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Pursell-Shanks Type Theorem for Orbit Spaces of G-Manifolds

By

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§ 0. Introduction

Pursell and Shanks [8] proved that a Lie algebra isomorphism between Lie algebras of all C^∞ vector fields with compact support on paracompact connected C^∞ manifolds M and N yields a diffeomorphism between the manifolds M and N. Similar results hold for some other structures on manifolds. Indeed, Omori [6] proved the corresponding results in the case of volume structures, symplectic structures, contact structures and fibering structures with compact fibers. The case of complex structures was studied by Amemiya [1]. Koriyama [5] proved that in the case of Lie algebras of vector fields with invariant submanifolds.

Recently, Fukui [4] studies the case of Lie algebras of G-invariant C^∞ vector fields with compact support on paracompact free smooth G-manifolds when G is a compact connected semi-simple Lie group. The corresponding result is no longer true when G is not semi-simple or G does not act freely.

In this paper, we consider Pursell-Shanks type theorem for orbit spaces of smooth G-manifolds in the case of G a compact Lie group. For a smooth G-manifold M, the orbit space M/G inherits a smooth structure by defining a function on M/G to be smooth if it pulls back to a smooth function on M, and the Zariski tangent space of M/G can be defined. This smooth structure of the orbit space was studied by Schwarz [9], [11], Bierstone [2], Poénaru [7] and Davis [3]. Schwarz [10] defined a Lie algebra $\mathfrak{X}(M/G)$ of smooth vector fields on the orbit

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space M/G, and proved $\pi_*(\mathfrak{X}_G(M)) = \mathfrak{X}(M/G)$, where $\mathfrak{X}_G(M)$ is the Lie algebra of all G-invariant C^∞ vector fields with compact support on M and π: $M \to M/G$ is a natural projection.

The purpose of this paper is to prove the following:

Theorem. *Let G and G' be compact Lie groups. Let M and N be connected paracompact smooth G-manifold and G'-manifold without boundary, respectively. There exists a Lie algebra isomorphism Φ: $\mathfrak{X}(M/G) \to \mathfrak{X}(N/G')$ if and only if there exists a strata preserving diffeomorphism σ: $M/G \to N/G'$ such that $\Phi = \sigma_*$.*

Main part of the proof of our theorem is to find maximal ideals of $\mathfrak{X}(M/G)$. By the theorem of Schwarz, maximal ideals of $\mathfrak{X}(M/G)$ are induced from those of $\mathfrak{X}_G(M)$. To determine the maximal ideals of $\mathfrak{X}_G(M)$, we use the parallel method to those of Pursell-Shanks [8] and Koriyama [5].

§ 1. The Tangent Space of an Orbit Space

In this paper, we consider C^∞ smooth category. Let G and G' be compact Lie groups. Let M and N be connected paracompact smooth G-manifold and G'-manifold without boundary, respectively. Put $\overline{M} = M/G$, $\overline{N} = N/G'$. The orbit space $\overline{M}$ has an induced smooth structure such that a function f: $\overline{M} \to R$ is smooth if the composition $M \xrightarrow{\pi} \overline{M} \xrightarrow{f} R$ is smooth, where π is the natural projection. Let $C^\infty(\overline{M})$ denote the set of all smooth functions on $\overline{M}$. A map h: $\overline{M} \to \overline{N}$ is smooth if, $f \circ h \in C^\infty(\overline{M})$ for any $f \in C^\infty(\overline{N})$, and we say that h is diffeomorphic if h^{-1} is also smooth.

We can define a tangent space of the orbit space as usual. A tangent vector v of $\overline{M}$ at p is a correspondense assigning to any smooth functions f, g around p real numbers $v(f)$, $v(g)$ with the following conditions:

(1) $v(\lambda f + \mu g) = \lambda v(f) + \mu v(g)$ for $\lambda, \mu \in R$,

(2) $v(fg) = v(f) g(p) + f(p) v(g)$.

Put $\tau(\overline{M}) = \bigcup_{p \in \overline{M}} \tau_p(\overline{M})$.

Given $a \in M$, let G_a denote the isotropy group at a and V_a be a linear slice at a. Then V_a is a G_a-module. Put $p=\pi(a)$ and $\overline{V}_p = V_a/G_a$. Then $\overline{V}_p$ is an open neighborhood of p in $\overline{M}$.

Proposition 1.1 (cf. Davis [3], Proposition 2.3).

(1) $\tau_p(\overline{M}) \cong \tau_p(\overline{V}_p)$.

(2) *Let* $\overline{\mathfrak{M}}_p$ *denote the germs of smooth functions on* $\overline{V}_p$ *which vanish at* p. *Then* $\tau_p(\overline{V}_p) \cong \mathrm{Hom}\,(\overline{\mathfrak{M}}_p/\overline{\mathfrak{M}}_p^2, R)$.

Let H be a compact Lie group and let V be an H-module. By a theorem of Hilbert ([13], p. 275), the algebra of H-invariant polynomials $R[V]^H$ is finitely generated.

Theorem 1.2 (Schwarz [9]). *Let* $\{\theta_1, \cdots, \theta_s\}$ *be a set of generators for* $R[V]^H$, *and let* $\theta = (\theta_1, \cdots, \theta_s): V \to R^s$. *Then*

(1) $\theta^* C^\infty(R^s) = C^\infty_H(V)$.

(2) *The orbit map* $\bar{\theta}: V/H \to R^s$ *of* θ *is a topological embedding.*

Proposition 1.3 (cf. Davis [3] Lemma 2.1). *Let* $R[V]^H_0$ *denote the algebra of H-invariant polynomials which vanish at* 0. *Then*

(1) $\overline{\mathfrak{M}}_0/\overline{\mathfrak{M}}_0^2 \cong R[V]^H_0/(R[V]^H_0)^2$.

(2) *If* $\{\theta_1, \cdots, \theta_s\}$ *is a minimal set of generators for* $R[V]^H_0$, *then the dimension of* $\tau_0(V/H)$ *is* s.

§ 2. Smooth Vector Fields on an Orbit Space

Let $X: \overline{M} \to \tau(\overline{M})$ be a section. For any $f \in C^\infty(\overline{M})$, we can define a function $X(f): \overline{M} \to R$ by $X(f)(p) = X_p(f)$. If $X(f) \in C^\infty(\overline{M})$ for any $f \in C^\infty(\overline{M})$, then we say X is a smooth vector field on $\overline{M}$. Let $\mathfrak{D}(\overline{M})$ denote the Lie algebra of all smooth vector fields on $\overline{M}$. Let $DC^\infty(\overline{M})$ denote the set of all derivations of $C^\infty(\overline{M})$. Using Theorem 1.2 (1) we have:

Proposition 2.1. $\mathfrak{D}(\overline{M})$ *is isomorphic to* $DC^\infty(\overline{M})$ *as a Lie algebra.*

The orbit space $\overline{M}$ is stratified by its orbit type.

Definition 2.2 (Schwarz [11]). A smooth vector field X on $\overline{M}$ is said to be strata preserving if $X_p \in \tau_p(\sigma_p)$ for any $p \in \overline{M}$, where σ_p denotes the stratum of $\overline{M}$ containing p. Let $\mathfrak{X}(\overline{M})$ denote the set of all strata preserving smooth vector fields with compact support on $\overline{M}$. $\mathfrak{X}(\overline{M})$ is a Lie subalgebra of $DC^\infty(\overline{M})$. Let $\mathfrak{X}_G(M)$ denote the set of all G-invariant smooth vector fields with compact support on M. There is a Lie algebra homomorphism $\pi_*: \mathfrak{X}_G(M) \to DC^\infty(\overline{M})$ defined by $\pi_*(X)(\bar{f}) = X(f)$, where $f \in C_G^\infty(M)$ and $\bar{f}$ is the orbit map of f.

