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Homotopy Classification of Connected Sums of Sphere Bundles over Spheres, II

By

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Introduction

In the classification problems of manifolds, the connected sums of sphere bundles over spheres appear frequently. For example, we can see those in [6], [7], and [13]. Motivated by those, in the preceding paper [8], we classified the connected sums consisting of sphere bundles over spheres which admit cross-sections up to homotopy equivalence.

In this paper, as promised previously, we investigate the general case. And, under some assumptions on dimensions, i.e. in metastable range, we obtain a necessary and sufficient condition for two connected sums of sphere bundles over spheres to be homotopy equivalent, by extending the results of James-Whitehead [10] and using the handlebody theory of Wall [14] and Ishimoto [5]. Applications of the main theorem to special cases will appear in the subsequent paper.

Let B_i, $i=1, 2,\cdots, r$, be p-sphere bundles over q-spheres (p, $q>1$), and let $\bar{B}_i$, $i=1, 2,\cdots, r$, be the associated $(p+1)$-disk bundles. It is understood that each B_i, or $\bar{B}_i$, also denotes the total space of the bundle and has the oriented differentiable structure induced from those of the fibre and the base space. If $p\geqq q$, each B_i admits a cross-section, and the homotopy classification of the connected sums of such bundles has been completed in [8]. So, we assume that $p<q-1$. The torsion case that $p=q-1$ is excluded from this paper and the problem is still open. We denote the characteristic element of B_i by $\alpha(B_i)$ or simply by α_i and we put $\varepsilon_i=\pi_*(\alpha_i)$, where $\pi_*\colon \pi_{q-1}(SO_{p+1})\to\pi_{q-1}(S^p)$ is the homomorphism induced from the projection $\pi\colon SO_{p+1}\to SO_{p+1}/SO_p=S^p$.

The boundary connected sum $\natural_{i=1}^r \bar{B}_i$ can be considered as a handlebody of

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$\mathscr{H}(m+1, r, q)$, $m=p+q$, and the connected sum $\#_{i=1}^{r} B_i$ is its boundary. In general, $\#_{i=1}^{r} B_i$ may have various representations into the connected sums of p-sphere bundles over q-spheres up to diffeomorphism. In fact, we can observe it using the handlebody theory as follows.

Let W be a handlebody of $\mathscr{H}(m+1, r, q)$, $m=p+q$, and assume that $2p>q>1$. Let $\phi: H\times H\to\pi_q(S^{p+1})$, $H=H_q(W)$, be the pairing defined by Wall [14], and let $\alpha: H\to\pi_{q-1}(SO_{p+1})$ be the map assigning to each $x\in H \cong\pi_q(W)$ the characteristic element of the normal bundle of the imbedded q-sphere which represents x. α is a quadratic form with the associated homomorphism $\partial\circ\phi$, where $\partial: \pi_q(S^{p+1})\to\pi_{q-1}(SO_{p+1})$ belongs to the homotopy exact sequence of the fibering $SO_{p+1}\to SO_{p+2}\to S^{p+1}$. ([14], p. 257). A base $\{w_1, w_2,\cdots, w_r\}$ of the free abelian group H is called *admissible* if $\phi(w_i, w_j)=0$ for all i, j $(i\neq j)$. If H has an admissible base $\{w_1, w_2,\cdots, w_r\}$, then W can be represented as a boundary connected sum of $(p+1)$-disk bundles over q-spheres with the characteristic elements $\alpha(w_i)$, $i=1, 2,\cdots, r$. For, we can take the imbedded q-spheres which represent w_i, $i=1, 2,\cdots, r$, to be disjoint (cf. Ishimoto [5]). Hence, by tying the tubular neighbourhoods of such imbedded q-spheres with thin bands in W, and by the h-cobordism theorem, we know that W is diffeomorphic $(m>4)$ to such a boundary connected sum of disk bundles over spheres.

Thus, the representations of $W=\natural_{i=1}^{r} \bar{B}_i$ into the boundary connected sums of $(p+1)$-disk bundles over q-spheres correspond with the admissible bases of $H=H_q(W)$. Since $H_q(\partial W)\cong H_q(W)$ if $p\neq q-1$, q, we obtain various representations of $\partial W=\#_{i=1}^{r} B_i$ into the connected sums of p-sphere bundles over q-spheres associated with the admissible bases of $H\cong H_q(\partial W)$.

In Section 2, it is shown that Wall's pairing is a homotopy invariant of the boundary of the handlebody if $p\neq q-1$. That is,

Proposition 1. *Let W, W' be handlebodies of $\mathscr{H}(p+q+1, r, q)$ and assume that $2p>q>1$ and $p\neq q-1$. If there exists a homotopy equivalence $f: \partial W\to\partial W'$ which preserves orientation, then for the isomorphism $h=i'_*\circ f_* \circ i_*^{-1}: H_q(W)\to H_q(W')$, we have $\phi=\phi'\circ(h\times h)$, where ϕ, ϕ' are Wall's pairings of W, W' and i, i' are inclusion maps of ∂W, $\partial W'$ into W, W', respectively.*

If $p\geqq q$, the proposition is trivial since $\phi=\phi'=0$. Hence, it makes sense for $p\leqq q-1$. Note that $\phi(x, x)=(E\circ\pi_*)(\alpha(x))$ by [14], where E is the suspension homomorphism. Immediately we have the following.

Corollary 2. *Under the above assumptions on p, q, if ∂W has the homotopy* *type of* $\sharp_{i=1}^r B_i$, *a connected sum consisting of p-sphere bundles over q-spheres, then W is represented into a boundary connected sum of* $(p+1)$*-disk bundles over q-spheres, and hence* ∂W *into a connected sum of p-sphere bundles over q-spheres. Furthermore, if s bundles in* B_i, $i=1, 2,\cdots, r$, *admit cross-sections,* ∂W *is represented into a connected sum of p-sphere bundles over q-spheres in which s bundles admit cross-sections.*

Corollary 3. *Under the above assumptions on p, q, if* $H=H_q(W)$ *has no admissible bases, then* ∂W *never has the homotopy type of a connected sum of p-sphere bundles over q-spheres.*

Let ω be an element of $\pi_{q-1}(S^p)$. We have the following homomorphisms

$$\pi_{p+q-1}(S^{q-1}) \xrightarrow{\omega_*} \pi_{p+q-1}(S^p) \xleftarrow{J} \pi_{q-1}(SO_p) \xrightarrow{i_*} \pi_{q-1}(SO_{p+1}),$$

where ω_* is defined by the composition with ω, J is the J-homomorphism, and i_* is induced from the inclusion. Let $G(\omega)=i_*(J^{-1}(\mathrm{Im}\,\omega_*))$ (James-Whitehead [9]).

Let H, ϕ, α and $\varepsilon=\pi_*\circ\alpha$ be the invariants of $W=\natural_{i=1}^r \bar{B}_i$, where ε is a quadratic form with the associated homomorphism $\pi_*\circ\partial\circ\phi$. We note that if $p\neq q-1$, q, then $(H; \phi, \alpha)$ is determined from $\partial W=\sharp_{i=1}^r B_i$. In fact, $H=H_q(W) \cong H_q(\partial W)$, $\alpha(w_i)=\alpha(B_i)$, $i=1, 2,\cdots, r$, where $\{w_1,\cdots, w_r\}$ is the canonical basis of H represented by zero cross-sections of $\bar{B}_i$, $i=1, 2,\cdots, r$, and $\phi(w_i, w_j)=0$ if $i\neq j$, $\phi(w_i, w_i)=E\pi_*\alpha(B_i)$ for each i, j. Let B'_i, $i=1, 2,\cdots, r'$, be another set of p-sphere bundles over q-spheres $(p\neq q-1)$. If $\sharp_{i=1}^r B_i$ has the homotopy type of $\sharp_{i=1}^{r'} B'_i$, then $r=r'$ by those homological aspect. Therefore, we assume that $r=r'$ henceforth. Similarly define H', ϕ', α', and ε' for $W'=\natural_{i=1}^r \bar{B}'_i$. Let α_i, α'_i be the characteristic elements of B_i, B'_i respectively and put $\varepsilon_i=\pi_*(\alpha_i)$, $\varepsilon'_i=\pi_*(\alpha'_i)$, $i=1, 2,\cdots, r$. We obtain the following.

Theorem 4. *Let* $q/2<p<q-1$. *Then, the connected sums* $\sharp_{i=1}^r B_i$, $\sharp_{i=1}^r B'_i$ *are of the same oriented homotopy type if and only if* $\varepsilon_i=\varepsilon'_i$ *and* $\{\alpha_i\}=\{\alpha'_i\}$ *in* $\pi_{q-1}(SO_{p+1})/G(\varepsilon_i)=\pi_{q-1}(SO_{p+1})/G(\varepsilon'_i)$ *for* $i=1, 2,\cdots, r$ *"modulo representations". More precisely, they are of the same oriented homotopy type if and only if there exist the admissible bases* $\{w_1,\cdots, w_r\}$, $\{w'_1,\cdots, w'_r\}$ *of* H, H' *respectively such that*

(i) $\varepsilon(w_i)=\varepsilon'(w'_i)$, $i=1, 2,\cdots, r$, *(i.e.* $\varepsilon\cong\varepsilon'$*) and*

(ii) $\{\alpha(w_i)\}=\{\alpha'(w'_i)\}$ *in* $\pi_{q-1}(SO_{p+1})/G(\varepsilon(w_i))=\pi_{q-1}(SO_{p+1})/G(\varepsilon'(w'_i))$, $i=1, 2,\cdots, r$.

