if $p \leqq n-q-r$. In particular, if $K = \emptyset$, then

$$H^p(X, E) = 0, \qquad \text{for} \quad p \leqq n-q-r .$$

Proof. Similar to Theorem 2.2.

The original form of the following theorem has been proved by Akizuki-Nakano [2] and Girbau [7]. The present form has been partially proved by Nakano [20], Abdelkader [1], Ohsawa [24], and Takegoshi-Ohsawa [29].

Theorem 2.5. *Let (B, a) be a hermitian line bundle over a weakly 1-complete manifold X. Assume that (B, a) is semi-positive of type q outside a compact subset $K \subset X_d$ and that X is Kählerian outside K. Then $H^t(X, \Omega^s(B))$ are finite dimensional and the natural restriction maps*

$$H^t(X, \Omega^s(B)) \longrightarrow H^t(X_d, \Omega^s(B)),$$

are bijective, if $s+t \geqq n+q$. In particular, if $K = \emptyset$, then

$$H^t(X, \Omega^s(B)) = 0, \qquad \text{for} \quad s+t \geqq n+q .$$

Proof. Let $c > d$ and

$$(14) \qquad \begin{cases} ds_k^2 = \varepsilon ds^2 + \partial\bar{\partial}\chi_k(\Phi) + \Theta_a \\ a_k = a e^{-\chi_k(\Phi)} , \end{cases}$$

where ε is a positive number which is determined later. We have to prove that (X_d, X_c) is a pseudo-Runge pair with respect to the above metrics.

It is clear that $(*)$ is satisfied.

Let $x \in X_c - X_{d-\delta}$ be any point ($X_d \supset X_{d-\delta} \supset K$). We set

$$(15)\qquad \begin{cases} ds^2=\sum_{i=1}^{n}\tau_i^k\otimes\bar{\tau}_i^k \\ ds_k^2=\sum_{i=1}^{n}(\varepsilon+\Gamma_i^k)\tau_i^k\otimes\bar{\tau}_i^k \qquad \text{at} \quad x. \end{cases}$$

Let $\Gamma_1\geqq\cdots\geqq\Gamma_n\geqq 0$ be the eigenvalues of Θ_a at x with respect to ds^2. Then we have

$$(16)\qquad \Gamma_i^k\geqq\Gamma_i.$$

Let

$$(17)\qquad f=\sum_{\substack{i_1<\cdots<i_s\\ j_1<\cdots<j_t}} f^k_{i_1\cdots i_s\bar{j}_1\cdots\bar{j}_t}\tau^k_{i_1}\wedge\cdots\wedge\tau^k_{i_s}\wedge\bar{\tau}^k_{j_1}\wedge\cdots\wedge\bar{\tau}^k_{j_t},$$
$$f^k_{i_1\cdots i_s\bar{j}_1\cdots\bar{j}_t}\in B_x.$$

Then, (cf. [7])

$$(18)\qquad \sqrt{-1}(e(\Theta_{a_k})\Lambda_k-\Lambda_k e(\Theta_{a_k}))f$$
$$=\sum_{\substack{i_1<\cdots<i_s\\ j_1<\cdots<j_t}}\left(\sum_{\alpha=1}^{s}\frac{\Gamma^k_{i_\alpha}}{\Gamma^k_{i_\alpha}+\varepsilon}+\sum_{\beta=1}^{t}\frac{\Gamma^k_{j_\beta}}{\Gamma^k_{j_\beta}+\varepsilon}-\sum_{\gamma=1}^{n}\frac{\Gamma^k_{\gamma}}{\Gamma^k_{\gamma}+\varepsilon}\right)$$
$$\cdot\prod_{\alpha=1}^{s}\frac{1}{\Gamma^k_{i_\alpha}+\varepsilon}\prod_{\beta=1}^{t}\frac{1}{\Gamma^k_{j_\beta}+\varepsilon}$$
$$\cdot f_{i_1\cdots i_s\bar{j}_1\cdots\bar{j}_t}\tau_{i_1}\wedge\cdots\wedge\tau_{i_s}\wedge\bar{\tau}_{j_1}\wedge\cdots\wedge\bar{\tau}_{j_t}.$$

If $s+t\geqq n+q$, then

$$(19)\qquad \sum_{\alpha=1}^{s}\frac{\Gamma^k_{i_\alpha}}{\Gamma^k_{i_\alpha}+\varepsilon}+\sum_{\beta=1}^{t}\frac{\Gamma^k_{j_\beta}}{\Gamma^k_{j_\beta}+\varepsilon}-\sum_{\gamma=1}^{n}\frac{\Gamma^k_{\gamma}}{\Gamma^k_{\gamma}+\varepsilon}$$
$$\geqq\frac{(n-q+2)\Gamma^k_{n-q+1}}{\Gamma^k_{n-q+1}+\varepsilon}-\sum_{\gamma=1}^{n-q+1}\frac{\Gamma^k_{\gamma}}{\Gamma^k_{\gamma}+\varepsilon}\qquad \text{(cf. [29])}.$$

