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Linear Radon-Nikodym Theorems for States on a von Neumann Algebra

By

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Abstract

Several linear Radon-Nikodym theorems for states on a von Neumann algebra are obtained in the context of a one parameter family of positive cones introduced by H. Araki. Among other results, we determine when a normal state ϕ admits a linear Radon-Nikodym derivative with respect to a distinguished normal faithful state ϕ_0 in the sense of Sakai, that is, $\phi = h\phi_0 + \phi_0 h$ with a positive h in the algebra.

§ 0. Introduction

We consider a von Neumann algebra on a Hilbert space admitting a cyclic and separating vector. Making use of the associated modular operator, [9], Araki introduced a one parameter family of positive cones. Several Radon-Nikodym theorems are known in the context of positive cones ([1], [4], [5]) in which "Radon-Nikodym derivatives" reduce to the square roots of measure theoretic Radon-Nikodym derivatives provided that the algebra in question is commutative.

In this paper we obtain three linear Radon-Nikodym theorems (Theorem 1.5, 1.6, 1.7). Our proofs are very constructive so that we obtain explicit expressions of linear Radon-Nikodym derivatives.

Our main tools are relative modular operators and the function $\{\cosh(\pi t)\}^{-1}$ which was used by van Daele to obtain a simple proof of the fundamental theorem of the Tomita-Takesaki theory, [12].

§ 1. Notations and Main Results

Let $\mathscr{M}$ be a von Neumann algebra on a Hilbert space $\mathscr{H}$ with a unit cyclic

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and separating vector ξ_0 with the vector state $\phi_0=\omega_{\xi_0}$, and Δ, J be the associated modular operator and modular conjugation respectively, [9]. Fixing these throughout, we denote the modular automorphism group $\{\mathrm{Ad}\,\Delta^{it}\}_{t\in\boldsymbol{R}}$ on $\mathscr{M}$ simply by σ_t. Let $\mathscr{M}_0$ be a σ-weakly dense $*$-subalgebra consisting of every $x\in\mathscr{M}$ such that $t\in\boldsymbol{R}\to\sigma_t(x)\in\mathscr{M}$ extends to an entire function.

The following one parameter family of positive cones was introduced by Araki:

Definition 1.1 ([1]). For each $0\leqq\alpha\leqq 1/2$, we denote the closure of the positive cone $\Delta^{\alpha}\mathscr{M}_+\xi_0$ in $\mathscr{H}$ by $P^{\alpha}(=P^{\alpha}_{\phi_0})$. These cones enjoy the following properties:

Proposition 1.2 ([1]). *For each* $0\leqq\alpha\leqq 1/2$, *we have*

(i) $P^{\alpha}=JP^{(1/2)-\alpha}=(P^{(1/2)-\alpha})'$ $(=\{\zeta\in\mathscr{H};\ (\zeta\mid\xi)\geqq 0,\ \xi\in P^{(1/2)-\alpha}\}$, *the dual cone*),

(ii) $P^{\alpha}\subseteqq\mathscr{D}(\Delta^{(1/2)-2\alpha})$ *and* $\Delta^{(1/2)-2\alpha}\zeta=J\zeta$, $\zeta\in P^{\alpha}$,

(iii) *the map*: $\xi\in P^{1/4}\to\omega_{\xi}\in\mathscr{M}_*^+$ *is bijective.*

By (iii) in the above proposition, each $\phi\in\mathscr{M}_*^+$ admits a unique implementing vector in $P^{1/4}$, the natural cone, which we will denote by ξ_{ϕ}, that is, $\phi=\omega_{\xi_{\phi}}$. Then a positive self-adjoint operator $\Delta_{\phi\phi_0}$ on $\mathscr{H}$ with a form core $\mathscr{M}\xi_0$ satisfying $J\Delta_{\phi\phi_0}^{1/2}x\xi_0=x^*\xi_{\phi}$, $x\in\mathscr{M}$, is known as the relative modular operator (of ϕ with respect to ϕ_0). Also, a partial isometry $\Delta_{\phi\phi_0}^{it}\Delta_{\phi_0}^{-it}=(D\phi;D\phi_0)_t$, $t\in\boldsymbol{R}$, in $\mathscr{M}$ is known as the Radon-Nikodym cocycle (of ϕ with respect to ϕ_0). (See [3] or § 1 of [5] for full details.)