If all B_i, B'_i, $i=1, 2,\cdots, r$, admit cross-sections, then $\phi=\phi'=0$. So, any bases of H, H' are admissible, α, α' are the homomorphisms, and $\varepsilon=\varepsilon'=0$. Furthermore, $G(0)=i_*J^{-1}(0)$ induces $i_*\pi_{q-1}(SO_p)/G(0)\cong J\pi_{q-1}(SO_p)/P\pi_q(S^p)$, where $P=[\ ,\ \iota_p]$ and ι_p is the orientation generator of $\pi_p(S^p)$ (cf. [10], p. 152). Hence, we have Theorem 1 of [8] for $p<q-1$.

Proposition 1 is proved in Section 2, and using it Theorem 4 is proved in Section 4 and Section 5.

§1. Cell Structure and Linking Elements

Let $W=D^{m+1}\cup_{\{\varphi_i\}}\{\cup_{i=1}^r D_i^q\times D_i^{p+1}\}$ be a handlebody of $\mathscr{H}(m+1, r, q)$, $m=p+q$, p, $q>1$, where φ_i: $\partial D_i^q\times D_i^{p+1}\to\partial D^{m+1}$, $i=1, 2,\cdots, r$, are the disjoint imbeddings. Let $Y=S^m-\cup_{i=1}^r \operatorname{Int}\varphi_i(S_i^{q-1}\times D_i^{p+1})$. Then $\partial W=Y\cup_{\{\bar\varphi_i\}}\{\cup_{i=1}^r D_i^q\times S_i^p\}$, where $\bar\varphi_i=\varphi_i\,|\,S_i^{q-1}\times S_i^p$, $i=1, 2,\cdots, r$. Let $\tilde S_i^p\subset\operatorname{Int} Y$ be the imbedded p-sphere slightly moved from $x_i\times S_i^p$, $x_i\in\partial D_i^q$, where $i=1, 2,\cdots, r$. We join $\tilde S_i^p$, $i=1, 2,\cdots, r$, by r arcs in Int Y from a fixed point and take a thin closed neighbourhood N. N has the homotopy type of $\vee_{i=1}^r S_i^p$.

By the Alexander duality theorem, we have

$$H_i(N)\cong H_i(Y) \qquad \text{if} \quad i<m-1\,,$$

and, since N, Y are simply connected,

$$\pi_i(N)\cong\pi_i(Y) \qquad \text{if} \quad i<m-2\,,$$

where the isomorphisms are induced from the inclusion map. So that, $H_i(Y, N)\cong 0$ for $i<m-1$, and therefore by the homology exact sequence of $(\partial W, Y, N)$, we have

$$H_i(\partial W, N)\cong H_i(\partial W, Y) \qquad \text{if} \quad i<m-1\,.$$

Here, by the excision theorem,

$$H_i(\partial W, Y)\cong\begin{cases} \boldsymbol{Z}+\cdots+\boldsymbol{Z} & \text{if} \quad i=q,\ m \\ 0 & \text{otherwise,}\end{cases}$$

and $[D_i^q\times y_i]$, $y_i\in S_i^p$, $i=1, 2,\cdots, r$, form a basis of $H_q(\partial W, Y)$. Hence, noting that N, Y and ∂W are simply connected and $H_i(\partial W, N)\cong H_i(\partial W, Y)\cong 0$ for $i<q$, we know

$$\pi_q(\partial W, N)\cong H_q(\partial W, N)\,,$$
$$\pi_q(\partial W, Y)\cong H_q(\partial W, Y)\,,$$

by the Hurewicz isomorphism theorem.

Let $V=\partial W-\operatorname{Int} D^m$ and we may assume that $N\subset \operatorname{Int} V$. Then, by the homology exact sequence of $(\partial W, V, N)$,

$$H_i(V, N)\cong H_i(\partial W, N) \qquad \text{if} \quad i<m\,,$$

and similarly as above,

$$\pi_q(V, N)\cong H_q(V, N)\,.$$

Thus, we have the following commutative diagram

$$\begin{array}{ccccc}
H_q(V, N) & \xrightarrow{\cong} & H_q(\partial W, N) & \xrightarrow{\cong} & H_q(\partial W, Y) \\
\uparrow{\scriptstyle\cong} & & \uparrow{\scriptstyle\cong} & & \uparrow{\scriptstyle\cong} \\
\pi_q(V, N) & \xrightarrow{\cong} & \pi_q(\partial W, N) & \xrightarrow{\cong} & \pi_q(\partial W, Y) \\
& {\scriptstyle\partial}\searrow & \downarrow{\scriptstyle\partial} & & \downarrow{\scriptstyle\partial} \\
& & \pi_{q-1}(N) & \xrightarrow{\cong} & \pi_{q-1}(Y) \\
& & \uparrow{\scriptstyle\cong} & \nearrow{\scriptstyle\cong} & \\
& & \pi_{q-1}(\bigvee_{i=1}^{r} S_i^p)\,, & &
\end{array}$$

where the horizontal isomorphisms are all induced from the inclusion maps.

We note that $H_{m-1}(\partial W, N)\cong 0$ by the homology exact sequence of $(\partial W, N)$ and $H_m(V, N)\cong 0$. So that, $H_i(V, N)\cong 0$ if $i\neq q$. Let $m>5$. Then, by [12] or applying Theorem 7.6 and 7.8 of [11] to the triad $(V'; \partial N, S^{m-1})$, where $V'=V-\operatorname{Int} N$, we obtain the q-handles T_i in V', $i=1, 2,\cdots, r$, such that the homology classes $[T_i]$, $i=1, 2,\cdots, r$, form the basis of $H_q(V', \partial N)\cong H_q(V, N)$ which corresponds to the basis $\{[D_i^q\times y_i],\ i=1, 2,\cdots, r\}$ of $H_q(\partial W, Y)$. We may identify V with $N\cup T_1\cup T_2\cup\cdots\cup T_r$.

Henceforth, we assume that $2p>q>1$ and $m>5$. Let $\lambda_j=\sum_{i=1}^r \lambda_{ij} \in \pi_{q-1}(\vee_{i=1}^r S_i^p)=\sum_{i=1}^r \pi_{q-1}(S_i^p)$ be the linking element of the link $\{\cup_{i=1}^r \varphi_i(S_i^{q-1}\times o)\}\cup \varphi_j(S_j^{q-1}\times y_j)\subset S^m$ defined by $\varphi_j(S_j^{q-1}\times y_j)\subset S^m-\cup_{i=1}^r \varphi_i(S_i^{q-1}\times o)\simeq Y$. $\lambda_{ij}\in\pi_{q-1}(S_i^p)$ coincides with the linking element of the link $\varphi_i(S_i^{q-1}\times o)\cup\varphi_j(S_j^{q-1}\times o)\subset S^m$ defined by $\varphi_j(S_j^{q-1}\times o)\subset S^m-\varphi_i(S_i^{q-1}\times o)\simeq S_i^p$ $(i\neq j)$. λ_{jj} coincides with the linking element of the link $\varphi_j(S_j^{q-1}\times o)\cup\varphi_j(S_j^{q-1}\times y_j)\subset S^m$ and is called the self-linking element of $\varphi_j(S_j^{q-1}\times o)$. Note that $\lambda_{jj}=\pi_*\alpha(w_j)$, where w_j is the basis element of $H_q(W)\cong H_q(W, D^{m+1})$ determined by $[D_j^q\times o]$. Let $v_j\in\pi_{q-1}(Y)$ be the homotopy class of $\varphi_j\,|\,S_j^{q-1}\times y_j$, $j=1, 2,\cdots, r$. Then, in the above diagram, v_j corresponds to λ_j for $j=1, 2,\cdots, r$. Hence, by commutativity of the diagram, the attaching map of the q-axis of T_j is given by λ_j, $j=1, 2,\cdots, r$. Thus, we have

Lemma 1.1. *Let* $W=D^{m+1}\cup_{\{\varphi_i\}}\{\cup_{i=1}^r D_i^q\times D_i^{p+1}\}$ *be a handlebody of* $\mathscr{H}(m+1, r, q)$, *where* $m=p+q$ *and* φ_i: $\partial D_i^q\times D_i^{p+1}\to\partial D^{m+1}$, $i=1, 2,\cdots, r$, *are disjoint imbeddings. We assume that* $2p>q>1$ *and* $(p, q)\neq(2, 3)$. *Then,* ∂W *has the homotopy type of*

$$(\bigvee_{i=1}^r S_i^p)\cup(\bigcup_{j=1}^r D_j^q)\cup D^m$$

and the attaching map of each D_j^q *is given by* $\lambda_j=\sum_{i=1}^r \lambda_{ij}\in\pi_{q-1}(\vee_{i=1}^r S_i^p)=\sum_{i=1}^r \pi_{q-1}(S_i^p)$, *where each* λ_{ij} *is the linking element of the link* $\varphi_i(S_i^{q-1}\times o)\cup\varphi_j(S_j^{q-1}\times o)\subset S^m$ $(i\neq j)$ *and* λ_{jj} *is the self-linking element of* $\varphi_j(S_j^{q-1}\times o)\subset S^m$, $i, j=1, 2,\cdots, r$.