Hence, if $2n\varepsilon<\inf_{x\in X_c-X_{d-\delta}}\Gamma_{n-q+1}$,

$$(20)\qquad \sqrt{-1}\langle(e(\Theta_{a_k})\Lambda_k-\Lambda_k e(\Theta_{a_k}))f,f\rangle$$
$$\geqq\frac{n+q-1}{2n+1}\langle f,f\rangle,\qquad \text{if}\quad s+t\geqq n+q,$$

where the inner products are with respect to ds_k^2 and a_k. Therefore, similarly as before, we obtain the corresponding basic estimates. The verification of (***) is the same as in the proof of Theorem 1.3. q.e.d.

Similarly we obtain

Theorem 2.6. *Let (B, a) be a hermitian line bundle over a weakly 1-complete manifold X. Assume that (B, a) is semi-negative of type q outside a* *compact subset* $K \subset X_d$, *X is Kählerian outside K, and the rank of $\partial\bar{\partial}(e^{\Phi})$ is $\leqq r$. Then $H^t(X, \Omega^s(B))$ are finite dimensional and the natural restriction* maps

$$H^t(X, \Omega^s(B)) \longrightarrow H^t(X_d, \Omega^s(B))$$

are bijective, if $s+t \leqq n-q-r$. In particular, if $K=\emptyset$,

$$H^t(X, \Omega^s(B))=0\,, \qquad \textit{for} \quad s+t \leqq n-q-r\,.$$

Proof. Let $c>d$ and

$$\text{(21)} \qquad \begin{cases} ds_k^2=\varepsilon ds^2+\partial\bar{\partial}\chi_k(\Phi)-\Theta_a \\ a_k=ae^{-\varepsilon\chi_k(\Phi)}\,, \end{cases}$$

where ε is a positive number which is determined later. We shall show that (X_d, X_c) is a pseudo-Runge pair with respect to the above metrics. It is clear that $(*)$ is satisfied. Let $x \in X_c - X_{d-\delta}$ be any point, where a positive number δ is so chosen that $X_d \supset X_{d-\delta} \supset K$. We have

$$\text{(22)} \qquad \begin{aligned} \Theta_{a_k} &= \Theta_a+\partial\bar{\partial}\chi_k(\Phi) \\ &= -ds_k^2+\varepsilon ds^2+(1+\varepsilon)\partial\bar{\partial}\chi_k(\Phi)\,. \end{aligned}$$

Let $f \in (B \otimes T_X^{s,t})_x$, then

$$\begin{aligned} \text{(23)} \quad & \langle\sqrt{-1}(e(\Theta_{a_k})\Lambda_k-\Lambda_k e(\Theta_{a_k}))f, f\rangle_k \\ &= \langle\sqrt{-1}(-e(ds_k^2)\Lambda_k+\Lambda_k e(ds_k^2))f, f\rangle_k \\ &\quad +\langle\sqrt{-1}(e(\varepsilon ds^2+(1+\varepsilon)\partial\bar{\partial}\chi_k(\Phi))\Lambda_k-\Lambda_k e(\varepsilon ds^2+(1+\varepsilon)\partial\bar{\partial}\chi_k(\Phi))f, f\rangle_k \\ &= (n-s-t)\langle f, f\rangle_k+\langle\sqrt{-1}(e(\varepsilon ds^2+(1+\varepsilon)\partial\bar{\partial}\chi_k(\Phi))\Lambda_k \\ &\quad -\Lambda_k e(\varepsilon ds^2+(1+\varepsilon)\partial\bar{\partial}\chi_k(\Phi)))f, f\rangle_k\,, \qquad \text{at} \quad x\,. \end{aligned}$$

Here we denote by $\langle\ ,\ \rangle_k$ the pointwise inner product with respect to ds_k^2 and a_k. We put

$$\text{(24)} \qquad \begin{cases} ds_k^2=\sum_{i=1}^{n} \tau_i^k \otimes \bar{\tau}_i^k \\ \varepsilon ds^2+(1+\varepsilon)\partial\bar{\partial}\chi_k(\Phi)=\sum_{i=1}^{n} (1+\gamma_i^k(\varepsilon))\tau_i^k \otimes \bar{\tau}_i^k, \qquad \text{at} \quad x\,. \end{cases}$$

