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A Coherency Theorem for Direct Images with Proper Supports in the Case of a 1-Convex Map

By

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Introduction

Let $f: X \to S$ be a morphism of complex spaces and $\mathcal{F}$ a coherent analytic sheaf on X. If f is proper, then the higher direct image sheaves $R^i f_* \mathcal{F}$ are coherent for all $i \geqq 0$ by Grauert [8]. Generalizations of this theorem to certain classes of non-proper morphisms have also been obtained by several authors, among which we shall mention the following results, being of particular interest to us here (cf. also [22]). Let $q \geqq 0$ be an integer. Then: a) If f is q-concave, then $R^i f_* \mathcal{F}$ is coherent for $i \leqq \operatorname{codh} \mathcal{F} - \dim S - q - 1$ [17]. b) If f is q-convex, then $R^i f_! \mathcal{F}$ is coherent for $i \leqq \operatorname{codh} \mathcal{F} - \dim S - q$ [4], where $f_!$ denotes the direct image with proper supports. c) If f is $(1, 1)$-convex-concave, then $R^i f_* \mathcal{F}$ is coherent for $1 \leqq i \leqq \operatorname{codh} \mathcal{F} - \dim S - 2$ [15]. (Note that here and in what follows the terms 'q-convex' and 'q-concave' are used in such a way that when S reduces to a point, they coincide with the notion of 'fortement q-pseudoconvexe' and 'fortement q-pseudoconcave' respectively of Andreotti-Grauert [1]. Hence they should be called '$(q-1)$-convex' and '$(q-1)$-concave' respectively in the terminology of [4], [17], [18] etc.) Though these results are best possible as they stand, we could expect to improve the bounds for i under the additional assumption that $\mathcal{F}$ is f-flat, and in fact in such a way that the condition is stable under base change. The latter fact would indeed be useful in certain applications (cf. e.g. § 5

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below) and is in general not satisfied in the condition of the above mentioned results since the quantity codh $\mathcal{F} - \dim S$ is not stable under base change. More precisely we raise the following:

Conjecture. Let $0 \leqq p$, $q < +\infty$. Let $f: X \to S$ be a (p, q)-convex-concave map of complex spaces and $\mathcal{F}$ a coherent analytic sheaf on X. Suppose that there exists a closed subset K of X such that K is proper over S via f and $\mathcal{F}$ is f-flat on $U = X - K$. Let $r = \mathrm{codh}_{(f|_U)}(\mathcal{F}|_U)$. Then 1) $R^i f_* \mathcal{F}$ is coherent for $p \leqq i \leqq r - q - 1$, and 2) $R^i f_! \mathcal{F}$ is coherent for $q \leqq i \leqq r - p$.

For the precise definition of a (strongly) (p, q)-convex-concave map we refer to [21] or [18] modulo the above remark, the case $p=0$ (resp. $q=0$) being understood to be the pure q-concave (resp. pure p-convex) case. On the other hand, in general for a morphism of complex spaces g: $Y \to T$ and a coherent analytic sheaf $\mathcal{G}$ on Y, $\mathrm{codh}_g \mathcal{G}$ is defined by $\mathrm{codh}_g \mathcal{G} = \inf_{y \in Y} (\mathrm{codh}_y \mathcal{G}_{g(y)})$. Hence when S is nonsingular, or more generally, is Cohen-Macaulay, the above results a), b), c) give the conjecture under the respective assumptions (cf. (4.1)).

Now the main purpose of this paper is to prove 2) of the above conjecture in a special case where f is 1-convex, making use of the fact that f is then a proper modification of a Stein morphism. We first recall the precise definition of a 1-convex map. Let $f: X \to S$ be a morphism of complex spaces. Then we call f a *1-convex map* if there exists a real C^∞-map $\varphi: X \to (-\infty, c^*)$, $-\infty < c^* \leqq +\infty$, called the *exhaustion function* for f, such that 1) the set $\{x \in X; \varphi(x) \leqq c\}$ is proper over S via f for every $c < c^*$ and 2) there exists a real number $c_\# < c^*$, called a *convexity bound* for φ, such that φ is strictly plurisubharmonic on $X_{c_\#} = \{x \in X; \varphi(x) > c_\#\}$. In this case for each $d < c^*$ we set $X^d = \{x \in X; \varphi(x) < d\}$, $f^d = f|_{X^d}: X^d \to S$ and $f_{c_\#} = f|_{X_{c_\#}}: X_{c_\#} \to S$. Then the following holds ture.

Theorem. *Let $f: X \to S$ be a 1-convex map with exhaustion function $\varphi: X \to (-\infty, c^*)$ and $\mathcal{F}$ a coherent analytic sheaf on X. Let $c_\# \in (-\infty, c^*)$ be a convexity bound for φ. Suppose that $\mathcal{F}$ is*

f-flat on $X_{c_\#}$. Then for $i<\operatorname{codh}_{f_{c_\#}}(\mathcal{F}|_{X_{c_\#}})$, $R^i f_! \mathcal{F}$ is coherent and the natural map $R^i f_!^d \mathcal{F} \to R^i f_! \mathcal{F}$ is isomorphic for $d \in (c_\#, c^*)$.

Moreover combining the above theorem with the result of [6], in the 1-concave and $(1, 1)$-convex-concave cases we can also improve the result of Ling [15] (cf. c) above) toward the above conjecture (cf. Corollary 4. 4 below).

Now we shall give a brief outline of the paper. First in Section 1 using the method of Siu-Trautmann [23] we prove a certain refinement of a result of Andreotti-Grauert [1], which is essentially a generalization of [1, Proposition 12] (cf. also its proof) to the case of a possibly singular parameter space. Then in Section 2 again by the method of [23] we show the coherency of certain direct image sheaves $R^i\pi_{B*}\mathcal{F}$ with supports in B where $\pi: S\times K(b)\to S$ is the natural projection, $B=S\times K(a)^-$ for some $a<b$ and $\mathcal{F}$ is a coherent analytic sheaf on $S\times K(b)$ which is π-flat on $S\times(K(b)-\{O\})$ (cf. Notation below). Also we prove results on isomorphy of $R^i\pi_{B*}\mathcal{F}$ when a varies. Next in Section 3 some lemmas are proved which is needed for preliminary reductions of Theorem; Lemma 3. 2 is used to reduce the case of a general 1-convex map to the case of a Stein 1-convex map with a special property, and then, the latter case is further reduced to the case where f is isomorphic to the projection π above by Lemma 3. 3. Once f is the projection, then by the refinement of Andreotti-Grauert's result together with the isomorphy of $R^i\pi_{B*}\mathcal{F}$, both mentioned above, we can finally reduce Theorem to the above coherency result of $R^i\pi_{B*}\mathcal{F}$. These reductions, and hence the proof of Theorem, are given in Section 4. Finally in Section 5 we obtain a relative version of the vanishing theorem of Grauert-Riemenschneider [9] as an application of the above coherency theorem.

Notation. Let $f: X\to S$ be a morphism of complex spaces and F an analytic sheaf on X. 1) For any morphism $\alpha: T\to S$ of complex spaces we write $X_T=X\times_S T$, $f_T=f\times_S T$: $X_T\to T$ and $\mathcal{F}_T=\pi_T^*\mathcal{F}$ where $\pi_T: X_T \to X$ is the natural projection. In particular if $T=\{s\}$ is a point of S, then we write X_s, f_s, $\mathcal{F}_s$ instead of $X_{\{s\}}$, $f_{\{s\}}$, $\mathcal{F}_{\{s\}}$ respectively. On the other hand, if $T=U$ is an open subset of S we often write $X(U)$ instead of X_U. 2) Let B be a closed subset of X. Then we shall denote by $R^i f_{B*}\mathcal{F}$ the $\mathcal{O}_S$-module defined by the presheaf $U \to H^i_{B_U}(X(U), \mathcal{F})$ where U is any open subset of S and $B_U = B \cap X(U)$. $R^i f_{B*}\mathcal{F}$ can also be defined as the i-th derived functor of the functor $f_{B*}\mathcal{F}$ with $f_{B*}\mathcal{F} = R^0 f_{B*}\mathcal{F}$ as above. 3) For any integer k we write $S_k(\mathcal{F}) = \{x \in \mathcal{F}\,;\, \mathrm{codh}_x \mathcal{F} \leqq k\}$ and $S_k(X) = S_k(\mathcal{O}_X)$, where $\mathrm{codh}_x \mathcal{F} = +\infty$ if $x \notin \mathrm{supp}\,\mathcal{F}$, supp denoting the support. 4) For a subset M of X, M^- denotes the topological closure of M in X. 5) Let $b = (b_1, \cdots, b_N) \in \boldsymbol{R}^N_+$ for some $N > 0$ where $\boldsymbol{R}_+ = \{c \in \boldsymbol{R}; c > 0\}$. Then $K(b) = \{(w_1, \cdots, w_N) \in \boldsymbol{C}^N; |w_i| < b_i, 1 \leqq i \leqq m\}$.

§ 1. Surjectivity Lemma of Andreotti-Grauert for a Flat (1, 1)-Complete Map

Let D_1 (resp. D_2) be a domain of $\boldsymbol{C}^n$ (resp. $\boldsymbol{C}^N$). Set $m = n + N$ and $D = D_1 \times D_2 \subseteq \boldsymbol{C}^n \times \boldsymbol{C}^N = \boldsymbol{C}^m$. Let $\pi: D \to D_1$ be the natural projection. Let $\varphi: D_2 \to \boldsymbol{R}$ be a C^∞ strictly plurisubharmonic function on D_2 and $\tilde{\varphi} = \varphi p_2$ where $p_2: D \to D_2$ is the natural projection. Let $\xi = (\xi_1, \xi_2) \in D$, $\xi_1 \in D_1$ and $\xi_2 \in D_2$, and $c = \tilde{\varphi}(\xi)$. Let $Y = \{z \in D; \tilde{\varphi}(z) > c\}$ and $Z = D - Y = \{\tilde{\varphi}(z) \leqq c\}$.

Lemma 1.1. *Let A be an analytic subset of D_1 and $a = \dim_{\xi_1} A$. Let $Z_A = Z \cap \pi^{-1}(A)$. Then there exists a fundamental system of Stein neighborhoods $\{Q\}$ of ξ in D such that $H^i_{Z_A}(Q, \mathcal{O}_D) = 0$, $i < m - a$.*

Proof. α. First we assume that A is smooth at ξ_1. If $a = n$, then the result follows from [1, Prop. 12] since $\tilde{\varphi}$ is clearly strictly $(n+1)$-pseudoconvex on D in the sense of [1]. So we assume that $n > a$. Take local coordinates $w_1, \cdots, w_n$ of D_1 at ξ_1 in such a way that $w_{a+1} = \cdots = w_n = 0$ is a system of defining equations of A at ξ_1. Let $\psi = \sum_{i=a+1}^{n} |w_i|^2$ and $\tilde{\psi} = \psi\pi$. We may assume that $\Delta_1 = \{|w_i| < 1, 1 \leqq i \leqq n\} \Subset D_1$ and A is smooth in Δ_1. Similarly take local coordinates $z_1, \cdots, z_N$ of D_2 around ξ_2 in such a way that $\Delta_2 = \{|z_j| < 1\} \Subset D_2$. Let $\Delta = \Delta_1 \times \Delta_2$. For $k = 2, 3, \cdots$ we define $\tilde{\varphi}_k = \tilde{\varphi} - (1/k) + k\tilde{\psi}$, $U'_k = \{\tilde{\varphi}_k(x) > c\}$ and $\Delta_{1k} = \{|w_i| < 1 - (1/k)\}$, $\Delta_{2k} = \{|z_j| < 1 - (1/k)\}$. Put $\Delta_k = \Delta_{1k} \times \Delta_{2k}$ and $U_k = U_k' \cap \Delta_k$.

