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Representation Theorems of Cohomology on Weakly 1-Complete Manifolds

By

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Contents

Introduction

Vanishing, finiteness and harmonic representation of cohomology on complex manifolds are one of the most interesting problems in analytic geometry (for example, see [2], [5], [9], [13], [19], [20], [38], [45]), and here we are concerned especially with cohomology on noncompact complex manifolds. Namely, in this paper, we deal with cohomology groups with coefficients in a locally free sheaf of rank one on domains with pseudoconvex boundaries and weakly 1-complete manifolds. We say

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that a complex manifold X is weakly 1-complete if there exists a C^∞-exhausting plurisubharmonic function on X. S. Nakano pointed out the importance of the concept in [31] (or [11]) by solving the inverse problem of monoidal transformations and the main point was some vanishing theorem of cohomology groups for positive line bundles (see [32], [33]). Then the theorem was generalized in various aspects (see [1], [12], [37], [40], [41], [42], [43], [44]). Recently, in [34], [35] and [37] T. Ohsawa showed the finiteness and isomorphic theorems of cohomology groups for line bundles on weakly 1-complete manifolds which are positive outside compact subsets. In these articles, the main method was the $\bar{\partial}$-operator theory without boundary condition, which was originated by Andreotti and Vesentini's works [3], [4] and [5], while in this paper, we want to make use of the $\bar{\partial}$-operator theory with boundary condition, which was studied by Kohn and Hörmander (for example, see [9], [17], [22], [23], [24], etc.). But of course we have to recreate from these authors because of weak pseudoconvexity of our domain. Hence we present, in Chapter I, some calculation i.e. a complex tensor calculus for Kähler manifolds with boundary. Although it is more or less a routine calculation, it is worth to do so because, in the opinion of the author, the complex tensor calculus is one of the important methods to treat the cohomology groups from differential geometric view point. The result is a simple generalization of a formula for compact complex Kähler manifolds due to K. Kodaira [20] and is more or less known.

The main purpose of this paper is to make Ohsawa's results more exact in the following sense. When we let X be a weakly 1-complete manifold with a C^∞-exhausting plurisubharmonic function Φ on X, we shall show that for a line bundle B on X which is positive outside a compact subset K of X, the p-th cohomology group $H^p(X_c, \mathcal{O}(B\otimes K_X))$ $(p\geqq 1)$ is finite dimensional and represented by the space of harmonic forms which are obtained as the harmonic part of $B\otimes K_X$-valued differential forms of type $(0,p)$ being smooth up to the boundary ∂X_c (see Theorem 3.9). Here we write $X_c=\{x\in X|\Phi(x)<c\}$ and $\partial X_c=\{x\in X|\Phi(x)=c\}$ for some non-critical value c of Φ with $c>\sup_{x\in K}\Phi(x)$ and K_X is the canonical line bundle of X. This seems to be the analogy of the case of compact or q-convex complex manifolds (see [9], [17], [19]).

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Chapter I. Differential Analysis on Complex Manifolds and Functional Analysis

§ 1. 1. A Complex Tensor Calculus for Kähler Manifolds with Boundary

Let M be an n-dimensional complex manifold and let X be a relatively compact domain on M with smooth boundary ∂X. In this paper, this means that there exist a neighborhood W of ∂X and a real valued function Φ of class C^∞ on W such that i) $X \cap W = \{x \in W | \Phi(x) < 0\}$, ii) the gradient of Φ does not vanish on ∂X. From now on, we assume that M is provided with a kähler metric ds^2. Let $\{U_i\}$ be a coordinate cover of M and let $(z_i^1, \cdots, z_i^n)$ be local coordinates on U_i. We use the notation $\partial_{i,\alpha} = \dfrac{\partial}{\partial z_i^\alpha}$ and $\partial_{i,\bar\alpha} = \dfrac{\partial}{\partial \bar z_i^\alpha}$. (We sometimes omit i for simplicity.) We set

$$ds^2 = \sum_{\alpha,\beta=1}^{n} g_{i,\alpha\bar\beta} dz_i^\alpha d\bar z_i^\beta . \tag{1.1}$$

The Kähler form is defined by

$$\Omega = \sqrt{-1} \sum_{\alpha,\beta=1}^{n} g_{i,\alpha\bar\beta} dz_i^\alpha \wedge d\bar z_i^\beta .$$

With respect to this metric, we can define a connection $\{\omega_i\}$, $\omega_i = (\omega_{i,}{}_\alpha^\beta)$ for the holomorphic tangent bundle TM on M:

$$\omega_{i,}{}_\alpha^\beta = \sum_{\gamma=1}^{n} \Gamma_{i,}{}_{\gamma\alpha}^\beta dz_i^\gamma \quad \text{where} \quad \Gamma_{i,}{}_{\gamma\alpha}^\beta = \sum_{\sigma=1}^{n} g_i^{\bar\sigma\beta} \partial_\gamma g_{i,\alpha\bar\sigma} . \tag{1.2}$$

The Riemann curvature tensor is defined by

$$R_{i,}{}_{\beta\bar\nu\lambda}^\alpha = \partial_{\bar\nu} \Gamma_{i,}{}_{\lambda\beta}^\alpha \tag{1.3}$$

and also we set

$$R_{i,\bar\alpha\beta\bar\nu\lambda} = \sum_{\mu=1}^{n} g_{i,\mu\bar\alpha} R_{i,}{}_{\beta\bar\nu\lambda}^\mu . \tag{1.4}$$

As for the conjugates of the above, we define

$$(1.5)\qquad \Gamma_{i,\bar{\beta}\bar{\gamma}}^{\bar{\alpha}}=\overline{\Gamma_{i,\beta\gamma}^{\alpha}},\quad R_{i,\bar{\beta}\nu\bar{\lambda}}^{\bar{\alpha}}=\overline{R_{i,\beta\bar{\nu}\lambda}^{\alpha}}\quad\text{and}\quad R_{i,\alpha\bar{\beta}\nu\bar{\lambda}}=\overline{R_{i,\bar{\alpha}\beta\bar{\nu}\lambda}}\,.$$

The Ricci curvature is defined by

$$(1.6)\qquad R_{i,\bar{\nu}\lambda}=\sum_{\beta=1}^{n}R_{i,\beta\bar{\nu}\lambda}^{\beta}\,.$$

Since ds^2 is a Kähler metric, in the same manner as in [21], p. 109, Theorem 5. 1 and p. 117, Proposition 6. 2, we have

$$(1.7)\qquad \Gamma_{i,\beta\gamma}^{\alpha}=\Gamma_{i,\gamma\beta}^{\alpha}$$

and

$$(1.8)\qquad R_{i,\bar{\alpha}\beta\bar{\nu}\lambda}=R_{i,\bar{\alpha}\lambda\bar{\nu}\beta}=R_{i,\bar{\nu}\beta\bar{\alpha}\lambda}=R_{i,\bar{\nu}\lambda\bar{\alpha}\beta}\,.$$

As in [21], p. 111–112 and p. 118, Proposition 6. 4, it is easily verified that

$$(1.9)\qquad \sum_{\lambda=1}^{n}\Gamma_{i,\lambda\alpha}^{\lambda}=\partial_{\alpha}\log g_i$$

and

$$(1.10)\qquad R_{i,\bar{\nu}\lambda}=\partial_{\bar{\nu}}\partial_{\lambda}\log g_i\quad\text{where}\quad g_i=\det(g_{i,\alpha\bar{\beta}})\,.$$

Let $C^{\infty}(TM)$ (resp. $C^{\infty}(\overline{TM})$) be the sheaf of germs of the C^{∞}-sections of the holomorphic tangent bundle TM (resp. conjugate tangent bundle $\overline{TM}$) and let $\Gamma(M, C^{\infty}(TM))$ (resp. $\Gamma(M, C^{\infty}(\overline{TM}))$) be the space of global sections of $C^{\infty}(TM)$ (resp. $C^{\infty}(\overline{TM})$).

We define covariant differentiation $\nabla_{\alpha}, \nabla_{\bar{\beta}}$ induced by (2) on $\Gamma(M, C^{\infty}(TM))$ and $\Gamma(M, C^{\infty}(\overline{TM}))$ as follows:

$$(1.11)\qquad \left.\begin{aligned}\nabla_{i,\lambda}\xi_i^{\alpha}&=\partial_{\lambda}\xi_i^{\alpha}+\sum_{\beta=1}^{n}\Gamma_{i,\lambda\beta}^{\alpha}\xi_i^{\beta}\\ \nabla_{i,\bar{\lambda}}\xi_i^{\alpha}&=\partial_{\bar{\lambda}}\xi_i^{\alpha}\end{aligned}\right\}\quad\text{for}\quad \xi=\sum_{\alpha=1}^{n}\xi_i^{\alpha}\Big(\frac{\partial}{\partial z_i^{\alpha}}\Big)\in\Gamma(M, C^{\infty}(TM)),$$

$$\left.\begin{aligned}\nabla_{i,\lambda}\eta_i^{\bar{\alpha}}&=\partial_{\lambda}\eta_i^{\bar{\alpha}}\\ \nabla_{i,\bar{\lambda}}\eta_i^{\bar{\alpha}}&=\partial_{\bar{\lambda}}\eta_i^{\bar{\alpha}}+\sum_{\beta=1}^{n}\Gamma_{i,\bar{\lambda}\bar{\beta}}^{\bar{\alpha}}\eta_i^{\bar{\beta}}\end{aligned}\right\}\quad\text{for}\quad \eta=\sum_{\beta=1}^{n}\eta_i^{\bar{\alpha}}\Big(\frac{\partial}{\partial\bar{z}_i^{\alpha}}\Big)\in\Gamma(M, C^{\infty}(\overline{TM})).$$

Then we remark that

$$(1.12)\qquad \nabla_{i,\lambda}g_{i,\alpha\bar{\beta}}=0\,,\quad \nabla_{i,\lambda}g_i^{\bar{\beta}\alpha}=0\,.$$

For any open subset Y of M, let $C^{p,q}(Y)$ be the space of differential forms of type (p, q) and of class C^∞ on Y and let $C_0^{p,q}(Y)$ be the space of the forms in $C^{p,q}(Y)$ with compact supports. Let $C^{p,q}(\overline{X})$ be the image of the restriction mapping from $C^{p,q}(M)$ to $C^{p,q}(X)$. We denote the length of the gradient of Φ with respect to (1.1) by $|\mathrm{grad}\,\Phi|$, then $|\mathrm{grad}\,\Phi|^2 = \sum_{\alpha,\beta=1}^{n} g_i^{\bar\beta\alpha}\partial_\alpha\Phi\partial_{\bar\beta}\Phi$ and, from the hypothesis, we may assume that $|\mathrm{grad}\,\Phi|>0$ on a neighborhood W of ∂X. We define a function f of class C^∞ on W by

$$f=\Phi/|\mathrm{grad}\,\Phi| \quad \text{on} \quad W.$$

Then we obtain

$$\text{(1.13)} \qquad \begin{aligned} &\text{i)} && f\equiv 0 \quad \text{on} \quad \partial X, \\ &\text{ii)} && df=|\mathrm{grad}\,\Phi|^{-1}d\Phi \quad \text{on} \quad \partial X, \\ &\text{iii)} && |\mathrm{grad}\, f|=1 \quad \text{on} \quad W. \end{aligned}$$

We separate z_i^α into the real and imaginary parts: $z_i=x_i^{2\alpha-1}+\sqrt{-1}x_i^{2\alpha}$ $(\alpha=1, 2, \cdots, n)$ and set $\partial_{i,k}=\dfrac{\partial}{\partial x_i^k}$ $(k=1, 2, \cdots, 2n)$. Let dV be the volume element of M with respect to (1.1). Then we have, by direct calculations,

$$\text{(1.14)} \qquad dV=\bigwedge^{n} \Omega/n\,!=2^n g_i dx_i^1\wedge dx_i^2\wedge\cdots\wedge dx_i^{2n}\,.$$

Let dS be the volume element of the real differential manifold ∂X of real dimension $2n-1$. Since (1.13) implies that $\mathrm{grad}\, f$ is the outward unit normal on ∂X, we have

$$\text{(1.15)} \qquad dV=df\wedge dS \quad \text{on} \quad \partial X.$$

We consider a vector field $\sum_{\alpha=1}^{n}\psi_i^\alpha\dfrac{\partial}{\partial z_i^\alpha}+\sum_{\beta=1}^{n}\psi_i^{\bar\beta}\dfrac{\partial}{\partial \bar z_i^\beta}$ on a neighborhood of $\overline{X}$. Then $\sum_{\alpha=1}^{n}\nabla_{i,\alpha}\psi_i^\alpha+\sum_{\beta=1}^{n}\nabla_{i,\bar\beta}\psi_i^{\bar\beta}$ is called the divergence of the vector field $\tilde\psi=\{\psi^\alpha, \psi^{\bar\beta}\}$.