Here $\gamma_i^k(\varepsilon)$ are the eigenvalues of $\Theta_a+\varepsilon\partial\bar{\partial}\chi_k(\Phi)$ with respect to ds_k^2, and we arrange them as follows:

$$\text{(25)} \qquad \gamma_1^k(\varepsilon) \leqq \cdots \leqq \gamma_n^k(\varepsilon)\,.$$

It is easily seen that $\gamma_i^k(\varepsilon) \leqq \varepsilon$ and that

$$\text{(26)} \qquad -1 \leqq \gamma_i^k(\varepsilon) \leqq -1+C\varepsilon \quad \text{for} \quad 1 \leqq i \leqq n-q+1-r\,,$$

where C is a constant which does not depend on k or x. Hence,

$$
\begin{aligned}
(27)\qquad & \sum_{\alpha=1}^{s}(1+\gamma_{i_\alpha}^k(\varepsilon))+\sum_{\beta=1}^{t}(1+\gamma_{j_\beta}^k(\varepsilon))-\sum_{u=1}^{n}(1+\gamma_u^k(\varepsilon)) \\
& \quad (=s+t-n+\sum_{\alpha=1}^{s}\gamma_{i_\alpha}^k(\varepsilon)+\sum_{\beta=1}^{t}\gamma_{j_\beta}^k(\varepsilon)-\sum_{u=1}^{n}\gamma_u^k(\varepsilon)) \\
& \geqq s+t-n-s-t+n-q+1-r-\{C(n-q+1-r)+q+r-1\}\varepsilon \\
& = -q-r+1-\{C(n-q+1-r)+q+r-1\}\varepsilon .
\end{aligned}
$$

Combining (23) with (27), we obtain the basic estimate for $s+t\leqq n-q-r$. The rest of the proof is the same as in the proof of Theorem 1.3. q. e. d.

A hermitian vector bundle (E, h) over a complex manifold X of dimension n is said to be *weakly semi-positive* if, for any point $x\in X$ and for any $v\in E_x-\{0\}$, $h(\Theta_h(v))(\bar{v})$ is a positive semi-definite hermitian form on $T_{X,x}$. If the rank of $h(\Theta_h(v))(\bar{v})$ is at least $n-q+1$, then (E, h) is said to be *weakly semi-positive of type* q. This definition is originally due to Griffiths [8] (cf. also Skoda [26]).

Let $P(E^*)$ and L be as in Section 1. Then we have the following isomorphism (cf. Le Potier, C. R. Acad. Sc. Paris, 276 (1973) pp. 535–537).

$$(28)\qquad H^t(X, \Omega^s(E)) \longrightarrow H^t(P(E^*), \Omega^s(L)) .$$

From (2.36) in [9], L is semi-positive of type q if E is weakly semi-positive of type q. Thus combining the isomorphism (21) with Theorem 2.5, we obtain

Theorem 2.7. *Let (E, h) be a hermitian vector bundle of rank r over a weakly 1-complete manifold X. Assume that (E, h) is weakly semi-positive of type q outside a compact subset $K\subset X_d$ and there is a Kähler metric on $X-K$. Then $H^t(X, \Omega^s(E))$ are finite dimensional and the natural restriction maps*

$$H^t(X, \Omega^s(E)) \longrightarrow H^t(X_d, \Omega^s(E))$$

are bijective, if $s+t\geqq n+r+q-1$. In particular, if $K=\emptyset$,

$$H^t(X, \Omega^s(E))=0 , \qquad \text{for}\quad s+t\geqq n+r+q-1 .$$

The counterpart of Theorem 2.6 is left to the reader.

The relation between semi-positivity and weak semi-positivity, except for the trivial implication, has been first revealed by Demailly and Skoda [5]. We restate here their theorem as follows.

Theorem 2.8. *If (E, h) is weakly semi-positive (resp. weakly 1-positive), then $((\det E)\otimes E, (\det h)\otimes h)$ is semi-positive (resp. 1-positive).*

It follows trivially from their inequality

$$\sum_{j,k,l,m} a_{jklm} x_{jl} \bar{x}_{km} + \sum_{j,k,l,m} a_{jjlm} x_{kl} \bar{x}_{km} \geqq \sum_{j,l,m} a_{jjlm} x_{jl} \bar{x}_{jm}$$

(cf. p. 307 in [5]) that if (E, h) is weakly semi-positive of type q, then $((\det E)\otimes E$, $(\det h)\otimes h)$ is semi-positive of type q.

Remark 1. In the above theorems we have proved also approximation theorems and harmonic representation theorems. From the harmonic representation theorem, we can deduce cohomology vanishing theorems under somewhat weaker assumptions on the curvature form of (E, h) (cf. Takegoshi [28]).