Then U_k form an increasing sequence of open subsets of $\Delta - Z_A = (\Delta - \pi^{-1}(A)) \cap Y_\Delta$ such that $U_k \Subset U_{k+1}$ and $\Delta - Z_A = \bigcup_k U_k$. Consider the following assertion:

$('')$ $H^i(U_k, \mathcal{O}_D) = 0$, $0 < i < m-(a+1)$ and $H^0(\Delta_k, \mathcal{O}_D) \cong H^0(U_k, \mathcal{O}_D)$ if $m-(a+1) \geqq 1$.

If we show $('')$ for each k, then as in the proof of Lemma 2 of [1, p. 222] it follows from [1, Prop. 9] that the assertion $('')$ is also true for $\Delta - Z_A$, i.e., $('')$ is true with U_k and Δ_k replaced respectively by $\Delta - Z_A$ and Δ. Since $('')$ for $\Delta - Z_A$ is equivalent to $H^i_{Z_A}(\Delta, \mathcal{O}_D) = 0$, $i < m-a$, and since Δ with varying w_i and z_j form a fundamental system of Stein neighborhoods of ξ, 1) follows. It remains to prove $('')$. Let $\lambda: \Delta_1 \to A$ be defined by the projection along the linear subspace defined by $w_1 = \cdots = w_a = 0$ regarding A as a subdomain of $C^a(w_1, \cdots, w_a)$. Let $\rho: U_k \to A$ be the map induced by $\tilde{\lambda} = \lambda\pi|_\Delta: \Delta \to A$. Consider ρ naturally as a family of subdomains of $C^{N+n-a} = C^{N+n-a}(z_1, \cdots, z_N, w_{a+1}, \cdots, w_n)$ over A in the sense of [1, § 3]. Then since $\tilde{\varphi}_k$ is strictly plurisubharmonic when restricted to each fiber $\Delta_{kt} = \tilde{\lambda}^{-1}(t) \cap \Delta_k$, $t \in A$, of $\tilde{\lambda}$ we have $H^i(U_{kt}, \mathcal{O}_{U_{kt}}) = 0$, $0 < i < m-a-1$ and $H^0(\Delta_{kt}, \mathcal{O}_D) \cong H^0(U_{kt}, \mathcal{O}_D)$ by [1, Lemma 2, p. 222], where $U_{kt} = \rho^{-1}(t)$. From this $('')$ follows just as in part β of the proof of [1, Prop. 12].

β. Next in the general case let A_0 be the singular locus of A and $A' = A - A_0$. Let U be any Stein neighborhood of ξ_1 and $U' = U - A_0$. Then A' is closed in U'. Let Δ_2 be any relatively compact polydisc in D_2 centered at ξ_2. Then the proof in α shows that for each $w \in U'$ there exists a fundamental system of Stein neighborhoods $\{N\}$ of w in U' such that $H^i_{Z_{A'}}(N \times \Delta_2, \mathcal{O}_D) = 0$, and hence that $R^i\bar{\pi}_{Z_{A'}*}\mathcal{O}_D = 0$ on U', $i < m-a$, where $Z_{A'} = Z \cap \pi^{-1}(A')$ and $\bar{\pi} = \pi|_{D_1 \times \Delta_2}$. Then from the standard spectral sequence $E_2^{p,q} := H^p(U', R^q\bar{\pi}_{Z_{A'}*}\mathcal{O}_D) \Rightarrow H^{p+q}_{Z_{A'}}(U' \times \Delta_2, \mathcal{O}_D)$ we have $H^i_{Z_{A'}}(U \times \Delta_2, \mathcal{O}_D) \cong H^i_{Z_{A'}}(U' \times \Delta_2, \mathcal{O}_D) = 0$, $i < m-a$. Since U and Δ_2 were arbitrary, from this follows the lemma by induction on a as in the proof of [23, Prop. 1. 12]. Q.E.D.

Now let V be an analytic subspace of D_1 and set $W = V \times D_2 \subseteqq D$. Let $\mathcal{F}$ be a coherent analytic sheaf on W, identified with its extension by zero to the whole D. Let $\pi_W = \pi|_W: W \to V$ and $r = \operatorname{codh}_\xi \mathcal{F}_{\xi_1}$ (cf. Notation).

Proposition 1.2. *Suppose that* $\mathcal{F}$ *is* π_W*-flat. Then there exists a fundamental system of Stein neighborhoods* $\{Q\}$ *of* ξ *in* D *such that the following is true:* 1) *If* $r\geqq 2$, *then the restriction map* $H^0(Q, \mathcal{F})\to H^0(Q\cap Y, \mathcal{F})$ *is isomorphic, and* 2) $H^i(Q\cap Y, \mathcal{F})=0$, $0<i<r-1$.

Proof. Since Q is Stein, it follows that 1) and 2) together are equivalent to the following: $H^i_Z(Q, \mathcal{F})=0$, $i\leqq r-1$. Since $\mathcal{F}$ is π_W-flat, there is an exact sequence

$$0\to \mathcal{O}_W^{p_l}\to\cdots\to\mathcal{O}_W^{p_0}\to\mathcal{F}\to 0$$

in a neighborhood of ξ where $l\leqq N-r$ (cf. [6]). Hence by descending induction on r we can reduce the problem to proving the following: (*) $H^i_Z(Q, \mathcal{O}_W)=0$, $i\leqq N-1$ with Q as above (cf. [1, § 15]). To show (*) we follow the method of Siu-Trautmann [23, § 1]; in the notation of Lemma 1.1 we can prove successively the following two assertions.

1) $H^i_{Z_A}(Q, \mathcal{G})=0$, $i<\mathrm{codh}_\xi\mathcal{G}-a$ for any coherent analytic sheaf $\mathcal{G}$ on D.

2) Let $\mathcal{H}$ be any coherent analytic sheaf on D_1 and $\mathcal{G}=\pi^*\mathcal{H}$. Let $q\geqq 0$ be an integer. Suppose that $\dim_{\xi_1}A\cap S_{k+q+1-N}(\mathcal{H})\leqq k$ for every integer k. Then $H^i_{Z_A}(Q, \mathcal{G})=0$, $i\leqq q$.

In fact, 1) follows from Lemma 1.1 by induction on $\mathrm{codh}_\xi\mathcal{G}$ by exactly the same way as in the proof of Proposition 1.13 of [23]. Similarly the implication 1)→2) follows by the same method as in the proof of Theorem 1.14 b)→c) of [23], using induction on a and noting that $\mathrm{codh}_\xi\mathcal{G}=N+\mathrm{codh}_{\xi_1}\mathcal{H}$. Finally 2) implies (*) as follows. Let $A=D_1$ and $\mathcal{H}=\mathcal{O}_S$ in 2) so that $\mathcal{G}=\mathcal{O}_W$ and $Z_A=Z$. Further if we let $q=N-1$, then the assumption reduces to $\dim_{\xi_1}S_k(\mathcal{H})\leqq k$ for all k, which is always true (cf. [23]). Thus (*) follows. Q.E.D.

Let $f: X\to S$ be a morphism of complex spaces. Then we call f a (1, 1)*-complete map* if there is a C^∞ strictly plurisubharmonic function $\varphi: X\to(c_*, c^*)$, $-\infty\leqq c_*<c^*\leqq+\infty$, called the exhaustion function for f, such that for any $c_*<c_1<c_2<c^*$ the restriction of f to $\{c_1\leqq\varphi\leqq c_2\}$ is proper. For $c_*<c_1<c_2<c^*$ we write $X^{c_2}_{c_1}=\{x\in X; c_1<\varphi(x)<c_2\}$, and

$f_{c_1}^{c_2}=f|X_{c_1}^{c_2}$. The following is a relative form of a result of Andreotti-Grauert [1] analogous to [15, Prop. 2. 4. 3] (cf. also [20, Prop. 11. 12]) under flatness assumption.

Proposition 1. 3. *Let $f: X\to S$ be a $(1, 1)$-complete map of complex spaces with exhaustion function $\varphi: X\to (c_*, c^*)$ and $\mathcal{F}$ an f-flat coherent analytic sheaf on X. Let $r=\operatorname{codh}_f\mathcal{F}$. 1) Let $c_*\leqq c'\leqq d'<d''\leqq c''\leqq c^*$. Then for each $s\in S$ there exists a fundamental system $\{U\}$ of Stein neighborhoods U of s such that the restriction map $H^i(X_{c'}^{c''}(U), \mathcal{F})\to H^i(X_{d'}^{d''}(U), \mathcal{F})$ is surjective for $1\leqq i\leqq r-2$. 2) Let $c_*\leqq c\leqq d<\tilde{c}\leqq c^*$. If $r\geqq 2$, then the restriction map $\Gamma(X_c^{\tilde{c}}(U), \mathcal{F})\to\Gamma(X_d^{\tilde{c}}(U), \mathcal{F})$ is surjective with U as above.*

Proof. It is enough to show that for a suitable $\{U\}$ as above the restriction maps $r_1: H^i(X_{c'}^{c''}(U), \mathcal{F})\to H^i(X_{c'}^{d''}(U), \mathcal{F})$, $1\leqq i\leqq r-2$, and $r_2: H^i(X_{c'}^{d''}(U), \mathcal{F})\to H^i(X_{d'}^{d''}(U), \mathcal{F})$, $0\leqq i\leqq r-2$, are surjective, where $c'\leqq d'\leqq d''\leqq c''$ are as in 1). First of all the surjectivity of r_1 can be proved just as in Andreotti-Grauert [1, Prop. 16] (cf. also [4, Lemme 1]) and the proof is omitted. On the other hand, in view of [1, p. 241, Lemma] the surjectivity of r_2 can be reduced to Proposition 1. 2, just in the same way as the proof of [20, Prop. 11. 12] is reduced to (the proof of) [20, Prop. 11. 8]. Q.E.D.

Corollary 1. 4. *The restriction map $R^if_{c'*}^{c''}\mathcal{F}\to R^if_{d'*}^{d''}\mathcal{F}$ is surjective for $1\leqq i\leqq r-2$ in 1) and $f_{c*}^{\tilde{c}}\mathcal{F}\to f_{d*}^{\tilde{c}}\mathcal{F}$ is surjective if $r\geqq 2$ in 2).*

Remark. The same proof applies without any change to the case of a (p, q)-convex-concave map to yield the surjectivity of $H^i(X_{c'}^{c''}(U), \mathcal{F})\to H^i(X_{d'}^{d''}(U), \mathcal{F})$ for $p\leqq i\leqq r-q-1$.