Remark. In each additional case for $m=4, 5$, the lemma holds trivially since ∂W is represented as a connected sum of p-sphere bundles over q-spheres which admit cross-sections.

§2. Proof of Proposition 1

Let W, W' be the handlebodies of $\mathscr{H}(m+1, r, q)$, $m=p+q$, and assume that $1<p<q-1$ or $p>q>1$. Let $W'=D'^{m+1}\cup_{\{\varphi'_i\}}\{\cup_{i=1}^r D_i'^q\times D_i'^{p+1}\}$ be a representation, where φ'_i: $\partial D_i'^q\times D_i'^{p+1}\to\partial D'^{m+1}$, $i=1, 2,\cdots, r$, are disjoint imbeddings. By the assumption on p, q, we know that $H_k(\partial W')\cong 0$ if $k\neq 0, p, q, m$, $H_p(\partial W')$ has the basis $u'_i=[x'_i\times S_i'^p]$, $i=1, 2,\cdots, r$, and $H_q(\partial W')$ has the basis v'_j, $j=1, 2,\cdots, r$, which corresponds to $[D_j'^q\times o]\in H_q(W', D'^{m+1})$, $j=1, 2,\cdots, r$, under the isomorphisms induced from the inclusion maps $H_q(\partial W')\cong H_q(W')\cong H_q(W', D'^{m+1})$. We call $\{u'_1,\cdots, u'_r\}$, $\{v'_1,\cdots, v'_r\}$ to be the bases associated with the handles of W'.

Lemma 2.1. *For any homotopy equivalence* f: $\partial W\to\partial W'$ *which preserves orientation, there exists a representation* $W=D^{m+1}\cup_{\{\varphi_i\}}\{\cup_{i=1}^r D_i^q\times D_i^{p+1}\}$, *where* φ_i: $\partial D_i^q\times D_i^{p+1}\to\partial D^{m+1}$, $i=1, 2,\cdots, r$, *are disjoint imbeddings, such that* $f_*(u_i)=u'_i$, $f_*(v_j)=v'_j$ *for* $i, j=1, 2,\cdots, r$. *Here,* $\{u_1,\cdots, u_r\}$, $\{v_1,\cdots, v_r\}$ *are the bases of* $H_p(\partial W)$, $H_q(\partial W)$ *respectively associated with the handles of* W.

Proof. Let $\tilde{u}_i=f_*^{-1}(u'_i)$, $\tilde{v}_j=f_*^{-1}(v'_j)$, $i, j=1, 2,\cdots, r$, and let $\tilde{w}_j=i_*(\tilde{v}_j)$, $j=1, 2,\cdots, r$, where i_*: $H_q(\partial W)\cong H_q(W)$ is induced from the inclusion map.

We represent W by the basis $\{\tilde{w}_1,\cdots,\tilde{w}_r\}$ (cf. Milnor [11], Theorem 7.6). So, we have a representation $W=D^{m+1}\cup_{\{\varphi_i\}}\{\cup_{i=1}^r D_i^q\times D_i^{p+1}\}$. Then, clearly $i_*(v_j)=\tilde{w}_j=i_*(\tilde{v}_j)$ and therefore $\tilde{v}_j=v_j$, $j=1,2,\cdots,r$. Furthermore, $\tilde{u}_i\cdot\tilde{v}_j=\delta_{ij}$, $i,j=1,2,\cdots,r$, and $u_i\cdot\tilde{v}_j=\partial[x_i\times D_i^{p+1}]\cdot\tilde{v}_j=[x_i\times D_i^{p+1}]\cdot(i_*(\tilde{v}_j))=[x_i\times D_i^{p+1}]\cdot\tilde{w}_j=\delta_{ij}$, $i,j=1,2,\cdots,r$. Hence, $\tilde{u}_i=u_i$, $i=1,2,\cdots,r$.

Now, we prove Proposition 1. If $p\geqq q$, the assertion holds trivially. So, we assume that $2p>q>1$ and $p<q-1$. Let $f\colon\partial W\to\partial W'$ be a homotopy equivalence which preserves orientation. Let $W=D^{m+1}\cup_{\{\varphi_i\}}\{\cup_{i=1}^r D_i^q\times D_i^{p+1}\}$ be the representation given by Lemma 2.1. Then, by Lemma 1.1, we have the following diagram commutative up to homotopy.

$$\begin{array}{ccc}
\partial W & \xrightarrow[\simeq]{f} & \partial W' \\
\uparrow{\scriptstyle\simeq} & & \uparrow{\scriptstyle\simeq} \\
(\bigvee_{i=1}^r S_i^p)\cup_{\{\lambda_j\}}(\bigcup_{j=1}^r D_j^q)\cup D^m & \xrightarrow[\simeq]{g} & (\bigvee_{i=1}^r S_i'^p)\cup_{\{\lambda_j'\}}(\bigcup_{j=1}^r D_j'^q)\cup D'^m
\end{array}$$

It may be assumed that $g((\vee_{i=1}^r S_i^p)\cup_{\{\lambda_j\}}(\cup_{j=1}^r D_j^q))\subset(\vee_{i=1}^r S_i'^p)\cup_{\{\lambda_j'\}}(\cup_{j=1}^r D_i'^q)$ and each $g\,|\,S_i^p$ is the identity (S_i^p, $S_i'^p$ are copies of S^p) since $f_*(u_i)=u_i'$, $i=1,2,\cdots,r$. Hence, we have the following commutative diagram, where we put $X=(\vee_{i=1}^r S_i^p)\cup_{\{\lambda_j\}}(\cup_{j=1}^r D_j^q)\cup D^m$ and $X'=(\vee_{i=1}^r S_i'^p)\cup_{\{\lambda_j'\}}(\cup_{j=1}^r D_j'^q)\cup D'^m$.

$$\begin{array}{ccc}
H_q(\partial W) & \xrightarrow{f_*} & H_q(\partial W') \\
\uparrow{\scriptstyle\cong} & & \uparrow{\scriptstyle\cong} \\
H_q(X) & \xrightarrow{g_*} & H_q(X') \\
\downarrow{\scriptstyle\cong} & & \downarrow{\scriptstyle\cong} \\
H_q(X,\bigvee_{i=1}^r S_i^p) & \xrightarrow{\bar{g}_*} & H_q(X',\bigvee_{i=1}^r S_i'^p) \\
\uparrow{\scriptstyle\cong} & & \uparrow{\scriptstyle\cong} \\
\pi_q(X,\bigvee_{i=1}^r S_i^p) & \xrightarrow{\bar{g}_*} & \pi_q(X',\bigvee_{i=1}^r S_i'^p) \\
\downarrow{\scriptstyle\partial} & & \downarrow{\scriptstyle\partial} \\
\pi_{q-1}(\bigvee_{i=1}^r S_i^p) & \xrightarrow{(g|\bigvee_{i=1}^r S_i^p)_*=1} & \pi_{q-1}(\bigvee_{i=1}^r S_i'^p)
\end{array}$$

Note that each v_j, v_j' correspond to $[D_j^q]\in H_q(X,\vee_{i=1}^r S_i^p)$, $[D_j'^q]\in H_q(X',\vee_{i=1}^r S_i'^q)$ respectively. $\{D_j^q\}\in\pi_q(X,\vee_{i=1}^r S_i^p)$, $\{D_j'^q\}\in\pi_q(X',\vee_{i=1}^r S_i'^p)$ correspond to $[D_j^q]$, $[D_j'^q]$ under the Hurewicz isomorphisms. Then, since $f_*(v_i)$

$=v'_j$, $j=1, 2,\cdots, r$, we know that $\lambda_j=\partial\{D_j^q\}=\partial\bar{g}_*\{D_j^q\}=\partial\{D_j'^q\}=\lambda'_j$, $j=1, 2,\cdots, r$. So that, $\lambda_{ij}=\lambda'_{ij}$ for all $i, j=1, 2,\cdots, r$. On the other hand, $E\lambda_{ij}=\phi(w_i, w_j)$, $E\lambda'_{ij}=\phi'(w'_i, w'_j)$ by Lemma 7 of Wall [14], where E is the suspension homomorphism, $w_j=i_*(v_j)$, $w'_j=i'_*(v'_j)$, $j=1, 2,\cdots, r$, and i_*, i'_* are the isomorphisms induced from inclusion maps. Therefore, $\phi(w_i, w_j)=\phi'(w'_i, w'_j)$, $i, j=1, 2,\cdots, r$, and this yields $\phi(w_i, w_j)=\phi'\circ(h\times h)(w_i, w_j)$, $i, j=1, 2,\cdots, r$, where $h=i'_*\circ f_*\circ i_*^{-1}$. This completes the proof.