Remark 2. Let $\pi\colon X\to\mathbf{C}^N$ be a proper holomorphic map from a weakly 1-complete manifold X of dimension n. Assume that X is embeddable into $\mathbf{P}^{N'}$ as a locally closed analytic submanifold, and that $\dim \pi^{-1}(x)\leqq n-r$ for $x\neq 0$. Then the latter part of Theorem 2.5 (Nakano's vanishing theorem), combined with Hodge-Lieberman-Rossi-Fujiki's decomposition theorem for strongly pseudoconvex manifolds, (cf. [6], [15]) implies that the decomposition of $H^p(X, \mathbf{C})$ into the direct sum of $H^t(X, \Omega^s)$ and the symmetry $\dim H^t(X, \Omega^s)$ $=\dim H^s(X, \Omega^t)$ are valid for degrees $s+t$, $p\geqq 2n-r+1$. Therefore the topology of the degenerate set of a holomorphic map from a projective variety is very restricted. Such phenomenon can be observed on Kähler manifolds, too (cf. [25]), which may suggest a further meaning of our isomorphism theorems.

Appendix

Let the notations be as in Chapter 1. We shall prove here Theorem 1.3 following [31] and [32]. We have to deal with operators on T_X^r rather than those on $T_X^{p,q}$, so we regard L, Λ, and so on, as operators on T_X^r.

Definition 1. An r-form u is said to be primitive if $\Lambda u=0$.

From now on r-forms (resp. primitive r-forms) are denoted by f^r (resp. $u^{(r)}$), unless otherwise stated. For the proofs of Theorem 2 to Theorem 6, the reader is referred to [31] and [32].

Theorem 2. i) *If $u^{(r)}$ is a nonzero primitive form, then $r\leqq n$.*

ii) *If $u^{(r)}$ is primitive with $r\leqq n$, then*

$$\left\{\begin{aligned} &L^{n-r+1}u^{(r)}=0\\ &\Lambda^k L^k u^{(r)}=\frac{k!(n-r)!}{(n-r-k)!}u^{(r)} \qquad \textit{for} \quad k\leqq n-r\,. \end{aligned}\right. \tag{1}$$

Theorem 3. i) *For any* f^r *with* $r \leqq n$,

(2) $$f^r = u^{(r)} + Lu^{(r-2)} + \cdots + L^k u^{(r-2k)}, \quad k \leqq [r/2].$$

ii) *For any* f^r *with* $r > n$,

(3) $$f^r = L^{r-n} u^{(2n-r)} + L^{r-n+1} u^{(2n-r-2)} + \cdots + L^{r-n+k} u^{(2n-r-2k)}, \\ k \leqq [n-(r/2)].$$

Furthermore the decompositions i) and ii) are unique.

Corollary 4. *If* $L^{n-r+k} f^r = 0$, $n-r+k \geqq 0$, *then* $u^{(i)} = 0$, $i \leqq r-2k$, *where* $u^{(i)}$ *are as in i) or ii).*

Corollary 5. *There is an operator* $P(n, r, i)$ *belonging to the algebra generated by* Λ *and* L *over the field of rational numbers, satisfying*

(4) $$P(n, r, i) f^r = u^{(i)},$$

where $u^{(i)}$ *are as in i) or ii).*

Theorem 6. *For any* $u^{(r)}$,

(5) $$\bar{*} L^k u^{(r)} = \begin{cases} (-1)^{\frac{r(r+1)}{2}} \dfrac{k!}{(n-r-k!)} L^{n-r-k} \bar{C} u^{(r)}, & k \leqq n-r \\ 0, & k > n-r. \end{cases}$$

From these theorems we can deduce

Proposition 7. *For any* $u^{(p)}$ *of class* C^∞,

(6) $$\begin{aligned} &(d\Lambda - \Lambda d) L^k u^{(p)} \\ &= \{ -C^{-1} \bar{*} d \bar{*} C + a_{n,p,k} L^{k+1} \Lambda^3 e(d\omega) \Lambda^k \\ &\quad + b_{n,p,k} (\Lambda L + (k-1)(k-1-n+p)) e(d\omega) L^{k-2} \Lambda^k \\ &\quad + c_{n,p,k} \bar{C}^{-1} \bar{*} e(d\omega) L^{n-k-p-1} \Lambda^k \} L^k u^{(p)}, \end{aligned}$$

where

(A) $$a_{n,p,k} = \left\{ 2n-2p-k+4 + \frac{(n-k-p+2)(n-k-p+1)}{k+1} \right\} \times \frac{(n-p+1)(n-p-k)!}{3!k!(n-p)!}$$

(B) $$b_{n,p,k} = -\frac{(n-p-k)!}{(k-1)!(n-p)!}$$

(C) $$c_{n,p,k} = -(-1)^{\frac{p(p-1)}{2}} \frac{n-p-k}{(n-p)!},$$

and we put $m! = 0$ *and* $L^m = 0$ *for negative* m.

$$(7)\qquad (e(d\Phi)\Lambda - \Lambda e(d\Phi))L^k u^{(p)} = -C^{-1}\bar{*}e(d\bar{\Phi})\bar{*}CL^k u^{(p)},$$

where Φ is a (complex valued) C^∞ function.