§ 2. Basic Coherency and Isomorphy Results

Let S be a complex space and $b\in\boldsymbol{R}_+^N$ for some $N>0$. Let $W=$

$S\times K(b)\subseteqq S\times \boldsymbol{C}^N$ (cf. Notation) and $\pi: W\rightarrow S$ the natural projection. Let $W'=W-(S\times\{0\})$ and $\pi':=\pi|_{W'}$. For $0\leqq a<b$, i.e., $a_i<b_i$ for $1\leqq i\leqq N$ we put $Q_a=S\times K(a)^-$ where $a=(a_1,\cdots,a_N)$ and $b=(b_1,\cdots,b_N)$. Here for $a=0$, i.e., $a_i=0$, $1\leqq i\leqq N$, we understand that $Q_0=S\times\{0\}$.

Proposition 2.1. *Let $\mathcal{F}$ be a coherent analytic sheaf on W. Suppose that $\mathcal{F}$ is π'-flat on W' and* $\mathrm{codh}_{\pi'}(\mathcal{F}|_{W'})\geqq r$ *for some $r\geqq 0$. Then for every $i\leqq r-1$ 1) the natural map $R^i\pi_{Q_{a'}*}\mathcal{F}\rightarrow R^i\pi_{Q_a*}\mathcal{F}$ is isomorphic for $0\leqq a'<a<b$ and 2) $R^i\pi_{Q_a*}\mathcal{F}$ is coherent for $0\leqq a<b$.*

Proof. α. Since the problem is local on S, we fix a point $s\in S$ and consider everything around s. First by Lemma 2 of [6] after restricting S around s if necessary, we may assume that there exists an exact sequence of coherent $\mathcal{O}_W$-modules

$$(1)\qquad 0\rightarrow\mathcal{L}\rightarrow\mathcal{O}_W^{p_{m-1}}\rightarrow\cdots\rightarrow\mathcal{O}_W^{p_0}\xrightarrow{\lambda}\mathcal{F}\rightarrow 0$$

such that $m=N-r$ and $\mathcal{L}$ is locally free on W'. (Taking b smaller and using the excision we may assume that (1) is defined on the whole W.) Using (1) we can readily reduce the proof by descending induction on r to showing that for $i\leqq N-1$ and $0\leqq a<b$, the natural map $R^i\pi_{Q_0*}\mathcal{L}\rightarrow R^i\pi_{Q_a*}\mathcal{L}$ is isomorphic and $R^i\pi_{Q_a*}\mathcal{L}$ is coherent. For this purpose, however, we have to treat a little more general situation. Let $\underline{d}:=(d_1,\cdots,d_q)$, $0\leqq q\leqq N$, be a q-tuple of integers with $d_i\geqq 1$. Let $L_{\underline{d}}$ be the subspace of $\boldsymbol{C}^N(z_1,\cdots,z_N)$ defined by the ideal $(z_1^{d_1},\cdots,z_q^{d_q})$. For $q=0$ we understand that $L_{\underline{d}}=\boldsymbol{C}^N$. Set $W_{\underline{d}}:=W\cap(S\times L_{\underline{d}})=S\times(K(b)\cap L'_{\underline{d}})$ and $W_{\underline{d}}:=W_{\underline{d}}-Q_0$. Then the above assertion is a special case $(q=0)$ of the following one:

(*) *Let $\mathcal{L}$ be a coherent analytic sheaf on $W_{\underline{d}}$ which is locally free on $W'_{\underline{d}}$. Then for $i<p:=N-q$ and $0\leqq a<b$, the natural map $R^i\pi_{Q_0*}\mathcal{L}\rightarrow R^i\pi_{Q_a*}\mathcal{L}$ is isomorphic and $R^i\pi_{Q_a*}\mathcal{L}$ is coherent.*

The rest of the proof is then devoted to prove (*).

β. Restricting S around s and taking b smaller (cf. the remark above) we may assume that S is a relatively compact subdomain of a Stein space S' and $\mathcal{F}$ is defined on $S'\times K(b)^-$. Fixing a we put $Q=Q_a$. Let $\mathcal{J}$ be the ideal sheaf of Q_0 in W. Then we shall first prove the following assertion:

(A) *Let U be any open subset of W. Then $\Gamma(U, \mathcal{J})^l H^i_Q(U, \mathcal{L}) = 0$, $i<p$, for some sufficiently large l which is independent of U.*

To prove (A), we have to consider a still more general situation. Let A be any analytic subset of S' and $Q_A = Q \cap \pi^{-1}(A)$. Identify S with Q_0 and consider A also as a subset of W. Let $\mathcal{J}_A$ be the ideal sheaf of A in W. Let S'_k be the union of those irreducible components of $S_k(\mathcal{L})$ which are not contained in A. Then (A) is a special case of the following assertion:

($\tilde{A}$) *If $\dim(A \cap S'_{k+\mu}) \leqq k$ for every k and some $\mu \geqq 0$, then $\Gamma(U, \mathcal{J}_A)^l H^i_{Q_A}(U, \mathcal{L}) = 0$, $i<\mu$, for a sufficiently large l which is independent of U.*

In fact, if in ($\tilde{A}$) we put $A=S$ and $\mu=p$, we get (A) since it is always true that $\dim S \cap S'_{k+p} = \dim S_k(S) \leqq k$ for every k. Here the first equality follows from the relation $S'_k = \pi^{-1}(S_{k-p}(S))$, which can be seen as follows: Since $\mathcal{L}$ is locally free on $W'_{\underline{d}}$, we have $S_k(\mathcal{L})|_{W'_{\underline{d}}} = S_k(W'_{\underline{d}}) = \pi'^{-1}(S_{k-p}(S))$. From this it follows that for any irreducible component $S_{k,\nu}$ of $S_k(\mathcal{L})$ with $S_{k,\nu} \cap W' \neq \emptyset$ there is a unique irreducible component $T_{k,\nu}$ of $S_{k-p}(S)$ such that $S_{k,\nu} = \pi^{-1}(T_{k,\nu})$. Conversely if $T_{k,\nu}$ is any irreducible component of $S_{k-p}(S)$, then $\pi^{-1}(T_{k,\nu})$ is an irreducible component of S'_k since it intersects with W' and coincides with an irreducible component of $S_k(W_{\underline{d}}) \cap W'$. Hence $S'_k = \pi^{-1}(S_{k-p})(S)$.

γ. We shall show ($\tilde{A}$). We may assume that S is an analytic subspace of a domain D_1 of $\mathbf{C}^n$ for some $n>0$ [12]. Let $D = D_1 \times K(b)$ and $m = n+N = \dim D$. Let $\mathcal{G}$ be a coherent analytic sheaf on D and $\mathcal{J}_\mu$ the ideal sheaf of $S_{\mu+a-1}(\mathcal{G})$ in D where $0 \leqq \mu \leqq m-a$ and $a = \dim A$. Then just as in the proof of [23, Lemma 3. 3] we deduce that $\Gamma(U', \mathcal{J}_\mu)^l H^i_{Q_A}(U', \mathcal{G}) = 0$, $i<\mu$, for a sufficiently large l which is independent of U', where U' is any open subset of D. Indeed, in view of the vanishing of $R^i\pi_{Q_A *}\mathcal{O}_D$, $0 \leqq i \leqq m-a$, (which follows from a lemma of Frenkel [23, (0. 14)] when A is nonsingular, and in the general case from this special case as in Lemma 1. 1 above), if we replace $\mathcal{F}$ and A by $\mathcal{G}$ and Q_A respectively there, the same argument works. From this (applied to $\mathcal{G} = \mathcal{L}$) together with the above description of the sets S'_k, we can prove ($\tilde{A}$) by the same method as in the proof of Lemma 3. 4 of [23], using the filtration of A by the subspaces $A_k := A \cap S'_{k+\mu}$.

δ. Finally we shall deduce (*) from (A). We proceed by induction on i, $0 \leqq i < p$. First we note that $\pi_{Q_0*}\mathcal{L} \cong \pi_{Q*}\mathcal{L}$ and $\pi_{Q*}\mathcal{L}$ is coherent. In fact since $\mathcal{L}$ is locally free on $W'_{\underline{d}}$, $\pi_{Q*}\mathcal{L} \cong \pi_{Q_0*}\mathcal{L} \cong \pi_*(\underline{\Gamma}_{Q_0}\mathcal{L})$, and the latter is coherent since $\underline{\Gamma}_{Q_0}(\mathcal{L})$ is coherent and $Q_0 \cong S$ (cf. [23, Prop. 1. 9]). So suppose that $p > i > 0$. For each $l > 0$ we have an exact sequence of $\mathcal{O}_W$-modules

$$(2) \qquad 0 \to \mathcal{K}_l \to \mathcal{L} \overset{\alpha_l}{\to} \mathcal{L} \to \mathcal{L}/z^l\mathcal{L} \to 0$$

where α_l is the multiplication by z^l, $z = z_{q+1}$, and $\mathcal{K}_l$ is the kernel of α_l. In particular since $\mathcal{L}$ is locally free on $W'_{\underline{d}}$, the support of $\mathcal{K}_l$ is contained in Q_0 and hence is finite over S, so that $\pi_{Q*}\mathcal{K}_l \cong \pi_*\mathcal{K}_l$ and is coherent, and that $R^i\pi_{Q*}\mathcal{K}_l \cong R^i\pi_*\mathcal{K}_l = 0$ for $i > 0$. On the other hand, since $z \in \Gamma(W, \mathcal{J})$, by (A) the map $R^i\pi_{Q*}\mathcal{L} \to R^i\pi_{Q*}\mathcal{L}$ induced by α_l are zero maps for $i < p$ and $l \gg 0$. Hence from (2) we get for $l \gg 0$ the exact sequence

$$(3) \qquad 0 \to R^{i-1}\pi_{Q*}\mathcal{L} \to R^{i-1}\pi_{Q*}\mathcal{L}/z^l\mathcal{L} \to R^i\pi_{Q*}\mathcal{L} \to 0, \quad 1 \leqq i < p.$$

Note that $\mathcal{L}/z^l\mathcal{L}$ is a coherent analytic sheaf on $W_{\underline{d}'} = W \cap L_{\underline{d}'}$ with $\underline{d}' = (d_1, \cdots, d_q, d_{q+1})$, $d_{q+1} = l$, which is locally free on $W'_{\underline{d}'} := W_{d'} - Q_0$. By induction both $R^{i-1}\pi_{Q*}\mathcal{L}$ and $R^{i-1}\pi_{Q*}\mathcal{L}/z^l\mathcal{L}$ are coherent since $i-1 < p-1 = N-(q+1)$. Hence from (3) follows the coherency of $R^i\pi_{Q*}\mathcal{L}$. Similarly comparing the exact sequence (3) with the corresponding ones with Q replaced by Q_0 (noting that a was arbitrary) we obtain the isomorphy of the map $R^i\pi_{Q_0*}\mathcal{F} \to R^i\pi_{Q*}\mathcal{F}$, $i < p$, by induction and five lemma. Q.E.D.