Lemma 1.1. *We have*

$$\text{(1.16)} \qquad \int_X\Big(\sum_{\alpha=1}^{n}\nabla_\alpha\psi^\alpha+\sum_{\beta=1}^{n}\nabla_{\bar\beta}\psi^{\bar\beta}\Big)dV=\int_{\partial X}\Big(\sum_{\alpha=1}^{n}\psi^\alpha\partial_\alpha f+\sum_{\beta=1}^{n}\psi^{\bar\beta}\partial_{\bar\beta}f\Big)dS\,,$$

where $\partial_\alpha=\dfrac{\partial}{\partial z_i^\alpha}$ *and* $\partial_{\bar\alpha}=\dfrac{\partial}{\partial \bar z_i^\alpha}$ *on* U_i $(\alpha=1, 2, \cdots, n)$.

Proof. From (1.13) and (1.15), Green's formula may be written in the following form:

$$\int_{X\cap U_i} \partial_{i,k} u \bar{v} dV = -\int_{X\cap U_i} u(\partial_{i,k} + \partial_{i,k} \log g_i) \bar{v} dV + \int_{\partial X\cap U_i} u\bar{v}\partial_{i,k} f dS$$

for any $u \in C_0^{0,0}(U_i)$, $v \in C^{0,0}(\bar{X})$ and $k=1, 2, \cdots, 2n$.

Since $\partial_{i,\alpha} = \partial_{i,2\alpha-1} + \sqrt{-1}\partial_{i,2\alpha}$ and $\partial_{i,\bar{\alpha}} = \partial_{i,2\alpha-1} - \sqrt{-1}\partial_{i,2\alpha}$, we have

(1.17)

i) $$\int_{X\cap U_i} \partial_{i,\alpha} u \bar{v} dV = -\int_{X\cap U_i} u\,\overline{(\partial_{i,\bar{\alpha}} + \partial_{i,\bar{\alpha}} \log g_i) v}\, dV + \int_{\partial X\cap U_i} u\bar{v}\partial_{i,\alpha} f dS$$

ii) $$\int_{X\cap U_i} \partial_{i,\bar{\beta}} u \bar{v} dV = -\int_{X\cap U_i} u\,\overline{(\partial_{i,\beta} + \partial_{i,\beta} \log g_i) v}\, dV + \int_{\partial X\cap U_i} u\bar{v}\partial_{i,\bar{\beta}} f dS$$

for any $u \in C_0^{0,0}(U_i)$, $v \in C^{0,0}(\bar{X})$ and $\alpha, \beta = 1, 2, \cdots, n$.

Let $\{\rho_i\}_{1\leqq i\leqq m}$ be a family of C^∞-functions on M such that $\operatorname{supp}\rho_i \subsetneqq U_i$, $0 \leqq \rho_i \leqq 1$ and $\sum \rho_i \equiv 1$ on $\bar{X}$. We set $\psi_i'^{\alpha} = \rho_i \psi_i^{\alpha}$ and $\psi_i'^{\bar{\beta}} = \rho_i \psi_i^{\bar{\beta}}$ respectively. Replacing u and v by $\psi_i'^{\alpha}$ and 1 in (1.17) i), we have

$$\int_{X\cap U_i} (\partial_{i,\alpha}\psi_i'^{\alpha} + \partial_{i,\alpha} \log g_i \psi_i'^{\alpha}) dV = \int_{\partial X\cap U_i} \psi_i'^{\alpha} \partial_{i,\alpha} f dS .$$

While,

$$\begin{aligned} &\sum_{\alpha=1}^{n} (\partial_{i,\alpha}\psi_i'^{\alpha} + \partial_{i,\alpha} \log g_i \psi_i'^{\alpha}) \\ &= \sum_{\alpha=1}^{n} \partial_{i,\alpha}\psi_i'^{\alpha} + \sum_{\alpha=1}^{n} (\sum_{\gamma=1}^{n} \Gamma_{i,\gamma\alpha}^{\ \gamma}) \psi_1'^{\alpha} \qquad \text{(Use (1.9).)} \\ &= \sum_{\alpha=1}^{n} (\partial_{i,\alpha}\psi_i'^{\alpha} + \sum_{\gamma=1}^{n} \Gamma_{i,\alpha\gamma}^{\ \alpha} \psi_i'^{\gamma}) \\ &= \sum_{\alpha=1}^{n} \nabla_{i,\alpha}\psi_i'^{\alpha}. \qquad \text{(Use (1.11).)} \end{aligned}$$

Therefore we have

$$\int_{X\cap U_i} (\sum_{\alpha=1}^{n} \nabla_{i,\alpha}\psi_i'^{\alpha}) dV = \int_{\partial X\cap U_i} (\sum_{\alpha=1}^{n} \psi_i'^{\alpha} \partial_{i,\alpha} f) dS.$$

For $\psi_i'^{\bar\beta}$, we have similarly

$$\int_{X\cap U_i}(\sum_{\beta=1}^{n}\nabla_{i,\bar\beta}\psi_i'^{\bar\beta})dV=\int_{\partial X\cap U_i}(\sum_{\beta=1}^{n}\psi_i'^{\bar\beta}\partial_{i,\bar\beta}f)dS\,.$$

Using $\psi_i^{\alpha}=\sum_{\beta=1}^{n}\frac{\partial z_i^{\alpha}}{\partial z_j^{\beta}}\psi_j^{\beta}, \nabla_{i,\alpha}=\sum_{\beta=1}^{n}\frac{\partial z_j^{\beta}}{\partial z_i^{\alpha}}\nabla_{j,\beta}$ and $\sum\rho_i\equiv 1$ on $\bar X$, we have the conclusion.

Remark 1.1. If ds^2 is a hermitian metric on M which is Kähler on a neighborhood U of ∂X and χ is a C^∞-function on M whose support is contained in U, then it is clear that the equation (1.16) holds for $\chi\tilde\psi=\{\chi\psi^{\alpha},\chi\psi^{\bar\beta}\}$ i.e.

$$(1.16)'\qquad \int_{X\cap U}(\sum_{\alpha=1}^{n}\nabla_{\alpha}(\chi\psi^{\alpha})+\sum_{\beta=1}^{n}\nabla_{\bar\beta}(\chi\psi^{\bar\beta}))dV$$
$$=\int_{\partial X}(\sum_{\alpha=1}^{n}\chi\psi^{\alpha}\partial_{\alpha}f+\sum_{\beta=1}^{n}\chi\psi^{\bar\beta}\partial_{\bar\beta}f)dS\,.$$

Let F be a holomorphic line bundle over M and suppose that F is defined by the system of transition functions $\{f_{ij}\}$ with respect to the coordinate cover $\{U_i\}_{i\in I}$. A hermitian metric on F with respect to this covering is given by the system of positive C^∞-functions $\{a_i\}$, each defined on U_i, such that $a_i\cdot a_j^{-1}=|f_{ij}|^2$ on $U_i\cap U_j$. (In this paper, we use the notation of a system of metric along the fibres in the sense of Kodaira [20], p. 1268, (1)). From now on, we fix a hermitian metric of F

$$(1.18)\qquad \{c_i\}\,.$$

With respect to the covering $\{U_i\}_{i\in I}$ a hermitian inner product $\langle\ ,\ \rangle$ of F is expressed by the metric $\{c_i\}$ i.e. for C^∞-sections $\xi=\{\xi_i\}$ and $\eta=\{\eta_i\}$ of F on M.

$$\langle\xi,\eta\rangle=c_i^{-1}\xi_i\bar\eta_i\,.$$

With respect to this metric, we can define a connection $\{\theta_i\}$ of F as follows:

$$(1.19)\qquad \theta_i=\sum_{\alpha=1}^{n}-\partial_{\alpha}\log c_i dz_i^{\alpha}\,.$$

The curvature tensor of the above connection is defined by

$$(1.20)\qquad \theta_{i,\alpha\bar{\beta}}=\partial_\alpha\partial_{\bar{\beta}}\log c_i.$$

The canonical line bundle K_M of M is defined by the system of transition functions

$$K_M=\{K_{M_{ij}}\}\quad\text{where}\quad K_{M_{ij}}=\frac{\partial(z_j^1,\cdots,z_j^n)}{\partial(z_i^1,\cdots,z_i^n)}\quad\text{on}\quad U_i\cap U_j.$$

We see that

$$|K_{M_{ij}}|^2=g_ig_j^{-1}\quad\text{on}\quad U_i\cap U_j.$$

Hence

$$(1.21)\qquad \{g_i\}$$

determines a metric of K_M.

Let $C^{p,q}(M,F)$ (resp. $C^{p,q}(X,F)$) be the space of F-valued differential forms of type (p,q) and of class C^∞ on M (resp. on X) and let $C^{p,q}(\bar{X},F)$ be the image of the restriction map from $C^{p,q}(M,F)$ to $C^{p,q}(X,F)$. Let $C_0^{p,q}(M,F)$ (resp. $C_0^{p,q}(X,F)$) be the space of the forms in $C^{p,q}(M,F)$ (resp. $C^{p,q}(X,F)$) with compact supports. It is clear that $C_0^{p,q}(X,F)$ is a subspace of $C^{p,q}(\bar{X},F)$. We express $\varphi=\{\varphi_i\}\in C^{p,q}(M,F)$ as $\varphi_i=\dfrac{1}{p!\,q!}\sum_{\alpha_1,\cdots,\alpha_p}\sum_{\beta_1,\cdots,\beta_q}\varphi_{i,\alpha_1,\cdots,\alpha_p,\bar{\beta}_1,\cdots,\bar{\beta}_q}dz_i^{\alpha_1}\wedge\cdots\wedge dz_i^{\alpha_p}\wedge d\bar{z}_i^{\beta_1}\wedge\cdots\wedge d\bar{z}_i^{\beta_q}$. For $\varphi\in C^{p,q}(M,F)$, we set

$$\varphi_i^{\bar{\alpha}_1,\cdots,\bar{\alpha}_p,\beta_1,\cdots,\beta_q}=\sum g_i^{\bar{\alpha}_1c_1}\cdots g_i^{\bar{\alpha}_pc_p}g_i^{\bar{d}_1\beta_1}\cdots g_i^{\bar{d}_q\beta_q}\varphi_{i,c_1,\cdots,c_p,\bar{d}_1,\cdots,\bar{d}_q}.$$

For simplicity, we write

$$\varphi_i^{\bar{A}_pB_q}=\sum g_i^{\bar{A}_pC_p}g_i^{\bar{D}_qB_q}\varphi_{i,C_p\bar{D}_q},$$

where $A_p=(\alpha_1,\cdots,\alpha_p)$, $B_q=(\beta_1,\cdots,\beta_q)$, $C_p=(c_1,\cdots,c_p)$ and $D_q=(d_1,\cdots,d_q)$.

We set

$$\langle\varphi,\psi\rangle=c_i^{-1}\sum_{A_p,B_q}\varphi_{i,A_p,\bar{B}_q}\overline{\psi_i^{\bar{A}_pB_q}}$$

where $A_p=(\alpha_1,\cdots,\alpha_p)$ and $B_q=(\beta_1,\cdots,\beta_q)$ run through the sets of multi-indices with $1\leqq\alpha_1<\alpha_2<,\cdots,<\alpha_p\leqq n$ and $1\leqq\beta_1<\beta_2<,\cdots,<\beta_q\leqq n$ respectively. Then we have

$$c_i^{-1}\varphi_i\wedge*\overline{\psi_i}=\langle\varphi,\psi\rangle dV,$$

where $*$ is the star operator with respect to the metric ds^2.

For φ and $\psi \in C^{p,q}(\overline{X}, F)$, we define

$$(\varphi, \psi) = \int_X \langle \varphi, \psi \rangle dV .$$

For any real valued C^∞-function Ψ on M, we define

$$(\varphi, \psi)_\Psi = \int_X \langle \varphi, \psi \rangle_\Psi dV ,$$

where $\langle \varphi, \psi \rangle_\Psi = \langle \varphi, \psi \rangle e^{-\Psi}$.

We set

$$\begin{aligned} &\|\varphi\|^2 = (\varphi, \varphi) \\ &\|\varphi\|_\Psi^2 = (\varphi, \varphi)_\Psi . \end{aligned} \tag{1.22}$$

We have the operator $\bar{\partial}: C^{p,q}(\overline{X}, F) \to C^{p,q+1}(\overline{X}, F)$ defined by $(\bar{\partial}\varphi)_i = \bar{\partial}\varphi_i$. Then formal adjoint operators ϑ and ϑ_Ψ are defined by

$$\begin{aligned} &(\bar{\partial}\varphi, \psi) = (\varphi, \vartheta\psi) \\ &(\bar{\partial}\varphi, \psi)_\Psi = (\varphi, \vartheta_\Psi\psi)_\Psi \end{aligned} \tag{1.23}$$

for any $\varphi \in C_0^{p,q}(X, F)$ and $\psi \in C_0^{p,q+1}(X, F)$.

