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Order Properties of a Class of Tensor Algebras

By

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Abstract

By means of an auxiliary coarser topology we study certain order properties of inductive tensor algebras over nuclear *LF*-spaces.

§ 1. Introduction, Notations and Statement of Results

A locally convex *-algebra $\mathcal{A}[\mathcal{J}]$ is a locally convex space equipped with a separately continuous multiplication

$$\times : \mathcal{A}[\mathcal{J}] \times \mathcal{A}[\mathcal{J}] \to \mathcal{A}[\mathcal{J}]$$

and a continuous involution

$$* : \mathcal{A}[\mathcal{J}] \to \mathcal{A}[\mathcal{J}].$$

The separate continuity of the multiplication means that for all $g \in \mathcal{A}$ the maps $f \rightsquigarrow g \times f$ and $f \rightsquigarrow f \times g$ from $\mathcal{A}[\mathcal{J}]$ to $\mathcal{A}[\mathcal{J}]$ are continuous. It is also assumed that $\mathcal{A}$ has a unit satisfying $\mathbb{1} = \mathbb{1}^*$.

An order structure is introduced on $\mathcal{A}_h = \{f \in \mathcal{A} : f = f^*\}$, the real subspace of hermitian elements of $\mathcal{A}$, by defining the cone of positive elements, $\overline{\mathcal{A}}_+$, to be the closure of the set

$$\mathcal{A}_+ = \{ \sum_{i=1}^{n} f_i^* \times f_i : f_i \in \mathcal{A},\ 1 \leq i \leq n,\ n \in \mathbb{N} \}.$$

The cone $\overline{\mathcal{A}}_+$ determines a transitive and reflexive partial order on $\mathcal{A}_h$, and we write $f \geq g$ whenever $f - g \in \overline{\mathcal{A}}_+$. When this order is antisymmetric the cone $\overline{\mathcal{A}}_+$ is called proper. Evidently $\overline{\mathcal{A}}_+$ is proper if and only if $\overline{\mathcal{A}}_+ \cap -\overline{\mathcal{A}}_+ = \{0\}$.

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If $f,\ g\in\mathcal{A}_h$ the order interval $[f, g]$ is defined by

$$[f, g]=\{k\in\mathcal{A}_h:\ f\leq k\leq g\}.$$

The subset $\{\mu f:\ \mu\in\boldsymbol{R}^+,\ f\in\overline{\mathcal{A}}_+\}$ of $\overline{\mathcal{A}}_+$ is said to be an extremal ray if $[0, f]=\{\lambda f:\ \lambda\in[0,1]\}$.

To $\overline{\mathcal{A}}_+$ we associate the dual cone of positive functionals

$$\mathcal{A}'_+=\{T\in\mathcal{A}[\mathcal{J}]' : T(f)\geq 0\quad \forall f\in\overline{\mathcal{A}}_+\}.$$

A positive functional T will be called strictly positive if

$$T(f)>0\quad \forall f\in\overline{\mathcal{A}}_+,\quad f\neq 0.$$

The algebras to be dealt with in this paper are the so called BU-algebras [1, 3]. Given a complex nuclear LF-space (strict inductive limit of Fréchet spaces) E, the BU-algebra over E is the locally convex direct sum tvs

$$\underline{E}=\bigoplus_{n=0}^{\infty}\overline{\otimes}^n E$$

where $n=0$ corresponds to $\boldsymbol{C}$ by convention and $\overline{\otimes}$ indicates the completion of the tensor product in the inductive tensor product topology ([12], p. 96 and p. 119, Exercise 22). The product with respect to which $\underline{E}$ is an algebra follows from its graded structure:

$$\text{if}\quad \underline{f}=(f_0, f_1,\cdots, f_r, 0, 0, \cdots),\ \underline{g}=(g_0, g_1,\cdots, g_s, 0, 0, \cdots)\in\underline{E}\,,$$

then

$$\underline{f}\times\underline{g}=(f_0g_0,\ f_0g_1+f_1g_0,\ \cdots,\ \sum_{i+j=p} f_i\otimes g_j,\ \cdots,\ f_r\otimes g_s, 0, 0, \cdots).$$

It is further assumed that a continuous involution, *, is defined on $\underline{E}$. In an obvious way this extends linearly to an involution

$$\underline{f}\rightsquigarrow\underline{f}^*\ \text{on}\ \underline{E},\ \text{with}\ (\lambda\underline{f}\times\underline{g})^*=\bar{\lambda}\underline{g}^*\times\underline{f}^*\,.$$

In [1, 3, 4] BU-algebras have been used to formulate some physical theories that deal with infinite systems.