In the first main theorem (Theorem 1.5), we need the following concept:

Definition 1.3 (One parameter family of orderings, [3]). For $\lambda>0$, we write $\phi\leqq\phi_0(\lambda)$ if the map: $t\in\boldsymbol{R}\to(D\phi;D\phi_0)_t$ extends to a bounded σ-weakly continuous function on $-\lambda\leqq\mathrm{Im}\,z\leqq 0$ which is analytic in the interior and $\|(D\phi;D\phi_0)_{-i\lambda}\|\leqq 1$. It is well-known that $\phi\leqq\phi_0(1/2)$ is equivalent to $\phi\leqq\phi_0$ in the usual ordering in $\mathscr{M}_*$, that is, $\phi(x)\leqq\phi_0(x)$, $x\in\mathscr{M}_+$.

The next lemma is our main tool in the paper.

Lemma 1.4. *Let* $f(z)$ *be a bounded continuous function on* $0\leqq\mathrm{Re}\,z\leqq 1$ *which is analytic in the interior. We then have*

$$f(1/2)=\int_{-\infty}^{\infty}\{f(it)+f(1+it)\}\{2\cosh(\pi t)\}^{-1}dt\,.$$

In fact, it is well-known (see p. 208, [8] for example) that the pair $(P_0$

$(\alpha+i\beta, t)=\sin(\pi\alpha)/2\{\cosh\pi(t-\beta)-\cos\pi\alpha\}$, $P_1(\alpha+i\beta, t)=\sin(\pi\alpha)/2\{\cosh\pi(t-\beta)+\cos\pi\alpha\}$) gives rise to the harmonic measure for the strip $0\leqq\mathrm{Re}\,z\leqq 1$. And both of $P_0(1/2, t)$ and $P_1(1/2, t)$ are exactly $\{2\cosh(\pi t)\}^{-1}$. Since we use this result repeatedly, we shall denote the function $\{2\cosh(\pi t)\}^{-1}$ simply by $F(t)$ throughout the paper.

We now state our three main results. The first one is a slight strengthening of the result in Section 6, [1]. However, more importantly, we obtain the explicit expression of Radon-Nikodym derivatives.

Theorem 1.5. *Let ϕ be a normal state on $\mathcal{M}$ and $0\leqq\alpha\leqq 1/2$. If $\phi\leqq l\phi_0$ $(\mathrm{Max}(\alpha, (1/2)-\alpha))$ for some $l>0$, the vector*

$$\zeta_\alpha=\int_{-\infty}^{\infty}F(t)\Delta^{(1-2\alpha)it}(D\phi;D\phi_0)^*_{-i((1/2)-\alpha)}J(D\phi;D\phi_0)^*_{-i\alpha}\xi_\phi dt$$

in P^α satisfies $\phi(x)=(x\zeta_\alpha\mid\xi_0)+(x\xi_0\mid\zeta_\alpha)$, $x\in\mathcal{M}$.