Theorem 2.3 (Schwarz [11]). *The image of the homomorphism $\pi_*: \mathfrak{X}_G(M) \to DC^\infty(\overline{M})$ is $\mathfrak{X}(\overline{M})$.*

§ 3. Maximal Ideals of $\mathfrak{X}(\overline{M})$

Let $a \in M$ and put $p = \pi(a) \in \overline{M}$. Let V_a be a linear slice at a. Then $N_a = G \times_{G_a} V_a$ is equivalent to a linear tubular neighborhood of the orbit $G(a)$ of a. Let $\tau(N_a)$ be the tangent bundle of the G-manifold N_a, and let $\Gamma_G(\tau(N_a))$ denote the set of all G-invariant smooth sections of $\tau(N_a)$. Let $\tau(V_a)$ be the tangent bundle of the G_a-manifold V_a, and let $\Gamma_{G_a}(\tau(V_a))$ denote the set of all G_a-invariant smooth sections of $\tau(V_a)$. Then we have canonical isomorphisms $\Gamma_G(\tau(N_a)) \cong \Gamma_{G_a}(\tau(N_a) | V_a)$ and $C_G^\infty(N_a) \cong C_{G_a}^\infty(V_a)$. It is easy to see the following:

Lemma 3.1. (1) *For any $X \in \Gamma_{G_a}(\tau(V_a))$, there exists $Y \in \mathfrak{X}_G(M)$ such that $Y = X$ on a G_a-invariant neighborhood U_a of a in V_a.*

(2) *For any $f \in C_{G_a}^\infty(V_a)$, there exists $F \in C_G^\infty(M)$ such that $F = f$ on a G_a-invariant neighborhood U_a of a in V_a.*

Put $\overline{M}_0 = \{q \in \overline{M}\,;\, X_q = 0 \text{ for any } X \in \mathfrak{X}(\overline{M})\}$, and put $\overline{M}_1 = \overline{M} - \overline{M}_0$.

Proposition 3.2. *$\overline{M}_0$ is discrete.*

Proof. For any $a \in M$, let $\{x_1, \cdots, x_n\}$ be a canonical coordinate of a linear slice V_a of a. We can assume G_a acts orthogonally on V_a. Then the radial vector field $X=\sum_{i=1}^{n} x_i\frac{\partial}{\partial x_i}$ is a G_a-invariant smooth vector field on V_a. Let $f\colon V_a\to R$ be a G_a-invariant smooth function defined by $f(x_1, \cdots, x_n)=x_1^2+\cdots+x_n^2$. By Lemma 3.1, there exist $Y\in\mathfrak{X}_G(M)$ and $F\in C_G^\infty(M)$ such that $Y=X$ and $F=f$ on a G_a-invariant neighborhood U_a of a in V_a, respectively. Put $\bar{U}_p=U_a/G_a$, $\bar{Y}=\pi_*(Y)$ and let $\bar{F}$ be the orbit map of F. Then $\bar{Y}(\bar{F})\neq 0$ on $\bar{U}_p-\{p\}$, and Proposition 3.2 follows.

Note that $\pi_a\colon N_a\to G(a)$ is a G-vector bundle. The tangent bundle $\tau(N_a)$ of N_a is isomorphic to $\pi_a^*(\tau(G(a))\oplus\xi_a$ as a G-vector bundle, where ξ_a is a bundle along the fibres of N_a. Let r_a be the composition

$$\mathfrak{X}_G(M)\xrightarrow{\text{restriction}}\Gamma_G(\tau(N_a))\cong\Gamma_{G_a}(\tau(N_a)\,|V_a)$$

$$\xrightarrow{\text{projection}}\Gamma_{G_a}(\xi_a|V_a)\cong\Gamma_{G_a}(\tau(V_a)).$$

It is easy to see that r_a is a Lie algebra homomorphism. Put $\Gamma_{G_a}(\tau(V_a))_0=\{X\in\Gamma_{G_a}(\tau(V_a))\,;X_a=0\}$. For $X\in\Gamma_{G_a}(\tau(V_a))_0$, we denote $j_a^r(X)$ the r-jet of X at a $(r=1,2,\cdots)$. Put $\Gamma_{G_a}(\tau(V_a))_0^k=\{X\in\Gamma_{G_a}(\tau(V_a))_0;\ j_a^r(X)=0$ for $1\leqq r\leqq k\}$ $(1\leqq k\leqq\infty)$.

For $q\in\bar{M}_0$, choose a point $b\in\pi^{-1}(q)$. Let $\mathfrak{gl}_{G_b}(V_b)$ denote the set of G_b-invariant endomorphisms of V_b. Note that, for $X\in\Gamma(\tau(V_b))$, $j_b^1(X)$ defines an element of $\mathfrak{gl}(V_b)$ as usual. It is easy to see that $j_b^1(X)\in\mathfrak{gl}_{G_b}(V_b)_b$ for $X\in\Gamma_{G_b}(\tau(V_b))$.

Lemma 3.3. *$j_b^1\colon\Gamma_{G_b}(\tau(V_b))\to\mathfrak{gl}_{G_b}(V_b)$ is an onto Lie algebra homomorphism.*

Proof. Since $\pi(b)=q\in\bar{M}_0$, $\Gamma_{G_b}(\tau(V_b))=\Gamma_{G_b}(\tau(V_b))_0$. Then $\Gamma_{G_b}(\tau(V_b))/\Gamma_{G_b}(\tau(V_b))_0^1\cong\mathfrak{gl}_{G_b}(V_b)$ and Lemma 3.3 follows.

By Proposition 3.2, $\bar{M}_0$ is discrete. Then $\bar{M}_0$ is a countable set $\{q_i;\ i\in I\}$. Choose a point $b_i\in\pi^{-1}(q_i)$ for each $q_i\in\bar{M}_0$. Put $J_i^1(M)=\mathfrak{gl}_{G_{b_i}}(V_{b_i})$ and put $J^1(M)=\prod_{i\in I}J_i^1(M)$ which consists of those elements having only finite number of non-zero factors. Then we have:

Corollary 3.4. *The composition*

$$J^1: \mathfrak{X}_G(M) \xrightarrow{\prod_{i\in I} r_i} \prod_{i\in I} \Gamma_{G_{b_i}}(\tau(V_{b_i})) \xrightarrow{\prod_{i\in I} j^1_{b_i}} J^1(M)$$

is an onto Lie algebra homomorphism.

G_{b_i}-module V_{b_i} is isomorphic to $\bigoplus_j d_{ij} W_{ij}$. Here d_{ij} is a non-negative integer and W_{ij} runs over the inequivalent irreducible G_{b_i}-modules. Let K_{ij} be the real numbers R, complex numbers C or quaternionic numbers H if $\dim_R \mathfrak{gl}_{G_{b_i}}(W_{ij}) = 1$, 2 or 4, respectively. Then $\mathfrak{gl}_{G_{b_i}}(V_{b_i}) \cong \bigoplus_j \mathfrak{gl}(d_{ij}, K_{ij})$.