In addition to their role in applications, BU-algebras are important because of the following structure theorem [2]. Every I^*-algebra is isomorphic to the quotient of a BU-algebra by a complemented *-invariant positive ideal. An I^*-algebra $\mathcal{A}[\mathcal{J}]$ is a locally convex complex unital *-algebra with a proper incomplete cone $\mathcal{A}_+$ and a nuclear LF-topology

$\mathcal{J}$. Such algebras are discussed in [1, 3].

The principal results of this paper are that, *for any BU-algebra* $\underset{\sim}{E}$.

(a) *the positive cone* $\bar{\underset{\sim}{E}}_+$ *of* E *is given explicitly by the set of convergent series:*

$$\bar{\underset{\sim}{E}}_+ = \{ \sum_{i=1}^{\infty} f_i^* \times f_i : f_i \in \underset{\sim}{E} \} ;$$

(b) *the order intervals of* $\underset{\sim}{E}$ *are compact and* $\bar{\underset{\sim}{E}}_+$ *is the closed convex hull of its extremal rays*;

(c) $\underset{\sim}{E}$ *has a strictly positive functional if and only if* E *has a continuous norm.*

This paper is organized as follows. In Section 2 we prove some preliminary lemmas which we feel to be of some independent interest. The main results are proven in Section 3, and we finish with a list of unsolved problems.

§ 2. Preliminary Results

Before proving our first lemma we need to introduce the following definition.

Definition. A seminorm P on a direct sum $\bigoplus_{n=0}^{\infty} E_n$ will be called graded if it is of the form $P = \sum_{n\geq 0} p_n$, where p_n is a seminorm on E_n. A locally convex topology on a tensor algebra will be called graded if it has a generating family of seminorms that are graded.

In general the multiplication in tensor algebras is not jointly continuous. For example the multiplication in a BU-algebra is jointly continuous if and only if E is an LB-space ([1], Corollary 1.13 or [3], Proposition 1.7). If E is a complex nuclear space with a continuous involution and $\hat{\otimes}^n E$ is the n-fold completed projective tensor product, the lemma below gives some properties of the finest graded topology $\mathcal{J}_\infty$ on $\underset{\sim}{E} \equiv \bigoplus_{n=0}^{\infty} \hat{\otimes}^n E$, coarser than the original topology, for which the multiplication is jointly continuous, i.e., for which the map

$$\times : \underset{\sim}{E}[\mathcal{J}_\infty] \times \underset{\sim}{E}[\mathcal{J}_\infty] \rightarrow \underset{\sim}{E}[\mathcal{J}_\infty]$$

is continuous.

Lemma 1. *Let E be a complex nuclear space with a continuous involution. Then the finest graded topology $\mathcal{J}_\infty$ on $\underset{\sim}{E}$, coarser than the original one, for which the multiplication is jointly continuous, has the following properties*:

(i) *the involution* $*: \underset{\sim}{E}[\mathcal{J}_\infty]\to\underset{\sim}{E}[\mathcal{J}_\infty]$ *is continuous and on* $\bigoplus_{n=0}^{N}\hat{\bigotimes}^n E$, *$N$ finite, $\mathcal{J}_\infty$ induces the original topology*;

(ii) $\underset{\sim}{E}[\mathcal{J}_\infty]$ *is a nuclear space*;

(iii) *the cone $\bar{\underset{\sim}{E}}_+$ is $\mathcal{J}_\infty$-normal*;

(iv) *the topology defined by the seminorms* $f\rightsquigarrow T(f^*\times f)^{1/2}$, *for all $\mathcal{J}_\infty$-continuous positive functionals T, is equal to $\mathcal{J}_\infty$*.