Let B_i, B'_i, $i=1, 2,\cdots, r$, be p-sphere bundles over q-spheres ($p, q>1$) with the characteristic elements α_i, α'_i, $i=1, 2,\cdots, r$, respectively. Then, $\#_{i=1}^r B_i$ has the homotopy type of $X=(\vee_{i=1}^r S_i^p)\cup(\cup_{i=1}^r D_i^q)\cup D^{p+q}$, and $\#_{i=1}^r B'_i$ the homotopy type of $X'=(\vee_{i=1}^r S_i'^p)\cup(\cup_{i=1}^r D_i'^q)\cup D'^{p+q}$, where each D_i^q, $D_i'^q$ are attached to S_i^p, $S_i'^p$ by $\varepsilon_i=\pi_*\alpha_i$, $\varepsilon'_i=\pi_*\alpha'_i$ respectively (cf. [8] § 1). Let $p<q-1$. Let $\{u_i;\ i=1, 2,\cdots, r\}$ be the basis of $H_p(\#_{i=1}^r B_i)$ represented by the fibres of B_i, $i=1, 2,\cdots, r$. Since $H_q(\#_{i=1}^r B_i)\cong H_q(\#_{i=1}^r \bar{B}_i)$, the zero cross-sections of $\bar{B}_i$, $i=1, 2,\cdots, r$, determine the basis $\{v_1,\cdots, v_r\}$ of $H_q(\#_{i=1}^r B_i)$. u_i corresponds to $[S_i^p]\in H_p(X)$ and v_i to $[D_i^q]\in H_q(X)$, $i=1, 2,\cdots, r$. Those are the bases associated with handles if we consider $\#_{i=1}^r \bar{B}_i$ to be a handlebody. Similarly define $\{u'_i;\ i=1, 2,\cdots, r\}$, $\{v'_i;\ i=1, 2,\cdots, r\}$ for $\#_{i=1}^r B'_i$. Then, the above diagram and a similar argument will show the following, where $\pi_{q-1}(S_i^p)$, $\pi_{q-1}(S_i'^p)$ are direct summands of $\pi_{q-1}(\vee_{i=1}^r S_i^p)$, $\pi_{q-1}(\vee_{i=1}^r S_i'^p)$ respectively, $i=1, 2,\cdots, r$.

Lemma 2.2. *Let $1<p<q-1$. If there exists a map f: $\#_{i=1}^r B_i\to\#_{i=1}^r B'_i$ such that $f_*(u_i)=u'_i, f_*(v_j)=v'_j$, $i, j=1, 2,\cdots, r$, then $\varepsilon_i=\varepsilon'_i$ for $i=1, 2,\cdots, r$. Here, if $p\geqq q$ the assertion is trivial.*

§ 3. Difference of Bundles

Let B_i, B'_i be p-sphere bundles over q-spheres with the characteristic elements α_i, α'_i respectively, $i=1, 2,\cdots, r$, and assume that $\varepsilon_i=\varepsilon'_i$, where $\varepsilon_i=\pi_*(\alpha_i)$, $\varepsilon'_i=\pi_*(\alpha'_i)$. Let S_i^p, p_i, and S_i^q be respectively the fixed fibre, the projection, and the base space of B_i. Define $S_i'^p$, p'_i, and $S_i'^q$ similarly for B'_i. In the disjoint union of B_i and B'_i, identify S_i^p with $S_i'^p$. Then, we have a p-sphere bundle over $S_i^q\vee S_i'^q$, where S_i^q, $S_i'^q$ are identified at $s_i=p_i(S_i^p)=p'_i(S_i'^p)$. Since B_i, B'_i are included in this bundle as subspaces, we may denote it by $B_i\cup B'_i$ (cf. [10], p. 156).

Let $g_i\colon S^q \to S_i^q \vee S_i'^q$ be a map representing $\iota_q^i - \iota_q'^i \in \pi_q(S_i^q \vee S_i'^q)$, where ι_q^i, $\iota_q'^i$ are the orientation generators of $\pi_q(S_i^q)$, $\pi_q(S_i'^q)$ respectively. The induced bundle $A_i = g_i^*(B_i \cup B_i')$ has the characteristic element $\alpha_i - \alpha_i'$ and admits a cross-section since $\pi_*(\alpha_i - \alpha_i') = \pi_*(\alpha_i) - \pi_*(\alpha_i') = \varepsilon_i - \varepsilon_i' = 0$ by the above assumption. Let $h_i\colon A_i \to B_i \cup B_i'$ be the bundle map which covers g_i. A fixed fibre $S_{A_i}^p$ of A_i is oriented so that $h_i | S_{A_i}^p\colon S_{A_i}^p \to S_i^p = S_i'^p$ is of degree 1, and A_i is oriented by the orientations of $S_{A_i}^p$ and S^q.

Let $S_{A_i}^q$ be the cross-section of A_i associated with $\xi_i \in \pi_{q-1}(SO_p)$ satisfying $i_*(\xi_i) = \alpha_i - \alpha_i'$. Then, $A_i = (S_{A_i}^p \vee S_{A_i}^q) \cup e_{A_i}^{p+q}$ and the attaching map is given by $\partial\tau_i = \iota_p^{A_i} \circ \eta_i + [\iota_q^{A_i}, \iota_p^{A_i}]$, where $\eta_i = J\xi_i$ and τ_i is the orientation generator of $\pi_{p+q}(A_i, S_{A_i}^p \vee S_{A_i}^q)$ (cf. [9]). Hence, by Lemma 1.1 of [8],

$$(3.1) \qquad \mathop{\#}_{i=1}^{r} A_i \simeq A = \{\bigvee_{i=1}^{r} (S_{A_i}^p \vee S_{A_i}^q)\} \cup e_{A_i}^{p+q},$$

and the attaching map of the $(p+q)$-cell is given by

$$(3.2) \qquad \partial\tau = \sum_{i=1}^{r} (\iota_p^{A_i} \circ \eta_i + [\iota_q^{A_i}, \iota_p^{A_i}]),$$

where τ is the orientation generator of $\pi_{p+q}(A, \vee_{i=1}^r (S_{A_i}^p \vee S_{A_i}^q))$ and $\pi_{p+q-1}(S_{A_i}^p \vee S_{A_i}^q)$, $i = 1, 2, \cdots, r$, are considered as direct summands of $\pi_{p+q-1}(\vee_{i=1}^r (S_{A_i}^p \vee S_{A_i}^q))$.

In $A_1 \# A_2 \# \cdots \# A_r$, join every $(p+q-1)$-sphere where connected sum is performed to the base points of the bundles neighboring at the $(p+q-1)$-sphere by suitably chosen arcs. If we crush the $(p+q-1)$-spheres and the arcs to a point, the yielding space can be considered as $\vee_{i=1}^r A_i$. Let $\nu\colon \#_{i=1}^r A_i \to \vee_{i=1}^r A_i$ be the collapsing map. Then, we have a map

$$h = (\bigvee_{i=1}^{r} h_i) \circ \nu\colon \mathop{\#}_{i=1}^{r} A_i \longrightarrow \bigvee_{i=1}^{r} (B_i \cup B_i').$$

$\#_{i=1}^r A_i$ can be replaced by the complex A of (3.1) and h may be assumed to preserve the base point. We denote the map by the same symbol h.

B_i has the cell structure $B_i = S_i^p \cup e_i^q \cup e_i^{p+q}$, where e_i^q is attached to S_i^p by $\iota_p^i \circ \varepsilon_i$. Here, ι_p^i is the orientation generator of $\pi_p(S_i^p)$. Let σ_i be the orientation generator of $\pi_{p+q}(B_i, S_i^p \cup e_i^q)$. Then, $\partial\sigma_i \in \pi_{p+q-1}(S_i^p \cup e_i^q)$ is represented by the attaching map of e_i^{p+q}. Similarly to Lemma 1.1 of [8], it is seen that

$$(3.3) \qquad \mathop{\#}_{i=1}^{r} B_i \simeq B = \{\bigvee_{i=1}^{r} (S_i^p \cup e_i^q)\} \cup e^{p+q},$$

where each e_i^q is attached by $\iota_p^i \circ \varepsilon_i$, and

(3.4) $$\partial\sigma=\partial\sigma_1+\partial\sigma_2+\cdots+\partial\sigma_r,$$

where σ is the orientation generator of $\pi_{p+q}(B, \vee_{i=1}^r(S_i^p\cup e_i^q))$ and each $\pi_{p+q-1}(S_i^p\cup e_i^q)$ is considered as a direct summand of $\pi_{p+q-1}(\vee_{i=1}^r(S_i^p\cup e_i^q))$. Let $B_i'=S_i'^p\cup e_i'^q\cup e_i'^{p+q}$ be the cell structure of B_i'. $e_i'^q$ is attached to $S_i'^p$ by $\iota_p'^i\circ\varepsilon_i'$, where $\iota_p'^i$ is the orientation generator of $\pi_p(S_i'^p)$. Let σ_i' be the orientation generator of $\pi_{p+q}(B_i', S_i'^p\cup e_i'^q)$.