For the special value of $\alpha=0$, the assumption in the theorem is exactly $\phi\leqq l\phi_0$ in the usual ordering. Furthermore, $\zeta_0\in P^0$ is written as $\zeta_0=h\xi_0$ with $h=\int_{-\infty}^{\infty}F(t)\sigma_t(|(D\phi;D\phi_0)_{-i/2}|^2)dt\in\mathcal{M}$ so that we have $\phi=h\phi_0+\phi_0 h$, which is Sakai's linear Radon-Nikodym theorem (Proposition 1, 24, 4, [7]). As the second main result, we prove

Theorem 1.6. *Let ϕ be a normal state on $\mathcal{M}$. Then ϕ admits a (unique) linear Radon-Nikodym derivative $h\in\mathcal{M}_+$, that is, $\phi=h\phi_0+\phi_0 h$, if and only if $\tilde{\phi}\leqq l\phi_0$ with some $l>0$. Here, $\tilde{\phi}\in\mathcal{M}_*^+$ is defined by*

$$\tilde{\phi}=\int_{-\infty}^{\infty}F(t)\phi\circ\sigma_t dt\,.$$

Furthermore, if this is the case, h is exactly $|(D\tilde{\phi};D\phi_0)_{-i/2}|^2=(D\tilde{\phi};D\phi_0)^*_{-i/2}\cdot(D\tilde{\phi};D\phi_0)_{-i/2}$. (See also Lemma 4.1.)

In the next result, we further assume that ϕ_0 is periodic in the sense that $\sigma_T=\mathrm{Id}$ for some $T>0$ ([10]). When $\phi=h\phi_0+\phi_0 h\in\mathcal{M}_*^+$ $(h\in\mathcal{M}_+)$, for each positive x in the centralizer $\mathcal{M}_{\phi_0}$ ([10]), we estimate

$$\begin{aligned}\phi(x)&=\phi_0(xh)+\phi_0(hx)=2\phi_0(x^{1/2}hx^{1/2})\\&\leqq 2\|h\|\phi_0(x)\,.\end{aligned}$$

Our third result asserts that the converse is also true.

Theorem 1.7. *Let ϕ be a normal state on $\mathcal{M}$. When the distinguished ϕ_0 is periodic, ϕ admits a linear Radon-Nikodym derivative as in the previous* *theorem if and only if* $\phi\leqq l\phi_0$ *on* $\mathcal{M}_{\phi_0}$, *the centralizer of* $\mathcal{M}_{\phi_0}$, *for some* $l>0$, *that is*, $\phi(x)\leqq l\phi_0(x)$ *for each positive* $x\in\mathcal{M}_{\phi_0}$.

The rest of the paper is devoted to the proofs of these three theorems. We denote a generic normal state on $\mathcal{M}$ by ϕ and the distinguished ϕ_0 is supposed to be periodic only in the proof of the last theorem.

§2. Proof of Theorem 1.5

In this section, we prove two lemmas from which Theorem 1.5 follows immediately.

Lemma 2.1. *Let* ζ *be a vector in* $\mathcal{H}$ *satisfying*

$$\phi(x)=(\Delta^{(1/2)-\alpha}x\xi_0\,|\,\zeta),\quad x\in\mathcal{M}.$$

Then ζ *belongs to* P^α. *If we set*

$$\eta=\int_{-\infty}^{\infty}F(t)\Delta^{(1-2\alpha)it}\zeta dt,$$

then η *belongs to* P^α *and satisfies*

$$\phi(x)=(x\eta\,|\,\xi_0)+(x\xi_0\,|\,\eta),\quad x\in\mathcal{M}.$$

Proof. The cone $\Delta^{(1/2)-\alpha}\mathcal{M}_+\xi_0$ being dense in $P^{(1/2)-\alpha}$, $\zeta\in P^\alpha$ follows from the positivity of ϕ and Proposition 1.2, (i). Also, since P^α is invariant under $\Delta^{(1-2\alpha)it}$ and $F(t)$ is positive for each $-\infty<t<\infty$, η belongs to P^α as well.