Proposition 3.5. $\mathfrak{gl}(d, R) \cong R \oplus \mathfrak{sl}(d, R)$,

$\mathfrak{gl}(d, C) \cong C \oplus \mathfrak{sl}(d, C)$ *and*

$\mathfrak{gl}(d, H) \cong R \oplus \mathfrak{sl}(d, H)$,

where $\mathfrak{sl}(d, K) = [\mathfrak{gl}(d, K), \mathfrak{gl}(d, K)]$ *for* $K = R, C$ *or* H.

Proof. Note that $\mathfrak{sl}(d, H) = \{X \in \mathfrak{gl}(n, H)\,;\ \mathrm{Re\ Tr}(X) = 0\}$ and $\mathfrak{sl}(d, H)$ is a simple Lie algebra. Other cases are similar to this.

Next we consider maximal ideals of $\Gamma_{G_a}(\tau(V_a))$ for $a \in M$ such that $\pi(a) = p \in \overline{M}_1$. First we need the following:

Lemma 3.6. *Let H be a compact Lie group and let V be an H-module. For $Y \in \Gamma(\tau(V))$, we define $\tilde{Y} \in \Gamma_H(\tau(V))$ by $\tilde{Y}_p = \int_H (h_* Y)_p\, dh$ for $p \in V$. Then $[X, \tilde{Y}] = \int_H h_*[X, Y]\, dh$ for $X \in \Gamma_H(\tau(V))$. Here $(h_* Y)_p = (dh)_{h^{-1}p} Y_{h^{-1}p}$.*

Proof. Let $\{x_1, \cdots, x_n\}$ be a canonical coordinate of V. For $p \in V$, $f \in C^\infty(V)$, we have:

$$\tilde{Y}_p(f) = \left(\int_H \left(\sum_{i=1}^n (h_* Y)_p(x_i) \left(\frac{\partial}{\partial x_i} \right)_p \right) dh \right)(f)$$

$$= \sum_{i=1}^n \left(\int_H (h_* Y)_p(x_i)\, dh \right) \left(\frac{\partial}{\partial x_i} \right)_p (f)$$

$$= \int_H \left(\sum_{i=1}^n (h_* Y)_p (x_i) \left(\frac{\partial f}{\partial x_i} \right)_p \right) dh$$

$$= \int_H (h_* Y)_p (f) dh .$$

Then $$[X, \tilde{Y}]_p (f) = X_p \left(\int_H (h_* Y)(f) dh \right) - \int_H (h_* Y)_p (Xf) dh$$

$$= \int_H (X_p (h_* Y)(f) - (h_* Y)_p (Xf)) dh$$

$$= \int_H [X, h_* Y]_p (f) dh$$

$$= \left(\int_H [X, h_* Y]_p dh \right) (f).$$

Lemma 3.7. *Suppose that $\mathfrak{N}$ is a proper ideal of $\Gamma_{G_a}(\tau(V_a))$ which contains $\Gamma_{G_a}(\tau(V_a))_0^\infty$ for $a \in M$ such that $\pi(a) = p \in \overline{M}_1$. Then $\mathfrak{N}$ is contained in $\Gamma_{G_a}(\tau(V_a))_0$.*

Proof. Suppose there exists $X \in \mathfrak{N}$ with $X_a \neq 0$. By Koriyama [5] Lemma 2.1, for any $Z \in \Gamma_{G_a}(\tau(V_a))$ there exist a G_a-invariant neighborhood U of a in V_a and $Y \in \Gamma(\tau(V_a))$ such that $[X, Y] = Z$ on U. Put $\tilde{Y} = \int_{G_a} g_* Y \, dg \in \Gamma_{G_a}(\tau(V_a))$. By Lemma 3.6, we have $[X, \tilde{Y}] = \int_{G_a} g_* [X, Y] dg = \int_{G_a} g_* Z \, dg = Z$ on U. Put $Z_1 = Z - [X, \tilde{Y}]$. Then $Z_1 \in \Gamma_{G_a}(\tau(V_a))_0^\infty$ which is contained in $\mathfrak{N}$. Since $\mathfrak{N}$ is an ideal, $Z \in \mathfrak{N}$. Thus $\mathfrak{N} = \Gamma_{G_a}(\tau(V_a))$ which is a contradiction to $\mathfrak{N}$ a proper ideal.

By Lemma 3.7, there exists a unique maximal ideal $\mathfrak{N}_a$ of $\Gamma_{G_a}(\tau(V_a))$ satisfying $\Gamma_{G_a}(\tau(V_a))_0^\infty \subset \mathfrak{N}_a \subset \Gamma_{G_a}(\tau(V_a))_0$. Put $\mathfrak{J}_a = \{X \in \mathfrak{X}_G(M) ; r_a(X) \in \mathfrak{N}_a\}$.

Proposition 3.8. *$\mathfrak{J}_a$ is a maximal ideal of $\mathfrak{X}_G(M)$.*

Proof. Put $V_a(\rho) = \{v \in V_a ; \|v\| < \rho\}$ for a positive number ρ. $\mathfrak{N}_a(\rho) = \{Y \in \Gamma_{G_a}(\tau(V_a)) ; \operatorname{supp} Y \subset V_a - V_a(\rho)\}$ is an ideal of $\Gamma_{G_a}(\tau(V_a))$ which is contained in $\Gamma_{G_a}(\tau(V_a))_0$. Then $\mathfrak{N}_a(\rho)$ is contained in $\mathfrak{N}_a$. It is clear that $\mathfrak{J}_a$ is an ideal of $\mathfrak{X}_G(M)$.

Let $\mathfrak{M}$ be a maximal ideal of $\mathfrak{X}_G(M)$ which contains $\mathfrak{J}_a$. Suppose that there exists $X\in\mathfrak{M}$ with $r_a(X)_a\neq 0$. Similarly as in the proof of Lemma 3.7, we can prove $\mathfrak{M}=\mathfrak{X}_G(M)$, which is a contradiction. Then $r_a(\mathfrak{M})$ is contained in $\Gamma_{G_a}(\tau(V_a))_0$. Combining $\mathfrak{N}_a(\rho)\subset\mathfrak{N}_a$ and Lemma 3.1, we see $r_a(\mathfrak{M})+\mathfrak{N}_a$ is an ideal of $\Gamma_{G_a}(\tau(V_a))$. Therefore $r_a(\mathfrak{M})$ is contained in $\mathfrak{N}_a$, and $\mathfrak{M}=\mathfrak{J}_a$. Thus Proposition 3.8 follows.

Put $\overline{\mathfrak{J}}_p=\pi_*(\mathfrak{J}_a)$ and $\mathfrak{X}(\overline{M})_p=\{X\in\mathfrak{X}(\overline{M})\,;\,X_p=0\}$. Then $\overline{\mathfrak{J}}_p$ is contained in $\mathfrak{X}(\overline{M})_p$, and $\overline{\mathfrak{J}}_p$ is a maximal ideal.