Proof. If $(p_\delta)_{\delta\in\Delta}$ is a generating family of seminorms for E it can be shown that

$$P_{\underset{\sim}{\gamma},\delta}=\sum_{n\geq 0}\gamma_n(p_\delta\otimes_\pi\cdots\otimes_\pi p_\delta),$$

where $\underset{\sim}{\gamma}=(\gamma_n)_{n\geq 0}$ varies over all sequences of non-negative numbers and $\delta\in\Delta$, is a generating family of seminorms for $\underset{\sim}{E}[\mathcal{J}_\infty]$. The assertions in (i) follow immediately from this.

To prove property (ii) it is convenient to replace the seminorms $P_{\underset{\sim}{\gamma},\delta}$ by the equivalent family of seminorms

$$P'_{\underset{\sim}{\gamma},\delta}=[\sum_{n\geq 0}\gamma_n(p_\delta\otimes_\pi\cdots\otimes_\pi p_\delta)^2]^{1/2}$$

and take the $(p_\delta)_{\delta\in\Delta}$ to be Hilbertian seminorms. This is always possible because E is nuclear. By nuclearity, for any p_δ, there exists a $p_\omega(\delta,\omega\in\Delta)$ such that p_ω dominates p_δ and the canonical injection $i_{\omega\delta}$ from $\mathcal{H}_\omega$, the Hilbert space completion of $E/p_\omega^{-1}(0)$, into $\mathcal{H}_\delta$ is Hilbert-Schmidt. Without loss of generality p_ω can be chosen so that the Hilbert-Schmidt norm of $i_{\omega\delta}$ is less than 1. Then it is not difficult to show that the natural injection $i_{\underset{\sim}{\gamma}\omega\delta}$ from $\mathcal{H}_{\underset{\sim}{\gamma},\omega}$, the Hilbert space completion of $\underset{\sim}{E}/P'^{-1}_{\underset{\sim}{\gamma},\omega}(0)$, into $\mathcal{H}_{\underset{\sim}{\gamma},\delta}$ is also Hilbert-Schmidt.

Property (iii) follows from the nuclearity of E by using the seminorms $P''_{\underset{\sim}{\gamma},\delta}=\sum_{n\geq 0}\gamma_n p_\delta\otimes_\varepsilon\cdots\otimes_\varepsilon p_\delta$ in [6], Satz 1.

Finally the nuclearity of $\underset{\sim}{E}[\mathcal{J}_\infty]$ and the $\mathcal{J}_\infty$-normality of $\bar{\underset{\sim}{E}}_+$ imply that for every $\mathcal{J}_\infty$-continuous seminorm P there is a summable sequence of positive numbers $(\lambda_n)_{n\geq1}$ and a $\mathcal{J}_\infty$-equicontinuous sequence of positive functionals $(T_n)_{n\geq1}$ such that

$$P(f)^2 \leq \sum_{n\geq1} \lambda_n |T_n(f)|^2$$

([10], Théorème 3). By the Cauchy-Schwarz inequality for positive functionals,

$$\sum_{n\geq1} \gamma_n |T_n(f)|^2 \leq \sum_{n\geq1} \lambda_n T_n(\mathbb{1}) T_n(f^* \times f).$$

Therefore $T \equiv \sum_{n\geq1} \lambda_n T_n(\mathbb{1}) T_n$ is a $\mathcal{J}_\infty$-continuous positive functional such that $P(f)^2 \leq T(f^* \times f)$. Property (iv) follows from this inequality, the $\mathcal{J}_\infty$-continuity of the involution and the $\mathcal{J}_\infty$-joint continuity of the multiplication. ∎

The $\mathcal{J}_\infty$ topology of a BU-algebra $\underline{E}$ also satisfies Lemma 1, because it has $\{P_{\underline{r},\delta}\}$ as a generating family of seminorms. It is also worth mentioning that if $\mathcal{J}$ is the original topology on $\underline{E}$ and T is a $\mathcal{J}$-continuous positive functional, then the seminorm $f \rightsquigarrow T(f^* \times f)^{1/2}$ is $\mathcal{J}$-continuous. But the analog of part (iv) of Lemma 1 for $\mathcal{J}$ holds when E is a Fréchet nucler space if and only if E is isomorphic to a closed subspace of s, the Fréchet space of rapidly decreasing sequences (Yngvason [17], Satz 4.8 and private communication); when E is a nuclear LF-space such that $E \otimes_\iota E \neq E \otimes_\pi E$ it never holds ([3], Proposition 2.8).