Let $K=\vee_{i=1}^r(S_i^p\cup e_i^q\cup e_i'^q)$ be the subcomplex of $\vee_{i=1}^r(B_i\cup B_i')$, where e_i^q, $e_i'^q$ are attached to $S_i^p=S_i'^p$ by $\iota_p^i\circ\varepsilon_i$ and $\iota_p'^i\circ\varepsilon_i'$ respectively. Then, it may be assumed that $h(\vee_{i=1}^r(S_{A_i}^p\vee S_{A_i}^q))\subset K$ for the map $h\colon A\to\vee_{i=1}^r(B_i\cup B_i')$. Let $\bar h\colon(A, \vee_{i=1}^r(S_{A_i}^p\vee S_{A_i}^q))\to(\vee_{i=1}^r(B_i\cup B_i'), K)$. From the construction of h, we know

(3.5) $$\bar h_*(\tau)=(\bar\sigma_1-\bar\sigma_1')+(\bar\sigma_2-\bar\sigma_2')+\cdots+(\bar\sigma_r-\bar\sigma_r'),$$

where $\bar\sigma_i$ is the image of σ_i by the homomorphism induced from the inclusion $(B_i, S_i^p\cup e_i^q)\subset(B_i\cup B_i', S_i^p\cup e_i^q\cup e_i'^q)$, $i=1, 2,\cdots, r$, $\bar\sigma_i'$ is similar, and $\pi_{p+q}(B_i\cup B_i', S_i^p\cup e_i^q\cup e_i'^q)$, $i=1, 2,\cdots, r$, are considered as the direct summands of $\pi_{p+q}(\vee_{i=1}^r(B_i\cup B_i'), K)$. Let $\delta_i=\partial\sigma_i$, and let $\bar\delta_i$ be the image of δ_i by the homomorphism induced from the inclusion $S_i^p\cup e_i^q\subset S_i^p\cup e_i^q\cup e_i'^q$, $i=1, 2,\cdots, r$. Define δ_i', $\bar\delta_i'$ similarly, $i=1, 2,\cdots, r$. Here, $\pi_{p+q-1}(S_i^p\cup e_i^q\cup e_i'^q)$, $i=1, 2,\cdots, r$, are understood as direct summands of $\pi_{p+q-1}(\vee_{i=1}^r(S_i^p\cup e_i^q\cup e_i'^q))$. Then,

$$\partial\bar h_*\tau=\partial\sum_{i=1}^r(\bar\sigma_i-\bar\sigma_i')=\sum_{i=1}^r(\partial\bar\sigma_i-\partial\bar\sigma_i')=\sum_{i=1}^r(\bar\delta_i-\bar\delta_i')=\sum_{i=1}^r\bar\delta_i-\sum_{i=1}^r\bar\delta_i',$$

and by (3.2),

$$\partial\bar h_*\tau=h_*\partial\tau=\sum_{i=1}^r h_*(\iota_p^{A_i}\circ\eta_i+[\iota_q^{A_i}, \iota_p^{A_i}]).$$

Hence, we have

(3.6) $$\sum_{i=1}^r\bar\delta_i-\sum_{i=1}^r\bar\delta_i'=\sum_{i=1}^r h_*(\iota_p^{A_i}\circ\eta_i+[\iota_q^{A_i}, \iota_p^{A_i}]).$$

§4. Proof of the Necessity for Theorem 4

Let B_i, B_i', $i=1, 2,\cdots, r$, be p-sphere bundles over q-spheres with the characteristic elements α_i, α_i' respectively and assume that $q/2<p<q-1$. Let $f\colon\#_{i=1}^r B_i\to\#_{i=1}^r B_i'$ be a homotopy equivalence which preserves orientation.

Assertion 1. *There exists another expression of* $\#_{i=1}^r B_i$ *into a connected* *sum of p-sphere bundles over q-spheres* $\sharp_{i=1}^{r}\tilde{B}_i$ *such that in the cell decompositions* $\sharp_{i=1}^{r}\tilde{B}_i\simeq\tilde{B}=\{\vee_{i=1}^{r}(\tilde{S}_i^p\cup\tilde{e}_i^q)\}\cup\tilde{e}^{p+q}$ *and* $\sharp_{i=1}^{r}B_i'\simeq B=\{\vee_{i=1}^{r}(S_i'^p\cup e_i'^q)\}\cup e'^{p+q}$, $f_*: H_*(\tilde{B})\to H_*(B')$ *satisfies* $f_*([\tilde{S}_i^p])=[S_i'^p]$, $i=1, 2,\cdots, r$ *and* $\bar{f}_*([\tilde{e}_j^q])=[e_j'^q]$, $j=1, 2,\cdots, r$, *where f may be assumed to satisfy* $f(\vee_{i=1}^{r}\tilde{S}_i^p)\subset\vee_{i=1}^{r}S_i'^p$ *and* $\bar{f}$: $(\tilde{B}, \vee_{i=1}^{r}\tilde{S}_i^p)\to(B', \vee_{i=1}^{r}S_i'^p)$ *is the relativization of f.*

Proof. Let $W=\natural_{i=1}^{r}\bar{B}_i$, $W'=\natural_{i=1}^{r}\bar{B}_i'$, and let $\{u_1',\cdots, u_r'\}$, $\{v_1',\cdots, v_r'\}$ be the bases of $H_p(\partial W')$, $H_q(\partial W')$ respectively associated with the handles of W'. Then, by Lemma 2.1, there exists a representation of W into such a handlebody that $f_*(u_i)=u_i'$, $f_*(v_j)=v_j'$, $i, j=1, 2,\cdots, r$, where $\{u_1,\cdots, u_r\}$, $\{v_1,\cdots, v_r\}$ are bases of $H_p(\partial W)$, $H_q(\partial W)$ respectively associated with the new handles of W. Of course, $\{w_j'=i_*'v_j';\, j=1, 2,\cdots, r\}$, the basis of $H_q(W')$ is admissible since w_j', $j=1, 2,\cdots, r$, are represented by zero cross-sections of $\bar{B}_j'$, $j=1, 2,\cdots, r$. Hence, by Proposition 1, the basis of $H_q(W)$, $\{w_j=i_*v_j;\, j=1, 2,\cdots, r\}$ is admissible. Therefore, again W can be represented into a boundary connected sum of $(p+1)$-disk bundles over q-spheres $\natural_{i=1}^{r}\tilde{\bar{B}}_i$ and ∂W into a connected sum of p-sphere bundles over q-spheres $\sharp_{i=1}^{r}\tilde{B}_i$. We note that in the above cell-decompositions, u_i, v_j, u_i', and v_j' correspond to $[\tilde{S}_i^p]$, $[\tilde{e}_j^q]$, $[S_i'^p]$, and $[e_j'^q]$ respectively for each i, j. This completes the proof.

Assertion 2. *Under the cell decompositions* $\sharp_{i=1}^{r}B_i\simeq B=\{\vee_{i=1}^{r}(S_i^p\cup e_i^q)\}\cup e^{p+q}$ *and* $\sharp_{i=1}^{r}B_i'\simeq B'=\{\vee_{i=1}^{r}(S_i'^p\cup e_i'^q)\}\cup e'^{p+q}$, *if* $f_*: H_*(B)\to H_*(B')$ *satisfies* $f_*([S_i^p])=[S_i'^p]$, $\bar{f}_*([e_j^q])=[e_j'^q]$ *for* $i, j=1, 2,\cdots, r$, *then* $\varepsilon_i=\varepsilon_i'$, $i=1, 2,\cdots, r$, *where* $\varepsilon_i=\pi_*\alpha_i$, $\varepsilon_i'=\pi_*\alpha_i'$, *and* $\{\alpha_i\}=\{\alpha_i'\}$, $i=1, 2,\cdots, r$, *in* $\pi_{q-1}(SO_{p+1})/G(\varepsilon_i)=\pi_{q-1}(SO_{p+1})/G(\varepsilon_i')$.

Proof. The former half of the assertion is known immediately from Lemma 2.2. To prove the latter half, we apply Section 3. By the assumption, we may assume that f maps each S_i^p identically onto $S_i'^p$ (S_i^p and $S_i'^p$ can be identified by means of $\iota_p'^i\circ(\iota_p^i)^{-1}$) and $f(\vee_{i=1}^{r}(S_i^p\cup e_i^q))\subset\vee_{i=1}^{r}(S_i'^p\cup e_i'^q)$. Let $f^0=f|\vee_{i=1}^{r}(S_i^p\cup e_i^q)$ and let ρ: $K\to\vee_{i=1}^{r}(S_i'^p\cup e_i'^q)$ be the retraction defined by $\rho|\vee_{i=1}^{r}(S_i^p\cup e_i^q)=f^0$ and $\rho|\vee_{i=1}^{r}(S_i^p\cup e_i'^q)=$identity. Then, we have the following commutative diagram.

$$
\begin{array}{ccccc}
\pi_q(\bigvee_{i=1}^{r} S_i^p) & \xrightarrow{k_*^0} & \pi_q(K) & \xrightarrow{l_*^0} & \pi_q(K, \bigvee_{i=1}^{r} S_i^p) \\
\downarrow{\scriptstyle \rho_*=1} & & \downarrow{\scriptstyle \rho_*} & & \downarrow{\scriptstyle \bar{\rho}_*} \\
\pi_q(\bigvee_{i=1}^{r} S_i'^p) & \xrightarrow{k_*'} & \pi_q(\bigvee_{i=1}^{r}(S_i'^p\cup e_i'^q)) & \xrightarrow{l_*'} & \pi_q(\bigvee_{i=1}^{r}(S_i'^p\cup e_i'^q), \bigvee_{i=1}^{r} S_i'^p)\,,
\end{array}
\tag{4.1}
$$

where k^0, l^0, k', and l' are inclusion maps and $\bar{\rho}$ is the relativization of ρ.