Lemma 3.9. (1) *$\overline{\mathfrak{J}}_p$ is an infinite codimensional maximal ideal of $\mathfrak{X}(\overline{M})$ for $p\in\overline{M}_1$.*

(2) *For a maximal ideal $\mathfrak{L}$ of $J^1(M)$, put $\mathfrak{M}=(J^1)^{-1}(\mathfrak{L})$. Then $\mathfrak{M}$ is a finite codimensional maximal ideal of $\mathfrak{X}_G(M)$.*

Proof. (1) *For* $a\in\pi^{-1}(p)$, there exists $X\in\Gamma_{G_a}(\tau(V_a))$ with $X_a\neq 0$. Then there exists a G_a-invariant local one parameter group of transformations ϕ; $(-\varepsilon,\varepsilon)\times U\to V_a$ defined on a G_a-invariant neighborhood U such that $\frac{\partial\phi}{\partial t}(t,u)=X_{\phi(t,u)}$. Let θ: $(-\varepsilon,\varepsilon)\to V_a$ be a map defined by $\theta(t)=\phi(t,a)$. Since $X_a\neq 0$, θ is an embedding for a sufficiently small number ε. Let W be a G_a-invariant normal space of $\theta((-\varepsilon,\varepsilon))$ at a in V_a. Then we may assume that ϕ: $(-\varepsilon,\varepsilon)\times W\to V_a$ is a G_a-invariant embedding. Let $\{w_1,\cdots,w_{n-1}\}$ be a canonical coordinate of W. We have a local coordinate $\{x_1,\cdots,x_n\}$ of V_a around a neighborhood $U_1=\phi((-\varepsilon,\varepsilon)\times W)$ of a given by $x_1(\phi(t,w_1,\cdots,w_{n-1}))=t$, $x_i(\phi(t,w_1,\cdots,w_{n-1}))=w_{i-1}$ for $i=2,\cdots,n$. Note that $X=\frac{\partial}{\partial x_1}$ on U_1.

By Lemma 3.1. there are $X_1\in\mathfrak{X}_G(M)$ and $f\in C_G^\infty(M)$ such that $X_1=X$ and $f=x_1$ on a neighborhood $U_2\subset U_1$ of a in V_a, respectively. Let $Y\in\mathfrak{J}_a$ and $r_a(Y)=\sum_{i=1}^n\xi_i\frac{\partial}{\partial x_i}$ on U_2. Then $r_a[X_1,Y]=\left[\frac{\partial}{\partial x_1},\sum_{i=1}^n\xi_1\frac{\partial}{\partial x_i}\right]$ $=\sum_{i=1}^n\frac{\xi_i\partial}{\partial x_1}\frac{\partial}{\partial x_i}$ on U_2. Since $r_a[X_1,Y]\in\mathfrak{N}_a$, we have $\frac{\partial\xi_i}{\partial x_1}(a)=0$ for $i=1,\cdots,n$. Inductively we have $\frac{\partial^k\xi_i}{\partial x_1{}^k}(a)=0$ for $i=1,\cdots,n$ and $k=1,2,$ $\cdots$. Let α: $\mathfrak{X}_G(M)\to R[[x_1]]$ be an R-module homomorphism defined by $\alpha(Z)=\sum_{k=1}^\infty\frac{\partial^k\xi_1}{\partial x_1{}^k}(a)\,x_1{}^k$ if $r_a(Z)=\sum_{i=1}^n\xi_i\frac{\partial}{\partial x_i}$ on U. Since $\alpha(\mathfrak{J}_a)=0$, the above map α induces an R-module homomorphism $\beta: \mathfrak{X}_G(M)/\mathfrak{J}_a \to R[[x_1]]$. Note that $\alpha(f^j X_1) = j!\, x_1^j$ for $j=1, 2, \cdots$, and $\dim(\text{Image}\,\beta) = \infty$. Since $\mathfrak{J}_a \supset \operatorname{Ker} \pi_*$, we have $\dim \mathfrak{X}(\overline{M})/\overline{\mathfrak{J}}_p = \infty$.

(2) There is an index $i \in I$ such that $\mathfrak{L}$ does not contain $J_i^1(M)$. Since $\mathfrak{L}$ is a maximal ideal, $\mathfrak{L} + J_i^1(M) = J^1(M)$. Then $\mathfrak{X}_G(M)/\mathfrak{M} \cong J^1(M)/\mathfrak{L} \cong J_i^1(M)/(\mathfrak{L} \cap J_i^1(M))$. Since $J_i^1(M)$ is finite dimensional, $\mathfrak{M}$ is finite codimensional. This completes the proof of Lemma 3.9.

Proposition 3.10. *Let $\overline{\mathfrak{M}}$ be a maximal ideal of $\mathfrak{X}(\overline{M})$. Then $\overline{\mathfrak{M}} = \overline{\mathfrak{J}}_p$ for $p \in \overline{M}_1$ or $\pi_*^{-1}(\overline{\mathfrak{M}}) = (J^1)^{-1}(\mathfrak{L})$ for some maximal ideal $\mathfrak{L}$ of $J^1(M)$.*

Proposition 3.10 plays a key role to prove our theorem. We shall prove Proposition 3.10 in Section 6.

§ 4. Stone Topology of Maximal Ideals of $\mathfrak{X}(\overline{M})$

Let $\overline{M}^*$ be the set of all maximal ideals of $\mathfrak{X}(\overline{M})$. $\overline{M}^*$ is determined by Proposition 3.10.

Definition 4.1 (Stone topology of $\overline{M}^*$, cf. Pursell-Shanks [8]). The Stone topology on $\overline{M}^*$ is defined by closure operator CL as follows:

(1) $\mathrm{CL}(\phi) = \phi$.

(2) If $B \neq \phi$ is a subset of $\overline{M}^*$, then $\mathrm{CL}(B) = \{\mathfrak{M} \in \overline{M}^*; \mathfrak{M} \supset \bigcap_{\mathfrak{N} \in B} \mathfrak{N}\}$.

Let $S(\overline{M}_0)$ be the set of all subsets of $\overline{M}_0$. Let τ_M (or simply τ) be a map from $\overline{M}^*$ to $\overline{M}_1 \cup S(\overline{M}_0)$ defined as follows:

(1) $\tau(\overline{\mathfrak{J}}_p) = p$ for $p \in \overline{M}_1$.

(2) If $\mathfrak{M}$ is a maximal ideal of $\mathfrak{X}(\overline{M})$ such that $J^1(\pi_*{}^{-1}(\mathfrak{M}))$ is a maximal ideal of $J^1(M)$, then $\tau(\mathfrak{M}) = \{q_i \in \overline{M}_0;\ J^1(\pi_*{}^{-1}(\mathfrak{M}))$ does not contain $J_{q_i}^1(\overline{M})\}$. (Making use of Proposition 3.5, we see that $\tau(\mathfrak{M})$ does not consist of a single point.)

For a subset A of $\overline{M}$, we denote the closure of A in $\overline{M}$ by $\mathrm{cl}(A)$.

Lemma 4.2. *If* $\mathrm{cl}(A)$ *is contained in* $\overline{M}_1$, *then* $\mathrm{CL}(\tau^{-1}(A))$

$=\tau^{-1}(\mathrm{cl}(A))$.

Proof. First we shall prove "$\subset$". Let $\mathfrak{M}\in \mathrm{CL}(\tau^{-1}(A))$. Assume that $\tau(\mathfrak{M})$ is not contained in $\mathrm{cl}(A)$.

In the case $\mathfrak{M}=\overline{\mathfrak{J}}_p$ for $p\in\overline{M}_1$: We can find $X\in\mathfrak{X}(\overline{M})$ such that $X_p\neq 0$ and $\mathrm{supp}\, X\cap \mathrm{cl}(A)=\phi$. Then $X\in \bigcap_{p'\in A}\overline{\mathfrak{J}}_{p'}\subset\mathfrak{M}$, which is a contradiction to $X_p\neq 0$.