Admitting Proposition 7, Theorem 1.3 is proved as follows:

Proof of Theorem 1.3. Combining (6) with Theorem 3 and Corollary 5, for any f^r of class C^∞,

$$(8)\qquad (d\Lambda - \Lambda d)f^r = -C^{-1}\bar{*}d\bar{*}Cf^r + Tf^r,$$

where T is a linear operator belonging to the algebra over the field of rational numbers generated by L, Λ, $e(d\omega)$, $\bar{C}^{-1}$, and $\bar{*}$. Similarly we have

$$(9)\qquad (e(d\Phi)\Lambda - \Lambda e(d\Phi))f^r = -C^{-1}\bar{*}e(d\bar{\Phi})\bar{*}Cf^r.$$

Decomposing (8) and (9) into types, we have

$$(10)\qquad \partial\Lambda - \Lambda\partial = -\sqrt{-1}(\bar{*}\bar{\partial}\bar{*} + T_1),$$

$$(11)\qquad \bar{\partial}\Lambda - \Lambda\bar{\partial} = \sqrt{-1}(\bar{*}\partial\bar{*} + T_2) = \sqrt{-1}(\bar{\vartheta} + T_2),$$

$$(12)\qquad e(\partial\Phi)\Lambda - \Lambda e(\partial\Phi) = -\sqrt{-1}\bar{*}e(\bar{\partial}\bar{\Phi})\bar{*},$$

and

$$(13)\qquad e(\bar{\partial}\Phi)\Lambda - \Lambda e(\bar{\partial}\Phi) = \sqrt{-1}\bar{*}e(\partial\bar{\Phi})\bar{*},$$

where T_1 and T_2 are the components of T.

Thus in particular we have proved the latter part of (13) in Theorem 1.3.

By (12), we have

$$(14)\qquad e(h^{-1}(\partial h))\Lambda - \Lambda e(h^{-1}(\partial h)) = -\sqrt{-1}\bar{*}e(\bar{h}^{-1}(\bar{\partial}\bar{h}))\bar{*}.$$

On the other hand,

$$(15)\qquad \vartheta_E = -\bar{*}\bar{h}^{-1}\bar{\partial}\bar{h}\bar{*} = -\bar{*}\bar{\partial}\bar{*} - \bar{*}\bar{h}^{-1}e(\bar{\partial}\bar{h})\bar{*}.$$

Combining these equalities we obtain Theorem 1.3. q.e.d.

Proof of Proposition 7. We prove only (6). The proof of (7) is similar. First we note that

$$(16)\qquad L^{n-p+2}du^{(p)} = dL^{n-p+2}u^{(p)} - (n-p+2)e(d\omega)L^{n-p+1}u^{(p)}.$$

From Theorem 2 ii), the right hand side of (16) is zero, so that by Corollary 4, we have the primitive decomposition

$$du^{(p)}=\eta^{(p+1)}+L\eta^{(p-1)}+L^2\eta^{(p-3)}\,. \tag{17}$$

Hence we have

$$\begin{aligned}(d\Lambda-\Lambda d)L^k u^{(p)}&\\ &=d\Lambda L^k u^{(p)}-\Lambda L^k du^{(p)}-k\Lambda e(d\omega)L^{k-1}u^{(p)}\\ &=d\Lambda L^k u^{(p)}-\Lambda L^k(\eta^{(p+1)}+L\eta^{(p-1)}+L^2\eta^{(p-3)})\\ &\quad -k\Lambda e(d\omega)L^{k-1}u^{(p)}\,.\end{aligned} \tag{18}$$

By Theorem 2 ii),

$$\Lambda L^k u^{(p)}=-k(k-1-n+p)L^{k-1}u^{(p)}\,, \tag{19}$$

$$\Lambda L^k\eta^{(p+1)}=-k(k-n+p)L^{k-1}\eta^{(p+1)}\,, \tag{20}$$

$$\Lambda L^{k+1}\eta^{(p-1)}=-(k+1)(k-1-n+p)L^k\eta^{(p-1)}\,, \tag{21}$$

and

$$L^{k+2}\eta^{(p-3)}=-(k+2)(k-2-n+p)L^{k+1}\eta^{(p-3)}\,. \tag{22}$$

Hence,

$$\begin{aligned}d\Lambda L^k u^{(p)}&\\ &=-k(k-1-n+p)dL^{k-1}u^{(p)}\\ &=-k(k-1-n+p)\{L^{k-1}(\eta^{(p+1)}+L\eta^{(p-1)}+L^2\eta^{(p-3)})\\ &\quad+(k-1)e(d\omega)L^{k-2}u^{(p)}\}\,.\end{aligned} \tag{23}$$