Let j: $(S_i^p \cup e_i^q,\ S_i^p) \to (\vee_{i=1}^r (S_i^p \cup e_i^q),\ \vee_{i=1}^r S_i^p)$, $\bar{m}$: $(S_i^p \cup e_i^q,\ S_i^p) \to (K,\ \vee_{i=1}^r S_i^p)$ be inclusion maps and let j', $\bar{m}'$ be similar for $(S_i'^p \cup e_i'^q,\ S_i'^p)$. Let $\kappa_q^i \in \pi_q(S_i^p \cup e_i^q,\ S_i^p) \cong H_q(S_i^p \cup e_i^q,\ S_i^p)$ be the generator corresponding to $[e_i^q]$ and define $\kappa_q'^i \in \pi_q(S_i'^p \cup e_i'^q,\ S_i'^p)$ similarly. Then, we have

$$\bar{\rho}_* \bar{m}'_*(\kappa_q'^i) = j'_* \kappa_q'^i, \quad \bar{\rho}_* \bar{m}_*(\kappa_q^i) = j'_* \kappa_q'^i. \tag{4.2}$$

The former is clear since $\bar{\rho} \circ \bar{m}' = j'$. The latter is known from the following commutative diagram including the factorization of $\bar{m}_*$.

$$\begin{array}{ccccc}
\bar{m}_*: \pi_q(S_i^p \cup e_i^q,\ S_i^p) & \xrightarrow{j_*} & \pi_q(\bigvee_{i=1}^r (S_i^p \cup e_i^q),\ \bigvee_{i=1}^r S_i^p) & \longrightarrow & \pi_q(K,\ \bigvee_{i=1}^r S_i^p) \\
 & & \downarrow \bar{f}_* & \swarrow \bar{\rho}_* & \\
\pi_q(S_i'^p \cup e_i'^q,\ S_i'^p) & \xrightarrow{j'_*} & \pi_q(\bigvee_{i=1}^r (S_i'^p \cup e_i'^q),\ \bigvee_{i=1}^r S_i'^p), & &
\end{array}$$

where $\bar{f}_*(j_* \kappa_q^i) = j'_* \kappa_q'^i$ since $\bar{f}_*[e_i^q] = [e_i'^q]$ by the assumption.

We apply the homomorphism h_*: $\pi_q(\vee_{i=1}^r (S_{A_i}^p \vee S_{A_i}^q)) \to \pi_q(K)$. $l_*^0 h_* \iota_q^{A_i} = \bar{m}_* \kappa_q^i - \bar{m}'_* \kappa_q'^i$ is clear from the definition of h. Hence, by (4.1) and (4.2),

$$\begin{aligned}
l'_*(\rho_* h_* \iota_q^{A_i}) &= \bar{\rho}_*(l_*^0 h_* \iota_q^{A_i}) = \bar{\rho}_*(\bar{m}_* \kappa_q^i - \bar{m}'_* \kappa_q'^i) \\
&= \bar{\rho}_* \bar{m}_* \kappa_q^i - \bar{\rho}_* \bar{m}'_* \kappa_q'^i = j'_* \kappa_q'^i - j'_* \kappa_q'^i = 0.
\end{aligned}$$

So that,

$$\rho_* h_* \iota_q^{A_i} = k'_* \theta'_i \qquad \text{for some} \quad \theta'_i \in \pi_q(\bigvee_{j=1}^r S_j'^p). \tag{4.3}$$

Then, applying ρ_* to (3.6) and by (4.3),

$$\begin{aligned}
\sum_{i=1}^r \rho_* \bar{\delta}_i - \sum_{i=1}^r \rho_* \bar{\delta}'_i &= \sum_{i=1}^r \rho_* h_*(\iota_p^{A_i} \circ \eta_i + [\iota_q^{A_i},\ \iota_p^{A_i}]) \\
&= \sum_{i=1}^r (\rho_* h_* \iota_p^{A_i} \circ \eta_i + [\rho_* h_* \iota_q^{A_i},\ \rho_* h_* \iota_p^{A_i}]) \\
&= \sum_{i=1}^r (k'_* \iota_p'^i \circ \eta_i + [k'_* \theta'_i,\ k'_* \iota_p'^i]) = \sum_{i=1}^r k'_*(\iota_p'^i \circ \eta_i + [\theta'_i,\ \iota_p'^i]).
\end{aligned}$$

On the other hand, let $\sigma \in \pi_{p+q}(B,\ \vee_{i=1}^r (S_i^p \cup e_i^q))$, $\sigma' \in \pi_{p+q}(B',\ \vee_{i=1}^r (S_i'^p \cup e_i'^q))$ be the orientation generators and let $\delta = \partial \sigma$, $\delta' = \partial \sigma'$. Since f is of degree 1

$$f_*^0 \delta = f_*^0 \partial \sigma = \partial f_* \sigma = \partial \sigma' = \delta',$$

and therefore,

$$\begin{aligned}
\sum_{i=1}^r \rho_* \bar{\delta}_i - \sum_{i=1}^r \rho_* \bar{\delta}'_i &= \sum_{i=1}^r f_*^0 \delta_i - \sum_{i=1}^r \delta'_i = f_*^0(\sum_{i=1}^r \delta_i) - \sum_{i=1}^r \delta'_i \\
&= f_*^0 \delta - \delta' = 0.
\end{aligned}$$

Thus, we have

$$(4.4)\qquad \sum_{i=1}^{r} k'_*(\iota'^i_p\circ\eta_i+[\theta'_i,\ \iota'^i_p])=0\,.$$

(i) Now, we assume that $2p>q+1$. Then, $\pi_q(\vee_{i=1}^r S'^p_i)\cong\pi_q(S'^p_1)\oplus\cdots\oplus\pi_q(S'^p_r)$ and we have the unique summation $\theta'_i=\sum_{j=1}^r\theta'_{ij}$, $\theta'_{ij}\in\pi_q(S'^p_j)$, $j=1, 2,\cdots, r$. Let $\theta'_{ij}=\iota'^j_p\circ\theta_{ij}$, $\theta_{ij}\in\pi_q(S^p)$, for $j=1, 2,\cdots, r$. Then,

$$\begin{aligned}\iota'^i_p\circ\eta_i+[\theta'_i,\ \iota'^i_p]&=\iota'^i_p\circ\eta_i+\sum_{j=1}^r[\theta'_{ij},\ \iota'^i_p]\\ &=\iota'^i_p\circ\eta_i+\sum_{j=1}^r[\iota'^j_p\circ\theta_{ij},\ \iota'^i_p]\\ &=\iota'^i_p\circ(\eta_i+[\theta_{ii},\ \iota^i_p])+\sum_{j\neq i}[\iota'^j_p\circ\theta_{ij},\ \iota'^i_p]\,,\end{aligned}$$

and by Barcus-Barratt [1] or G. W. Whitehead [15],

$$[\iota'^j_p\circ\theta_{ij},\ \iota'^i_p]=[\iota'^j_p,\ \iota'^i_p]\circ(-1)^{p+q}E^{p-1}\theta_{ij}\,,$$

where θ_{ij}, $j=1, 2,\cdots, r$, are the suspension elements. Hence, we have

$$(4.5)\qquad \iota'^i_p\circ\eta_i+[\theta'_i,\ \iota'^i_p]=\iota'^i_p\circ(\eta_i+[\theta_{ii},\ \iota^i_p])+\sum_{j\neq i}[\iota'^j_p,\ \iota'^i_p]\circ(-1)^{p+q}E^{p-1}\theta_{ij}\,.$$

Let $a_i=\iota'^i_p\circ(\eta_i+[\theta_{ii},\ \iota^i_p])$, $\beta_{ji}=(-1)^{p+q}E^{p-1}\theta_{ij}+(-1)^qE^{p-1}\theta_{ji}$, and $b_{ji}=[\iota'^j_p,\ \iota'^i_p]\circ\beta_{ji}$, where $a_i\in\pi_{p+q-1}(S'^p_i)\subset\pi_{p+q-1}(\vee_{i=1}^r S'^p_i)$ and $b_{ji}\in\pi_{p+q-1}(S'^p_j\vee S'^p_i)\subset\pi_{p+q-1}(\vee_{i=1}^r S'^p_i)$, $i, j=1, 2,\cdots, r$. Then, by (4.4) and (4.5), we know

$$(4.6)\qquad \sum_{i=1}^r k'_*a_i+\sum_{i<j}k'_*b_{ij}=0,\qquad \text{for}\quad k'_*:\pi_n(\bigvee_{i=1}^r S'^p_i)\longrightarrow\pi_n(\bigvee_{i=1}^r(S'^p_i\cup e'^q_i))\,,$$
$$n=p+q-1\,.$$

Here, each k'_*a_i belongs to the direct summand $\pi_{p+q-1}(S'^p_i\cup e'^q_i)$. Now, assume temporarily that every k'_*b_{ij} belongs to another direct summand independent of $\pi_{p+q-1}(S'^p_i\cup e'^q_i)$, $i=1, 2,\cdots, r$. This is the fact which will be shown in Assertion 3. Then, (4.6) yields $k'_*a_i=0$ for $i=1, 2,\cdots, r$, and by the commutative diagram

$$\begin{array}{ccc}\pi_{p+q-1}(S^p) & \xrightarrow{k_*} & \pi_{p+q-1}(S^p\underset{\varepsilon'_i}{\cup} e'^q_i)\\ \cong\Big\downarrow \iota'^i_p\circ & & \cong\Big\downarrow \mu_*\\ \pi_{p+q-1}(S'^p_i) & \xrightarrow{k'_*} & \pi_{p+q-1}(S'^p_i\underset{\iota'^i_p\circ\varepsilon'_i}{\cup} e'^q_i)\,,\end{array}$$

where k_* is induced from the inclusion map and μ_* is the canonical isomorphism, we have