In the case $\pi_*{}^{-1}(\mathfrak{M})=(J^1)^{-1}(\mathfrak{L})$ for a maximal ideal $\mathfrak{L}$ of $J^1(M)$: There exists $Y\in\mathfrak{X}_G(M)$ such that $J^1(Y)\notin\mathfrak{L}$. Let $\{i_1, \cdots, i_k\}$ be a set $\{i\in I;\ j^1_{b_i}(r_{b_i}(Y))\neq 0\}$. There exists $\psi\in C^\infty_G(M)$ such that $\psi=1$ on a neighborhood of b_{i_j} $(j=1, \cdots, k)$ and $\psi=0$ on π^{-1} $(\mathrm{cl}(A))$. Put $X=\psi Y$. Then $J^1(X)=J^1(Y)$, and $\pi_*(X)\notin\mathfrak{M}$. Moreover $\pi_*(X)\in\bigcap_{p'\in A}\overline{\mathfrak{J}}_{p'}\subset\mathfrak{M}$, which is a contradiction.

Next we shall prove "$\supset$". Note that an ideal $\bigcap_{\tau(\mathfrak{N})\in A}\mathfrak{N}=\bigcap_{p'\in A}\overline{\mathfrak{J}}_{p'}$ is contained in $\mathfrak{X}(\overline{M})_p$ for any $p\in\mathrm{cl}(A)$. Then $\bigcap_{\tau(\mathfrak{N})\in A}\mathfrak{N}$ is contained in a maximal ideal $\overline{\mathfrak{J}}_p$ and $\overline{\mathfrak{J}}_p\in\mathrm{CL}(\tau^{-1}(A))$ for any $p\in\mathrm{cl}(A)$. This completes the proof of Lemma 4.2.

If Φ: $\mathfrak{X}(\overline{M})\to\mathfrak{X}(\overline{N})$ is a Lie algebra isomorphism, then $\Phi^*: \overline{M}^*\to\overline{N}^*$ is homeomorphic. Combining Lemma 3.9 and Proposition 3.10, $\Phi^*(\tau_M{}^{-1}(\overline{M}_1))=\tau_N{}^{-1}(\overline{N}_1)$ and $\Phi^*(\tau_M{}^{-1}(\overline{M}_0))=\tau_N{}^{-1}(\overline{N}_0)$. By Lemma 4.2, we have

Corollary 4.3. *If Φ: $\mathfrak{X}(\overline{M})\to\mathfrak{X}(\overline{N})$ is a Lie algebra isomorphism, there exists a homeomorphism σ: $\overline{M}_1\to\overline{N}_1$ defined by $\sigma(p)=\tau_N(\Phi^*(\overline{\mathfrak{J}}_p))$.*

We shall extend the homeomorphism σ: $\overline{M}_1\to\overline{N}_1$ to a homeomorphism from $\overline{M}$ to $\overline{N}$.

Lemma 4.4. *Let U be a neighborhood of $q\in\overline{M}_0$ such that* $\mathrm{cl}(U)\cap\overline{M}_0=\{q\}$. *Then* $\mathrm{CL}(\tau^{-1}(U))=\tau^{-1}(\mathrm{cl}(U))$.

The proof of Lemma 4.4 is similar to that of Lemma 4.2.

Proposition 4.5. *The map $\sigma:\overline{M}_1\to\overline{N}_1$ is extended to a homeomorphism from $\overline{M}$ to $\overline{N}$.*

Proof. For $q\in\overline{M}_0$, let $b\in M$ such that $\pi(b)=q$. Let V_b be a linear slice at b. Put $U_q=V_b/G_b$, $U_q^0=U_q-\{q\}$. Since $q\in\overline{M}_0$, it is easy to see that U_q^0 is connected. Note that $\bigcap_{\tau(\mathfrak{N})\in U_q^0}\mathfrak{N}=\bigcap_{\tau(\mathfrak{N})\in U_q}\mathfrak{N}$. By Lemma 4.4,

$$\begin{aligned}\mathrm{CL}(\tau_M^{-1}(U_q^0))&=\mathrm{CL}(\tau_M^{-1}(U_q))\\&=\tau_M^{-1}(\mathrm{cl}(U_q))=\tau_M^{-1}(\mathrm{cl}(U_q^0)).\end{aligned}$$

Since $\Phi^*:\overline{M}^*\to\overline{N}^*$ is homeomorphic and since $\sigma\circ\tau_M=\tau_N\circ\Phi^*$,

$$\begin{aligned}\mathrm{CL}(\tau_N^{-1}(\sigma(U_q^0)))&=\mathrm{CL}(\Phi^*(\tau_M^{-1}(U_q^0)))\\&=\Phi^*(\mathrm{CL}(\tau_M^{-1}(U_q^0)))=\Phi^*(\tau_M^{-1}(\mathrm{cl}(U_q))).\end{aligned}$$

There exists a maximal ideal $\mathfrak{M}\in\overline{M}^*$ such that $\tau_M(\mathfrak{M})=q$. Then $\tau_N(\mathrm{CL}(\tau_N^{-1}(\sigma(U_q^0))))\cap\overline{N}_0$ contains $\tau_N(\Phi(\mathfrak{M}))$, and, by Lemma 4.2, $\mathrm{cl}(\sigma(U_q^0))\cap\overline{N}_0\neq\phi$. Since $\sigma(U_q^0)$ is connected, $\mathrm{cl}(\sigma(U_q^0))\cap\overline{N}_0=\{q'\}$ for some $q'\in\overline{N}_0$. Let $\sigma(q)=q'$ for $q\in\overline{M}_0$. Then it is clear that σ: $\overline{M}\to\overline{N}$ is homeomorphic.

§ 5. Proof of Theorem

In this section we shall prove our theorem. Let Φ: $\mathfrak{X}(\overline{M})\to\mathfrak{X}(\overline{N})$ be a Lie algebra isomorphism. By Proposition 4.5, we have a homeomorphism σ: $\overline{M}\to\overline{N}$ such that $\sigma(p)=\tau_N(\Phi(\overline{\mathfrak{J}}_p))$ for $p\in\overline{M}_1$.

Proposition 5.1. *σ: $\overline{M}_1\to\overline{N}_1$ is diffeomorphic.*

In order to prove Proposition 5.1, we need the following lemma.

Lemma 5.2. *For $X\in\mathfrak{X}(\overline{M})$ and $p\in\overline{M}_1$, $X_p\neq 0$ if and only if $[X,\mathfrak{X}(\overline{M})]+\overline{\mathfrak{J}}_p=\mathfrak{X}(\overline{M})$.*

Proof. Let $a\in M$ such that $\pi(a)=p$. Assume that $X_p\neq 0$. There exists $\widehat{X}\in\mathfrak{X}_G(M)$ such that $\widehat{X}_a\neq 0$. By the similar argument as the proof of Lemma 3.7, we can prove that $[\widehat{X}, \mathfrak{X}_G(M)]+\mathfrak{J}_a=\mathfrak{X}_G(M)$, and $[X, \mathfrak{X}(\overline{M})]+\overline{\mathfrak{J}}_p=\mathfrak{X}(\overline{M})$. Conversely, suppose that $X_p=0$. Moreover we shall assume that $[X, \mathfrak{X}(\overline{M})]+\overline{\mathfrak{J}}_p=\mathfrak{X}(\overline{M})$. There exists $\widehat{X}\in\mathfrak{X}_G(M)$ such that $\pi_*(\widehat{X})=X$. Put $\widehat{X}'=r_a(\widehat{X})\in\Gamma_{G_a}(\tau(V_a))$. Then $[\widehat{X}', \Gamma_{G_a}(\tau(V_a))]+\mathfrak{N}_a=\Gamma_{G_a}(\tau(V_a))$ and $\widehat{X}'_a=0$. Let $V_a(0)$ and $V_a(1)$ be the trivial and non-trivial direct summand of the G_a-module V_a, respectively. Then $j^1_a(\widehat{X}')$ can be expressed as $A\oplus B$, where $A\in\mathfrak{gl}(V_a(0))$ and $B\in\mathfrak{gl}_{G_a}(V_a(1))$. Since $[\widehat{X}', \Gamma_{G_a}(\tau(V_a))]+\mathfrak{N}_a=\Gamma_{G_a}(\tau(V_a))$, we can prove that A is invertible. Then we have $[\widehat{X}', \Gamma_{G_a}(\tau(V_a))_0]+\mathfrak{N}_a=\Gamma_{G_a}(\tau(V_a))_0$, which implies that a linear mapping