Lemma 1.2. *If* $\varphi \in C^{p,q}(\overline{X}, F)$ *and* $\psi \in C^{p,q+1}(\overline{X}, F)$, *we have*

i) $(\bar{\partial}\varphi, \psi)_\Psi = (\varphi, \vartheta_\Psi\psi)_\Psi$

$$+ \int_{\partial X} (c_i \exp(\Psi) q!)^{-1} \sum \varphi_{i, C_p \bar{D}_q} \overline{g_i^{\bar{C}_p E_p} g_i^{\bar{F}_q D_q} \Big(\sum_{\alpha=1}^n \psi_{i,}{}^{\alpha}{}_{E_p \bar{F}_q} \frac{\partial f}{\partial z_i^\alpha} \Big)} dS ,$$

ii) $(\vartheta_\Psi\psi)_{i, A_p \bar{B}_q} = -\sum_{\beta=1}^n (\partial_\beta + \partial_\beta \log(c_i^{-1} e^{-\Psi} g_i)) \psi_{i,}{}^{\beta}{}_{A_p, \bar{B}_q}$

for any real valued C^∞*-function* Ψ *on* M.

Remark 1.2. In this lemma, ds^2 need not be a Kähler metric.

Proof. Take elements u and v of $C^{0,0}(\overline{X}, F)$. Applying a family of C^∞-functions $\{\rho_i\}$ on M which was taken as in the proof of Lemma 1.1, to u and v, we have the following formula similar to (1.17):

$$\int_X c^{-1} \partial_{\bar{\alpha}} u \bar{v} \, e^{-\Psi} dV = \tag{1.24}$$

$$-\int_X c^{-1}u\,\overline{(\partial_\alpha+\partial_\alpha\log(c^{-1}e^{-\Psi}g))v}\,dV+\int_{\partial X}c^{-1}u\bar{v}\,e^{-\Psi}\partial_{\bar{\alpha}}f\,dS$$

for any real valued C^∞-function Ψ on M and $\alpha=1, 2, \cdots, n$. Here we omitted i for simplicity and, in this proof, we do so. For an element φ of $C^{p,q}(\overline{X}, F)$, φ and $\bar{\partial}\varphi$ are represented as

$$\varphi=\frac{1}{q!}\sum_{\beta_1,\cdots,\beta_q}\varphi_{A_p,\bar{\beta}_1,\cdots,\bar{\beta}_q}dz^{A_p}\wedge d\bar{z}^{\beta_1}\wedge\cdots\wedge d\bar{z}^{\beta_q}$$

and

$$\bar{\partial}\varphi=\frac{1}{(q+1)!}\sum_{\beta_0,\cdots,\beta_q}(\bar{\partial}\varphi)_{A_p,\bar{\beta}_0,\cdots,\bar{\beta}_q}dz^{A_p}\wedge d\bar{z}^{\beta_0}\wedge\cdots\wedge d\bar{z}^{\beta_q}$$

where $A_p=(\alpha_1, \cdots, \alpha_p)$ and $dz^{A_p}=dz^{\alpha_1}\wedge\cdots\wedge dz^{\alpha_p}$ and so on.

Since $(\bar{\partial}\varphi)_{A_p,\bar{\beta}_0,\cdots,\bar{\beta}_q}=\sum_{k=0}^{q}(-1)^{p+k}\partial_{\bar{\beta}_k}\varphi_{A_p,\bar{\beta}_0,\cdots,\hat{\bar{\beta}}_k,\cdots,\bar{\beta}_q}$, we have

$$(\bar{\partial}\varphi, \psi)_\Psi=\int_X\frac{1}{(q+1)!}c^{-1}\sum_{A_p}\sum_{B_{q+1}}(\bar{\partial}\varphi)_{A_p,\bar{B}_{q+1}}\overline{\psi^{\bar{A}_pB_{q+1}}}e^{-\Psi}2^n g\,dx^1\wedge\cdots\wedge dx^{2n}$$

$$=\int_X\frac{2^n}{q!}\sum_{A_p\beta,B_q}\sum\partial_{\bar{\beta}}\varphi_{A_p\bar{B}_q}\overline{\psi^{\beta\bar{A}_pB_q}}(c^{-1}e^{-\Psi}g)\,dx^1\wedge\cdots\wedge dx^{2n}$$

(by (1.24))

$$=-\int_X\frac{2^n}{q!}\sum_{A_p\beta,B_q}\sum\varphi_{A_p\bar{B}_q}\overline{\partial_\beta(\psi^{\beta\bar{A}_pB_q}c^{-1}e^{-\Psi}g)}\,dx^1\wedge\cdots\wedge dx^{2n}$$

$$+\int_{\partial X}(ce^{\Psi}q!)^{-1}\sum_{A_p,B_q}\varphi_{A_p,\bar{B}_q}\overline{(\sum_{\beta=1}^{n}\psi^{\beta\bar{A}_pB_q}\partial_\beta f)}\,dS$$

$$=(\varphi, \vartheta_\Psi\psi)_\Psi$$

$$+\int_{\partial X}(ce^{\Psi}q!)^{-1}\sum_{A_p,B_q}\varphi_{A_p,\bar{B}_q}\overline{\sum_{C_p,D_q}g^{\bar{A}_pC_p}g^{\bar{D}_qB_q}(\sum_{\beta=1}^{n}\psi^{\beta}{}_{C_p\bar{D}_q}\partial_\beta f)}\,dS.$$

In the last line, the equality of the first term holds by definition. Hence we have i) and ii). Q.E.D.

From now on, we consider the following subspace of $C^{p,q}(\overline{X}, F)$.

(1.25) $$B^{p,q}(\overline{X}, F)=\{\varphi\in C^{p,q}(\overline{X}, F)\mid\sum_{\alpha=1}^{n}\varphi_{i,}{}^{\alpha}{}_{C_p\bar{D}_{q-1}}\partial_{i,\alpha}\Phi=0$$

on ∂X for every multi-index C_p and $D_{q-1}\}$.

Lemma 1.3. *If* $\varphi\in C^{p,q}(\overline{X}, F)$ *and* $\psi\in B^{p,q+1}(\overline{X}, F)$, *we have*

$(\bar{\partial}\varphi, \psi)_{\Psi} = (\varphi, \vartheta_{\Psi}\psi)_{\Psi}$ *for any real valued* C^{∞}*-function* Ψ *on* M.

Proof. Since df and $d\Phi$ are proportional on ∂X, our lemma follows immediately from Lemma 1.2. Q.E.D.

Let $\mathcal{T}_{p,q}(M, F)$ (resp. $\mathcal{T}_{p,q}(M)$) be the space of F-valued (resp. scalar) tensor fields of type (p, q). The connections (1.2) and (1.19) give rise to covariant differentiations $\nabla_{\alpha}, \nabla_{\alpha}^{(c)}$ of type $(1, 0)$ and $\nabla_{\bar{\beta}}$ of type $(0, 1)$ in $\mathcal{T}_{p,q}(M, F)$ and $\mathcal{T}_{p,q}(M)$ as follows:

(1.26)

$$\text{i)}\quad \nabla_{i,\alpha}\varphi_{i,\alpha_1,\cdots,\alpha_p,\bar{\beta}_1,\cdots,\bar{\beta}_q} = \partial_{i,\alpha}\varphi_{i,\alpha_1,\cdots,\alpha_p,\bar{\beta}_1,\cdots,\bar{\beta}_q} - \sum_{t=1}^{p}\sum_{\tau=1}^{n}\Gamma_{i,\alpha\alpha_t}^{\tau}\times\varphi_{i,\alpha_1,\cdots,\overset{t}{\overset{\downarrow}{\tau}},\cdots,\alpha_p,\bar{\beta}_1,\cdots,\bar{\beta}_q}$$

$$\text{ii)}\quad \nabla_{i,\bar{\beta}}\varphi_{i,\alpha_1,\cdots,\alpha_p,\bar{\beta}_1,\cdots,\bar{\beta}_q} = \partial_{i,\bar{\beta}}\varphi_{i,\alpha_1,\cdots,\alpha_p,\bar{\beta}_1,\cdots,\bar{\beta}_q} - \sum_{t=1}^{q}\sum_{\tau=1}^{n}\Gamma_{i,\bar{\beta}\bar{\beta}_t}^{\bar{\tau}}\times\varphi_{i,\alpha_1,\cdots,\alpha_p,\bar{\beta}_1,\cdots,\overset{t}{\overset{\downarrow}{\bar{\tau}}},\cdots,\bar{\beta}_q}$$

for every $\varphi\in\mathcal{T}_{p,q}(M)$.

(1.27)

$$\text{i)}\quad \nabla_{i,\alpha}^{(c)}\varphi_{i,\alpha_1,\cdots,\alpha_p,\bar{\beta}_1,\cdots,\bar{\beta}_q} = \nabla_{i,\alpha}\varphi_{i,\alpha_1,\cdots,\alpha_p,\bar{\beta}_1,\cdots,\bar{\beta}_q} - \partial_{i,\alpha}\log c_i\times\varphi_{i,\alpha_1,\cdots,\alpha_p,\bar{\beta}_1,\cdots,\bar{\beta}_q}$$

$$\text{ii)}\quad \nabla_{i,\bar{\beta}}\varphi_{i,\alpha_1,\cdots,\alpha_p,\bar{\beta}_1,\cdots,\bar{\beta}_q} = \partial_{i,\bar{\beta}}\varphi_{i,\alpha_1,\cdots,\alpha_p,\bar{\beta}_1,\cdots,\bar{\beta}_q} - \sum_{t=1}^{q}\sum_{\tau=1}^{n}\Gamma_{i,\bar{\beta}\bar{\beta}_t}^{\bar{\tau}}\times\varphi_{i,\alpha_1,\cdots,\alpha_p,\bar{\beta}_1,\cdots,\overset{t}{\overset{\downarrow}{\bar{\tau}}},\cdots,\bar{\beta}_q}$$

for every $\varphi\in\mathcal{T}_{p,q}(M, F)$.

Then, by using the term of covariant differentiations, we can describe the operators $\bar{\partial}$, ϑ as follows (see [21], p. 110, Proposition 5.2 and p. 122, Proposition 6.7):

(1.28)

$$\bar{\partial}\varphi_{i,C_p,\bar{a}_0,\cdots,\bar{a}_q} = \sum_{\mu=0}^{q}(-1)^{\mu+p}\nabla_{i,\bar{a}_\mu}\varphi_{i,C_p,\bar{a}_0,\cdots,\hat{\bar{a}}_\mu,\cdots,\bar{a}_q}$$

$$\vartheta\varphi_{i,C_p,\bar{a}_1,\cdots,\bar{a}_{q-1}} = -\sum_{\alpha,\beta=1}^{n}g_i^{\bar{\beta}\alpha}\nabla_{i,\alpha}^{(c)}\varphi_{i,\bar{\beta},C_p,\bar{a}_1,\cdots,\bar{a}_{q-1}}$$

for any $\varphi\in\mathcal{T}_{p,q}(M, F)$.

Lemma 1.4. *For an element* $\varphi=\{\varphi_{i,D_p}\}$ $(D_p=(d_1,\cdots,d_p))\in\mathcal{T}_{0,p}(M,F)$, *we set*

$$\xi_i^{\beta}=\sum_{\gamma=1}^{n}c_i^{-1}\nabla_{i,\bar\gamma}\varphi_{i,}{}^{\beta}{}_{\bar D_{p-1}}\overline{\varphi_i^{\gamma D_{p-1}}}\quad(\beta=1,2,\cdots,n)$$

$$\eta_i^{\bar\gamma}=\sum_{\beta=1}^{n}c_i^{-1}\nabla^{(c)}_{i,\beta}\varphi_{i,}{}^{\beta}{}_{\bar D_{p-1}}\overline{\varphi_i^{\gamma D_{p-1}}}\quad(\gamma=1,2,\cdots,n)$$

then $\xi=\{\xi_i^{\beta}\}$ *and* $\eta=\{\eta_i^{\bar\gamma}\}$ *are vector fields on* M.