Using the previous notation, let $\mathcal{J}$ be the ideal sheaf of Q_0 in W. For every $\pi_*\mathcal{O}_W$-module $\mathcal{H}$ and every integer $k > 0$ we define the submodule $\mathcal{H}\langle k\rangle$ of $\mathcal{H}$ by the submodule defined by the local sections annihilated by $\pi_*\mathcal{J}^k \subseteqq \pi_*\mathcal{O}_W$;

$$\mathcal{H}\langle k\rangle_s = \{\alpha \in \mathcal{H}_s;\ (\pi_*\mathcal{J}^k)_s\alpha = 0\}; \quad s \in S:$$

In particular for $0 \leqq a < b$ we can speak of the $\pi_*\mathcal{O}_W$-submodule $R^i\pi_{Q_a*}\mathcal{F}\langle k\rangle$ of $R^i\pi_{Q_a*}\mathcal{F}$. On the other hand, we note that if $\mathcal{H}$ is coherent as an $\mathcal{O}_S$-module via the natural inclusion $\mathcal{O}_S \subseteqq \pi_*\mathcal{O}_W$ then $\mathcal{H}\langle k\rangle$ also is a coherent $\mathcal{O}_S$-module. In fact, by the definition of $\mathcal{J}$ we can find, for each k, elements $g_1, \cdots, g_t \in \Gamma(W, \mathcal{J}^k)$ such that g_i, when considered as sections of $\pi_*\mathcal{J}^k$, generate $\pi_*\mathcal{J}^k$ as an $\pi_*\mathcal{O}_W$-module at each point of S. Then we obtain an exact sequence of $\pi_*\mathcal{O}_W$-modules $0 \to \mathcal{H}\langle k\rangle \to \mathcal{H} \to \mathcal{H}^{\oplus t}$, where the last arrow is defined by $\mathcal{H} \ni h \to (g_i h) \in \mathcal{H}^{\oplus t}$. Hence $\mathcal{H}\langle k\rangle$ also is a coherent $\mathcal{O}_S$-module.

Proposition 2. 2. *The notations and assumption being as in Proposition* 2. 1 *the following holds true*: 1) *The natural map* $R^r\pi_{Q_{a'}*}\mathcal{F}\langle k\rangle \to R^r\pi_{Q_a*}\mathcal{F}\langle k\rangle$ *is isomorphic for* $0 \leqq a' < a < b$ *and* 2) $R^r\pi_{Q_a*}\mathcal{F}\langle k\rangle$ *is coherent for* $0 \leqq a < b$.

Proof. Fixing $0 \leqq a' < a < b$ we put $Q = Q_a$ and $Q' = Q_{a'}$. We shall prove the proposition by descending induction on $r \leqq N$. Suppose first that $r < N$. Consider the exact sequence (1) in the proof of Proposition 2. 2 and let $\mathcal{G} = \operatorname{Ker} \lambda$. Then we have the long exact sequence on S

$$\to R^r\pi_{Q*}\mathcal{O}_W^{p_0} \to R^r\pi_{Q*}\mathcal{F} \xrightarrow{\nu} R^{r+1}\pi_{Q*}\mathcal{G} \to R^{r+1}\pi_{Q*}\mathcal{O}_W^{p_0} \to$$

associated to the short exact sequence $0 \to \mathcal{G} \to \mathcal{O}_W^{p_0} \xrightarrow{\nu} \mathcal{F} \to 0$. Let $\mathcal{H}_k = \nu^{-1}(R^{r+1}\pi_{Q*}\mathcal{G}\langle k\rangle)$. Then we get the exact sequence

$$(4) \qquad \to R^r\pi_{Q*}\mathcal{G} \to R^r\pi_{Q*}\mathcal{O}_W^{p_0} \to \mathcal{H}_k \to R^{r+1}\pi_{Q*}\mathcal{G}\langle k\rangle \to R^{r+1}\pi_{Q*}\mathcal{O}_W^{p_0}\langle k\rangle\,.$$

Then $R^r\pi_{Q*}\mathcal{G}$ and $R^r\pi_{Q*}\mathcal{O}_W^{p_0}$ are coherent by Proposition 2. 1 and $R^{r+1}\pi_{Q*}\mathcal{G}\langle k\rangle$, and $R^{r+1}\pi_Q\mathcal{O}_W^{p_0}\langle k\rangle$ also are coherent by induction since $\mathcal{G}|_{W'}$ is π'-flat and $\operatorname{codh}(\mathcal{G}|_{W'}) \geqq r+1$. Hence $\mathcal{H}_k$ also is coherent. On the other hand, we have $R^r\pi_{Q*}\mathcal{F}\langle k\rangle \subseteqq \mathcal{H}_k$ and hence $R^r\pi_{Q*}\mathcal{F}\langle k\rangle = \mathcal{H}_k\langle k\rangle$. Hence the coherency of $R^r\pi_{Q*}\mathcal{F}\langle k\rangle$ follows from that of $\mathcal{H}_k$ by the remark just before the proposition. As for 1), since for $\mathcal{G}' := \mathcal{G}$ or $\mathcal{O}_W^{p_0}$, we have $R^{r+1}\pi_{Q'*}\mathcal{G}'\langle k\rangle \cong R^{r+1}\pi_{Q*}\mathcal{G}'\langle k\rangle$ by induction and have $R^r\pi_{Q'*}\mathcal{G}' \cong R^r\pi_{Q*}\mathcal{G}$ by Proposition 2. 1, comparing the sequences (4) for Q' and Q we obtain by the five lemma that $\mathcal{H}'_k \cong \mathcal{H}_k$ and hence $\mathcal{H}'\langle k\rangle \cong \mathcal{H}\langle k\rangle$ as desired where $\mathcal{H}'_k$ is the $\mathcal{H}_k$ in (4) with Q replaced by Q'.

Thus it remains to consider the case $r = N$. In this case $\mathcal{F}$ is locally free on W'. As in the proof of previous proposition we have then to deal with the more general situation described there. Using the notation there for any coherent analytic sheaf $\mathcal{L}$ on $W_{\underline{d}}$, $\underline{d}=(d_1, \cdots, d_q)$, which is locally free on $W'_{\underline{d}}$ we prove the following assertions; 1) $R^p\pi_{Q'*}\mathcal{L}\langle k\rangle \cong R^p\pi_{Q*}\mathcal{L}\langle k\rangle$ and 2) $R^p\pi_{Q*}\mathcal{L}\langle k\rangle$ is coherent, by induction on $p=N-q$. First, if $p=0$, then $W_{\underline{d}}$ is finite over S so that 1) and 2) are immediate to see. So assume that $p>0$. Then as we have deduced (3) from (2) in the proof of the previous proposition, for $p\geqq 1$ we have from (2) an exact sequence of $\mathcal{O}_S$-modules

$$\to R^{p-1}\pi_{Q*}\mathcal{L}\xrightarrow{\gamma}R^{p-1}\pi_{Q*}\mathcal{L}/z^k\mathcal{L}\xrightarrow{\delta}R^p\pi_{Q*}\mathcal{L}\xrightarrow{\alpha_k}R^p\pi_{Q*}\mathcal{L}\to$$

for every sufficiently large k, with γ injective for $p\geqq 2$. Since α_k is defined by the multiplication by z^k, $z=z_{q+1}$, $R^p\pi_{Q*}\mathcal{L}\langle k\rangle$ is in the image of δ. Moreover by (A) in the proof of the previous proposition $R^{p-1}\pi_{Q*}\mathcal{L}$ is annihilated by $\pi_*\mathcal{J}^{k_1}$ for some $k_1>0$. Hence $R^p\pi_{Q*}\mathcal{L}\langle k\rangle$ is contained in the image $\mathcal{M}_k$ of $(R^{p-1}\pi_Q\mathcal{L}/z^k\mathcal{L})\langle k+k_1\rangle$ by δ. Hence $R^p\pi_{Q*}\mathcal{L}\langle k\rangle =\mathcal{M}_k\langle k\rangle$. On the other hand, since $\mathcal{M}_k\cong(R^{p-1}\pi_{Q*}\mathcal{L}/z^k\mathcal{L})\langle k+k_1\rangle/\mathrm{Im}\,\gamma$ and $(R^{p-1}\pi_{Q*}\mathcal{L}/z^k\mathcal{L})\langle k+k_1\rangle$ (resp. $R^{p-1}\pi_{Q*}\mathcal{L}$) is coherent by induction (resp. by $(*)$ in the proof of the previous proposition), $\mathcal{M}_k$ is coherent, where we note that $R^{p-1}\pi_{Q*}\mathcal{L}=(R^{p-1}\pi_{Q*}\mathcal{L})\langle k+k_1\rangle$. Thus $R^p\pi_{Q*}\mathcal{L}\langle k\rangle$ is coherent as above. Moreover since $R^{p-1}\pi_{Q'*}\mathcal{L}\to R^{p-1}\pi_{Q*}\mathcal{L}$ (resp. $(R^{p-1}\pi_{Q*}\mathcal{L}/z^k\mathcal{L})\langle k+k_1\rangle\to(R^{p-1}\pi_{Q*}\mathcal{L}/z^k\mathcal{L})\langle k+k_1\rangle$) is isomorphic by $(*)$ in the proof of the previous proposition (resp. by induction) the natural homomorphism $\mathcal{M}'_k\to\mathcal{M}_k$ is isomorphic where $\mathcal{M}'_k$ is defined analogously to $\mathcal{M}_k$ for Q'. Hence $R^p\pi_{Q'*}\mathcal{L}\langle k\rangle\to R^p\pi_{Q*}\mathcal{L}\langle k\rangle$ also is isomorphic. Thus the proposition is proved for $k\gg 0$. Since $R^p\pi_{Q*}\mathcal{L}\langle k\rangle =(R^p\pi_{Q*}\mathcal{L}\langle k'\rangle)\langle k\rangle$ for $k'>k$ in general, the general case follows from the above special case. Q.E.D.

§ 3. Some Lemmas

Let X be a complex space and F a $\boldsymbol{C}$-subalgebra of $\Gamma(X, \mathcal{O}_X)$. We recall briefly the theory of F-quotient of X. For more detail see Wiegmann [24]. We denote by the same letter F the equivalence relation defined by F on (the underlying topological space of) X; $x\sim y\Leftrightarrow h(x)=h(y)$ for all $h\in F$ where $x, y\in X$. Let $Y=X/F$ be the topological quotient space of X by F and $\bar{\sigma}: X\to Y$ the quotient map. Then we define the sheaf of local rings $\mathcal{O}_Y$ on Y as follows;

$\mathcal{O}_{Y,y} = \varinjlim \{A(U)\,;\, U$ open neighborhoods of $y\}, y \in Y$
where $A(U)$ is the subalgebra of $\Gamma(\bar{\sigma}^{-1}(U), \mathcal{O}_X)$ such that $h \in \Gamma(\bar{\sigma}^{-1}(U), \mathcal{O}_X)$ is in $A(U)$ if and only if there exist a convergent power series $\sum c_{i_1 \cdots i_d} T_1^{i_1} \cdots T_d^{i_d}$, $c_{i_1 \cdots i_d} \in \mathbf{C}$, in d indeterminates T_i for some $d > 0$ and elements $f_1, \cdots, f_d \in F$ with $f_i(x) = 0$ for all $x \in \bar{\sigma}^{-1}(y)$ such that $\sum c_{i_1 \cdots i_d} f_1^{i_1} \cdots f_d^{i_d}$ converges and equals h on $\bar{\sigma}^{-1}(U)$. Then we have the natural surjective morphism of local ringed spaces $\sigma: X \to Y$, and then regarding $\Gamma(Y, \mathcal{O}_Y)$ as a subalgebra of $\Gamma(X, \mathcal{O}_X)$ via σ we have the natural inclusion $F \subseteqq \Gamma(Y, \mathcal{O}_Y)$.