$$\beta\colon \mathfrak{gl}_{G_a}(V_a)/j^1_a(\mathfrak{N}_a)\to\mathfrak{gl}_{G_a}(V_a)/j^1_a(\mathfrak{N}_a),$$

defined by $\beta(C+j^1_a(\mathfrak{N}_a))=[j^1_a(\widehat{X}'), C]+j^1_a(\mathfrak{N}_a)$ for $C\in\mathfrak{gl}_{G_a}(V_a)$, is isomorphic. But this is impossible because $j^1_a(\widehat{X}')\in j^1_a(\mathfrak{N}_a)$. This completes the proof of Lemma 5.2.

Proof of Proposition 5.1. Let f be any smooth function on $\overline{N}$. Put $g=f\circ\sigma$. We have $fY-f(\sigma(p))Y\in\mathfrak{X}(\overline{N})_{\sigma(p)}$ for any $Y\in\mathfrak{X}(\overline{N})$, $p\in\overline{M}_1$ and hence, using Lemma 5.2, $\Phi^{-1}(fY)-g(p)\Phi^{-1}(Y)\in\mathfrak{X}(\overline{M})_p$ for any $p\in\overline{M}_1$. Thus we have $\Phi^{-1}(fY)=g\Phi^{-1}(Y)$ for any $Y\in\mathfrak{X}(\overline{N})$. For any $p\in\overline{M}_1$, there exist $Y\in\mathfrak{X}(\overline{N})$ and $h\in C^\infty(\overline{M})$ such that $\Phi^{-1}(Y)(h)\neq0$ on a neighborhood U of p in $\overline{M}$. Then $g=\Phi^{-1}(fY)(h)(\Phi^{-1}(Y)(h))^{-1}$ on U, and g is smooth on U. Thus $f\circ\sigma$ is smooth on $\overline{M}_1$ for any $f\in C^\infty(\overline{N})$, and σ is smooth on $\overline{M}_1$. Similarly σ^{-1} is smooth on $\overline{N}_1$, and Proposition 5.1 follows.

Now we shall prove that $\sigma\colon\overline{M}\to\overline{N}$ is diffeomorphic. By Proposition 5.1, it is sufficient that σ is smooth at $q\in\overline{M}_0$. Let f be any smooth function on $\overline{N}$, and put $g=f\circ\sigma$. As in the proof of Proposition 5.1, $g\Phi^{-1}(Y)=\Phi^{-1}(fY)$ for any $Y\in\mathfrak{X}(\overline{N})$. Since Φ is a Lie algebra isomorphism, $gX(h)\in C^\infty(\overline{M})$ for any $X\in\mathfrak{X}(\overline{M})$, $h\in C^\infty(\overline{M})$. Let b be a point of M such that $\pi(b)=q$, and let V be a linear slice at b. Let H be the isotropy subgroup at b, and put $\overline{V}=V/H$. It suffices to prove the following:

Proposition 5.3. *Let g be a continuous function on $\overline{V}$ such that*

$gX(h)\in C^\infty(\overline{V})$ for any $X\in\Gamma(\tau(\overline{V}))$, $h\in C^\infty(\overline{V})$. Then g is a smooth function.

Let $\{\theta_1, \cdots, \theta_s\}$ be a minimal set of homogeneous generators of $R[V]_0^H$ (see Davis [3], Lemma 4.6). Let $\{x_1,\cdots, x_n\}$ be a canonical coordinate of V such that H acts orthogonally on this coordinate. Since $q\in\overline{M}_0$, $\Gamma(\tau(\overline{V}))=\Gamma(\tau(\overline{V}))_0$. It is easy to see that $\deg\theta_i>1$ for $i=1, \cdots, s$. Then we can assume $\theta_1=x_1^2+\cdots+x_n^2$. Let X be a radial vector field $\sum_{i=1}^n x_i\frac{\partial}{\partial x_i}$. Then $X(\theta_i)=(\deg\theta_i)\theta_i$ for $i=1, \cdots, s$, and Proposition 5.3 follows from the following:

Proposition 5.4. *Let g be an H-invariant continuous function on V such that $\theta_i g\in C_H^\infty(V)$ for $i=1, \cdots, s$. Then g is an H-invariant smooth function.*

Proof. Since $\theta_1 g\in C_H^\infty(V)$, it is sufficient to prove that g is smooth at 0. Put $g_1=\theta_1 g\in C_H^\infty(V)$. First we consider the case $s=1$. In this case $\overline{V}$ is a half line R_+, it follows from Theorem 1.2 that $\bar{g}_1=x\bar{g}$ is a smooth function on R_+, where $\bar{g}_1$ and $\bar{g}$ are functions on R_+ such that $g_1=\bar{g}_1\circ\theta_1$ and $g=\bar{g}\circ\theta_1$, respectively. By Koriyama [5] Lemma 6.2, $\bar{g}\in C^\infty(R_+)$, and hence $g\in C_H^\infty(V)$.

Now we consider the case $s\geqq 2$. Put $R[V]_i^H=\{h\in R[V]^H;\ \deg h=i\}$. From Taylor's formula, for an integer $m\geqq 2$, there exist $P_m(x)\in\sum_{2\leqq i\leqq m}R[V]_i^H$ and $R_m(x)\in\sum_{|I|=m+1}x^I C^\infty(V)$ such that $g_1(x)=P_m(x)+R_m(x)$, where $x^I=x_1^{i_1}\cdots x_n^{i_n}$ and $|I|=i_1+\cdots+i_n$. Put $g_2=\theta_2\cdot g_1$, $k=\deg\theta_2+m$. Then $\theta_2 P_m\in\sum_{2\leqq i\leqq k}R[V]_i^H$ and $\theta_2 R_m(x)\in\sum_{|I|=k+1}x^I C^\infty(V)$. Let $\Delta=\sum_{i=1}^n\frac{\partial^2}{\partial x_i^2}$ be the Laplacian, and put $H^k(V)=\{f\in R[V]_k;\ \Delta f=0\}$. Here we need the following result:

Theorem 5.5 (cf. [13] § 14).

$$R[V]_i=\theta_1 R[V]_{i-2}\oplus H^i(V) \text{ for an integer } i\geqq 2.$$

From Theorem 5.5, there exist $Q_m\in\sum_{i\leqq k-2}R[V]_i$ and $T_m\in\sum_{i\leqq k}H^i(V)$ such that $\theta_2 P_m=\theta_1 Q_m+T_m$.

Proposition 5.6. $T_m=0$ $(m=2, 3, \cdots)$.

Proof. It is easy to see the following:

(1) For $f\in H^l(V)$, $\Delta^p(f/\theta_1)=N(p, l)f/\theta_1^{p+1}$, where $N(p, l)=-[4pl+2pn-4(4(p-1)^2+12(p-1)+8)]$.