Proof. We recall the following relations:

$$\text{i)}\quad G_i={}^t\left(\frac{\partial z_j^{\lambda}}{\partial z_i^{\tau}}\right)G_j\overline{\left(\frac{\partial z_j^{\tau}}{\partial z_i^{\sigma}}\right)},\quad G_i^{-1}=\overline{\left(\frac{\partial z_i^{\sigma}}{\partial z_j^{\tau}}\right)}G_i^{-1}{}^t\left(\frac{\partial z_i^{\tau}}{\partial z_j^{\lambda}}\right)$$

(1.29) where $G_i=(g_{i,\alpha\bar\beta})$ and $G_j^{-1}=(g_i{}^{\bar\beta\alpha})$

$$\text{ii)}\quad \nabla_{i,\bar k}=\sum_{\lambda=1}^{n}\overline{\left(\frac{\partial z_j^{\lambda}}{\partial z_i^{k}}\right)}\nabla_{j,\bar\lambda},\quad \nabla^{(c)}_{i,k}=\sum_{\lambda=1}^{n}\left(\frac{\partial z_j^{\lambda}}{\partial z_i^{k}}\right)\nabla^{(c)}_{j,\lambda}\quad\text{(see [21], p. 108).}$$

We prove the case $p=2$. The proofs of other cases are similar. We set

$$\xi_i=\sum_{h=1}^{n}\xi_i^{h}\left(\frac{\partial}{\partial z_i^{h}}\right)=\sum_{h=1}^{n}\Big(\sum_{k=1}^{n}c_i^{-1}\nabla_{i,\bar k}\varphi_{i,}{}^{h}{}_{\bar\lambda}\overline{\varphi_i^{k\lambda}}\Big)\left(\frac{\partial}{\partial z_i^{h}}\right)\quad\text{on } U_i.$$

Since $\{\varphi_{i,}{}^{h}{}_{\bar\lambda}\}$ (resp. $\{\varphi_i^{k\lambda}\}$) is a C^∞-section of $F\otimes TM\otimes\overline{TM}^*$ (resp. $F\otimes TM\otimes TM$), we have

$$\left.\begin{aligned}\varphi_{i,}{}^{h}{}_{\bar\lambda}&=f_{ij}\sum_{\alpha,\tau=1}^{n}\frac{\partial z_i^{h}}{\partial z_j^{\tau}}\overline{\frac{\partial z_j^{\alpha}}{\partial z_i^{\lambda}}}\varphi_{j,}{}^{\tau}{}_{\bar\alpha}\\ \varphi_i^{k\lambda}&=f_{ij}\sum_{\alpha,\sigma=1}^{n}\frac{\partial z_i^{k}}{\partial z_j^{\tau}}\frac{\partial z_i^{\lambda}}{\partial z_j^{\alpha}}\varphi_j^{\sigma\alpha}\end{aligned}\right\}\quad\text{on } U_i\cap U_j.$$

Then, using (1.29), we have

$$\begin{aligned}\xi_i&=\sum_{h,k}\nabla_{i,\bar k}\varphi_{i,}{}^{h}{}_{\bar\lambda}\overline{\varphi_i^{k\lambda}}\left(\frac{\partial}{\partial z_i^{h}}\right)\\ &=c_i^{-1}|f_{ij}|^2\sum_{h,k}\left(\sum_{\beta=1}^{n}\overline{\frac{\partial z_j^{\beta}}{\partial z_i^{k}}}\nabla_{j,\bar\beta}\left(\sum_{\alpha,\tau}\frac{\partial z_i^{h}}{\partial z_j^{\tau}}\overline{\frac{\partial z_j^{\alpha}}{\partial z_i^{\lambda}}}\varphi_{j,}{}^{\tau}{}_{\bar\alpha}\right)\right)\\ &\qquad\times\overline{\left(\sum_{\alpha,\sigma}\frac{\partial z_i^{k}}{\partial z_j^{\sigma}}\frac{\partial z_i^{\lambda}}{\partial z_j^{\alpha}}\varphi_j^{\sigma\alpha}\right)}\left(\sum_{\tau}\frac{\partial z_j^{\tau}}{\partial z_i^{h}}\frac{\partial}{\partial z_j^{\tau}}\right)\\ &=c_j^{-1}\sum_{h,k}\left(\sum_{\alpha,\beta,\tau}\overline{\frac{\partial z_j^{\beta}}{\partial z_i^{k}}}\frac{\partial z_i^{h}}{\partial z_j^{\tau}}\overline{\frac{\partial z_j^{\alpha}}{\partial z_i^{\lambda}}}\nabla_{j,\bar\beta}\varphi_{j,}{}^{\tau}{}_{\bar\alpha}\right)\left(\sum_{\alpha,\sigma}\overline{\frac{\partial z_i^{k}}{\partial z_j^{\sigma}}}\overline{\frac{\partial z_i^{\lambda}}{\partial z_j^{\alpha}}}\overline{\varphi_j^{\sigma\alpha}}\right)\left(\sum_{\tau}\frac{\partial z_j^{\tau}}{\partial z_i^{h}}\frac{\partial}{\partial z_j^{\tau}}\right)\end{aligned}$$

$$=\sum_{\tau,\beta}\nabla_{j,\bar\beta}\varphi_{j,}{}^{\tau}{}_{\bar\alpha}\overline{\varphi_j{}^{\beta\alpha}}\frac{\partial}{\partial z_j^{\tau}}=\xi_j\,.$$

Hence we have $\xi_i=\xi_j$ on $U_i\cap U_j$. Therefore $\xi=\{\xi_i{}^{\beta}\}$ is a vector field on M. We set

$$\eta_i=\sum_{h=1}^{n}\eta_i{}^{\bar h}\left(\frac{\partial}{\partial\bar z_i^h}\right)=\sum_{h=1}^{n}\left(\sum_{k=1}^{n}c_i^{-1}\nabla^{(c)}_{i,k}\varphi_{i,}{}^{k}{}_{\bar\lambda}\overline{\varphi_i{}^{h\lambda}}\right)\left(\frac{\partial}{\partial\bar z_i^h}\right)\quad\text{on } U_i.$$

Then we have $\eta_i=\eta_j$ on $U_i\cap U_j$ and so $\eta=\{\eta_i{}^{\bar\gamma}\}$ is a vector field on M. Q.E.D.

Proposition 1.5. *We have*

(1.30) $\|\bar\partial\varphi\|^2+\|\vartheta\varphi\|^2$

$$=\int_{\partial X}(c_i|\operatorname{grad}\Phi|)^{-1}\sum_{\substack{D_{p-1}=(d_1,\cdots,d_{p-1})\\ d_1<d_2<\cdots<d_{p-1}}}\sum_{\beta,\gamma=1}^{n}\partial_\beta\partial_{\bar\gamma}\Phi\varphi_{i,}{}^{\beta}{}_{\bar D_{p-1}}\overline{\varphi_i{}^{\gamma D_{p-1}}}dS$$

$$+\|\bar\nabla\varphi\|^2+\int_X c_i^{-1}\sum_{\substack{D_{p-1}=(d_1,\cdots,d_{p-1})\\ d_1<d_2<\cdots<d_{p-1}}}\sum_{\alpha,\gamma=1}^{n}(\theta_{i,\bar\gamma}{}^{\bar\alpha}-R_{i,\bar\gamma}{}^{\bar\alpha})\varphi_{i,\bar\alpha\bar D_{p-1}}\overline{\psi_i{}^{\gamma D_{p-1}}}dV$$

for $\varphi\in B^{0,p}(\bar X,F)$ $(p\geqq 1)$, *where* $\theta_{i,\bar\gamma}{}^{\bar\alpha}=\sum_{\beta=1}^{n}g_i{}^{\bar\alpha\beta}\theta_{i,\beta\bar\gamma}$, $R_{i,\bar\gamma}{}^{\bar\alpha}=\sum_{\beta=1}^{n}g_i{}^{\bar\alpha\beta}R_{i,\bar\gamma\beta}$ *and*

$$\|\bar\nabla\varphi\|^2=\int_X c_i^{-1}\sum_{\substack{D_p=(d_1,\cdots,d_p)\\ d_1<\cdots<d_p}}\sum_{\beta,\gamma=1}^{n}g_i{}^{\bar\beta\gamma}\nabla_{\bar\beta}\varphi_{i,\bar D_p}\overline{\nabla_{\bar\gamma}\varphi_i{}^{D_p}}dV\,.$$

Proof. For an element φ of $B^{0,p}(\bar X,F)$, we set

$$\tilde\xi=\{\xi^{\beta}=\sum_{\substack{\gamma\\ D_{p-1}=(d_1,\cdots,d_{p-1})}}c_i^{-1}\nabla_{i,\bar\gamma}\varphi_{i,}{}^{\beta}{}_{\bar D_{p-1}}\overline{\varphi_i{}^{\gamma D_{p-1}}},\ \xi^{\bar\beta}=0:\beta=1,2,\cdots,n\}$$

$$\tilde\eta=\{\eta^{\gamma}=0,\ \eta^{\bar\gamma}=\sum_{\substack{\beta\\ D_{p-1}=(d_1,\cdots,d_{p-1})}}c_i^{-1}\nabla^{(c)}_{i,\beta}\varphi_{i,}{}^{\beta}{}_{\bar D_{p-1}}\overline{\varphi_i{}^{\gamma D_{p-1}}}:\gamma=1,2,\cdots,n\}.$$

Then, by Lemma 1.4, $\tilde\xi$ and $\tilde\eta$ are vector fields on a neighborhood of $\bar X$. We calculate the divergences of $\tilde\xi$ and $\tilde\eta$.

$$\sum_{\beta=1}^{n}\nabla_\beta\xi^\beta=\sum_{\beta=1}^{n}\nabla_\beta\left(c_i^{-1}\sum_{\gamma,D_{p-1}}\nabla_{\bar\gamma}\varphi_{i,}{}^{\beta}{}_{\bar D_{p-1}}\overline{\varphi_i{}^{\gamma D_{p-1}}}\right)$$

$$=c_i^{-1}\sum_{\substack{\beta,\gamma\\ D_{p-1}}}(\nabla_\beta-\partial_\beta\log c_i)\nabla_{\bar\gamma}\varphi_{i,}{}^{\beta}{}_{\bar D_{p-1}}\overline{\varphi_i{}^{\gamma D_{p-1}}}$$

$$+c_i^{-1}\sum_{\substack{\beta,\gamma\\ D_{p-1}}}\nabla_{\bar\gamma}\varphi_{i,}{}^{\beta}{}_{\bar D_{p-1}}\overline{\nabla_{\bar\beta}\varphi_i{}^{\gamma D_{p-1}}}$$

$$= c_i^{-1} \sum_{\substack{\beta,\gamma \\ D_{p-1}}} \nabla_{\bar\gamma} \nabla_\beta^{(c)} \varphi_{i,}{}^{\beta}{}_{\bar D_{p-1}} \overline{\varphi_i^{\gamma D_{p-1}}} \quad (\text{Use } \nabla_\beta^{(c)} = \nabla_\beta - \partial_\beta \log c_i.)$$

$$+ c_i^{-1} \sum_{\substack{\beta,\gamma \\ D_{p-1}}} \nabla_{\bar\gamma} \varphi_{i,}{}^{\beta}{}_{\bar D_{p-1}} \overline{\nabla_{\bar\beta} \varphi_i^{\gamma D_{p-1}}} \cdots\cdots\cdots *)$$

$$+ c_i^{-1} \sum_{\substack{\beta,\gamma \\ D_{p-1}}} [\nabla_\beta^{(c)}, \nabla_{\bar\gamma}] \varphi_{i,}{}^{\beta}{}_{\bar D_{p-1}} \overline{\varphi_i^{\gamma D_{p-1}}} \cdots\cdots\cdots **).$$

We calculate the commutator. Since $\nabla_\beta^{(c)} = \nabla_\beta - \partial_\beta \log c_i$, for every $D_p = (d_1, \cdots, d_p)$, we have

$$[\nabla_\beta^{(c)}, \nabla_{\bar\gamma}] \varphi_{i,\bar D_p} = [\nabla_\beta, \nabla_{\bar\gamma}] \varphi_{i,\bar D_p} + \theta_{i,\beta\bar\gamma} \varphi_{i,\bar D_p}.$$

Using (1.26), we have

$$\nabla_{\bar\gamma} \nabla_\beta \varphi_{i,\bar D_p} = \partial_{\bar\gamma} \partial_\beta \varphi_{i,\bar D_p} - \sum_{\mu=1}^{p} \sum_{\tau=1}^{n} \Gamma_{i,\bar\gamma \bar d_\mu}^{\bar\tau} \partial_\beta \varphi_{i,\bar d_1,\cdots,\overset{\mu}{\overset{\downarrow}{\bar\tau}},\cdots,\bar d_p}$$

and

$$\nabla_\beta \nabla_{\bar\gamma} \varphi_{i,\bar D_p} = \partial_\beta \partial_{\bar\gamma} \varphi_{i,\bar D_p} - \sum_{\mu=1}^{p} \sum_{\tau=1}^{n} \partial_\beta \Gamma_{i,\bar\gamma \bar d_\mu}^{\bar\tau} \varphi_{i,\bar d_1,\cdots,\overset{\mu}{\overset{\downarrow}{\bar\tau}},\cdots,\bar d_p}$$

$$- \sum_{\mu=1}^{p} \sum_{\tau=1}^{n} \Gamma_{i,\bar\gamma \bar d_\mu}^{\bar\tau} \partial_\beta \varphi_{i,\bar d_1,\cdots,\overset{\mu}{\overset{\downarrow}{\bar\tau}},\cdots,\bar d_p}.$$