Combining (18) with (23), we obtain

$$\begin{aligned}(d\Lambda-\Lambda d)L^k u^{(p)}&\\ &=kL^{k-1}\eta^{(p+1)}-(n-p-k+1)L^k\eta^{(p-1)}\\ &\quad-(2n-2p-k+4)L^{k+1}\eta^{(p-3)}\\ &\quad-(k\Lambda e(d\omega)L^{k-1}+k(k-1)(k-1-n+p)e(d\omega)L^{k-2}u^{(p)}\\ &=kL^{k-1}\eta^{(p+1)}-(n-p-k+1)L^k\eta^{(p-1)}\\ &\quad-(2n-2p-k+4)L^{k+1}\eta^{(p-3)}\\ &\quad-\frac{(n-p-k)!}{(k-1)!(n-p)!}\{\Lambda L+(k-1)(k-1-n+p)\}e(d\omega)L^{k-2}\Lambda^k L^k u^{(p)}\,.\end{aligned} \tag{24}$$

On the other hand, by Theorem 6, we have

(25)
$$\begin{aligned}
&-C^{-1}\bar{*}d\bar{*}CL^{k}u^{(p)}\\
&=-\bar{C}^{-1}\bar{*}d\bar{*}L^{k}\bar{C}u^{(p)}\\
&=-\bar{C}^{-1}*d\bar{*}(-1)^{\frac{p(p+1)}{2}}\frac{k!}{(n-k-p)!}\bar{*}L^{n-k-p}u^{(p)}\\
&=-\bar{C}^{-1}\bar{*}d(-1)^{\frac{p(p-1)}{2}}\frac{k!}{(n-k-p)!}L^{n-k-p}u^{(p)}\\
&=-(-1)^{\frac{p(p-1)}{2}}\frac{k!}{(n-k-p)!}\bar{C}^{-1}\bar{*}L^{n-k-p}(\eta^{(p+1)}+L\eta^{(p-1)}\\
&\quad+L^{2}\eta^{(p-3)})+(n-k-p)e(d\omega)L^{n-k-p-1}u^{(p)}\,.
\end{aligned}$$

Applying Theorem 6 again, we have

(26)
$$\begin{aligned}
&-C^{-1}\bar{*}d\bar{*}CL^{k}u^{(p)}\\
&=-(-1)^{\frac{p(p-1)}{2}}\frac{k!}{(n-k-p)!}\bar{C}^{-1}\Big\{(-1)^{\frac{(p+1)(p+2)}{2}}\cdot\\
&\qquad\frac{(n-k-p)!}{(k-1)!}L^{k-1}\bar{C}\eta^{(p+1)}\\
&\quad+(-1)^{\frac{p(p-1)}{2}}\frac{(n-k-p+2)!}{(k+1)!}L^{k}\bar{C}\eta^{(p-1)}\\
&\quad+(-1)^{\frac{(p-2)(p-3)}{2}}\frac{(n-k-p+2)!}{(k+1)!}L^{k+1}\bar{C}\eta^{(p-3)}\\
&\quad+(n-k-p)\bar{*}e(d\omega)L^{n-k-p-1}u^{(p)}\Big\}\\
&=kL^{k-1}\eta^{(p+1)}-(n-p-k+1)L^{k}\eta^{(p-1)}\\
&\quad+\frac{(n-k-p+2)(n-k-p+1)}{k+1}L^{k+1}\eta^{(p-3)}\\
&\quad-(-1)^{\frac{p(p-1)}{2}}\frac{k!}{(n-k-p-1)!}\bar{C}^{-1}\bar{*}e(d\omega)L^{n-k-p-1}u^{(p)}\\
&=kL^{k-1}\eta^{(p+1)}-(n-p-k+1)L^{k}\eta^{(p-1)}\\
&\quad+\frac{(n-k-p+2)(n-k-p+1)}{k+1}L^{k+1}\eta^{(p-3)}\\
&\quad-(-1)^{\frac{p(p-1)}{2}}\bar{C}^{-1}\bar{*}e(d\omega)\frac{n-p-k}{(n-p)!}L^{n-k-p-1}\Lambda^{k}L^{k}u^{(p)}\,.
\end{aligned}$$