To prove the final equality, we may and do assume $x\in\mathcal{M}_0$. Firstly we observe

$$\begin{aligned}(x\xi_0\,|\,\eta)&=\int_{-\infty}^{\infty}F(t)(x\xi_0\,|\,\Delta^{-(1-2\alpha)it}\zeta)dt\\&=\int_{-\infty}^{\infty}F(t)(\Delta^{(1-2\alpha)it}x\xi_0\,|\,\zeta)dt\\&=\int_{-\infty}^{\infty}F(t)(\Delta^{(1/2)-\alpha}\sigma_{(1-2\alpha)t+i((1/2)-\alpha)}(x)\xi_0\,|\,\zeta)dt\\&=\int_{-\infty}^{\infty}F(t)\phi(\sigma_{(1-2\alpha)t+i((1/2)-\alpha)}(x))dt.\end{aligned}$$

Secondly, by Proposition 1.2, (ii), we observe

$$\begin{aligned}(x\eta\,|\,\xi_0)&=(J\Delta^{(1/2)-2\alpha}\eta\,|\,x^*\xi_0)=(J\Delta^{(1/2)-2\alpha}\eta\,|\,J\Delta^{1/2}x\xi_0)\\&=(\Delta^{1/2}x\xi_0\,|\,\Delta^{(1/2)-2\alpha}\eta)=(\Delta^{1-2\alpha}x\xi_0\,|\,\eta)\\&=\int_{-\infty}^{\infty}F(t)(\Delta^{1-2\alpha}x\xi_0\,|\,\Delta^{-(1-2\alpha)it}\zeta)dt\end{aligned}$$

$$=\int_{-\infty}^{\infty} F(t)(\Delta^{1-2\alpha+(1-2\alpha)it}x\xi_0 \mid \zeta)dt$$
$$=\int_{-\infty}^{\infty} F(t)(\Delta^{(1/2)-\alpha}\sigma_{(1-2\alpha)t-i((1/2)-\alpha)}(x)\xi_0 \mid \zeta)dt$$
$$=\int_{-\infty}^{\infty} F(t)\phi(\sigma_{(1-2\alpha)t-i((1/2)-\alpha)}(x))dt\,.$$

Hence, Lemma 1.4 applied to $f(z)=\phi(\sigma_w(x))$, $w=-i(1-2\alpha)(z-(1/2))$, yields:

$$(x\xi_0 \mid \eta)+(x\eta \mid \xi_0)=\phi(\sigma_0(x))=\phi(x)\,. \qquad \text{Q.E.D.}$$

Lemma 2.2. *If* $\phi \leqq l\phi_0$ (Max $(\alpha,\ (1/2)-\alpha)$), *then we have*

$$\phi(x)=(\Delta^{(1/2)-\alpha}x\xi_0 \mid \zeta), \quad x\in\mathcal{M}$$

with

$$\zeta=(D\phi;\ D\phi_0)^*_{-i((1/2)-\alpha)}J(D\phi;\ D\phi_0)^*_{-i\alpha}\xi_\phi\,.$$

Proof. We simply compute

$$(\Delta^{(1/2)-\alpha}x\xi_0 \mid \zeta)=((D\phi;\ D\phi_0)_{-i((1/2)-\alpha)}\Delta^{(1/2)-\alpha}x\xi_0 \mid J(D\phi;\ D\phi_0)^*_{-i\alpha}\xi_\phi)$$
$$=((D\phi;\ D\phi_0)_{-i((1/2)-\alpha)}\Delta^{(1/2)-\alpha}x\xi_0 \mid \Delta^{1/2}_{\phi\phi_0}(D\phi;\ D\phi_0)_{-i\alpha}\xi_0)\,.$$

Using the uniqueness of analytic continuation, we can easily prove

$$(D\phi;\ D\phi_0)_{-i((1/2)-\alpha)}\Delta^{(1/2)-\alpha}x\xi_0=\Delta^{(1/2)-\alpha}_{\phi\phi_0}x\xi_0\,,$$
$$(D\phi;\ D\phi_0)_{-i\alpha}\xi_0=\Delta^{\alpha}_{\phi\phi_0}\xi_0.$$