Hence

$$[\nabla_\beta, \nabla_{\bar\gamma}] \varphi_{i,\bar D_p} = - \sum_{\mu=1}^{p} \sum_{\tau=1}^{n} R_{i,\bar d_\mu \beta\bar\gamma}^{\bar\tau} \varphi_{i,\bar d_1,\cdots,\overset{\mu}{\overset{\downarrow}{\bar\tau}},\cdots,\bar d_p}. \qquad (\text{Use } (1.3).)$$

Therefore we have

$$[\nabla_\beta^{(c)}, \nabla_{\bar\gamma}] \varphi_{i,\bar D_p} = - \sum_{\mu=1}^{p} \sum_{\tau=1}^{n} R_{i,\bar d_\mu \beta\bar\gamma}^{\bar\tau} \varphi_{i,\bar d_1,\cdots,\overset{\mu}{\overset{\downarrow}{\bar\tau}},\cdots,\bar d_p} + \theta_{i,\beta\bar\gamma} \varphi_{i,\bar D_p}$$

$$\text{for every } D_p = (d_1, \cdots, d_p).$$

So

$$**) = c_i^{-1} \sum_{\alpha,\beta,\gamma} g_i^{\bar\alpha\beta} [\nabla_\beta^{(c)}, \nabla_{\bar\gamma}] \varphi_{i,\bar\alpha \bar D_{p-1}} \overline{\varphi_i^{\gamma D_{p-1}}} \qquad (\text{Use } (1.12).)$$

$$= - c_i^{-1} \sum_{\beta,\gamma,\alpha} \sum g_i^{\bar\alpha\beta} \Big(\sum_{\mu=1}^{p-1} \sum_{\tau=1}^{n} R_{i,\bar d_\mu \beta\bar\gamma}^{\bar\tau} \varphi_{i,\bar\alpha \bar d_1,\cdots,\overset{\mu}{\overset{\downarrow}{\bar\tau}},\cdots,\bar d_{p-1}} \Big) \overline{\varphi_i^{\gamma D_{p-1}}}$$

$$+ c_i^{-1} \sum_{\alpha,\beta,\gamma} g_i^{\bar\alpha\beta} \theta_{i,\beta\bar\gamma} \varphi_{i,\bar\alpha \bar D_{p-1}} \overline{\varphi_i^{\gamma D_{p-1}}}$$

$$= - c_i^{-1} \sum_{\alpha,\beta,\gamma,\tau} g_i^{\bar\alpha\beta} R_{i,\bar\alpha\beta\bar\gamma}^{\bar\tau} \varphi_{i,\bar\tau \bar D_{p-1}} \overline{\varphi_i^{\gamma D_{p-1}}}$$

$$-c_i^{-1}\sum_{\alpha,\beta,\gamma,\tau} g_i^{\bar\alpha\beta}R_{i,\bar d_\mu\beta\bar\gamma}^{\ \bar\tau}\varphi_{i,\bar\alpha\bar d_1,\dots,\overset{\mu}{\downarrow}{\bar\tau},\dots,\bar d_{p-1}}\overline{\varphi_i^{\gamma D_{p-1}}}$$

$$+c_i^{-1}\sum_{\alpha,\gamma}\theta_{i,\bar\gamma}^{\ \alpha}\varphi_{i,\bar\alpha\bar D_{p-1}}\cdot\overline{\varphi_i^{\gamma D_{p-1}}}$$

From the Kähler property of ds^2, we have

$$R_{i,\bar d_\mu\bar\gamma}^{\ \bar\tau\ \bar\alpha}=\sum_\beta g_i^{\bar\alpha\beta}R_{i,\bar d_\mu\beta\bar\gamma}^{\ \bar\tau}=R_{i,\bar d_\mu\bar\gamma}^{\ \bar\alpha\ \bar\tau}\qquad\text{(see 1.8)}.$$

Moreover we remark that $\varphi_{i,\bar\alpha\bar d_1,\dots,\overset{\mu}{\downarrow}{\bar\tau},\dots,\bar d_{p-1}}=-\varphi_{i,\bar\tau\bar d_1,\dots,\overset{\mu}{\downarrow}{\bar\alpha},\dots,\bar d_{p-1}}$. Hence the second term is zero.

So we have

$$**)=-c_i^{-1}\sum_{\gamma,\tau}\Big(\sum_{\alpha,\beta}g_i^{\bar\alpha\beta}R_{i,\bar\alpha\beta\bar\gamma}^{\ \bar\tau}\Big)\varphi_{i,\bar\tau\bar D_{p-1}}\overline{\varphi_i^{\gamma D_{p-1}}}$$

$$+c_i^{-1}\sum_{\alpha,\gamma}\theta_{i,\bar\gamma}^{\ \bar\alpha}\varphi_{i,\bar\alpha\bar D_{p-1}}\overline{\varphi_i^{\gamma D_{p-1}}}.$$

On the other hand,

$$\sum_{\alpha,\beta=1}^{n}g_i^{\bar\alpha\beta}R_{i,\bar\alpha\beta\bar\gamma}^{\ \bar\tau}=\sum_{\alpha,\beta,\lambda}g_i^{\bar\alpha\beta}g_i^{\bar\tau\lambda}R_{i,\lambda\bar\alpha\beta\bar\gamma}$$

$$=\sum_{\alpha,\lambda}g_i^{\bar\tau\lambda}\overline{\sum_\beta g_i^{\bar\beta\alpha}R_{i,\bar\beta\alpha\bar\lambda\gamma}}\qquad\text{(Use (1.5) and (1.8).)}$$

$$=\sum_{\alpha,\lambda}g_i^{\bar\tau\lambda}\overline{R_{i,\alpha\bar\lambda\gamma}^{\ \alpha}}$$

$$=\sum_\lambda g_i^{\bar\tau\lambda}R_{i,\bar\gamma\lambda}\qquad\text{(Use (1.6).)}$$

$$=R_{i,\bar\gamma}^{\ \bar\tau}.$$

Hence we have

$$**)=(p-1)!\,c_i^{-1}\sum_{d_1<\cdots<d_{p-1}}\sum_{\alpha,\gamma=1}^{n}(\theta_{i,\bar\gamma}^{\ \bar\alpha}-R_{i,\bar\gamma}^{\ \bar\alpha})\varphi_{i,\bar\alpha\bar D_{p-1}}\overline{\varphi_i^{\gamma D_{p-1}}}.$$

Since $\bar\partial\varphi_{i,\bar D_{p+1}}=\sum_{\mu=0}^{p}(-1)^\mu\nabla_{\bar d_\mu}\varphi_{i,\bar d_0,\dots,\widehat{\bar d_\mu},\dots,\bar d_p}$,

$$(\bar\partial\varphi)_i^{D_{p+1}}=\sum_{E_{p+1}}g_i^{\bar E_{p+1},D_{p+1}}(\bar\partial\varphi)_{i,\bar E_{p+1}}$$

$$=\sum g_i^{\bar e_0d_0}\cdots g_i^{\bar e_\tau d_\tau}\cdots g_i^{\bar e_pd_p}\Big(\sum_{\tau=0}^{p}(-1)^\tau\nabla_{\bar e_\tau}\varphi_{i,\bar e_0,\dots,\widehat{\bar e_\tau},\dots,\bar e_p}\Big)$$

$$=\sum_{\tau=0}^{p}(-1)^\tau g_i^{\bar e_\tau d_\tau}\nabla_{\bar e_\tau}\varphi_i^{d_0,\dots,\widehat{d_\tau},\dots,d_p}.\qquad\text{(Use (1.12).)}$$

Hence

$$\langle\bar\partial\varphi,\bar\partial\varphi\rangle=c_i^{-1}\sum_{\substack{D_{p+1}=(d_0,\cdots,d_p)\\ d_0<\cdots<d_p}}(\bar\partial\varphi)_{i,\bar D_{p+1}}\overline{(\bar\partial\varphi)_i^{D_{p+1}}}$$

$$=c_i^{-1}\sum\Big(\sum_{\mu=0}^{p}(-1)^{\mu}\nabla_{\bar d_\mu}\varphi_{i,\bar d_0,\cdots,\hat{\bar d}_\mu,\cdots,\bar d_p}\Big)$$

$$\times\overline{\Big(\sum_{\tau=0}^{p}(-1)^{\tau}g_i^{\bar e_\tau d_\tau}\nabla_{\bar e_\tau}\varphi_i^{d_0,\cdots,\hat d_\tau,\cdots,d_p}\Big)}$$

$$=c_i^{-1}\sum_{\substack{\beta\notin D_p=(d_1,\cdots,d_p)\\ d_1<\cdots<d_p}}g_i^{\bar\beta\gamma}\nabla_{\bar\beta}\varphi_{i,\bar D_p}\overline{\nabla_{\bar\gamma}\varphi_i^{D_p}}$$

(the terms with $\mu=\tau$)

$$+c_i^{-1}\sum_{\mu\neq\tau}(-1)^{\mu+\tau}g_i^{\bar d_\tau e_\tau}\nabla_{d_\mu}\varphi_{i,\bar d_0,\cdots,\bar d_\tau,\cdots,\hat{\bar d}_\mu,\cdots,\bar d_p}$$

$$\times\overline{\nabla_{\bar e_\tau}\varphi_i^{d_0,\cdots,\hat d_\tau,\cdots,d_\mu,\cdots,d_q}}$$ (the terms with $\mu\neq\tau$).

Since $\mu+\tau+(\mu+\tau-1)\equiv1 \bmod 2$, we have

$$\langle\bar\partial\varphi,\bar\partial\varphi\rangle=c_i^{-1}\sum_{\substack{\beta\notin D_p\\ d_1<\cdots<d_p}}g_i^{\bar\beta\gamma}\nabla_{\bar\beta}\varphi_{i,\bar D_p}\overline{\nabla_{\bar\gamma}\varphi_i^{D_p}}$$

$$-c_i^{-1}\sum_{\substack{\alpha\neq\beta\\ \alpha,\beta\notin D_{p-1}=(d_1,\cdots,d_{p-1})\\ d_1<\cdots<d_{p-1}}}g_i^{\bar\beta\gamma}\nabla_{\bar\alpha}\varphi_{i,\bar\beta\bar D_{p-1}}\overline{\nabla_{\bar\gamma}\varphi_i^{\alpha D_{p-1}}}\,.$$

Add terms with $\beta\in D_p$ to the first sum on the right-hand side and add terms with $\alpha=\beta$ to the second sum, then the difference remains unchanged and we have

$$\langle\bar\partial\varphi,\bar\partial\varphi\rangle=c_i^{-1}\sum_{\substack{\beta,\gamma\\ D_p=(d_1,\cdots,d_p)\\ d_1<\cdots<d_p}}g_i^{\bar\beta\gamma}\nabla_{i,\bar\beta}\varphi_{i,\bar D_p}\overline{\nabla_{\bar\gamma}\varphi_i^{D_p}}$$

$$-c_i^{-1}\sum_{\substack{\beta,\gamma\\ D_{p-1}=(d_1,\cdots d_{,p-1})\\ d_1<\cdots<d_{p-1}}}\nabla_{i,\bar\beta}\varphi_{i,}{}^{\gamma}{}_{\bar D_{p-1}}\overline{\nabla_{i,\bar\gamma}\varphi_i^{\beta D_{p-1}}}\,.$$

We set

$$\langle\bar\nabla\varphi,\bar\nabla\varphi\rangle=c_i^{-1}\sum_{\substack{D_p\\ d_1<\cdots<d_p}}\sum_{\beta,\gamma}g_i^{\bar\beta\gamma}\nabla_{\bar\beta}\varphi_{i,\bar D_p}\overline{\nabla_{\bar\gamma}\varphi_i^{D_p}}\,.$$

Then we have

$$\langle\bar\partial\varphi,\bar\partial\varphi\rangle=\langle\bar\nabla\varphi,\bar\nabla\varphi\rangle-\sum_{\substack{\beta,\gamma\\ D_{p-1}\\ d_1<\cdots<d_{p-1}}}\nabla_{i,\bar\beta}\varphi_{i,}{}^{\gamma}{}_{\bar D_{p-1}}\overline{\nabla_{i,\bar\gamma}\varphi_i^{\beta D_{p-1}}}\,,$$

and so

$$*)=(p-1)!\{\langle\bar\nabla\varphi,\bar\nabla\varphi\rangle-\langle\bar\partial\varphi,\bar\partial\varphi\rangle\}.$$