Therefore,

(27)
$$\begin{aligned}
&(d\Lambda-\Lambda d)L^{k}u^{(p)}\\
&=-C^{-1}\bar{*}d\bar{*}CL^{k}u^{(p)}+a'_{n,p,k}L^{k+1}\eta^{(p-3)}\\
&\quad+b_{n,p,k}\{\Lambda L+(k-1)(k-1-n+p)\}e(d\omega)L^{k-2}\Lambda^{k}L^{k}u^{(p)}\\
&\quad+c_{n,p,k}\bar{C}^{-1}\bar{*}e(d\omega)L^{n-k-p-1}\Lambda^{k}L^{k}u^{(p)}\,,
\end{aligned}$$

where

(A′) $$a'_{n,p,k} = -2n+2p+k-4-\frac{(n-k-p+2)(n-k-p+1)}{k+1},$$

(B) $$b_{n,p,k} = -\frac{(n-p-k)!}{(k-1)!(n-p)!},$$

and

(C) $$c_{n,p,k} = -(-1)^{\frac{p(p-1)}{2}}\frac{n-p-k}{(n-p)!}.$$

It remains to express $\eta^{(p-3)}$ in terms of $L^k u^{(p)}$. In virtue of Theorem 3 and Corollary 4, we can put

(28) $$e(d\omega)u^{(p)} = v^{(p+3)} + Lv^{(p+1)} + L^2 v^{(p-1)} + L^3 v^{(p-3)}.$$

Since

(29) $$\begin{aligned} L^{n-p+1}du^{(p)} &= (L^{n-p+1}d - dL^{n-p+1})u^{(p)} \\ &= -(n-p+1)L^{n-p}e(d\omega)u^{(p)}, \end{aligned}$$

we have

(30) $$\begin{aligned} &L^{n-p+1}\eta^{(p+1)} + L^{n-p+2}\eta^{(p-1)} + L^{n-p+3}\eta^{(p-3)} \\ &= -(n-p+1)(L^{n-p}v^{(p+3)} + L^{n-p+1}v^{(p+1)} + L^{n-p+2}v^{(p-1)} \\ &\quad + L^{n-p+3}v^{(p-3)}). \end{aligned}$$

By the uniqueness of the primitive decomposition,

(31) $$\eta^{(p-3)} = -(n-p+1)v^{(p-3)}.$$

On the other hand,

(32) $$\begin{aligned} v^{(p-3)} &= \frac{(n-p)!}{3!(n-p+3)!}\Lambda^3 L^3 v^{(p-3)} \\ &= \frac{(n-p)!}{3!(n-p+3)!}\Lambda^3 e(d\omega)u^{(p)} \\ &= \frac{(n-p-k)!}{3!(n-p+3)!}\Lambda^3 e(d\omega)\Lambda^k L^k u^{(p)}. \end{aligned}$$

Thus,

(33) $$\eta^{(p-3)} = -\frac{(n-p+1)(n-p-k)!}{3!(n-p-3)!k!}\Lambda^3 e(d\omega)\Lambda^k L^k u^{(p)}.$$

Putting (33) into (27), we obtain the proposition. q.e.d.

References

[1] Abdelkader, O., Annulation de la cohomologie d'une variété kählérienne faiblement 1-complète à valeur dans un fibré vectoriel holomorphe semi-positif, *C. R. Acad. Sc. Paris,* **290** (1980), 75–78.

[2] Akizuki, Y. and Nakano, S., Note on Kodaira-Spencer's proof of Lefschetz theorems, *Proc. Jap. Acad.,* **30** (1954), 266–272.

[3] Andreotti, A. and Grauert, H., Théorème de finitude pour la cohomologie des espaces complexes, *Bull. Soc. Math. France,* **90** (1962), 193–259.

[4] Andreotti, A. and Vesentini, E., Carleman estimates for the Laplace-Beltrami equation on complex manifolds, *Inst. Hautes Etudes Sci., Publ. Math.,* **25** (1965), 81–130.

[5] Demailly, J.-P. and Skoda, H., Relations entre les notions de positivités de P. A. Griffiths et de S. Nakano pour les fibrés vectoriels, *Seminaire Pierre Lelong-Henri Skoda, (Analyse) Années* 1978/79, *Lecture notes in mathematics,* **822** (1980), 304–309.

[6] Fujiki, A., Hodge to de Rham spectral sequence on a strongly pseudoconvex manifold, *preprint.*

[7] Girbau, J., Sur le théorème de Le Potier d'annulation de la cohomologie, *C. R. Acad. Sc. Paris,* **283** (1976), 355–358.