Since $(D\phi;\ D\phi_0)_{-i\alpha}\xi_0$ belongs to $\mathcal{M}\xi_0 \subseteqq \mathcal{D}(\Delta^{1/2}_{\phi\phi_0})$, we have $\xi_0\in\mathcal{D}(\Delta^{(1/2)+\alpha}_{\phi\phi_0})$ and

$$(\Delta^{(1/2)-\alpha}x\xi_0 \mid \zeta)=(\Delta^{(1/2)-\alpha}_{\phi\phi_0}x\xi_0 \mid \Delta^{(1/2)+\alpha}_{\phi\phi_0}\xi_0)$$
$$=(\Delta^{1/2}_{\phi\phi_0}x\xi_0 \mid \Delta^{1/2}_{\phi\phi_0}\xi_0)=(Jx^*\xi_\phi \mid \xi_\phi)$$
$$=(\xi_\phi \mid x^*\xi_\phi)=\phi(x)\,. \qquad \text{Q.E.D.}$$

§3. Proof of Theorem 1.6

The proof of Theorem 1.6 is divided into three lemmas.

Lemma 3.1. *Assume that* $\phi=h\phi_0+\phi_0 h\in\mathcal{M}^+_*$ *with a positive* $h\in\mathcal{M}$. *Then for each* $x\in\mathcal{M}$ *we have*

$$\phi_0(x\sigma_{-i/2}(h))=\int_{-\infty}^{\infty} F(t)\phi(\sigma_{-t}(x))dt$$
$$=\int_{-\infty}^{\infty} F(t)\phi(\sigma_t(x))dt$$

Here, the left hand side makes sense due to the K.M.S. condition, [9].

Proof. By the invariance $\phi_0 \circ \sigma_t = \phi_0$, we compute

$$\int_{-\infty}^{\infty} F(t)\phi(\sigma_{-t}(x))dt = \int_{-\infty}^{\infty} \{\phi_0(\sigma_{-t}(x)h) + \phi_0(h\sigma_{-t}(x))\}F(t)dt$$
$$= \int_{-\infty}^{\infty} \{\phi_0(x\sigma_t(h)) + \phi_0(\sigma_t(h)x)\}F(t)dt .$$

Thus, the result follows from Lemma 1.4 and the fact that $\phi_0(x\sigma_z(h))$ is bounded analytic on $-1 \leqq \mathrm{Im}\, z \leqq 0$ and $\phi_0(x\sigma_{-i+t}(h)) = \phi_0(\sigma_t(h)x)$ (the K.M.S. condition). Q.E.D.

Lemma 3.2. *For x, $h \in \mathcal{M}_+$, we have*

$$0 \leqq \phi_0(x\sigma_{-i/2}(h)) \leqq \|h\| \phi_0(x) .$$

Proof. We notice that

$$\phi_0(x\sigma_{-i/2}(h)) = (x\Delta^{1/2}h\xi_0 \mid \xi_0)$$
$$= (xJhJ\xi_0 \mid \xi_0)$$

due to the positivity of h. Since $0 \leqq JhJ \leqq \|h\|$ and they commute with $x \in \mathcal{M}_+$, we have

$$0 \leqq (xJhJ\xi_0 \mid \xi_0) \leqq \|h\| \, (x\xi_0 \mid \xi_0) = \|h\| \phi_0(x) . \qquad \text{Q.E.D.}$$

Lemma 3.3. *When*

$$\tilde{\phi} = \int_{-\infty}^{\infty} F(t)\phi \circ \sigma_t dt \leqq l\phi_0 ,$$

we have $\phi = h\phi_0 + \phi_0 h$ with $h = |(D\tilde{\phi};\, D\phi_0)_{-i/2}|^2 \in \mathcal{M}_+$.