Our next lemma deals with the problem of extension of positive functionals in tensor algebras.

Lemma 2. (i) *Let E be a complex nuclear space with a continuous involution and, for finite N, suppose that $T = (T_0, T_1, \cdots, T_{2N}, 0, 0, \cdots)$ is a continuous linear functional on $\underline{E}$ which is positive on $\bigoplus_{n=0}^{2N} \hat{\otimes}^n E$. Then there is a $\mathcal{J}_\infty$-continuous positive functional S such that for all $\varepsilon > 0$ one can find a sequence of positive type $(\alpha_n)_{n\geq0}$, with $\max_{0\leq n\leq 2N} |\alpha_n| < \varepsilon$, so that $T + S_{(\alpha_n)}$ is a positive functional. Here $S_{(\alpha_n)} \equiv (\alpha_0 S_0, \alpha_1 S_1, \cdots, \alpha_n S_n, \cdots)$.*

(ii) *Let E be a complex nuclear space with a continuous involution and G a closed $*$-invariant subspace of E. If T is a $\mathcal{J}_\infty$-*

continuous positive functional on $\underset{\sim}{G}$, *then it has a positive extension to* $\underset{\sim}{E}$.

Proof. (i) This will follow as in [17], pp. 17–18, Lemma, if there are a continuous seminorm P and a positive functional S such that $|T(f^*\times g)|\leq P(f)P(g)$ and $P(f)^2\leq S(f^*\times f)$. Since T is $\mathcal{J}_\infty$-continuous and $\underset{\sim}{\bar{E}}_+$ is $\mathcal{J}_\infty$-normal (Lemma 1 (iii)), there are $\mathcal{J}_\infty$-continuous positive functionals T_1, T_2 such that $T=T_1-T_2$ ([12], p. 220, Corollary 3). Using in part the Cauchy-Schwarz inequality for positive functionals we get $|T(f^*\times g)|\leq P(f)P(g)$, where $P(f)\equiv T_1(f^*\times f)^{1/2}+T_2(f^*\times f)^{1/2}$ is a $\mathcal{J}_\infty$-continuous seminorm (Lemma 1 (iv)). Finally by Lemma 1 (iv) there is a $\mathcal{J}_\infty$-continuous positive functional S such that $P(f)^2 \leq S(f^*\times f)$.

(ii) This is proven in [18], Theorem 8, when T is dominated by a seminorm of the form $\sum_{n\geq 0}\gamma_n p\otimes_\pi\cdots\otimes_\pi p$, where p is a *-invariant Hilbertian norm on E. When E does not have a continuous norm a similar proof goes through if we work with the spaces $E/p^{-1}(0)$ and $G/p^{-1}(0)$ instead of E and G, respectively. ∎

Our last lemma is partly a generalization of the following result of Schmüdgen ([13], Section 3, Lemma 2): Let E be a complex Fréchet space with a continuous involution $f\rightsquigarrow f^*$. Then the closure of the cone

$$(E\otimes E)_+=\{\sum_{i=1}^n f_i^*\otimes f_i\colon f_i\in E,\ 1\leq i\leq n,\ n\in\mathbf{N}\}$$

in $E\widehat{\otimes}_\varepsilon E$. the completion of the tensor product in the injective topology, is

$$\{\sum_{i=1}^\infty f_i^*\otimes f_i\colon f_i\in E\}.$$

Lemma 3. *Let E be a complex nuclear LF-space with a continuous involution $f\rightsquigarrow f^*$. Then the closure of the cone*

$$(E\otimes E)_+=\{\sum_{i=1}^n f_i^*\otimes f_i\colon f_i\in E,\ 1\leq i\leq n,\ n\in\mathbf{N}\}$$

in $E\overline{\otimes}E$ *is*

$$\{\sum_{i=1}^{\infty} f_i^*\otimes f_i : f_i \in E\}.$$

Proof. By [15], p. 134, Exercise 13. 4, E has a sequence of definition $\{E_i\}_{i\geq 1}$, consisting of $*$-invariant subspaces. It has been shown in [1], Proposition A. 49, that $E\overline{\otimes}E$ is a nuclear LF-space with a sequence of definition $\{E_i\widehat{\otimes}E_i\}_{i\geq 1}$. (As E_i is nuclear, we have omitted the subscript ε in $E_i\widehat{\otimes}_\varepsilon E_i$).