(2) For $F_I\in C^\infty(V)$ with $|I|=k+1$, put $F(x)=\sum_{|I|=k+1}F_I(x)x^I$. Then we have $\Delta^p(F/\theta_1)=\sum_{q=1}^{p+1}F^{p,q}/\theta_1^q$ for some $F^{p,q}(x)\in\sum_{|I|=k+2q-2p-1}x^IC^\infty(V)$.

Now we assume that $T_m=\sum_{i=d}^k T_m{}^i$ such that $T_m{}^i\in H^i(V)$ and $T_m{}^d\neq0$. Let d_1 be an integer such that $d=2d_1+1$ or $2d_1+2$. To see Proposition 5.5, we can assume that k is an even integer, and hence $k\geqq 2d_1+2$. By the definitions of g_1 and g_2, $\theta_2 g=g_2/\theta_1=Q_m+T_m/\theta_1+\theta_2R_m/\theta_1$. Since $\theta_2 g$ is a smooth map, $T_m/\theta_1+\theta_2R_m/\theta_1$ is also a smooth map.

Put $F=\theta_2R_m$. Applying (2), we have $\Delta^{d_1}(F/\theta_1)=\sum_{q=1}^{d_1+1}F^{d_1,q}/\theta_1^q$ for some $F^{d_1,q}(x)\in\sum_{|I|=k+2q-2d_1-1}\in x^IC^\infty(V)$. Let $a=(a_1, \cdots, a_n)\in V$ with $a\neq0$ and ξ be a positive real number. Since $k-2d_1-1>0$, it is easy to see that

(3) $$\lim_{\xi\to0}\Delta^{d_1}(F/\theta_1)|_{x=\xi a}=0.$$

It follows from (1) that

(4) $$\lim_{\xi\to0}\Delta^{d_1}(T_m{}^i/\theta_1)|_{x=\xi a}=0 \quad \text{for} \quad i\geqq 2d_1+2.$$

We can write

$$T_m{}^{2d_1+1}(x)=\sum_{|I|=2d_1+1}\lambda_I x^I \quad \text{for} \quad \lambda_I\in R,$$

$$T_m{}^{2d_1+2}(x)=\sum_{|J|=2d_1+2}\mu_J x^J \quad \text{for} \quad \mu_J\in R,$$

Then $\Delta^{d_1}(T_m{}^{2d_1+1}(x)/\theta_1+T_m{}^{2d_1+2}(x)/\theta_1)|_{x=\xi a}=(N(d_1, 2d_1+1)\sum\lambda_I a_I)/(\xi\|a\|^{2d_1+2})+(N(d_1, 2d_1+2)\sum\mu_J a_J)/\|a\|^{2d_1+2}$. Note that $N(d_1, 2d_1+1)\neq0$, $N(d_1, 2d_1+2)\neq0$. Since $T_m/\theta_1+F/\theta_1$ is smooth, it follows from (3) that the limit $\lim_{\xi\to0}\Delta^{d_1}(T_m/\theta_1)|_{x=\xi a}$ exists. From (4) we have $\lambda_I=\mu_J=0$ for any I, J. Then $T_m{}^d=0$, which is a contradiction. Therefore $T_m=0$.

Proof of Proposition 5.4 *continued.* From Proposition 5.6, we have $\theta_2P_m=\theta_1Q_m$. Since $\{\theta_1, \cdots, \theta_s\}$ is a minimal set of generators, there exists an H-invariant polynomial P_m' such that $P_m=\theta_1P_m'$. Then $g=g_1/\theta_1=P_m'+R_m/\theta_1$ for $R_m(x)\sum_{|I|=m+1}x^IC^\infty(V)$ $(m=2, 3, \cdots)$, and g is a smooth map. This completes the proof of Proposition 5.4.

To complete the proof of our theorem, we shall prove that $\Phi=\sigma_*$. Similar way as in the proof of Proposition 5.1, for any $f\in C^\infty(M)$, $X\in\mathfrak{X}(\overline{M})$, we have $\Phi(fX)=(f\circ\sigma^{-1})\Phi(X)$. Then $\Phi(X)(f\circ\sigma^{-1})\Phi(X)$ $=\Phi(X(f)X)=\Phi([X,fX])=[\Phi(X),\Phi(fX)]=[\Phi(X),(f\circ\sigma^{-1})\Phi(X)]$ $=\Phi(X)(f\circ\sigma^{-1})\Phi(X)$. Hence $\Phi(X)(f\circ\sigma^{-1})\Phi(X)=\Phi(X)(f\circ\sigma^{-1})\Phi(X)$, and we see $\Phi(X)(f\circ\sigma^{-1})=X(f)\circ\sigma^{-1}$. Then, for any $g\in C^\infty(\overline{N})$, $X\in\mathfrak{X}(\overline{M})$, $\Phi(X)(g)=X(g\circ\sigma)\circ\sigma^{-1}=\sigma_*(X)(g)$, and hence $\Phi=\sigma_*$. This completes the proof of our theorem.

Remark. We can prove that σ is strata preserving.

§ 6. Proof of Proposition 3.10

In this section we shall prove Proposition 3.10. The proof is parallel to those of Pursell-Shanks [8] and Koriyama [5]. We start with some lemmas.

Lemma 6.1 (Sternberg [12]). *Let X be a radial vector field on R^n defined by $X=\sum_{i=1}^n x_i\frac{\partial}{\partial x_i}$. Let Y be a smooth vector field on R^n such that $j_0^1(X)=j_0^1(Y)$. Then there exists a local coordinate system $(y_1,\cdots,y_n)$ defined on a neighborhood U of 0 such that $Y=\sum_{i=1}^n y_i\frac{\partial}{\partial y_i}$ on U.*

Lemma 6.2. *Let $\overline{\mathfrak{M}}$ be an ideal of $\mathfrak{X}(\overline{M})$ such that for any $p\in\overline{M}_1$ there exists $\overline{Y}\in\overline{\mathfrak{M}}$ such that $\overline{Y}_p\neq 0$. Then $\overline{\mathfrak{M}}$ contains an ideal $\overline{\mathfrak{J}}_1=\{\overline{X}\in\mathfrak{X}(\overline{M})\,;\operatorname{supp}\overline{X}\subset\overline{M}_1\}$.*

Proof. Put $\mathfrak{M}=\pi_*^{-1}(\overline{\mathfrak{M}})$, $\mathfrak{J}_1=\pi_*^{-1}(\overline{\mathfrak{J}}_1)$. We shall prove that $\mathfrak{M}$ contains $\mathfrak{J}_1$. Similarly as in the proof of Lemma 3.9, for any point $a\in\pi^{-1}(\overline{M}_1)$ there exist a local coordinate system $(x_1,\cdots,x_n)$ around a G_a-invariant neighborhood U_a of a in V_a and $Y\in\mathfrak{M}$ such that $r_a(Y)=\frac{\partial}{\partial x_1}$ on U_a. x_1 is extended to a G-invariant smooth function f on M.

Put $Y_1=Y-r_a(Y)$ on V_a. Then $\pi_*(Y_1)=0$ on $\overline{V}_a$. Using a partition of unity, we can find $Y_2\in\operatorname{Ker}\pi_*\subset\mathfrak{M}$ such that $Y_2=Y_1$ on U_a. Put $Y_3=Y-Y_2$. Then $Y_3=r_a(Y)=\frac{\partial}{\partial x_1}$ on U_a and $Y_3\in\mathfrak{M}$.