We call X *F-convex* if for every compact $K \subseteqq X$ its F-convex hull

$$\hat{K}_F := \{x \in X;\, |h(x)| \leqq \sup\{|h(y)|;\, y \in K\} \quad \text{for all} \quad h \in F\}$$

is again compact. Then Wiegmann [24] proves the following:

If X is F-convex, then $Y = (Y, \mathcal{O}_Y)$ has the natural structure of a Stein complex space such that σ is a proper surjective morphism of complex spaces.

In this case we call Y, or σ, the *F-quotient* of X. When $F = \Gamma(X, \mathcal{O}_X)$, F-quotient is called the *Remmert quotient* of X, and then we have $\mathcal{O}_Y \cong \sigma_* \mathcal{O}_X$.

Let $f: X \to S$ be a 1-convex map of complex spaces with exhaustion function $\varphi: X \to (-\infty, c^*)$. Then in what follows we shall use the following notation: $X_c = \{x \in X;\, \varphi(x) > c\}$, $X^c = \{x \in X;\, \varphi(x) < c\}$ for $c < c^*$ and $X_{c_1}^{c_2} = \{x \in X;\, c_1 < \varphi(x) < c_2\}$ for $c_1 < c_2 < c^*$. Suppose now that S is Stein. Then by Knorr-Schneider [14] or by Siu [21] X is $\Gamma(X, \mathcal{O}_X)$-convex. Let $\sigma_0: X \to Y_0$ be the Remmert quotient of X. Since $f^* \Gamma(S, \mathcal{O}_S) \subseteqq \Gamma(X, \mathcal{O}_X) \cong \Gamma(Y, \mathcal{O}_Y)$ and S is Stein, we have the natural morphism $g_0: Y_0 \to S$ such that $g_0 \sigma_0 = f$.

Lemma 3.1. *Let $f: X \to S$ be as above. Let $\mathcal{J}$ be the coherent sheaf of ideals of $\mathcal{O}_X$ such that the support A of $\mathcal{O}_X / \mathcal{J}$ is proper over S with respect to f. Let $F = F(\mathcal{J})$ be the subalgebra of $\Gamma(X, \mathcal{O}_X)$ generated by $\Gamma(X, \mathcal{J})$ and $f^* \Gamma(S, \mathcal{O}_S)$. Then X is F-convex.*

Proof. Let $\sigma_0: X \to Y_0$ be the Remmert quotient of X and $\mathcal{J}' = \sigma_{0*} \mathcal{J}$.

Then $\mathcal{J}'$ is a coherent sheaf of ideals of $\mathcal{O}_{Y_0}$ such that the support of $\mathcal{O}_{Y_0}/\mathcal{J}'$ coinsides with $A':=\sigma_0(A)$. Since σ_0 is proper, it then suffices to show that Y_0 is $F(\mathcal{J}')$-convex. This allows us to assume from the beginning that X is Stein. Then, if we denote by $r: \mathcal{O}_X \to \mathcal{O}_{X_{red}}$ the natural quotient homomorphism and set $\mathcal{J}_1 = r(\mathcal{J})$, then $\Gamma(X, \mathcal{J}) \to \Gamma(X_{red}, \mathcal{J}_1)$ is surjective, where X_{red} is the underlying reduced subspace of X. Hence we may further assume that X is reduced. Now note that since A is proper over S, $A \subseteqq X^c$ for some $c < c^*$. Then by Narasimhan [16, Theorem 1] there is a holomorphic map $h := (h_1, \cdots, h_N) : X \to \boldsymbol{C}^N$ for some N such that $h^{-1}(0) = A$ and that $h|_{X-X^c}$ is proper. Moreover the proof shows that we can assume that $h_i \in \Gamma(X, \mathcal{J})$. Then $H := \{h_1, \cdots, h_N\} \subseteqq F$ so that if K is any compact subset of X, then $\hat{K}_F \subseteqq \hat{K}_H$. On the other hand, if we put $r_i = \sup\{|h_i(x)|; x \in K\}$, then $\hat{K}_H = h^{-1}(\Delta(r))$ where $\Delta(r)$ is the closed polydisc of multi-radius $r = (r_i)$ in $\boldsymbol{C}^N$ with center the origin. Hence by the property of h mentioned above $\hat{K}_H \cap (X - X^c)$ is compact. Thus it remains to show that $\hat{K}_F \cap X^c$ is relatively compact in X. In fact, since S is Stein and $f^*\Gamma(S, \mathcal{O}_S) \subseteqq F$, $\hat{L}_F := f(\hat{K}_F)$ is compact, and hence $\hat{K}_F \cap X^c (\subseteqq f^{-1}(\hat{L}_F) \cap X^c)$ is relatively compact in X. Q.E.D.

Let $f: X \to S$ and $\mathcal{J}$ be as in Lemma 3.1 and $F = F(\mathcal{J})$. Let $\sigma: X \to Y$ be the resulting F-quotient of X. Since $f^*\Gamma(S, \mathcal{O}_S) \subseteqq F$ and Y and S are Stein, there is a unique Stein morphism $g: Y \to S$ such that $g\sigma = f$. We call the map σ together with the map g, the *$\mathcal{J}$-quotient* of X, or of f.

Let $f: X \to S$ be a 1-convex map of complex spaces. Then each fiber X_s, $s \in S$, of f is a 1-convex space. Let E_s be the exceptional set of X_s, i.e., the maximal compact analytic subset of X_s of positive dimension. Then $E = \bigcup_{s \in S} E_s$ has the natural structure of a (reduced) analytic subset of X, called the *relative exceptional set* for f. Moreover if $\sigma_0: X \to Y_0$ is the Remmert quotient of X, then σ_0 induces an isomorphism of $X - E$ and $Y - \sigma_0(E)$ (cf. [14]).

Lemma 3.2. *Let $f: X \to S$ be a 1-convex map of complex spaces with S Stein. Let A be an analytic subset of X such that $f|_A: A \to S$*

is proper and that A contains the relative exceptional set E for f. Then there exist a Stein morphism $g: Y \to S$ and a proper surjective S-morphism $\sigma: X \to Y$ such that $\sigma|_{X-A}: X-A \to Y-\sigma(A)$ is isomorphic, and $\sigma(A)$ is mapped isomorphically onto a subspace of S by g,

Proof. Let $\mathcal{J}$ be the ideal sheaf of A in X. Then by our assumption on A and by Lemma 3.1 we can take the $\mathcal{J}$-quotient of f. So let $\sigma: X \to Y$ with $g: Y \to S$ be the $\mathcal{J}$-quotient. First, if X is Stein, then $\Gamma(X, \mathcal{J})$ separates points of $X-A$ and give local coordinates at each point of $X-A$, so that from the above construction of $\mathcal{O}_Y$ it follows readily that $\sigma|_{X-A}: X-A \cong Y-\sigma(A)$. The general case can be reduced easily to this case by using the Remmert quotient as in the proof of the previous lemma if we note that $E \subseteqq A$. Now we remark that since $\Gamma(X, \mathcal{J})$ is an ideal of $\Gamma(X, \mathcal{O}_X)$, every $h \in F$ can be written in the form $h = h_1 + h_2$, with $h_1 \in \Gamma(X, \mathcal{J})$ and $h_2 \in f^*\Gamma(S, \mathcal{O}_S)$, and hence that in the definition of $\mathcal{O}_Y$ we can assume that each f_i belongs to either $\Gamma(X, \mathcal{J})$ or $f^*\Gamma(S, \mathcal{O}_S)$. Then since $\Gamma(Y, \sigma_*\mathcal{J}) \cong \Gamma(X, \mathcal{J})$ and $\sigma_*\mathcal{J}$ is generated by global sections on Y, Y being Stein, again from the definition of $\mathcal{O}_Y$ it follows that $\bar{\mathcal{J}} := \sigma_*\mathcal{J}$ may be regarded naturally as an ideal of $\mathcal{O}_Y$, and that if we denote by A' the subspace of Y defined by $\bar{\mathcal{J}}$, then g induces an isomorphism of A' with $g(A')$. (Note that since $\Gamma(X, \mathcal{J})$ cannot separate any two points of A, $g|_{A'}: A' \to S$ is clearly injective.) Finally A' is reduced as well as A so that $A' = \sigma(A)$, the latter given with the reduced structure. This proves the lemma.

Remark. When $f(A) = S$, from the above proof we get the following exact sequence of $\mathcal{O}_S$-modules

$$0 \to \bar{\mathcal{J}} \to \mathcal{O}_Y \to \mathcal{O}_S \to 0\,.$$

Lemma 3.3. *Let $f: X \to S$ be a Stein 1-convex map with an exhaustion function $\varphi: X \to (-\infty, c^*)$ with convexity bound $c_\# \in (-\infty, c^*)$. Let A be an analytic subset of $X^{c_\#}$ which is mapped isomorphically onto a subspace of S by f. Let $s \in S$ and $c_\# < c' < c_1 < c_2 < c'' < c^*$ be arbitrary. Then there exist $a < b$ in $\mathbb{R}^{+N}$ for some $N > 0$, a relatively compact subdomain P_b of X, a neighborhood U of s in S and* *a holomorphic embedding* $\Phi: P_b \to U \times K(b)$, *satisfying the following conditions*: *Let* $Q_a = \Phi^{-1}(U \times K(a)^-)$ *and* $B^d = \{x \in X; \varphi(x) \leqq d\}$ *for* $d \in (c_\#, c^*)$. *Then* 1) $B^{c'}(U) \subseteqq Q_a \subseteqq B^{c_1}(U) \subseteqq B^{c_2}(U) \subseteqq P_b \subseteqq X^{c''}$, 2) $\tau\Phi = f|_{P_b}$, $\tau: S \times K(b) \to S$ *being the natural projection and* 3) $\Phi^{-1}(U \times \{0\}) = A(U)$.