We are going to show that

$$\overline{(E\otimes E)_+} = \bigcup_{i=1}^{\infty} \overline{(E_i\otimes E_i)_+}$$

and the result will then follow from Schmüdgen's Lemma. By [16], Theorem 2. 15,

$$\overline{(E\otimes E)_+} = \{h \in E\overline{\otimes}E : T(h) \geq 0,\ \forall T \in (E\otimes E)'_+\}.$$

It is straightforward to show that all the extremal rays of the dual cone $(E\otimes E)'_+$ are of the form $l^*\otimes l$, where $l\in E'$ and $l^*(f) \equiv \overline{l(f^*)}$. Since $E\overline{\otimes}E = \bigcup_{j=1}^{\infty} (E_i\widehat{\otimes}E_i)$, every element u of $E\overline{\otimes}E$ has a representation of the form

$$u = \sum_{j=1}^{\infty} \lambda_j f_j\otimes g_j,$$

where $\sum_{j=1}^{\infty} |\lambda_j| < +\infty$ and $\{f_j\}_{j\geq 1}$, $\{g_j\}_{j\geq 1}$ are null sequences in E ([12], p. 94, Theorem 6. 4). A simple polarization argument then shows that the cone $\overline{(E\otimes E)_+}$ is generating, i.e., the smallest subspace containing it is $E\overline{\otimes}E$. Consequently the order intervals associated to the dual cone $(E\otimes E)'_+$ are bounded in the $\sigma((E\overline{\otimes}E)', E\overline{\otimes}E)$ topology and therefore compact, because the strong dual $(E\overline{\otimes}E)'$ of $E\overline{\otimes}E$ is Montel ([1], Proposition 1. 2 or [3], Lemma 1. 2). Since $(E\overline{\otimes}E)'$ is dual nuclear and complete ([1], Proposition 1. 2 or [3], Lemma 1. 2), the compactness of its order intervals implies that $(E\otimes E)'_+$ is the closed convex hull of its extremal rays ([14], Théorème 1, Corollaire), because $(E\otimes E)'_+$ is proper ($\overline{(E\otimes E)_+}$ is generating) and closed. Therefore

$$\overline{(E\otimes E)_+} = \{h \in E\overline{\otimes}E : (l^*\otimes l)(h) \geq 0,\ \forall l \in E'\}.$$

Similarly

$$\overline{(E_i \otimes E_i)}_+ = \{k \in E_i \hat{\otimes} E_i : (l_i^* \otimes l_i)(k) \geq 0, \ \forall l_i \in E_i'\}.$$

By the Hahn-Banach extension theorem we then get

$$\overline{(E \otimes E)}_+ \cap (E_i \hat{\otimes} E_i) = \overline{(E_i \otimes E_i)}_+$$

which finishes the proof of the lemma. ∎

§ 3. Main Results

Theorem 1. *Let* $\underset{\sim}{E}[\mathcal{J}]$ *be a BU-algebra. Then*

$$\bar{\underset{\sim}{E}}^+ = \{\sum_{i=1}^{\infty} f_i^* \times f_i : f_i \in \underset{\sim}{E}\}.$$