[8] Griffiths, P. A., The extension problem in complex analysis II; embedding with positive normal bundle, *Amer. J. of Math.,* **88** (1966), 366–446.

[9] ———, Hermitian differential geometry, Chern classes, and positive vector bundles, *Global Analysis, papers in honor of K. Kodaira,* edited by D. C. Spencer and S. Iyanaga, Univ. of Tokyo Press and Princeton Univ. Press 1969, 185–251.

[10] Hörmander, L., L^2 estimates and existence theorems for the $\bar{\partial}$ operator, *Acta Math.,* **113** (1965), 89–152.

[11] ———, *An Introduction to Complex Analysis in Several Variables,* North-Holland Publishing Company, 1973.

[12] Kazama, H., Approximation theorem and application to Nakano's vanishing theorem for weakly 1-complete manifolds, *Mem. Fac. Sci. Kyushu Univ.,* **27** (1973), 221–240.

[13] Kodaira, K., On Kähler varieties of restricted type, *Ann. Math.,* **60** (1954), 28–48.

[14] Leistner, D., Der Endlichkeitssatz mit Abschätzung für pseudokonvexe und pseudokonkave Räume, *Dissertation Regensburg,* 1974.

[15] Lieberman, D. and Rossi, H., Deformations of strongly pseudo-convex manifolds, *Rencontre sur l'analyse complexe à plusieurs variables et les systèmes surdéterminés, les presses de l'université de Montréal,* 1975, 119–165.

[16] Morrey, C. B., *Multiple integrals in the calculus of variations,* Springer, 1966.

[17] Nakano, S., On complex analytic vector bundles, *J. M. S. Jap.,* **7** (1955), 1–12.

[18] ———, On the inverse of monoidal transformation, *Publ. RIMS, Kyoto Univ.,* **6** (1970/71), 483–502.

[19] ———, Vanishing theorems for weakly 1-complete manifolds, *Number Theory, Algebraic Geometry and Commutative Algebra, in honor of Y. Akizuki,* Kinokuniya, Tokyo, 1973, 169–179.

[20] ———, On weakly 1-complete manifolds, *Manifolds-Tokyo* (Proceedings of the international conference on manifolds and related topics in topology), 1973, 323–327.

[21] ———, Vanishing theorems for weakly 1-complete manifolds, II, *Publ. RIMS, Kyoto Univ.,* **10** (1974), 101–110.

[22] Nakano, S. and Rhai, T. S., Vector bundle version of Ohsawa's finiteness theorems, *Math. Japonica*, **24** (1980), 657–664.
[23] Ohsawa, T., Finiteness theorems on weakly 1-complete manifolds, *Publ. RIMS, Kyoto Univ.*, **15** (1979), 853–870.
[24] ———, On $H^{p,q}(X, B)$ of weakly 1-complete manifolds, *Publ. RIMS, Kyoto Univ.*, **17** (1981), 113–126.
[25] ———, A reduction theorem for cohomology groups of very strongly q-convex Kähler manifolds, *Invent. Math.*, **63** (1981), 335–354.
[26] Skoda, H., Remarques a propos des théorèmes d'annulation pour les fibrés semi-positifs, *Seminaire Pierre Lelong-Henri Skoda (Analyse)* 1978/79, *Lecture Notes in Math.*, **822** (1980), 252–257.
[27] Suzuki, O., Simple proofs of Nakano's vanishing theorem and Kazama's approximation theorem for weakly 1-complete manifolds, *Publ. RIMS, Kyoto Univ.*, **11** (1975), 201–211.
[28] Takeogoshi, K., A generalization of vanishing theorems for weakly 1-complete manifolds, *Publ. RIMS, Kyoto Univ.*, **17** (1981), 311–330.
[29] Takegoshi, K. and Ohsawa, T., A vanishing theorem for $H^p(X, \Omega^q(B))$ on weakly 1-complete manifolds, *Publ. RIMS, Kyoto Univ.*, **17** (1981), 723–733.
[30] Vesentini, E., *Lectures on Levi convexity of complex manifolds and cohomology vanishing theorems,* Institute of Fundamental Research, Bombay, 1967.
[31] Weil, A., *Introduction à l'etude des variétés kähleriennes,* Act. Sci. Ind. 1267, Hermann, 1958.
[32] Wells, R. O., *Differential analysis on complex manifolds,* Prentice-Hall, Englewood Cliffs, N. J., 1973.

Added in proof: O. Abdelkader has proved in "Un théorème d'approximation pour les formes à valeurs dans un fibré semi-positif, *C. R. Acad. Sci.* **293** (1981), 513–515" the result $H^t(X, \Omega^s(E))=0$ in our Theorem 2.7 by a different method, but still using the approximation argument as in [12].