Proof. The lemma is essentially [15, Lemma 3.1.1] except for 3) (cf. Remark below). In particular we may assume that $A \neq \emptyset$ since otherwise 3) follows immediately from that lemma. First we consider the case $S = \{s\}$. In particular A consists of a single point $a \in X$. Take and fix any $c'' < c_3 < c^*$. Then by the same technique as in parts (A) and (B) of the proof of [15, Lemma 3.1.1] we can find $f_1, \cdots, f_q \in \Gamma(X^{c_3}, \mathcal{O}_X)$ such that $F := (f_1, \cdots, f_q)$ defines an embedding of X^{c_3} into $\boldsymbol{C}^q$ and, further, if we put $Q_R = \{x \in X^{c_3}; |f_i(x)| < R_i\}$ for a suitable $R := (R_1, \cdots, R_q)$, $R_i > 1$, and define $W = \{x \in X; |f_i(x)| < 1\}$, then $B^{c'} \Subset W \Subset X^{c_1} \Subset B^{c_2} \Subset Q_R \Subset X^{c''}$. It then suffices to check that these f_i can be taken in such a way that $f_i(a) = 0$ (cf. the proof in the general case below). In fact this follows readily from the construction in [15] together with the following remark: For every $c' < c < c_3$, B^c is $\Gamma(X^{c_3}, \mathfrak{m}_a)$-convex, where $\mathfrak{m}_a$ is the maximal ideal at a. *Proof.* Let $x_0 \in X^{c_3} - B^c$. Since B^c is $\Gamma(X^{c_3}, \mathcal{O}_X)$-convex [12, IX, C8] we can find $g \in \Gamma(X^{c_3}, \mathcal{O}_X)$ such that $r := \sup\{|g(x)|; x \in B^c\} < |g(x_0)|$. Let $D_r = \{z \in \boldsymbol{C}; |z| \leqq r\}$. Then we can find $\xi \in \Gamma(\boldsymbol{C}, \mathcal{O}_{\boldsymbol{C}})$ such that $|\xi(g(x_0))| \geqq 3$ and $\sup\{|\xi(z)|; z \in D_r\} \leqq 1$ (cf. the proof of Theorem 1.3.1 (b)$\to$(c) in [13, p. 8]). Hence if we put $\tilde{g} = \xi(g)$ then we have $3 \sup_{x \in B^c} |\tilde{g}(x)| \leqq |\tilde{g}(x_0)|$. Put $g' = \tilde{g} - \tilde{g}(a) \in \Gamma(X^{c_3}, \mathfrak{m}_a)$. Then $\sup_{x \in B^c} |g'(x)| < |g'(x_0)|$ since $a \in B^c$. Hence B^c is $\Gamma(X^{c_3}, \mathfrak{m}_a)$-convex.

Next in the general case apply the above consideration to the pair $(X_s, \varphi|_{X_s})$ and obtain $f_1, \cdots, f_q \in \Gamma(X_s^{c_3}, \mathfrak{m}_a)$ having the properties described above, where $\mathfrak{m}_a$ is the maximal ideal of $\mathcal{O}_{X_s}$ at $a := A \cap X_s$. Restricting S, we may assume that S, and hence X^{c^*} also, are Stein. Then X^{c_3} also is Stein (cf. [21]). Take $g_i \in \Gamma(X^{c_3}, \mathcal{J})$ extending $f_i \in \Gamma(X_s^{c_3}, \mathfrak{m}_a)$ where $\mathcal{J}$ is the ideal sheaf of A in X. Let $P_R = \{x \in X^{c_3}; |g_i(x)| < R_i\}$ and $P_1 = \{x \in X^{c_3}; |g_i(x)| < 1\}$. Then if we take a neighborhood U of s in S sufficiently small, we have $B^{c'}(U) \subseteqq P_1 \cap X^{c''}(U) \subseteqq X^{c_1}(U) \subseteqq B^{c_2}(U)$

$\subseteqq P_R \cap X^{c''}(U) \subseteqq X^{c''}(U)$. Put $b=R$, $a=1:=(1,\cdots,1)$, and $P_b=P_R\cap X^{c''}(U)$. Define $\Phi: P_b \to U\times K(b)$ by $\Phi=(f,(g_1,\cdots,g_q))$. Then Φ is proper and hence by (the proof of) [11, VIII, Lemma 2.2] Φ is an embedding if we restrict U to a smaller neighborhood of s since $\Phi|_{X_s}$ is one. Moreover $Q_a=(P_1\cap X^{c''}(U))^-$, the closure being taken in P_b, and the conditions 1) and 2) are immediately verified. Further by our construction $A\subseteqq \Phi^{-1}(U\times\{0\})$. Then adding $g_{q+1},\cdots,g_t\in\Gamma(X^{c''}(U),\mathcal{J})$ to g_i which generate $\mathcal{J}$ on P_b with $|g_j(x)|<1$ on P_b (after eventual restriction of U) so that P_b and Q_a remain invariant, we finally obtain the desired embedding Φ with $\Phi^{-1}(U\times\{0\})=A(U)$. Q.E.D.

Remark. Let $P_{a,b}=P_b-Q_a=\Phi^{-1}(U\times(K(b)-K(a)^-))$. Then from the above proof it follows readily that for any $U''\Subset U'\Subset U$ we have $X_{c_1}^{c_2}(U'')\Subset P_{a,b}(U')\Subset X_{c'}^{c''}$.

§ 4. Proof of Theorem

Let $f: X\to S$ be a morphism of complex spaces and $\mathcal{F}$ a coherent analytic sheaf on X. Let $S_k(\mathcal{F},f)=\{x\in X; \operatorname{codh}_x\mathcal{F}_{f(x)}\leqq k\}$. Suppose that $\mathcal{F}$ is f-flat. Then $S_k(\mathcal{F},f)$ is an analytic subset of X [2]. Moreover the following equality holds [10, IV, 6.3.1]:

$$\operatorname{codh}_x\mathcal{F}=\operatorname{codh}_{f(x)}S+\operatorname{codh}_x\mathcal{F}_{f(x)},\quad x\in X. \tag{5}$$

(Thus the bounds for i in the conjecture in the introduction is obtained by replacing $\dim S$ by codh S in those appearing in the mentioned results a), b) and c). In particular both coincide when S is Cohen-Macaulay.)

In general let $A=\{x\in X;\ \mathcal{F}$ is not f-flat at $x\}$. Then A is an analytic subset of X by [5]. Let $U=X-A$.

Lemma 4.1. *The closure T_k of $S_k(\mathcal{F}|_U, f|_U)$ in X is analytic in X.*

Proof. Let $X_m=f^{-1}(S_m(S))$ and $V_m=X_m-X_{m-1}$. Let $T_{m,k}=V_m\cap S_{m+k}(\mathcal{F})$. Then by virtue of (5) we have $S_k(\mathcal{F}|_U, f|_U)=\bigcup_m T_{m,k}$. Hence it suffices to show that the closure $\overline{T}_{m,k}$ of $T_{m,k}$ in X_m is analytic for each k. In fact, $\overline{T}_{m,k}$ is easily seen to be the union of some of the irreducible components of $X_m \cap S_{m+k}(\mathcal{F})$. Q.E.D.

Let $f: X \to S$ be a 1-convex map of complex spaces with exhaustion function $\varphi: X \to (-\infty, c^*)$ with convexity bound $c_\# \in (c_*, c^*)$. Let $\mathcal{F}$ be a coherent analytic sheaf on X such that $\mathcal{F}$ is f-flat on $X_{c_\#}$ and $\mathrm{codh}_f \mathcal{F} \geqq r$ on $X_{c_\#}$ for some $r \geqq 0$.

Let $A_1 = \{x \in X; \mathcal{F}$ is not f-flat at $x\}$ and $A_2' = \{x \in X - A_1;$ $\mathrm{codh}_x \mathcal{F}_{f(x)} < r\}$. A_1 and A_2' are analytic subsets of X and $X - A_1$ respectively. Let A_2 be the closure of A_2' in X. Then A_2 is an analytic subset of X by Lemma 4.1. Let E be the relative exceptional set for f. Then we set $A = A_1 \cup A_2 \cup E$. By our assumption $A \subseteqq X^{c_\#}$ and hence is proper over S. As before, for $d \in (c_\#, c^*)$ we set $B^d = \{x \in X; \varphi(x) \leqq d\}$.

Theorem 4.2. *Let $c \in (c_\#, c^*)$, Then for $i \leqq r-1$ the following hold:* 1) *The natural map $R^i f_{A*} \mathcal{F} \to R^i f_{B^c*} \mathcal{F}$ is isomorphic and* 2) *$R^i f_{B^c*} \mathcal{F}$ is a coherent analytic sheaf on S.*

We need a lemma.

Lemma 4.3. *Suppose that f is Stein. Then for any $c_\# < c' < c$ the natural map $R^i f_{B^{c'}*} \mathcal{F} \to R^i f_{B^c*} \mathcal{F}$ is surjective for $i \leqq r-1$.*

Proof. Since f is Stein, the connecting homomorphism $R^{i-1}(f|_{X_d})_* \mathcal{F} \to R^i f_{B^d*} \mathcal{F}$ is surjective for any $c_\# < d$ and $i \geqq 1$. Hence the lemma follows from Corollary 1.4 when $i \geqq 1$. Thus it suffices to show that when $r > 0$, $f_{A*} \mathcal{F} \to f_{B^d*} \mathcal{F}$ is surjective for any $d > c_\#$. In fact let $\eta \neq 0$ be a section of $\mathcal{F}$ on $X(U)$ for some open subset $U \subseteqq S$ with its support $T = T(\eta)$ contained in $B^d(U)$. Suppose that $T \not\subseteqq A$ and take $s \in U$ with $T_s \not\subseteqq A_s$. T_s consists of a finite number of points since X_s is Stein. It then follows that $\mathrm{codh}_x \mathcal{F}_s = 0$ for $x \in (T_s - A_s) \neq \emptyset$, which is a contradiction. Hence $T \subseteqq A$. Q.E.D.

Proof of Theorem 4.2. Since the problem is local on S, we fix a point $s \in S$ and consider everything around s.

α. First we assume that f is Stein and A is mapped isomorphically onto a subspace of S. For the given c take $c_* < c' < c_1 < c < c_2 < c'' < c^*$ arbitrarily. Then with respect to this data we can apply Lemma 3.3 to f. So let $\Phi: P_b \to U \times K(b)$ and Q_a be as in that lemma. Since the problem is local on S, we may assume that $U=S$. By 1) of the lemma we have $Q_a \subseteqq B \subseteqq P_b$, where $B=B^c$. Hence by excision $R^i f_{B*} \mathcal{F} \cong R^i (f|_{P_b})_{B*} (\mathcal{F}|_{P_b})$. Thus replacing X by P_b, with P_b identified with its image in $S \times K(b)$ by Φ, and then $\mathcal{F}$ by its trivial extension to $S \times K(b)$ we may assume that $X = S \times K(b)$, f is the natural projection $S \times K(b) \to S$, and further that $A = Q_0 := S \times \{0\}$. We shall then show the following: (*) *the natural homomorphim* $R^i f_{Q_a *} \mathcal{F} \to R^i f_{B*} \mathcal{F}$, $i \leqq r-1$, *are isomorphic.* In fact, this would imply the theorem in our special case in view of Proposition 2.1. Now by 1) of Lemma 3.3 $B^{c'} \subseteqq Q_a \subseteqq B \subseteqq Q_{a'} \subseteqq X$ if we take a' sufficiently near to b with $a' < b$ (after eventual restriction of S). Then for (*) it suffices to show that the natural map $R^i f_{Q_a *} \mathcal{F} \to R^i f_{Q_{a'} *} \mathcal{F}$ (resp. $R^i f_{B^{c'} *} \mathcal{F} \to R^i f_{B*} \mathcal{F}$), $i \leqq r-1$, is injective (resp. surjective). Indeed, this follows from Proposition 2.1 (resp. Lemma 4.3). Thus (*) is proved. Note also that since the natural map $R^r f_{A*} \mathcal{F} \langle k \rangle \to R^r f_{Q_{a'} *} \mathcal{F} \langle k \rangle$ is injective by Proposition 2.2 (in the notation there with π replaced by f), $R^r f_{A*} \mathcal{F} \langle k \rangle \to R^r f_{B*} \mathcal{F} \langle k \rangle$ also is injective. Further $R^r f_{A*} \mathcal{F} \langle k \rangle$ is coherent by Proposition 2.2.