Proof. To prove $\phi(x) = \phi_0(xh) + \phi_0(hx)$, we may and do assume that $x \in \mathcal{M}_0$. At first we notice that

$$\begin{aligned}
&(D\tilde{\phi};\, D\phi_0)_{-i/2}\xi_0 = \xi_{\tilde{\phi}} ,\\
&(D\tilde{\phi};\, D\phi_0)_{-i/2}y\xi_0 = (D\tilde{\phi};\, D\phi_0)_{-i/2}J\Delta^{1/2}y^*\xi_0\\
&\qquad = (D\tilde{\phi};\, D\phi_0)_{-i/2}J\sigma_{-i/2}(y^*)J\xi_0\\
&\qquad = J\sigma_{-i/2}(y^*)J(D\tilde{\phi};\, D\phi_0)_{-i/2}\xi_0\\
&\qquad = J\sigma_{-i/2}(y^*)\xi_{\tilde{\phi}} \qquad (y \in \mathcal{M}_0) .
\end{aligned}$$

By using these facts, it is easily shown that

$$\phi_0(hx) + \phi_0(xh) = \tilde{\phi}(\sigma_{i/2}(x) + \sigma_{-i/2}(x))$$
$$= \int_{-\infty}^{\infty} F(t) \{\phi(\sigma_{i/2+t}(x)) + \phi(\sigma_{-i/2+t}(x))\} dt.$$

Thus, the result follows from Lemma 1.4 applied to the function

$$f(z)=\phi(\sigma_w(x)), \quad w=-i(z-2^{-1}). \qquad \text{Q.E.D.}$$

§4. Proof of Theorem 1.7

The next lemma is obtained in [6] in a slightly different set up. However, for the sake of completeness, we present its proof in our context.

Lemma 4.1. *We have*

$$\tilde{\phi}=\int_{-\infty}^{\infty} F(t)\phi\circ\sigma_t dt \leqq l\phi_0$$

if and only if for each $\alpha>0$ (*hence all* α) *there exists a positive* $c=c_\alpha$ *such that*

$$\int_{-\alpha}^{\alpha} \phi\circ\sigma_t dt \leqq c\phi_0 .$$

Proof. The "only if" part is trivial since $F(t)$ is a strictly positive even function and monotone decreasing on $[0, \infty)$. To show the "if" part, we assume that

$$\int_{-\alpha}^{\alpha} \phi\circ\sigma_t dt \leqq c\phi_0 .$$

We compute that

$$\begin{aligned}
\int_{-\infty}^{\infty} F(t)\phi\circ\sigma_t dt &= \sum_{n=-\infty}^{\infty} \int_{(2n-1)\alpha}^{(2n+1)\alpha} F(t)\phi\circ\sigma_t dt \\
&= \sum_{n=-\infty}^{\infty} \int_{-\alpha}^{\alpha} F(t+2n\alpha)\phi\circ\sigma_{t+2n\alpha} dt \\
&\leqq \sum_{n=-\infty}^{\infty} c\{\exp(\pi(2|n|+1)\alpha)\}^{-1}\phi_0\circ\sigma_{2n\alpha} \\
&= (\sum_{n=-\infty}^{\infty} c\{\exp(\pi(2|n|+1)\alpha)\}^{-1})\phi_0 .
\end{aligned}$$

Q.E.D.

Proof of the theorem. Let $T>0$ be the period of σ_t. It is easy to observe that

$$\int_{-T}^{T} \sigma_t dt = 2T\varepsilon ,$$

where ε is a normal projection of norm 1 from $\mathcal{M}$ onto the centralizer $\mathcal{M}_{\phi_0}$ satisfying $\phi_0\circ\varepsilon=\phi_0$. (See [2], [10], [11].) By the previous lemma and Theorem 1.6, we know that $\phi=h\phi_0+\phi_0 h$ if and only if $\phi\circ\varepsilon\leqq l'\phi_0=l'\phi_0\circ\varepsilon$, which is equivalent to $\phi\leqq l'\phi_0$ on $\mathcal{M}_{\phi_0}$. Q.E.D.

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