Therefore

$$\sum_{\beta=1}^{n} \nabla_\beta \xi^\beta = c_i^{-1} \sum \nabla_{\bar\gamma} \nabla_\beta^{(c)} \varphi_{i,}{}^{\beta}{}_{\bar D_{p-1}} \overline{\varphi_i{}^{\gamma D_{p-1}}} + *) + **)$$

$$= c_i^{-1} \sum \nabla_{\bar\gamma} \nabla_\beta^{(c)} \varphi_{i,}{}^{\beta}{}_{\bar D_{p-1}} \overline{\varphi_i{}^{\gamma D_{p-1}}}$$

$$+ (p-1)!\{c_i^{-1} \sum_{\substack{D_{p-1} \\ d_1<\cdots<d_{p-1}}} \sum_{\alpha,\gamma} (\theta_{i,}{}^{\bar\alpha}{}_{\bar\gamma} - R_{i,}{}^{\bar\alpha}{}_{\bar\gamma}) \varphi_{i,\bar\alpha \bar D_{p-1}} \overline{\varphi_i{}^{\gamma D_{p-1}}}$$

$$+ \langle \bar\nabla\varphi, \bar\nabla\varphi\rangle - \langle \bar\partial\varphi, \bar\partial\varphi\rangle\}.$$

On the other hand

$$\sum_{\gamma=1}^{n} \nabla_{\bar\gamma} \eta^{\bar\gamma} = c_i^{-1} \sum (\nabla_{\bar\gamma} - \bar\partial_{\bar\gamma} \log c_i) \nabla_\beta^{(c)} \varphi_{i,}{}^{\beta}{}_{\bar D_{p-1}} \overline{\varphi_i{}^{\gamma D_{p-1}}}$$

$$+ c_i^{-1} \sum \nabla_\beta^{(c)} \varphi_{i,}{}^{\beta}{}_{\bar D_{p-1}} \overline{\nabla_\gamma \varphi_i{}^{\gamma D_{p-1}}}$$

$$= c_i^{-1} \sum \nabla_{\bar\gamma} \nabla_\beta^{(c)} \varphi_{i,}{}^{\beta}{}_{\bar D_{p-1}} \overline{\varphi_i{}^{\gamma D_{p-1}}}$$

$$+ c_i^{-1} \sum \nabla_\beta^{(c)} \varphi_{i,}{}^{\beta}{}_{\bar D_{p-1}} \overline{\nabla_\gamma^{(c)} \varphi_i{}^{\gamma D_{p-1}}}.$$

Hence

$$\sum_{\gamma=1}^{n} \nabla_{\gamma} \eta^{\bar\gamma} = c_i^{-1} \sum \nabla_{\bar\gamma} \nabla_\beta^{(c)} \varphi_{i,}{}^{\beta}{}_{\bar D_{p-1}} \overline{\varphi_i{}^{\gamma D_{p-1}}}$$

$$+ (p-1)!\langle \vartheta\varphi, \vartheta\varphi\rangle. \qquad \text{(See (1.28).)}$$

Therefore, we have

$$\sum_{\beta=1}^{n} \nabla_\beta \xi^\beta - \sum_{\gamma=1}^{n} \nabla_{\bar\gamma} \eta^{\bar\gamma} = (p-1)!\{c_i^{-1} \sum_{\substack{D_{p-1} \\ d_1<\cdots<d_{p-1}}} \sum_{\alpha,\gamma} (\theta_{i,}{}^{\bar\alpha}{}_{\bar\gamma} - R_{i,}{}^{\bar\alpha}{}_{\bar\gamma}) \varphi_{i,\bar\alpha \bar D_{p-1}} \overline{\varphi_i{}^{\gamma D_{p-1}}}$$

$$+ \langle \bar\nabla\varphi, \bar\nabla\varphi\rangle - \langle \bar\partial\varphi, \bar\partial\varphi\rangle - \langle \vartheta\varphi, \vartheta\varphi\rangle\}.$$

We apply (1.16) of Lemma 1.1 to $\sum_{\beta=1}^{n} \nabla_\beta \xi^\beta - \sum_{\gamma=1}^{n} \nabla_{\bar\gamma} \eta^{\bar\gamma}$. We have

i) $$\int_X c_i^{-1} \sum_{\substack{D_{p-1} \\ d_1<\cdots<d_{p-1}}} \sum_{\alpha,\gamma} (\theta_{i,}{}^{\bar\alpha}{}_{\bar\gamma} - R_{i,}{}^{\bar\alpha}{}_{\bar\gamma}) \varphi_{i,\bar\alpha \bar D_{p-1}} \overline{\varphi_i{}^{\gamma D_{p-1}}} dV$$

$$+ \|\bar\nabla\varphi\|^2 - \|\bar\partial\varphi\|^2 - \|\vartheta\varphi\|^2$$

$$= \frac{1}{(p-1)!} \int_{\partial X} \left(\sum_{\beta=1}^{n} \xi^\beta \partial_{i,\beta} f - \sum_{\gamma=1}^{n} \eta^{\bar\gamma} \partial_{i,\bar\gamma} f\right) dS$$

and

$$\|\bar\nabla\varphi\|^2 = \int_X \langle \bar\nabla\varphi, \bar\nabla\varphi\rangle dV.$$

From $\varphi \in B^{0,p}(\overline{X}, F)$, we have

$$\text{ii)} \quad \sum_{\beta=1}^{n} \eta_i^{\bar{\gamma}} \partial_{i,\bar{\gamma}} f = (c_i |\operatorname{grad} \Phi|)^{-1} \sum_{\substack{\beta=1 \\ D_{p-1}}}^{n} \nabla_{\beta}^{(c)} \varphi_{i,}{}^{\beta}{}_{\bar{D}_{p-1}} \overline{(\sum_{\gamma=1}^{n} \varphi_i^{\gamma D_{p-1}} \partial_{i,\gamma} \Phi)}$$

$$= 0 \quad \text{on} \quad \partial X \qquad \text{(See (1.25).)}$$

and

$$\sum_{\beta=1}^{n} \xi_i^{\beta} \partial_{i,\beta} f = (c_i |\operatorname{grad} \Phi|)^{-1} \sum_{\gamma=1}^{n} (\sum_{\beta=1}^{n} \nabla_{\bar{\gamma}} \varphi_{i,}{}^{\beta}{}_{\bar{D}_{p-1}} \partial_{i,\beta} \Phi) \overline{\varphi_i^{\gamma D_{p-1}}} .$$

While, in a neighborhood of every point of ∂X, for any multi-index $D_{p-1} = (d_1, \cdots, d_{p-1})$, there exists a C^{∞}-section $\psi_{\bar{D}_{p-1}}$ of $\overset{p-1}{\Lambda} \overline{TM^*}$ such that

$$\sum_{\beta=1}^{n} \varphi_{i,}{}^{\beta}{}_{\bar{D}_{p-1}} \partial_{i,\beta} \Phi = \Phi \psi_{\bar{D}_{p-1}} .$$

Since $\sum_{\beta=1}^{n} \varphi_{i,}{}^{\beta}{}_{\bar{D}_{p-1}} \partial_{i,\beta} \Phi = f_{ij} \sum_{\beta=1}^{n} \varphi_{j,}{}^{\beta}{}_{\bar{D}_{p-1}} \partial_{j,\beta} \Phi$ on $U_i \cap U_j \cap W$, where W is the neighborhood of ∂X taken in (1.13), $\varphi_{\bar{D}_{p-1}} = \{\varphi_{i,\bar{D}_{p-1}} = \sum_{\beta=1}^{n} \varphi_{i,}{}^{\beta}{}_{\bar{D}_{p-1}} \partial_{i,\beta} \Phi\}$ is an element of $\mathcal{T}_{0,p-1}(W, F)$ for any multi-index D_{p-1}. Hence we can operate the covariant differentiation to $\varphi_{\bar{D}_{p-1}}$. Then we have

$$\sum_{\beta=1}^{n} \nabla_{\bar{\gamma}} \varphi_{i,}{}^{\beta}{}_{\bar{D}_{p-1}} \partial_{i,\beta} \Phi + \sum_{\beta=1}^{n} \varphi_{i,}{}^{\beta}{}_{\bar{D}_{p-1}} \partial_{i,\beta} \partial_{i,\bar{\gamma}} \Phi$$

$$= \psi_{\bar{D}_{p-1}} \partial_{i,\bar{\gamma}} \Phi + \Phi \nabla_{\bar{\gamma}} \psi_{\bar{D}_{p-1}} \quad \text{on } W.$$

We multiply it by $\overline{\varphi_i^{\gamma D_{p-1}}}$ and sum up with respect to the index γ. Since $\varphi \in B^{0,p}(\overline{X}, F)$, we obtain

$$\sum_{\gamma=1}^{n} (\sum_{\beta=1}^{n} \nabla_{\bar{\gamma}} \varphi_{i,}{}^{\beta}{}_{\bar{D}_{p-1}} \partial_{\beta} \Phi) \overline{\varphi_i^{\gamma D_{p-1}}} + \sum_{\beta,\gamma=1}^{n} \partial_{\beta} \partial_{\bar{\gamma}} \Phi \varphi_{i,}{}^{\beta}{}_{\bar{D}_{p-1}} \cdot \overline{\varphi_i^{\gamma D_{p-1}}} = 0$$

on ∂X. And so

$$\text{iii)} \quad \sum_{\beta=1}^{n} \xi_i^{\beta} \partial_{i,\beta} f = -(c_i |\operatorname{grad} \Phi|)^{-1} \sum_{\substack{\beta,\gamma=1 \\ D_{p-1}}}^{n} \partial_{\beta} \partial_{\bar{\gamma}} \Phi \varphi_{i,}{}^{\beta}{}_{\bar{D}_{p-1}} \overline{\varphi_i^{\gamma D_{p-1}}}$$

on ∂X.

Finally, from i) ii), and iii), we obtain

$$\|\bar{\partial}\varphi\|^2 + \|\vartheta\phi\|^2 = \int_{\partial X} (c_i |\operatorname{grad} \Phi|)^{-1} \sum_{\substack{D_{p-1} \\ d_1 < \cdots < d_{p-1}}} \sum_{\beta,\gamma} \partial_{\beta} \partial_{\bar{\gamma}} \Phi \cdot \varphi_{i,}{}^{\beta}{}_{\bar{D}_{p-1}} \cdot \overline{\varphi_i^{\gamma D_{p-1}}} dS$$

$$+ \|\bar{\nabla}\varphi\|^2 + \int_X c_i^{-1} \sum_{\substack{D_{p-1} \\ d_1 < \cdots < d_{p-1}}} \sum_{\alpha,\gamma} (\theta_{i,}{}^{\bar{\alpha}}{}_{\bar{\gamma}} - R_{i,}{}^{\bar{\alpha}}{}_{\bar{\gamma}}) \varphi_{i,\bar{\alpha}\bar{D}_{p-1}} \cdot \overline{\varphi_i^{\gamma D_{p-1}}} dV$$

for any $\varphi \in B^{0,p}(\overline{X}, F)$. Q.E.D.

Remark 1.3 Dr. A. Fujiki has pointed out that the equality of Proposition 1.5 had already been obtained by P. Griffiths in [14], p. 429, Theorem 7.2 and recently, the author noticed that A. Andreotti and E. Vesentini also had obtained some inequality for the elements of $B^{p,q}(\overline{X}, F)$ in [6]. In their formulae, the base metric ds^2 need not to be a Kähler metric.

§ 1.2. Identity of Weak and Strong Extensions of $\bar{\partial}$-Operator and Its Formal Adjoint ϑ

Let X be a relatively compact domain with smooth boundary on a complex manifold M and let $F \xrightarrow{\pi} M$ be a holomorphic line bundle on M. We fix a hermitian metric $d\sigma^2$ on M and a hermitian metric $\{a_i\}$ along the fibres of F. We use notations $\| \ , \ \|$, $\| \ , \ \|_\Psi$, ϑ and ϑ_Ψ and so on as in Section 1.1 with respect to $d\sigma^2$, $\{a_i\}$ and $\Psi \in C^{0,0}(\overline{X})$.