β. We consider the general case. As in part α we put $B=B^c$. Now A satisfies the condition of Lemma 3.2. So let $g: Y \to S$ and $\sigma: X \to Y$ be as in that lemma. Let $A' = \sigma(A)$ and $\mathcal{F}' = \sigma_* \mathcal{F}$. Then $\mathcal{F}'$ is g-flat on $Y - A'$ and $\mathrm{codh}_g(\mathcal{F}'|_{Y-A'}) \geqq r$. Then replacing S by a small neighborhood of s we can construct an exhaustion function φ' on Y which makes g 1-convex and with the following property; $\varphi' = \varphi \cdot \sigma^{-1}$ on $\sigma(X_{c\#})$ and $\sup\{\varphi'(y); y \in \sigma(B^{c\#})\} \leqq c$. Then if we set $B' = \{y \in Y; \varphi'(y) \leqq c\}$, then $\sigma^{-1}(B') = B$ and φ' is strictly plurisubharmonic in a neighborhood of the boundary of B' in Y. Then applying part α to $g: Y \to S$ with the exhaustion function φ' we get the following: For $i \leqq r-1$, $R^i g_{A'*} \mathcal{F}' \to R^i g_{B'*} \mathcal{F}'$ is isomorphic and $R^i g_{B'*} \mathcal{F}'$ is coherent. Moreover for $k > 0$, $\lambda: R^r g_{A'*} \mathcal{F}' \langle k \rangle \to R^r g_{B'*} \mathcal{F}' \langle k \rangle$ is injective and $R^r g_{A'*} \mathcal{F}' \langle k \rangle$ is coherent. Here $R^r g_{A'*} \mathcal{F}' \langle k \rangle$ is defined to be the $g_* \mathcal{O}_Y$-submodule of $R^r g_{A'*} \mathcal{F}'$ annihilated by $g_* \mathcal{J}'^k$ and $R^r g_{B'*} \mathcal{F}' \langle k \rangle$ is defined similarly where $\mathcal{J}'$ is the ideal sheaf of A' in Y, and the result follows from the last remark in part α.

On the other hand, $\sigma^{-1}(B')=B$ implies that $f_{B*}=g_{B'*}\sigma_*$. Hence $f_{B*}\mathcal{F}\cong g_{B'*}\mathcal{F}'$ and the theorem is proved for $i=0$. So we assume that $i>0$ and hence that $r\geqq 2$. Consider the spectral sequence $E_2^{p,q}:=R^pg_{B'*}R^q\sigma_*\mathcal{F}\Rightarrow R^{p+q}f_{B*}\mathcal{F}$ associated to the composite functor $f_{B*}=g_{B'*}\sigma_*$. First of all, since the support of $R^q\sigma_*\mathcal{F}$, $q>0$, is contained in $\sigma(E)\subseteqq A'$ and hence is finite over S, we have $R^pg_{B'*}(R^q\sigma_*\mathcal{F})\cong R^pg_*(R^q\sigma_*\mathcal{F})=0$ if $p>0$. Therefore we obtain the long exact sequence

$$\begin{aligned}(6)\qquad & 0\to R^1g_{B'*}\mathcal{F}'\to\cdots\to R^ig_{B'*}\mathcal{F}'\to R^if_{B*}\mathcal{F}\\ &\to g_*(R^i\sigma_*\mathcal{F})\to R^{i+1}g_{B'*}\mathcal{F}'\to .\end{aligned}$$

By the same reasoning as above $g_*(R^i\sigma_*\mathcal{F})$ is coherent for every $i\geqq 1$. Since $R^ig_{B'*}\mathcal{F}'$ is coherent for $i\leqq r-1$, we get from (6) that $R^if_{B*}\mathcal{F}$ is coherent for $i\leqq r-2$. Next consider the natural map into (6) from the exact sequence

$$\to R^ig_{A'*}\mathcal{F}'\to R^if_{A*}\mathcal{F}\to g_*(R^i\sigma_*\mathcal{F})\to R^{i+1}g_{A'*}\mathcal{F}'\to$$

which is obtained in the same way as (6) using the relation $f_{A*}=g_{A'*}\sigma_*$. Since $R^ig_{A'*}\mathcal{F}'\cong R^ig_{B'*}\mathcal{F}'$, $1\leqq i\leqq r-1$, $R^if_{A*}\mathcal{F}\cong R^if_{B*}\mathcal{F}$ for $i\leqq r-2$ by the five lemma. Thus it remains to consider the case $i=r-1$. Note first that $R^{r-1}\sigma_*\mathcal{F}$ has support in A' and hence $g_*(R^{r-1}\sigma_*\mathcal{F})$ is coherent and is annihilated by $g_*\mathcal{J}'^{k_0}$ for some $k_0>0$ in a neighborhood of s. Hence we have the following commutative diagram of exact sequences.

$$\begin{array}{ccccccc}\to R^{r-1}g_{A'*}\mathcal{F}' & \to & R^{r-1}f_{A*}\mathcal{F} & \to & g_*(R^{r-1}\sigma_*\mathcal{F}) & \to & R^rg_{A'*}\mathcal{F}'\langle k_0\rangle\\ \wr\downarrow & & \downarrow & & \| & & \downarrow\lambda\\ \to R^{r-1}g_{B'*}\mathcal{F}' & \to & R^{r-1}f_{B*}\mathcal{F} & \to & g_*(R^{r-1}\sigma_*\mathcal{F}) & \to & R^rg_{B'*}\mathcal{F}'\langle k_0\rangle .\end{array}$$

Since λ is injective, the isomorphism $R^{r-1}f_{A*}\mathcal{F}\cong R^{r-1}f_{B*}\mathcal{F}$ follows again by the five lemma. Then the coherency of $R^{r-1}f_{B*}\mathcal{F}$ follows from that of $R^{r-1}f_{A*}\mathcal{F}$, which in turn follows from the top line of the above diagram in view of what we have proved above. Q.E.D.

Proof of Theorem. By the definition of $R^if_!$ we have the natural isomorphism $\varinjlim_{c\in(c_\#,c^*)} R^if_{Bc*}\mathcal{F}\cong R^if_!\mathcal{F}$. By 1) of Theorem 4.2 all the homomorphisms in the inductive system $\{R^if_{Bc*}\mathcal{F}\}_{c\in(c_\#,c^*)}$ are isomorphic. Hence we have $R^if_{Bc*}\mathcal{F}\cong R^if_!\mathcal{F}$ for every $c\in(c_\#,c^*)$. From this follows the coherency of $R^i f_! \mathcal{F}$ by 2) of Theorem 4.2. Since for $d \in (c_\#, c^*)$ if we take any $c \in (c_\#, d)$, then we get that $R^i f_{d!} \mathcal{F} \cong R^i f_{B^c *} \mathcal{F} \cong R^i f_! \mathcal{F}$, Theorem follows.

Remark. From the proof it also follows that the natural map $R^i f_{A*} \mathcal{F} \to R^i f_! \mathcal{F}$, $i \leqq r-1$, is isomorphic, where A is as in Theorem 4.2.

Proposition 4.4. *Let $f: X \to S$ be a $(1,1)$-complete map of complex spaces with S Stein and with exhaustion function $\varphi: X \to (c_*, c^*)$. Let $\mathcal{F}$ be an f-flat coherent analytic sheaf on X. Let $r = \mathrm{codh}_f \mathcal{F}$. Suppose that f admits a Stein completion $\tilde{f}: \tilde{X} \to S$ (cf. [15]) and $\mathcal{F}$ admits a coherent extension $\tilde{\mathcal{F}}$ to $\tilde{X}$. Then: 1) The restriction map $R^i f_* \mathcal{F} \to R^i f_{d*} \mathcal{F}$ is isomorphic for $0 \leqq i \leqq r-2$ and $d \in (c_*, c^*)$ where $f_d = f|_{X_d}$. 2) $R^i f_* \mathcal{F}$ is coherent for $1 \leqq i \leqq r-2$. 3) The natural map $H^i(X, \mathcal{F}) \to H^0(S, R^i f_* \mathcal{F})$ is isomorphic for $0 \leqq i \leqq r-2$.*

Proof. For any $d \in (c_*, c^*)$ we construct a C^∞ extension $\tilde{\varphi}$ of $\varphi|_{X_{d'}}$ to $\tilde{X}$ for some $c < d' < d$ such that $\tilde{\varphi}(X - X_{d'}) \subseteqq (-\infty, d']$. Then $\tilde{f}$ becomes a 1-convex map with exhaustion function $\tilde{\varphi}$ with convexity bound d'. Then we can speak of B^d with respect to this $\tilde{\varphi}$. Then we have the standard exact sequence

$$(7)_d \qquad \to R^i \tilde{f}_{B^d *} \tilde{\mathcal{F}} \to R^i \tilde{f}_* \tilde{\mathcal{F}} \to R^i f_{d*} \mathcal{F} \to R^{i+1} \tilde{f}_{B^d *} \tilde{\mathcal{F}} \to$$

where $R^i \tilde{f}_* \tilde{\mathcal{F}} = 0$ since $\tilde{f}$ is Stein and $R^i \tilde{f}_{B^d *} \tilde{\mathcal{F}}$ is coherent for $1 \leqq i \leqq r-1$ by Theorem 4.2. Hence $R^i f_{d*} \mathcal{F} \cong R^{i+1} \tilde{f}_{B^d *} \tilde{\mathcal{F}}$ is coherent for $1 \leqq i \leqq r-2$. Moreover by 1) of Theorem 4.2, considering the natural morphism from $(7)_c$ to $(7)_d$ for $c_\# < c < d$, it follows from the five lemma that the restriction map $R^i f_{c*} \mathcal{F} \to R^i f_{d*} \mathcal{F}$ is isomorphic for $0 \leqq i \leqq r-2$. (For the given c and d take d' above in such a way that $d' < c$.) Then by [20, Lemma 4.1] $R^i f_* \mathcal{F} \to R^i f_{d*} \mathcal{F}$ also is isomorphic. In particular $R^i f_* \mathcal{F}$ is coherent for $1 \leqq i \leqq r-2$. This proves 1) and 2).