Proof. First we will prove the result when E is a Fréchet nuclear space. Note that in this case $\underset{\sim}{E}$ is isomorphic to $\underset{\sim}{E}$ ([5], chapter 1, p. 74). Since the algebra $\bigoplus_{n=0}^{\infty} \otimes_\pi^n E$ is dense in $\underset{\sim}{E}$, it follows that $\bar{\underset{\sim}{E}}_+ = \overline{K}_+$, where K_+ is the positive cone of the algebra $\bigoplus_{n=0}^{\infty} \otimes_\pi^n E$. Next it will be shown that the closure of K_+ is equal to its sequential closure. If $F_i \equiv \bigoplus_{n=0}^{2i} \hat{\otimes}^n E$, then $\underset{\sim}{E} = \bigcup_{i=1}^{\infty} F_i$ and therefore $\overline{K}_+ = \bigcup_{i=1}^{\infty} (\overline{K}_+ \cap F_i)$. Part (i) of Lemma 2 implies that the set of positive functionals on F_i that have a positive extension to $\underset{\sim}{E}$ is dense in the set of positive functionals on F_i. Consequently $\overline{K}_+ \cap F_i = \overline{K_+ \cap F_i}$ (cf. [16], Theorem 2.15). This completes the proof that the closure of K_+ is equal to its sequential closure because F_i is metrizable if i is finite.

We now show that if $\{u_n = \sum_{i=1}^{r_n} f_{i,n}^* \times f_{i,n} : n \in N\}$ is a convergent sequence in K_+, then the set $\{v_n = \sum_{i=1}^{r_n} f_{i,n}^* \otimes f_{i,n} : n \in N\}$ is bounded in $\underset{\sim}{E}[\mathcal{J}] \bar{\otimes} \underset{\sim}{E}[\mathcal{J}]$. If p is a $\mathcal{J}_\infty$-continuous seminorm then

$$(p \otimes_\pi p)(v_n) \leq \sum_{i=1}^{r_n} p(f_{i,n})^2 \leq \sum_{i=1}^{r_n} T(f_{i,n}^* \times f_{i,n}) = T(u_n)$$

by part (iv) of Lemma 1. Therefore $(v_n)_{n \geq 1}$ is bounded in $\underset{\sim}{E}[\mathcal{J}_\infty] \otimes_\pi \underset{\sim}{E}[\mathcal{J}_\infty]$. Since there is a finite i such that $(v_n)_{n \geq 1} \subset F_i \otimes F_i$, Lemma 1 (i) implies that $(v_n)_{n \geq 1}$ is bounded in $F_i \otimes_\pi F_i$. This set is also bounded in $\underset{\sim}{E}[\mathcal{J}] \bar{\otimes} \underset{\sim}{E}[\mathcal{J}]$, because this space is the strict inductive limit of $\{F_i \hat{\otimes} F_i\}_{i \geq 1}$ (cf. [1], Propositions A.49 and A.50). By the nuclearity of

$\underset{\sim}{E}[\mathcal{J}]\bar{\otimes}\underset{\sim}{E}[\mathcal{J}]$ ([1], Proposition A. 49), $(v_n)_{n\geq 1}$ has a convergent subsequence $(w_n)_{n\geq 1}$ ([12], p. 101, Corollary 2) and by Lemma 3, $\lim_n w_n = \sum_{i=1}^{\infty} f_i^* \otimes f_i$. The separate continuity of the multiplication in $\underset{\sim}{E}[\mathcal{J}]$ implies that the map $M: \underset{\sim}{E}[\mathcal{J}]\bar{\otimes}\underset{\sim}{E}[\mathcal{J}] \to \underset{\sim}{E}[\mathcal{J}]$; $M(f\otimes g) = f\times g$, is continuous ([1], Proposition 1.10 or [3], Proposition 1.7). Consequently

$$\lim_n u_n = \lim_n M(w_n) = M(\lim_n w_n) = \sum_{i=1}^{\infty} f_i^* \times f_i .$$

This finishes the proof of the Theorem when E is a nuclear Fréchet space.

Before dealing with the last part of the proof, note that by Lemma 2 (i), for any complex nuclear space E with a continuous involution, the $\mathcal{J}_\infty$-closure of $\underset{\sim}{E}_+$, $\bar{\underset{\sim}{E}}_+^{\mathcal{J}_\infty}$, is equal to $\bar{\underset{\sim}{E}}_+$. The same conclusion holds for BU-algebras $\underset{\sim}{E}$, because $\underset{\sim}{E}\subset\underset{\sim}{E}$ and both have the same positive functionals ([3], Proposition 2.8) and the same $\mathcal{J}_\infty$-continuous positive functionals (see first remark after Lemma 1).