(4.7) $$k_*(\eta_i+[\theta_{ii}, \iota_p])=0, \qquad i=1, 2,\cdots, r,$$

where $\eta_i=J\xi_i$, $\xi_i\in\pi_{q-1}(SO_p)$, and $i_*\xi_i=\alpha_i-\alpha'_i$, $i=1, 2,\cdots, r$. Here, $[\theta_{ii}, \iota_p]=-J\partial\theta_{ii}$, $\partial\colon \pi_q(S^p)\to\pi_{q-1}(SO_p)$. Let $\xi'_i=\xi_i-\partial\theta_{ii}\in\pi_{q-1}(SO_p)$. Then, $k_*J\xi'_i=k_*(J\xi_i-J\partial\theta_{ii})=k_*(\eta_i+[\theta_{ii}, \iota_p])=0$, $i=1, 2,\cdots, r$. Since $\operatorname{Ker} k_*=\operatorname{Im}(\varepsilon'_i)_*$, $(\varepsilon'_i)_*=\varepsilon'_i\circ\colon \pi_{p+q-1}(S^{q-1})\to\pi_{p+q-1}(S^p)$ by (3.2) of James-Whitehead [10], $J\xi'_i$ belongs to $\operatorname{Im}(\varepsilon'_i)_*$, and $i_*\xi'_i=i_*(\xi_i-\partial\theta_{ii})=i_*\xi_i=\alpha_i-\alpha'_i$. Hence, we know that $\alpha_i-\alpha'_i\in i_*(J^{-1}(\operatorname{Im}(\varepsilon'_i)_*))=G(\varepsilon'_i)$. That is, $\{\alpha_i\}=\{\alpha'_i\}$ in $\pi_{q-1}(SO_{p+1})/G(\varepsilon_i)=\pi_{q-1}(SO_{p+1})/G(\varepsilon'_i)$, $i=1, 2,\cdots, r$.

(ii) Let $2p=q+1$ $(p, q>1)$. Then, $\pi_q(\vee_{i=1}^r S_i'^p)=\sum_{j=1}^r \iota_p'^j\circ\pi_q(S^p)\oplus\sum_{j<k}[\iota_p'^j, \iota_p'^k]\circ\pi_q(S^{2p-1})$ by Hilton [4]. So that, we have the unique sum $\theta'_i=\sum_{j=1}^r \iota_p'^j\circ\theta_{ij}+\sum_{j<k}[\iota_p'^j, \iota_p'^k]\circ\theta_{ijk}$, where $\theta_{ij}\in\pi_q(S^p)$, $\theta_{ijk}\in\pi_q(S^{2p-1})\cong\boldsymbol{Z}$ for any i, j, k. Therefore,

$$[\theta'_i, \iota_p'^i]=\sum_{j=1}^r[\iota_p'^j\circ\theta_{ij}, \iota_p'^i]+\sum_{j<k}[[\iota_p'^j, \iota_p'^k]\circ\theta_{ijk}, \iota_p'^i].$$

By (7.4) of Barcus-Barratt [1],

$$[\iota_p'^j\circ\theta_{ij}, \iota_p'^i]=[\iota_p'^j, \iota_p'^i]\circ(-1)^{p-1}E^{p-1}\theta_{ij}+[\iota_p'^j, [\iota_p'^j, \iota_p'^i]]\circ(-1)^pE^{p-1}H_0(\theta_{ij}),$$

where H_0 is the Hopf-Hilton homomorphism and the second term vanishes if p is odd since θ_{ij} becomes a suspension element. And,

$$\begin{aligned}[][[\iota_p'^j, \iota_p'^k]\circ\theta_{ijk}, \iota_p'^i]&=[[\iota_p'^j, \iota_p'^k], \iota_p'^i]\circ E^{p-1}\theta_{ijk}\\ &=[\iota_p'^i, [\iota_p'^j, \iota_p'^k]]\circ(-1)^pE^{p-1}\theta_{ijk}.\end{aligned}$$

So that, we have

(4.8) $$\begin{aligned}[][\theta'_i, \iota_p'^i]=\iota_p'^i\circ[\theta_{ii}, \iota_p^i]&+\sum_{j\neq i}[\iota_p'^j, \iota_p'^i]\circ(-1)^{p-1}E^{p-1}\theta_{ij}\\ &+\sum_{j\neq i}[\iota_p'^j, [\iota_p'^j, \iota_p'^i]]\circ(-1)^pE^{p-1}H_0(\theta_{ij})\\ &+\sum_{j<k}[\iota_p'^i, [\iota_p'^j, \iota_p'^k]]\circ(-1)^pE^{p-1}\theta_{ijk}.\end{aligned}$$

Every Whitehead product of weight 3 is a linear combination of the Whitehead products $[\iota_p'^i, [\iota_p'^j, \iota_p'^k]]$ such that $i\geqq j<k$ by using the Jacobi identity (Hilton [4]). Hence,

(4.9) $$\begin{aligned}\sum_{i=1}^r(\iota_p'^i\circ\eta_i+[\theta'_i, \iota_p'^i])=\sum_{i=1}^r\iota_p'^i\circ(\eta_i+[\theta_{ii}, \iota_p^i])&+\sum_{j<i}[\iota_p'^j, \iota_p'^i]\circ\beta_{ji}\\ &+\sum_{i\geqq j<k}[\iota_p'^i, [\iota_p'^j, \iota_p'^k]]\circ\gamma_{ijk},\end{aligned}$$

where $\beta_{ji}\in\pi_{3p-2}(S^{2p-1})$ $(j<i)$ is defined as in (i) and γ_{ijk} $(i\geqq j<k)$ is a certain element of $\pi_{3p-2}(S^{3p-2})\cong\boldsymbol{Z}$.

Let $a_i=\iota_p'^i\circ(\eta_i+[\theta_{ii}, \iota_p^i])$, $b_{ji}=[\iota_p'^j, \iota_p'^i]\circ\beta_{ji}$ as in (i), and let $c_{ijk}=[\iota_p'^i, [\iota_p'^j, \iota_p'^k]]\circ\gamma_{ijk}$. Then, by (4.4) and (4.9), we know

$$\sum_{i=1}^{r} k'_*a_i+\sum_{i<j} k'_*b_{ij}+\sum_{i\geqq j<k} k'_*c_{ijk}=0, \tag{4.10}$$

where $k'_*: \pi_{p+q-1}(\vee_{i=1}^r S_i'^p)\to\pi_{p+q-1}(\vee_{i=1}^r (S_i'^p\cup e_i'^q))$, $q=2p-1$. Therefore, if we show that every k'_*b_{ij} and every k'_*c_{ijk} belong to the direct summands independent of $\pi_{p+q-1}(S_i'^p\cup e_i'^q)$, $i=1, 2,\cdots, r$, then $k'_*a_i=0$ for $i=1, 2,\cdots, r$ by (4.10), and we can complete the proof similarly as in (i).

Thus, the following will conclude the proof of Assertion 2.

Assertion 3. *Every k'_*b_{ij} $(i<j)$ and every k'_*c_{ijk} $(i\geqq j<k)$ are included in a direct summand of $\pi_{p+q-1}(\vee_{i=1}^r (S_i'^p\cup e_i'^q))$ which is independent of $\pi_{p+q-1}(S_i'^p\cup e_i'^q)$, $i=1, 2,\cdots, r$.*

Proof. Let $X_t=S_t'^p\cup e_t'^q$, $t=1, 2,\cdots, r$. Then, $\pi_n(\vee_{t=1}^r X_t)=\sum_{t=1}^r \pi_n(X_t)\oplus\partial\pi_{n+1}(\prod_{t=1}^r X_t, \vee_{t=1}^r X_t)$, $n=p+q-1$. We have the following commutative diagram.

$$\begin{array}{ccccccccc} S^{p+q-1} & \xrightarrow{\beta_{ij}} & S^{2p-1} & \xrightarrow{[\iota_p'^i,\iota_p'^j]} & S_i'^p\vee S_j'^p & \xrightarrow{k'} & X_i\vee X_j & \subset & \bigvee_{t=1}^r X_t \\ \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\ D^{p+q} & \xrightarrow{c(\beta_{ij})} & D^{2p} & \xrightarrow{\iota_p'^i\times\iota_p'^j} & S_i'^p\times S_j'^p & \xrightarrow{k'} & X_i\times X_j & \subset & \prod_{t=1}^r X_t, \end{array}$$

where vertical maps and k' are inclusions. Hence, k'_*b_{ij} belongs to $\partial\pi_{n+1}(\prod_{t=1}^r X_t, \vee_{t=1}^r X_t)$ which is independent of $\pi_n(X_t)$, $t=1, 2,\cdots, r$.