Let $X\in\mathfrak{J}_1$. To prove $X\in\mathfrak{M}$ we can assume that supp X is contained in $G\times_{G_a}V_a$ by using arguments of invariant partition of unity. Moreover, we can assume that $r_a(X)=X$ on U_a since $\operatorname{Ker}\pi_*$ is contained in $\mathfrak{M}$. Put $X=\sum_{i=1}^n\xi_i\frac{\partial}{\partial x_i}$ on U_a. Since x_1 is a G_a-invariant function, we see that ξ_1 is a G_a-invariant function. Similarly as in the proof of Koriyama [5] Lemma 2.13, we can prove that X is an element of $\mathfrak{M}$. This completes the proof of Lemma 6.2.

Put $\mathfrak{X}_G(M)_0^\infty=\bigcap_{i\in I}r_{b_i}{}^{-1}(\Gamma_{G_{b_i}}(\tau(V_{b_i}))_0^\infty)$.

Lemma 6.3. *Let $\overline{\mathfrak{M}}$ be an ideal of $\mathfrak{X}(\overline{M})$ such that $J^1(\pi_*{}^{-1}(\overline{\mathfrak{M}}))=J^1(M)$ and for any $p\in\overline{M}_1$ there exists an element $\overline{Y}\in\overline{\mathfrak{M}}$ such that $\overline{Y}_p\neq0$. Then an ideal $\mathfrak{M}=\pi_*{}^{-1}(\overline{\mathfrak{M}})$ of $\mathfrak{X}_G(M)$ contains an ideal $\mathfrak{X}_G(M)_0^\infty$.*

Proof. Since $J^1(\mathfrak{M})=J^1(M)$, there exists an element $X\in\mathfrak{M}$ such that $j^1_{b_i}(r_{b_i}(X))=j^1_{b_i}(\mathfrak{R}_i)$. Here $\mathfrak{R}_i$ is a radial vector field on V_{b_i} defined by $\mathfrak{R}_i=\sum_{j=1}^n y_j\frac{\partial}{\partial y_j}$ on V_{b_i}, where $\{y_1,\cdots,y_n\}$ is a canonical coordinate of V_{b_i}. Since $\mathfrak{M}$ contains $\operatorname{Ker}\pi_*$, we can assume that $X=\sum_{j=1}^n y_j\frac{\partial}{\partial y_j}$ on U_i.

As in the proof of Koriyama [5] Lemma 2.13, for any element $Z\in\mathfrak{X}_G(M)_0^\infty$, there exist vector fields Y_i, $i\in I$, such that $[X,Y_i]=r_{b_i}(Z)$ on a G_{b_i}-invariant neighborhood $W_i\subset U_i$ of b_i and supp $Y_i\subset U_i$. Put $\widetilde{Y}_i=\int_{G_{b_i}}g_*Y_i dg\in\Gamma_{G_{b_i}}(\tau(V_{b_i}))$. By Lemma 3.6, we have $[X,\widetilde{Y}_i]=r_{b_i}(Z)$ on W_i. By Lemma 3.1 (1), we can assume $\widetilde{Y}_i\in\mathfrak{X}_G(M)$ and supp $\widetilde{Y}_i$ is contained in $G\times_{G_{b_i}}U_i$. Since supp Z is compact, there is a finite index set $\{i_1,\cdots,i_k\}\subset I$ such that supp $Z\cap G\times_{G_{b_{i_j}}}V_{b_{i_j}}\neq\emptyset$. Put $\widetilde{Y}=\widetilde{Y}_{i_1}+\cdots+\widetilde{Y}_{i_k}$. Since $\operatorname{Ker}\pi_*$ is contained in $\mathfrak{M}$, there exists an element $Z_0\in\mathfrak{M}$ such that $Z_0=Z-r_{b_{i_j}}(Z)$ on W_{i_j} for $j=1,\cdots,k$. Then $Z_1=Z-Z_0-[X,\widetilde{Y}]$ is an element of $\mathfrak{J}_1$ in Lemma 6.2, and $Z_1\in\mathfrak{M}$. Thus we have $Z\in\mathfrak{M}$, and this completes the proof of Lemma 6.3.

Lemma 6.4. *Let* $\overline{\mathfrak{M}}$ *be a maximal ideal of* $\mathfrak{X}(\overline{M})$ *such that for any* $p\in\overline{M}_1$ *there exists an element* $\overline{Y}\in\overline{\mathfrak{M}}$ *such that* $\overline{Y}_p\neq 0$. Then $J^1(\mathfrak{M})$ *is a maximal ideal of* $J^1(M)$, where $\mathfrak{M}=\pi_*^{-1}(\overline{\mathfrak{M}})$.

Proof. Suppose $J^1(\mathfrak{M})=J^1(M)$. We shall prove that $\mathrm{Ker}\, J^1$ is contained in $\mathfrak{M}$. By Lemma 6.2, it is enough to prove that an element $Z\in\mathrm{Ker}\, J^1$ satisfying $\mathrm{supp}\, Z\subset G\times_{G_{b_i}}V_{b_i}$ is an element of $\mathfrak{M}$. Since $\mathrm{Ker}\,\pi_*$ is contained in $\mathfrak{M}$, we can assume $Z=r_{b_i}(Z)$ on V_{b_i}. As in the proof of Lemma 6.3, there exist $X\in\mathfrak{M}$ and a local coordinate system $(x_1,\cdots,x_n)$ defined on a G_{b_i}-invariant neighborhood U_i of b_i in V_{b_i} such that $X=\sum_{j=1}^n x_i\dfrac{\partial}{\partial x_i}$ on U_i.

From the proof of Koriyama [5] Lemma 2.10, there exists a smooth vector field Y on V_{b_i} such that $Z_1=Z-[X,Y]\in\Gamma(\tau(V_{b_i}))_0^\infty$. Put $\tilde{Y}=\int_{G_{b_i}}g_*Ydg$. Then $Z-[X,\tilde{Y}]=\int_{G_{b_i}}g_*Z_1dg\in\Gamma_{G_{b_i}}(\tau(V_{b_i}))_0^\infty$. By Lemma 3.1 (1), we can assume $\tilde{Y}\in\mathfrak{X}_G(M)$, and $Z-[X,\tilde{Y}]\in\mathfrak{X}_G(M)_0^\infty$. Then it follows from Lemma 6.3 that $Z\in\mathfrak{M}$. Thus $\mathrm{Ker}\, J^1$ is contained in $\mathfrak{M}$. Since $J^1(\mathfrak{M})=J^1(M)$, $\mathfrak{M}=\mathfrak{X}_G(M)$. This is a contradiction, and this completes the proof of Lemma 6.4.

Proof of Proposition 3.10. Let $\overline{\mathfrak{M}}$ be a maximal ideal of $\mathfrak{X}(\overline{M})$. If there exists a point $p\in\overline{M}_1$ such that $\overline{\mathfrak{M}}$ is contained in $\mathfrak{X}(\overline{M})_p$, then $\overline{\mathfrak{M}}=\overline{\mathfrak{J}}_p$. Suppose for any point $p\in\overline{M}_1$ there exists an element $\overline{X}\in\overline{\mathfrak{M}}$ such that $\overline{X}_p\neq 0$. By Lemma 6.4, $J^1(\pi_*^{-1}(\overline{\mathfrak{M}}))$ is a maximal ideal of $J^1(M)$. This completes the proof of Proposition 3.10.

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