To conclude the proof of the theorem assume now that E is a nuclear LF-space. If $\{E_i\}_{i\geq 1}$ is the *-invariant sequence of definition of E introduded in the proof of Lemma 3, then $\underset{\sim}{E} = \bigcup_{i=1}^{\infty} \underset{\sim}{E}_i$ (cf. [1], Propositions A. 49 and A. 50) and therefore $\bar{\underset{\sim}{E}}_+ = \bigcup_{i=1}^{\infty} (\bar{\underset{\sim}{E}}_+ \cap \underset{\sim}{E}_i)$. By Lemma 2 (ii) $\bar{\underset{\sim}{E}}_+^{\mathcal{J}_\infty} \cap \underset{\sim}{E}_i = \overline{\bar{\underset{\sim}{E}}_+ \cap \underset{\sim}{E}_i}^{\mathcal{J}_\infty} = \bar{\underset{\sim}{E}}_{i+}^{\mathcal{J}_\infty}$ which finally gives

$$\bar{\underset{\sim}{E}}_+ = \bar{\underset{\sim}{E}}_+^{\mathcal{J}_\infty} = \bigcup_{i=1}^{\infty} (\bar{\underset{\sim}{E}}_+^{\mathcal{J}_\infty} \cap \underset{\sim}{E}_i) = \bigcup_{i=1}^{\infty} \bar{\underset{\sim}{E}}_{i+}^{\mathcal{J}_\infty} = \bigcup_{i=1}^{\infty} \bar{\underset{\sim}{E}}_{i+} ,$$

finishing the proof of the theorem because $\underset{\sim}{E}_i$ is a nuclear Fréchet space. ∎

Theorem 2. *Let $\underset{\sim}{E}[\mathcal{J}]$ be a BU-algebra. Then its order intervals are compact, and $\bar{\underset{\sim}{E}}_+$ is the closed convex hull of its extremal rays.*

Proof. Since $\underset{\sim}{E}[\mathcal{J}]$ is a Montel space ([1], Proposition 1.2 or [3], Lemma 1.2) and the order intervals are closed, their compactness will follow from boundedness. Let $\{E_i\}_{i\geq 1}$ be a *-invariant sequence of definition of E and $G_i \equiv \bigoplus_{n=0}^{2i} \hat{\otimes}^n E_i$. As $\bar{\otimes}^n E$ and $\hat{\otimes}^n E$ induce the original topology on $\hat{\otimes}^n E_i$ ([1], Proposition A. 49 and [8], p. 119), the topology $\mathcal{J}_\infty$ induces the original topology on G_i and therefore by Lemma 1 (iii) and the first remark after Lemma 1, $\bar{\underline{E}}_+ \cap G_i$ is normal. Consequently if every order interval of $\underline{E}$ is contained in a subspace G_i, they will be bounded ([12], p. 216, Corollary). Let $f \in \bar{\underline{E}}_+$ and $0 \leq g \leq f$. Since $\underline{E} = \bigcup_{i=1}^{\infty} G_i$ (cf. [1], Propositions A. 49 and A. 50), there is an i such that $f \in G_i$. It will be shown that $g \in G_i$. If l is a hermitian linear functional on G_i, then there exist positive linear functionals T_1, T_2 on G_i such that $l = T_1 - T_2$ ([12], p. 220, Corollary 3). If $\tilde{T}_1$ and $\tilde{T}_2$ are the extensions of T_1 and T_2 to $\underset{\sim}{E}_i$, taking the value zero on $\bigoplus_{n=2i+1}^{\infty} \widehat{\otimes}^n E_i$, part (i) of Lemma 2 implies that there are positive functionals S_{T_1} and S_{T_2} on $\underset{\sim}{E}_i$, with $S_{T_j}(f) < 1$ $(j=1,2)$, such that $\tilde{T}_j + S_{T_j}$ $(j=1,2)$ are positive. By part (ii) of Lemma 2, $\tilde{T}_j + S_{T_j}$ and S_{T_j} $(j=1,2)$ have positive extensions V_{T_j} and U_{T_j} $(j=1,2)$, respectively, to $\underline{E}$, because $\underline{E} \subset \underset{\sim}{E}$. If $\tilde{l}(g) \equiv V_{T_1}(g) - V_{T_2}(g) + U_{T_2}(g) - U_{T_1}(g)$, we are going to prove that $l \rightsquigarrow |\tilde{l}(g)|$ is a continuous seminorm on the hermitian part $G'_{i,h}$ of G'_i. The reflexivity of G_i will then imply that $g \in G_i$. Since $G'_{i,h}$ is bornological (cf. [1], Proposition 1. 2 or [3], Lemma 1. 2) we need to verify that if B is a bounded subset of $G'_{i,h}$, then