Generally, every basic product of weight $\geqq 2$ belongs to $\partial\pi_{n+1}(\prod_{t=1}^r S_t'^p, \vee_{t=1}^r S_t'^p)$. In fact, in the splitting exact sequence

$$0\longrightarrow\pi_{n+1}(\prod_{t=1}^r S_t'^p, \bigvee_{t=1}^r S_t'^p)\xrightarrow{\partial}\pi_n(\bigvee_{t=1}^r S_t'^p)$$
$$\xrightarrow{i_*}\pi_n(\prod_{t=1}^r S_t'^p)\cong\sum_{t=1}^r \pi_n(S_t'^p)\longrightarrow 0,$$

such Whitehead products are mapped to zero. So, for any basic product $[\iota_p'^i, [\iota_p'^j, \iota_p'^k]]$ $(i\geqq j<k)$, there exists an element $\chi\in\pi_{n+1}(\prod_{t=1}^r S_t'^p, \vee_{t=1}^r S_t'^p)$ such that $[\iota_p'^i, [\iota_p'^j, \iota_p'^k]]=\partial\chi$. Therefore, we have the following commutative diagram.

$$\begin{array}{ccccccc} S^{p+q-1} & \xrightarrow{\gamma_{ijk}} & S^{3p-2} & \xrightarrow{[\iota_p'^i,[\iota_p'^j,\iota_p'^k]]} & \bigvee_{t=1}^{r} S_t'^p & \xrightarrow{k'} & \bigvee_{t=1}^{r} X_t \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ D^{p+q} & \xrightarrow{c(\gamma_{ijk})} & D^{3p-1} & \xrightarrow{\chi} & \prod_{t=1}^{r} S_t'^p & \xrightarrow{k'} & \prod_{t=1}^{r} X_t \end{array}$$

Hence, $k'_* c_{ijk}$ belongs to $\partial\pi_{n+1}(\prod_{t=1}^r X_t, \vee_{i=1}^r X_t)$. This completes the proof.

Assertion 4. *For "any" admissible basis $\{w'_1, \cdots, w'_r\}$ of $H' = H_q(\#_{i=1}^r \bar{B}'_i)$, there exists an admissible basis $\{w_1, \cdots, w_r\}$ of $H = H_q(\#_{i=1}^r \bar{B}_i)$ such that*

(i) $\varepsilon(w_i) = \varepsilon'(w'_i)$, $i = 1, 2, \cdots, r$.

(ii) $\{\alpha(w_i)\} = \{\alpha'(w'_i)\}$ *in* $\pi_{q-1}(SO_{p+1})/G(\varepsilon(w_i)) = \pi_{q-1}(SO_{p+1})/G(\varepsilon'(w'_i))$, $i = 1, 2, \cdots, r$.

Proof. There exists another expression of $\#_{i=1}^r B'_i$ into a connected sum of p-sphere bundles over q-spheres $\#_{i=1}^r \tilde{B}'_i$ such that in the cell decomposition $\#_{i=1}^r \tilde{B}'_i \simeq \{\vee_{i=1}^r (\tilde{S}_i'^p \cup \tilde{e}_i'^q)\} \cup \tilde{e}'^{p+q}$, each homology class $[\tilde{e}_i'^q]$ corresponds to w'_i, $i = 1, 2, \cdots, r$. Hence, Assertion 1 and Assertion 2 conclude the proof.

This completes the proof of the necessity for Theorem 4.

§5. Proof of the Sufficiency for Theorem 4

Let B_i, B'_i be p-sphere bundles over q-spheres $(2p > q > 1)$ with the characteristic elements α_i, α'_i respectively and let $\varepsilon_i = \pi_*(\alpha_i)$, $\varepsilon'_i = \pi_*(\alpha'_i)$, where $i = 1, 2, \cdots, r$. Let $\{w_1, \cdots, w_r\}$, $\{w'_1, \cdots, w'_r\}$ be the admissible bases of H, H' respectively, satisfying

(i) $\varepsilon(w_i) = \varepsilon'(w'_i)$, $i = 1, 2, \cdots, r$, and

(ii) $\{\alpha(w_i)\} = \{\alpha'(w'_i)\}$ in $\pi_{q-1}(SO_{p+1})/G(\varepsilon(w_i)) = \pi_{q-1}(SO_{p+1})/G(\varepsilon'(w'_i))$, $i = 1, 2, \cdots, r$.

By adopting the representations of the connected sums of given bundles using the admissible bases $\{w_1, \cdots, w_r\}$, $\{w'_1, \cdots, w'_r\}$, we may assume that w_i, w'_i are represented by zero cross-sections of $\bar{B}_i$, $\bar{B}'_i$ respectively, $i = 1, 2, \cdots, r$. Then, $\alpha(w_i) = \alpha_i$, $\alpha'(w'_i) = \alpha'_i$, $\varepsilon(w_i) = \varepsilon_i$, and $\varepsilon'(w'_i) = \varepsilon'_i$, $i = 1, 2, \cdots, r$. Hence, the proof is accomplished by directly extending that of James-Whitehead [10] ((1.5), p. 163).

Since $\alpha_i - \alpha'_i \in G(\varepsilon_i)$, there exists an element $\xi_i \in \pi_{q-1}(SO_p)$ such that $i_* \xi_i = \alpha_i - \alpha'_i$ and $J\xi_i \in \mathrm{Im}\,(\varepsilon_i)_*$, $i = 1, 2, \cdots, r$. By (3.2) of [10], the sequence

$$\pi_{p+q-1}(S^{q-1}) \xrightarrow{(\varepsilon_i)_*} \pi_{p+q-1}(S^p) \xrightarrow{(k_i)_*} \pi_{p+q-1}(S^p \underset{\varepsilon_i}{\cup} e^q)$$

is exact, where $(k_i)_*$ is induced from the inclusion. Hence, $J\xi_i \in \mathrm{Im}(\varepsilon_i)_* = \mathrm{Ker}(k_i)_*$, $i=1, 2,\cdots, r$. Let $B_i = S_i^p \cup e_i^q \cup e_i^{p+q}$, $B_i' = S_i'^p \cup e_i'^q \cup e_i'^{p+q}$ be the cell-decompositions given by (3.3) of [9], where e_i^q, $e_i'^q$ are attached by $\iota_p^i \circ \varepsilon_i$, $\iota_p'^i \circ \varepsilon_i'$ respectively. We identify S^p canonically with S_i^p, $S_i'^p$ so that $\iota_p^i = \iota_p = \iota_p'^i$. Since $\varepsilon_i = \varepsilon_i'$, there exists a homotopy equivalence $g_i: S^p \cup e_i^q \to S^p \cup e_i'^q$ such that $g_i | S^p = \mathrm{id}$. Let $\sigma_i \in \pi_{p+q}(B_i, S^p \cup e_i^q)$, $\sigma_i' \in \pi_{p+q}(B_i', S^p \cup e_i'^q)$ be the orientation generators and let $\delta_i = \partial\sigma_i$, $\delta_i' = \partial\sigma_i'$. Then, by (3.3) and Lemma (3.8) of [10],

(i) $(g_i)_*\delta_i - \delta_i' = (k_i)_* J\xi_i'$ for some $\xi_i' \in \pi_{q-1}(SO_p)$ such that $i_*\xi_i' = \alpha_i - \alpha_i'$, and

(ii) g_i can be chosen so that ξ_i' is a given element in $i_*^{-1}(\alpha_i - \alpha_i')$.

Hence, by taking ξ_i as ξ_i', there exists a homotopy equivalence $g_i: S^p \cup e_i^q \to S^p \cup e_i'^q$ such that $g_i | S^p = \mathrm{id}$ and $(g_i)_*\delta_i = \delta_i'$, where $i=1, 2,\cdots, r$.

In the cell-decompositions $\#_{i=1}^r B_i \simeq B = \{\vee_{i=1}^r (S_i^p \cup e_i^q)\} \cup e^{p+q}$, $\#_{i=1}^r B_i' \simeq B' = \{\vee_{i=1}^r (S_i'^p \cup e_i'^q)\} \cup e'^{p+q}$, let $\sigma \in \pi_{p+q}(B, \vee_{i=1}^r (S_i^p \cup e_i^q))$, $\sigma' \in \pi_{p+q}(B', \vee_{i=1}^r (S_i'^p \cup e_i'^q))$ be the orientation generators, and let $\delta = \partial\sigma$, $\delta' = \partial\sigma'$. Then,

$$\delta = \delta_1 + \delta_2 + \cdots + \delta_r, \quad \delta' = \delta_1' + \delta_2' + \cdots + \delta_r',$$

where it is understood that $\sum_{i=1}^r \pi_{p+q-1}(S_i^p \cup e_i^q) \subset \pi_{p+q-1}(\vee_{i=1}^r (S_i^p \cup e_i^q))$ and $\sum_{i=1}^r \pi_{p+q-1}(S_i'^p \cup e_i'^q) \subset \pi_{p+q-1}(\vee_{i=1}^r (S_i'^p \cup e_i'^q))$. Now, let

$$g = \bigvee_{i=1}^r g_i : \bigvee_{i=1}^r (S_i^p \cup e_i^q) \longrightarrow \bigvee_{i=1}^r (S_i'^p \cup e_i'^q).$$

Then, $g_*\delta = \sum_{i=1}^r g_*\delta_i = \sum_{i=1}^r (g_i)_*\delta_i = \sum_{i=1}^r \delta_i' = \delta'$, that is, $g_*\delta = \delta'$. Hence, g has an extension $f: B \to B'$ of degree 1. $f_*: H_n(B) \to H_n(B')$ is isomorphic for $n=0, p, p+q$, and for $n=q$ as is shown by the following diagram

$$\begin{array}{ccc} H^p(B) & \xleftarrow[\cong]{f^*} & H^p(B') \\ \cong\downarrow D & & \cong\downarrow D \\ H_q(B) & \xrightarrow[f_*]{} & H_q(B), \end{array}$$

where $f_* \circ (D \circ f^* \circ D^{-1}) = \mathrm{id}$ and D is the Poincaré duality isomorphism. Since B, B' are simply connected, f is a homotopy equivalence. This completes the proof.

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