Let $L^{p,q}(X, F)$ (resp. $L^{p,q}(X, F, \Psi)$) be the completion of $C_0^{p,q}(\overline{X}, F)$ with respect to the norm $\| \ , \ \|$ (resp. $\| \ , \ \|_\Psi$). Then, since $\| \ , \ \|$ and $\| \ , \ \|_\Psi$ are equivalent on X, $L^{p,q}(X, F, \Psi)$ coincides as a topological vector space with $L^{p,q}(X, F)$. In other words, $L^{p,q}(X, F, \Psi)$ is the Hilbert space with $L^{p,q}(X, F)$ as the underlying space and $(\ , \ )_\Psi$ as the inner product. Thus $L^{p,q}(X, F, \Psi)$ is understood as the pair $\{L^{p,q}(X, F), (\ , \ )_\Psi\}$.

Let $\bar{\partial}$: $L^{p,q}(X, F, \Psi) \to L^{p,q+1}(X, F, \Psi)$ be the maximal closed extension of the original $\bar{\partial}$ and let ϑ_Ψ: $L^{p,q+1}(X, F, \Psi) \to L^{p,q}(X, F, \Psi)$ be the maximal closed extension of the original ϑ_Ψ i.e. when we represent the domain of $\bar{\partial}$ (resp. ϑ_Ψ) by $D_{\bar{\partial}}^{p,q}$ (resp. $D_{\vartheta_\Psi}^{p,q}$),

(1.31) i) $\varphi \in D_{\bar{\partial}}^{p,q} \subset L^{p,q}(X, F, \Psi)$ *if and only if there exists an element* $u \in L^{p,q+1}(X, F, \Psi)$ *such that* $(\vartheta_\Psi \psi, \varphi)_\Psi = (\psi, u)_\Psi$ *for every* $\psi \in C_0^{p,q+1}(X, F)$,

ii) $\varphi \in D_{\vartheta_\Psi}^{p,q} \subset L^{p,q}(X, F, \Psi)$ *if and only if there exists an element* $v \in L^{p,q-1}(X, F, \Psi)$ *such that* $(\bar{\partial}\psi, \varphi)_\Psi = (\psi, v)_\Psi$ *for every* $\psi \in C_0^{p,q-1}(X, F)$.

Let $L^{p,q}(M, F)$ (resp. $L^{p,q}(M, F, \Psi)$) be the completion of

$C_0^{p,q}(M, F)$ with respect to the norm $\| \ , \ \|$ (resp. $\| \ , \ \|_\Psi$). Then diffrential operators $\bar{\partial}$ and ϑ_Ψ in $L^{\cdot,\cdot}(M, F, \Psi)$ are defined similarly.

From (1.23), we have $C_0^{p,q}(X, F) \subset D_{\bar{\partial}}^{p,q} \cap D_{\vartheta_\Psi}^{p,q}$. Hence $\bar{\partial}$ and ϑ_Ψ defined as above are, in the weak sense, closed densely defined operators. Next we consider the closure of the graph of $\bar{\partial}$: $C^{p,q}(\bar{X}, F) \to C^{p,q+1}(\bar{X}, F)$ in $L^{p,q}(X, F, \Psi) \times L^{p,q+1}(X, F, \Psi)$. Then, from the general theory of linear operators in Hilbert spaces (cf. [46], p. 70, Theorem 4.15), there exists a unique linear operator T: $L^{p,q}(X, F, \Psi) \to L^{p,q+1}(X, F, \Psi)$ such that i) the graph $G(T)$ of T is equal to the closure of the graph of $\bar{\partial}|_{C^{p,q}(\bar{X},F)}$, ii) the domain $D_T^{p,q}$ of T in $L^{p,q}(X, F, \Psi)$ is the image of $G(T)$ by the projection to the first factor.

Remark 1.4. Since $L^{p,q}(X, F, \Psi)$ coincides with $L^{p,q}(X, F)$ as a topological vector space) for any $\Psi \in C^{0,0}(\bar{X})$, T is determined independently of Ψ. T is called the closure of $\bar{\partial}|_{C^{p,q}(\bar{X},F)}$.

Since $C_0^{p,q}(X, F)$ is contained in D_T, T is a closed densely defined operator and from ii) it is clear that

(1.32) *$\varphi \in D_T^{p,q}$ if and only if there exists a sequence $\{\varphi_n\}$ of $C^{p,q}(\bar{X}, F)$ such that $\|\varphi_n - \varphi\|_\Psi^2$ and $\|T\varphi_n - T\varphi\|_\Psi^2$ tend to zero as $n \to +\infty$.*

Since $D_T^{p,q}$ is dense in $L^{p,q}(X, F, \Psi)$, if for a given $g \in L^{p,q+1}(X, F, \Psi)$, there exists an element $g^* \in L^{p,q}(X, F, \Psi)$ such that $(T\varphi, g)_\Psi = (\varphi, g^*)_\Psi$ for any $\varphi \in D_T^{p,q}$, g^* is uniquely determined by g. Hence the adjoint operator T_Ψ^*: $L^{p,q+1}(X, F, \Psi) \to L^{p,q}(X, F, \Psi)$ is determined by $T_\Psi^* g = g^*$. Then from (1.32), we have

(1.33) *g is contained in the domain $D_{T_\Psi^*}^{p,q+1}$ of T_Ψ^* if and only if there exists a positive constant C such that $|(\bar{\partial}\varphi, g)_\Psi| \leqq C\|\varphi\|_\Psi$ for any $\varphi \in C^{p,q}(\bar{X}, F)$.*

Since $C_0^{p,q+1}(X, F)$ is contained in $D_{T_\Psi^*}^{p,q+1}$, T_Ψ^* is a closed densely defined operator. Hence it holds that $T_\Psi^{**} = T$ (see [46], p. 90, Theorem 5.3) and moreover, from Lemma 1.3 and (1.32), we have

(1. 34) $$C^{p,q}(\overline{X}, F) \cap D^{p,q}_{T^*_\Psi} = B^{p,q}(\overline{X}, F) \quad and$$
$$T^*_\Psi = \vartheta_\Psi \quad on \quad B^{p,q}(\overline{X}, F).$$

Similarly, we can consider the adjoint operator $\bar{\partial}^*_\Psi$ of the maximal closed extension $\bar{\partial}$, then we have

(1. 35) *h is contained in the domain $D^{p,q+1}_{\bar{\partial}^*_\Psi}$ of $\bar{\partial}^*_\Psi$ if and only if there exists a positive constant C such that $|(\bar{\partial}\varphi, h)_\Psi| \leqq C\|\varphi\|_\Psi$ for any $\varphi \in D^{p,q}_{\bar{\partial}}$.*

Hence $\bar{\partial}^*_\Psi$ is a closed densely defined operator and so $\bar{\partial}^{**}_\Psi = \bar{\partial}$. T and $\bar{\partial}^*_\Psi$ are called the minimal (or strong) extensions of the original $\bar{\partial}$ and its formal adjoint ϑ_Ψ. From (1. 31), (1. 32), (1. 33) and (1. 35), we have

(1. 36) *$D^{p,q}_T \subset D^{p,q}_{\bar{\partial}}$ and $D^{p,q}_{\bar{\partial}^*_\Psi} \subset D^{p,q}_{T^*_\Psi} \subset D^{p,q}_{\vartheta_\Psi}$ in $L^{p,q}(X, F, \Psi)$ for any $\Psi \in C^{0,0}(\overline{X})$.*

Remark 1. 5. If T is the closure of $\bar{\partial}|_{C^{p,q}_0(X,F)}$, it holds that $T^*_\Psi = \vartheta_\Psi$ and $\bar{\partial}^*_\Psi$ is the closure of $\vartheta_\Psi|_{C^{p,q+1}_0(X,F)}$.

The following Proposition is due to Hörmander [17] Propositions 1. 2. 3 and 1. 2. 4 which summarize results of [10], [16] and [30].

Proposition 1. 6. i) *If $v \in D^{p,q}_{\vartheta_\Psi} \subset L^{p,q}(X, F, \Psi)$ and* supp v, supp *$\vartheta_\Psi v \subseteqq X$, then $v|_X \in D^{p,q}_{\bar{\partial}^*_\Psi} \subset L^{p,q}(X, F, \Psi)$ i.e. $(\vartheta_\Psi v)|_X = \bar{\partial}^*_\Psi(v|_X)$ in $L^{p,q-1}(X, F, \Psi)$.*
ii) *$C^{p,q}(\overline{X}, F)$ is dense in $D^{p,q}_{\bar{\partial}}$ with respect to the graph norm $(\| \ , \|^2_\Psi + \|\bar{\partial}, \|^2_\Psi)^{1/2}$.*
iii) *$B^{p,q}(\overline{X}, F)$ is dense in $D^{p,q}_{T^*_\Psi}$ (resp. $D^{p,q}_{\bar{\partial}} \cap D^{p,q}_{T^*_\Psi}$) with respect to the gragh norm $(\| \ , \|^2_\Psi + \|\vartheta_\Psi, \|^2_\Psi)^{1/2}$ (resp. $(\| \ , \|^2_\Psi + \|\bar{\partial}, \|^2_\Psi + \|\vartheta_\Psi, \|^2_\Psi)^{1/2}$).*

As a consequence of Proposition 1. 6, we have the following.

Proposition 1. 7. i) *$D^{p,q}_T = D^{p,q}_{\bar{\partial}}$ in $L^{p,q}(X, F, \Psi)$ i.e. $T = \bar{\partial}$*

ii) $D^{p,q}_{\bar{\partial}^*_\Psi} = D^{p,q}_{T^*_\Psi}$ *in* $L^{p,q}(X, F, \Psi)$ *i.e.* $T^*_\Psi = \bar{\partial}^*_\Psi$

iii) $\bar{\partial}^*_\Psi$ $(= T^*_\Psi)$ *is the restriction of the maximal closed extension* ϑ_Ψ *to* $D^{p,q}_{\bar{\partial}^*_\Psi}$.

iv) $D^{p,q}_{\bar{\partial}^*} = D^{p,q}_{\bar{\partial}^*_\Psi}$ *in* $L^{p,q}(X, F)$ *for any* $\Psi \in C^{0,0}(\bar{X})$.

Proof. i), ii) and iii) follow from (1.32), (1.33), (1.35), (1.36) and Proposition 1.6 ii), iii) immediately. From (1.34), it holds that $C^{p,q}(\bar{X}, F) \cap D^{p,q}_{\bar{\partial}^*_\Psi} = C^{p,q}(\bar{X}, F) \cap D^{p,q}_{\bar{\partial}^*}$ for any $\Psi \in C^{0,0}(\bar{X})$. Hence, by Proposition 1.6 iii), we obtain iv). Q.E.D.

We denote the range and nullity of $\bar{\partial}$ in $L^{p,q}(X, F, \Psi)$ by $R^{p,q}_{\bar{\partial}}$ and $N^{p,q}_{\bar{\partial}}$ respectively. $R^{p,q}_{\bar{\partial}^*_\Psi}$ and $N^{p,q}_{\bar{\partial}^*_\Psi}$ are defined similarly. We set

$$N^{p,q}_\Psi = N^{p,q}_{\bar{\partial}} \cap N^{p,q}_{\bar{\partial}^*_\Psi} \qquad \text{in} \quad L^{p,q}(X, F, \Psi).$$

§ 1.3. Basic Fact from Functional Analysis and Application

Let $(H_i, \| \ , \ \|_i)$ $(i = 1, 2, 3)$ be Hilbert spaces and let $H_1 \xrightarrow{T} H_2 \xrightarrow{S} H_3$ be closed densely defined operators satisfying $S \circ T = 0$, We denote the adjoint operator of T (resp. S) by T^* (resp. S^*). Then, as mentioned in Section 1.2, T^* and S^* are also closed densely defined operators and satisfy $T = T^{**}$ and $S = S^{**}$. We use notations D_T, R_T and N_T etc. as defined in Section 1.2.

Proposition 1.8 (*cf.* [17], *Theorems* 1.1.1 *and* 1.1.2 *and* [25], *p.* 210–*p.* 216). *A necessary and sufficient condition for* R_T *and* R_{S^*} *both to be closed is that there exists a positive constant C such that*

$$(1.37) \qquad \|f\|_2^2 \leqq C\{\|Sf\|_3^2 + \|T^*f\|_1^2 \qquad \text{for any} \quad f \in D_S \cap D_{T^*}$$

and that f *is orthogonal to the space* $N = N_S \cap N_{T^*}$. *When this is the case, we have the strong orthogonal decomposition of* H_2

$$(1.38) \qquad H_2 \cong R_T \oplus N \oplus R_{S^*}.$$

Furthermore, the operator $L = TT^* + S^*S$ *whose domain is* $D_L = \{f \in D_S \cap D_{T^*} : Sf \in D_{S^*}$ *and* $T^*f \in D_T\}$, *is self-adjoint and has a closed range in* H_2. *And* (1.38) *can be written more explicitly.*

(1.39) $\quad H_2 \cong R_{TT^*} \oplus N \oplus R_{S^*S}$ *where* $R_T = R_{TT^*}$ *and* $R_{S^*} = R_{S^*S}$.