We show 3). We may assume that $r \geqq 3$ and $i > 0$. Consider the Leray spectral sequence for f and $\mathcal{F}$; $E_2^{p,q} := H^p(S, R^q f_* \mathcal{F}) \Rightarrow H^{p+q}(X, \mathcal{F})$. Since $R^q f_* \mathcal{F}$ is coherent for $1 \leqq q \leqq r-2$ and S is Stein, $H^p(X, R^q f_* \mathcal{F}) = 0$, $p > 0$, $1 \leqq q \leqq r-2$. Hence it is enough to show that for all $p > 0$,

$H^p(S, f_*\mathcal{F})=0$, or equivalently, $H^p(S, f_{d*}\mathcal{F})=0$ for some $d\in(c_*, c^*)$ by 1). Consider the short exact sequence

$$0\to\tilde{f}_{B^d*}\tilde{\mathcal{F}}\to\tilde{f}_*\tilde{\mathcal{F}}\to f_{d*}\mathcal{F}\to R^1\tilde{f}_{B^d*}\tilde{\mathcal{F}}\to 0\ . \tag{8}$$

Since $R^i\tilde{f}_{B^d*}\tilde{\mathcal{F}}$ is coherent for $0\leqq i\leqq r-1$ and S is Stein, $H^p(S, R^i\tilde{f}_{B^d*}\tilde{\mathcal{F}})=0$, $p>0$. Hence, in view of (8), it suffices to show that $H^p(\tilde{S}, \tilde{f}_*\tilde{\mathcal{F}})=0$, $p>0$. Considering the Leray spectral sequence for $\tilde{f}$ and $\tilde{\mathcal{F}}$ and noting the fact that $\tilde{f}$ is Stein, we get that $H^p(S, \tilde{f}_*\tilde{\mathcal{F}})=H^p(\tilde{X}, \tilde{\mathcal{F}})=0$ since $\tilde{X}$ is Stein. Q.E.D.

Combining the above proposition with the main results of [6] we obtain the following:

Corollary 4.5. *Let $f: X\to S$ and $\mathcal{F}$ be as above. Let $o\in S$. Then the conclusion of Proposition* 4.3 *is true in a neighborhood of o if the following condition is satisfied: Let $(X_o)^\sim$ be the maximal Stein completion of X_o and $(\mathcal{F}_o)^\sim$ the maximal coherent completion of $\mathcal{F}_o$ to $(X_o)^\sim$ (cf.* [6]). *Then* $\operatorname{codh}(\mathcal{F}_o)^\sim\geqq 3$.

Remark. The above results are immediately generalized to the case of a $(1,1)$-convex-concave map. However for the conjecture in the introduction in this case it still remains to consider the case where $\operatorname{codh}\mathcal{F}_o\geqq 3$ and $\operatorname{codh}(\mathcal{F}_o)^\sim=2$. In view of the above proposition this case can be reduced to a conjecture on Stein completion of $(1,1)$-complete maps and coherent extension of sheaves (cf. [6]).

Corollary 4.6. *In the above proposition suppose further that $\tilde{\mathcal{F}}$ is $\tilde{f}$-flat and $\operatorname{codh}_{\tilde{f}}\tilde{\mathcal{F}}\geqq r$. Then $R^if_*\mathcal{F}=0$, $0<i\leqq r-2$, and the restriction map $\tilde{f}_*\tilde{\mathcal{F}}\to f_*\mathcal{F}$ is isomorphic.*

Proof. For $0<i\leqq r-2$, $R^if_*\mathcal{F}\cong R^{i+1}\tilde{f}_{B^d*}\tilde{\mathcal{F}}\cong R^{i+1}\tilde{f}_{A*}\tilde{\mathcal{F}}$ in the notation of the proof of Proposition 4.4 with A as in Theorem 4.2. On the other hand, our assumption implies that we can take $A=\emptyset$. Hence $R^if_*\mathcal{F}=0$. Further form (8) it follows that $\tilde{f}_*\tilde{\mathcal{F}}=f_*\mathcal{F}$. Q.E.D.

§ 5. A Relative Vanishing Theorem

Let $f: X \to S$ be a morphism of complex spaces and $\mathfrak{F}$ an f-flat coherent analytic sheaf on X. Let $q \geqq 0$ be an integer and $o \in S$ a point. If f is proper, then from the vanishing of $H^q(X_o, \mathfrak{F}_o)$ follows the vanishing of $R^q f_* \mathfrak{F}$ in a neighborhood of o (cf. [3]). We shall generalize this as follows.

Proposition 5.1. *Let $f: X \to S$, $\mathfrak{F}$, q and $o \in S$ be as above. Suppose that for every locally closed analytic subspace $T \subseteqq S$ the q-th direct image sheaf $R^q f_{T*} \mathfrak{F}_T$ (resp. $R^q f_{T!} \mathfrak{F}_T$) is coherent. Then if $H^q(X_o, \mathfrak{F}_o) = 0$ (resp. $H^q_c(X_o, \mathfrak{F}_o) = 0$), $R^q f_* \mathfrak{F} = 0$ (resp. $R^q f_! \mathfrak{F} = 0$) in a neighborhood of o.*

In view of [19] (resp. Theorem) the above proposition yields the following:

Corollary 5.2. *Let $f: X \to S$ and $\mathfrak{F}$ be as above.* 1) *Suppose that f is p-convex for some $p > 0$* (cf. [19] modulo the remark in the introduction). *Then if $H^q(X_o, \mathfrak{F}_o) = 0$ for some $0 \in S$ and some $q \geqq p$, $R^q f_* \mathfrak{F} = o$ in a neighborhood of o.* 2) *Suppose that f is 1-convex and $H^q_c(X_o, \mathfrak{F}_0) = 0$ for some $o \in S$ and some $q \leqq \operatorname{codh} \mathfrak{F}_o - 1$. Then $R^q f_! \mathfrak{F} = o$ in a neighborhood of o.*

When f is 1-convex, 1) is due to Riemenschneider (cf. Comment. Math. Helv., 51 (1976)). The following is a relative form of the vanishing theorem of Grauert and Riemenschneider [9] in the 1-convex case, which has an application to a stability problem of exceptional divisors [7].

Corollary 5.3. *Let $f: X \to S$ be a smooth 1-convex map of complex spaces and $\mathcal{E}$ a locally free coherent analytic sheaf on X. Suppose that for some $o \in S$ the vector bundle corresponding to $\mathcal{E}_o$ is semi-negative in the sense of Nakano (cf. [9]). Let $r = \dim X_o$. Then $R^q f_! \mathcal{E} = 0$ in a neighborhood of o for $q \leqq r-1$. In particular $R^q f_! \mathcal{O}_X$*

$=0$, $q\leqq r-1$, *in a neighborhood of* o.

Proof. Let $\varphi: X\to(-\infty, c^*)$ be an exhaustion function for the strongly pseudoconvex manifold X_o. Let $c_\#$ be a convexity bound for φ. Take $c\in(c_\#, c^*)$ in such a way that the boundary bX_o^c of $X_o^c:=(X_o)^c$ is smooth. Then by [9] $H_c^q(X_o^c, \mathcal{E}_o)=0$, $q\leqq r-1$. Hence $H_c^q(X_o, \mathcal{E}_o)\cong H_c^q(X_o^c, \mathcal{E}_o)=0$, $q\leqq r-1$. The corollary then follows from 2) of the above corollary.

Proof of Proposition 5.1. Let $m=\dim_o S$. We proceed by induction on m. Suppose first that $m=0$. Then just as in the proof of [3, p. 120, Cor. 3.5] we see more generally that for any coherent analytic sheaf $\mathcal{G}$ on S, $R^qf_*(\mathcal{F}\otimes_{\mathcal{O}_X}f^*\mathcal{G})=0$. So suppose that $m>0$. First consider the case where $\operatorname{codh}_o S>0$. Then we can find a neighborhood $o\in U$ and $t\in\Gamma(U, \mathfrak{m}_o)$ which is not a zero-divisor in $\mathcal{O}_{S,s}$ for any $s\in U$, where $\mathfrak{m}_o$ is the maximal ideal of $\mathcal{O}_S$ at 0. Since $\mathcal{F}$ is f-flat, we have an exact sequence $0\to\mathcal{F}\xrightarrow{\alpha}\mathcal{F}\to\mathcal{F}/t\mathcal{F}\to 0$ on $X(U)$ where α is defined by the multiplication by t. From this we obtain the long exact sequence

$$\to R^qf_*\mathcal{F}\xrightarrow{\alpha'}R^qf_*\mathcal{F}\to R^qf_*(\mathcal{F}/t\mathcal{F})\to$$

on U where α' is defined by the multiplication by t. Let S' be the subspace of S defined by $t=0$ and $f': X'\to S'$ the induced morphism. Then $\dim_o S'<m$. Hence by induction $R^qf_*(\mathcal{F}/t\mathcal{F})\cong R^qf'_*(\mathcal{F}/t\mathcal{F})=0$ in a neighborhood of 0. On the other hand, by virtue of the above exact sequence $R^qf_*\mathcal{F}/tR^qf_*\mathcal{F}$ injects into $R^qf_*(\mathcal{F}/t\mathcal{F})$ and hence itself vanishes. Since $R^qf_*\mathcal{F}$ is coherent, by Nakayama $R^qf_*\mathcal{F}=0$ as was desired. Next suppose that $\operatorname{codh}_0 S=0$. Then we may write uniquely $S=S_1\cup S_2$ in a neighborhood of 0 where $\dim_o S_1=0$ and $\operatorname{codh}_o S_2\geqq 1$ (cf. [6]). Let $X_i=X_{S_i}$, $\mathcal{F}_i=\mathcal{F}_{S_i}$ and $f_i: X_i\to S_i$ be the induced morphisms, $i=1, 2$. Let $\mathcal{J}$ be the ideal sheaf of $\mathcal{O}_S$ defining S_2 in S. Then we have the obvious short exact sequence $0\to\mathcal{J}\mathcal{F}\to\mathcal{F}\to\mathcal{F}_2\to 0$ and the resulting long exact sequence

$$\to R^qf_*\mathcal{J}\mathcal{F}\to R^qf_*\mathcal{F}\to R^qf_*\mathcal{F}_2\to.$$

First of all, $R^qf_*\mathcal{F}_2=R^qf_{2*}\mathcal{F}_2=0$ as we have proved above. Thus it suffices to show that $R^q f_* \mathcal{J}\mathcal{F}=0$. Since $\mathcal{J}$ is already an $\mathcal{O}_{S_1}$-module, by the flatness of $\mathcal{F}$ we get $\mathcal{J}\mathcal{F}\cong\mathcal{J}\otimes_{\mathcal{O}_S}\mathcal{F}\cong\mathcal{J}\otimes_{\mathcal{O}_{S_1}}\mathcal{O}_{S_1}\otimes_{\mathcal{O}_S}\mathcal{F}\cong\mathcal{J}\otimes_{\mathcal{O}_{S_1}}\mathcal{F}_1$. Then since $\dim_o S_1=0$ and $H^q(X_{1,o},\mathcal{F}_{1,o})=H^q(X_o,\mathcal{F}_o)=0$, by what we have remarked above $R^q f_{1*}(\mathcal{J}\otimes_{\mathcal{O}_{S_1}}\mathcal{F}_1)=0$. Hence $R^q f_*\mathcal{J}\mathcal{F}=0$ as was desired. Finally replacing $*$ by ! in the above argument we obtain the assertion for $R^q f_! \mathcal{F}$, too. Q.E.D.

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