$$\sup_{l \in B} |\tilde{l}(g)| < +\infty .$$

Now as G_i is barreled, B must be equicontinuous ([12], p. 127, Corollary). By the normality of the cone $\bar{\underline{E}}_+ \cap G_i$, there is an equicontinuous set of positive functionals C in $G'_{i,h}$ such that $B \subset C - C$ ([12], p. 220, Corollary 1). Therefore

$$\begin{aligned} \sup_{l \in B} |\tilde{l}(g)| &\leq \sup_{l \in B} (V_{T_1}(g) + V_{T_2}(g) + U_{T_1}(g) + U_{T_2}(g)) \\ &\leq \sup_{T_1, T_2 \in C} (T_1(f) + T_2(f) + 2S_{T_1}(f) + 2S_{T_2}(f)) \\ &\leq 4 + \sup_{T \in C} T(f) < +\infty , \end{aligned}$$

which finishes the proof of the first part of the Theorem.

The last part of the theorem follows from [14], Théorème 1, Corollaire, because $\underline{E}[\mathcal{G}]$ is dual nuclear and complete ([1], Proposition 1. 2 or [3], Lemma 1. 2), the order intervals are compact and $\bar{\underline{E}}_+$ is proper (Lemma 1 (iii), first remark after Lemma 1 and [12], p. 216, Corollary 1) and closed. ∎

Theorem 3. *A BU-algebra $\underline{E}$ has a strictly positive functional if and only if E has a continuous norm,*

Proof. If p is a continuous norm on E, then $P=\sum_{n\geq 0}^{\infty} p\otimes_{\pi}\cdots\otimes_{\pi}p$ is a $\mathcal{J}_{\infty}$-continuous norm on $\underline{E}$. By part (iv) of Lemma 1 there is a $\mathcal{J}_{\infty}$-continuous positive functional T such that

$$P(f)^2 \leq T(f^* \times f), \quad \forall f \in \underline{E}.$$

This inequality and Theorem 1 imply that T is strictly positive.

To prove the reverse implication note that if T is a strictly positive functional, then $f \rightsquigarrow T(f^* \times f)^{1/2}$ is a continuous norm on $\underline{E}$, as $\underline{E}$, is barreled ([9], Theorem 4.1). Obviously E also has a continuous norm. ∎

For examples of nuclear Fréchet spaces without a continuous norm see [11], Theorem 2.

We finish this paper with a short list of unsolved problems.

(1) *Is it true that $\underline{E}=\underset{\sim}{E}$ (equality as vector spaces only), for every nuclear LF-space E? Do they have the same bounded sets?*

(2) *Characterize explicitly the extremal rays of BU-algebras.*

(3) *A subspace I of a locally convex *-algebra $\mathcal{A}[\mathcal{J}]$ will be called state-related if*

$$I=\bigcap \{K(T) : T \in \mathcal{A}'_{+} \text{ and } K(T) \supset I\}$$

where $K(T)$ is the kernel of T. Characterize the subspaces of BU-algebras that are state-related. In particular is the Wightman kernel of Quantum Field Theory ([7]) a state-related subspace?

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2 (i) and constructive criticism.

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Note added in proof: Problem (1) has been answared in the negative (J. Alcántara: Some new results on topological tensor products, *The Open Univ. preprint*); and Hofmann has given an answer to (2) (Beschreibung der Extremalstraplen des Positivitätskegels in Tensoralgebren, *Leipzig preprint*.)