Remark 1.6. From the definition of D_L, the nullity of the operator L is equal to $N = N_S \cap N_{T^*}$.

Proposition 1.9 (*cf.* [17], *Theorem* 1.1.3). *Assume that from every sequence* $f_k \in D_S \cap D_{T^*}$ *with* $\|f_k\|_2$ *bounded and* $\|T^* f_k\|_1^2 \to 0$ *in* H_1, $\|S f_k\|_3^2 \to 0$ *in* H_3 *as* $k \to +\infty$, *one can select a strongly convergent subsequence. Then* (1.37) *holds and N is finite dimensional.*

Proposition 1.10 (*cf.* [17], *Theorem* 1.1.4). *Let P be a closed linear subspace of* H_2 *containing* R_T. *Assume that there exists a positive constant C such that*

$$\|f\|_2^2 \leq C\{\|Sf\|_3^2 + \|T^* f\|_1^2\} \qquad \text{for any} \quad f \in D_S \cap T_{T^*} \cap P. \tag{1.40}$$

Then i) *for every* $g \in P$ *satisfying* $Sg = 0$, *there exists* $h \in D_T \subset H_1$ *such that* $Th = g$,

ii) R_{T^*} *is closed in* H_1 *and for every* $g \in R_{T^*}$ *there exists* $h \in D_{T^*} \subset H_2$ *such that* $T^* h = g$ *and* $\|h\|_2^2 \leq C \|g\|_1^2$.

Let M, X and F be as in Section 1.2.

Proposition 1.11.

I) *If there exist in the degree* (p, q)

1) *a hermitian metric* ds^2 *on* M *and a hermitian metric* $\{c_i\}$ *along the fibres of* F,

2) *a positive constant* C_1,

and

3) *a proper compact subset* K *of* X *such that*

$$\|\varphi\|_{X \setminus K}^2 \leq C_1 \{\|\bar\partial \varphi\|^2 + \|\bar\partial^* \varphi\|^2 + \|\varphi\|_K^2\} \tag{1.41}$$

for any $\varphi \in D_{\bar\partial}^{p,q} \cap D_{\bar\partial^*}^{p,q} \subset L^{p,q}(X, F)$, *then*

i) $\dim_C N^{p,q} < +\infty$

ii) $L^{p,q}(X, F) \cong R_{\bar\partial\bar\partial^*}^{p,q} \oplus N^{p,q} \oplus R_{\bar\partial^*\bar\partial}^{p,q}$.

II) *If there exist in the degree* (p, q)

1) *a hermitian metric* ds^2 *on* M *and a hermitian metric* $\{c_i\}$ *along the fibres of* F.

2) *a positive constant* C_1,

3) *a proper compact subset* K *of* X, *which does not contain any connected component of* X, *such that*

$$\|\varphi\|^2_{X\setminus K}\leqq C_1\{\|\bar{\partial}\varphi\|^2+\|\bar{\partial}^*\varphi\|^2\} \tag{1.42}$$

for any $\varphi\in D^{p,q}_{\bar{\partial}}\cap D^{p,q}_{\bar{\partial}^*}\subset L^{p,q}(X, F)$, *then there exists a positive constant* C_2 *such that*

$$\|\varphi\|^2\leqq C_2\{\|\bar{\partial}\varphi\|^2+\|\bar{\partial}^*\varphi\|^2\} \tag{1.43}$$

for any $\varphi\in D^{p,q}_{\bar{\partial}}\cap D^{p,q}_{\bar{\partial}^*}\subset L^{p,q}(X, F)$.

Proof. Take any sequence $\{\varphi_m\}_{m\geqq 1}$ such that $\varphi_m\in D^{p,q}_{\bar{\partial}}\cap D^{p,q}_{\bar{\partial}^*}$, $\|\varphi_m\|^2\leqq 1$ and $\|\bar{\partial}\varphi_m\|^2$, $\|\bar{\partial}^*\varphi_m\|^2\to 0$ as $m\to+\infty$. Then we assert that there exists a subsequence $\{\varphi_{m_k}\}$ of $\{\varphi_m\}$ which converges strongly on X. From Proposition 1.6 iii), we may assume $\varphi_m\in B^{p,q}(\bar{X}, F)$. Let χ be a C^∞-function on M with compact support in X and $\chi\equiv 1$ on K. Since $\chi\varphi_m\in C^{p,q}_0(X, F)$, we have that

$$\begin{aligned}(\bar{\partial}(\chi\varphi_m), \bar{\partial}(\chi\varphi_m)) + (\vartheta(\chi\varphi_m), \vartheta(\chi\varphi_m)) + (\chi\varphi_m, \chi\varphi_m)\\ = (\square(\chi\varphi_m), \chi\varphi_m) + (\chi\varphi_m, \chi\varphi_m)\end{aligned}$$

is bounded by the assumption. From coerciveness of elliptic differential operator $\square=\bar{\partial}\vartheta+\vartheta\bar{\partial}$ on $C^{p,q}_0(X, F)$ (see for example [9], (2.2.1) Theorem) and Rellich's lemma (see for example [9], Appendix (A.1.6) Proposition and p. 122, 2 Sobolev norms on manifolds), it follows that $\{\varphi_m\}$ has a sequence $\{\varphi_{m_k}\}$ which is strongly convergent on the compact subset K of X. By (1.41), we conclude that $\{\varphi_{m_k}\}$ converges strongly on X. Therefore we can apply Propositions 1.8 and 1.9 for $H_i=L^{p,q+i-2}(X, F)$ $(i=1, 2, 3)$, $S=\bar{\partial}$ and $T^*=\bar{\partial}^*$. Hence we obtain i) and ii) of I).

In the case of II), the same assertion of I) holds. In particular, from Propositions 1.8 and 1.9, we have that there exists a constant

$C_2 \geqq 0$ such that

$$\|\varphi\|^2 \leqq C_2\{\|\bar{\partial}\varphi\|^2 + \|\bar{\partial}^*\varphi\|^2\} \tag{1.44}$$

for any $\varphi \in D_{\bar{\partial}}^{p,q} \cap D_{\bar{\partial}^*}^{p,q}$ with $\varphi \perp N^{p,q}$, while, each element φ in $N^{p,q}$ is a solution of the Laplace-Beltrami operator $\square = \bar{\partial}\vartheta + \vartheta\bar{\partial}$. Namely φ is a harmonic form with valued in F. In general, when E is a hermitian vector bundle over a connected complex hermitian manifold M, a harmonic form $\varphi \in \mathcal{H}^{p,q}(E)$ vanishes identically on M if it vanishes on a non-empty open subset of M (cf. [7], [38]). Now, from (1.42), φ vanishes identically on $X \backslash K$. Since any connected component of X is not contained in K, by the above unique continuation property, φ vanishes on each connected component and so φ vanishes identically on X. Hence $N^{p,q}$ is the null space. Combining this with (1.44), the proof is completed.

Remark 1.7. Since, from Proposition 1.7, $L = \bar{\partial}\bar{\partial}^* + \bar{\partial}^*\bar{\partial}$ is the restriction of the Laplace-Beltrami operator $\square = \bar{\partial}\vartheta + \vartheta\bar{\partial}$ to the domain of L, $N^{p,q}$ may not coincide with the space of all harmonic forms in $L^{p,q}(X, F)$.

Chapter II. Cohomology Groups on Domains with Pseudoconvex Boundaries

§ 2.1. Definitions

Let X be a relatively compact domain with boundary ∂X on a complex manifold M of complex dimension n.

Definition 2.1. X is said to be *a domain with pseudoconvex boundary* ∂X if for any point p of ∂X, there exist a neighborhood U of p and a real valued C^∞-function λ on U such that i) $U \cap X = \{x \in U \mid \lambda(x) < 0\}$, ii) the gradient of λ nowhere vanishes on $\partial X \cap U$, iii) the complex Hessian of λ is positive semi-definite when restricted to the complex tangent space of $\partial X \cap U$.

Remak 2.1. The relation between the above definition and other ones is the following.

The next three conditions are equivalent to one another:
I) *X is a domain with pseudoconvex boundary ∂X.*
II) *There exist a neighborhood W of ∂X and a real valued C^∞-function Φ on W such that* i) $X\cap W=\{x\in W|\Phi(x)<0\}$, ii) *the gradient of Φ nowhere vanishes on W,* iii) *the complex Hessian of Φ is positive semi-definite when restricted to the complex tangent space of ∂X.*
III) *X is a locally Stein domain with smooth boundary ∂X. Here we say X is locally Stein if for any point p of ∂X, there exists a neighborhood U of p such that $U\cap X$ is Stein.*

Outline of the proof. We can show I) $\Rightarrow$II) easily (see [15], the proof of p. 263, 4. Proposition). For II) $\Rightarrow$III), since III) is a local property, we may assume that $W=U$ is an open ball centered at the origin of $\boldsymbol{C}^n$ and the boundary $B=\{x\in U|\Phi(x)=0\}$ contains the origin and satisfies the property of II). Then the Euclidean metric function $d(x)$ from $x\in\{\Phi<0\}$ to B is C^∞ near the boundary B by the implicit function theorem and moreover, from [18], Theorems 2.6.7 and 2.6.12, $-\log d$ is C^∞-plurisubharmonic on $V_\varepsilon\cap\{\Phi<0\}$, where $V_\varepsilon=\{|z|^2<\varepsilon\}$ is an open ball contained in U. Then we may assume that $-\log d\geqq 0$. Hence $1/(\varepsilon-|z|^2)-\log d$ is a C^∞-strictly plurisubharmonic exhaustion function on $V_\varepsilon\cap\{\Phi<0\}$ and so $V_\varepsilon\cap\{\Phi<0\}$ is Stein. If X is locally Stein, III) $\Rightarrow$I) is due to E. E. Levi (see [15], p. 264, 1. Proposition).

Let $ds^2=\sum_{\alpha,\beta=1}^{n} g_{i,\alpha\bar\beta}dz_i^\alpha d\bar z_i^\beta$ be a hermitian metric on M and let $\sigma=\sum_{\alpha,\beta=1}^{n}\theta_{i,\alpha\bar\beta}dz_i^\alpha\wedge d\bar z_i^\beta$ be a C^∞-(1,1) form on M whose matrix of coefficients $\theta_{i,\alpha\bar\beta}$ is a hermitian matrix. We set

$$G_i=(g_{i,\alpha\bar\beta})\quad\text{and}\quad \Theta_i=(\theta_{i,\alpha\bar\beta}). \tag{2.1}$$

If J_{ij} are the transition functions $\left(\dfrac{\partial z_i^\alpha}{\partial z_j^\beta}\right)$ of the tangent bundle TM, then on $U_i\cap U_j$, $G_i={}^tJ_{ji}G_j\bar J_{ji}$ and $\Theta_i={}^tJ_{ji}\Theta_j\bar J_{ji}$ so that the coefficients of the characteristic polynomial $\det(G_i^{-1}\Theta_i-\lambda E)$ are C^∞-functions on M. The eigenvalues of $G_i^{-1}\Theta_i$ at each point x are real, let them be

(2.2) $$\varepsilon_1(x) \geqq \varepsilon_2(x) \geqq \cdots \geqq \varepsilon_n(x),$$

so that each ε_α is a real valued continuous function on M.

Definition 2.2. A holomorphic line bundle $F \xrightarrow{\pi} M$ is said to be *q-positive* (resp. *q-semi-positive*) with respect to a given hermitian metric ds^2 on a subset Y of M if there exist a coordinate cover $\{U_i\}_{i\in I}$ such that $\pi^{-1}(U_i)$ are trivial and a hermitian metric $\{a_i\}$ along the fibres of F such that for $G_i=(g_{i,\alpha\bar\beta})$ and $\Theta_i=\left(\dfrac{\partial^2 \log a_i}{\partial z_i^\alpha \partial \bar z_i^\beta}\right)$, $\varepsilon_{n-q+1}(x)+n\inf(0, \varepsilon_n(x))$ is positive (resp. non-negative) on Y.

Here $\varepsilon_\alpha(x)$ should be understood as in (2.2).

Remark 2.2. If F is q-positive (resp. q-semi-positive) with respect to ds^2 on Y, $G_i^{-1}\Theta_i$ has at least $n-q+1$ positive (resp. non-negative) eigenvalues on Y. In particular, if F is 1-positive (resp. 1-semi-positive) with respect to ds^2 on Y, the hermitian matrix Θ_i is positive-definite (resp. positive semi-definite) on Y. Since the inverse is true for any hermitian metric ds^2, we say simply that F is positive (resp. semi-positive) on Y instead that F is 1-positive (resp. 1-semi-positive).

§ 2.2. Basic Estimates

Let X be a (connected) relatively compact domain with smooth boundary ∂X and let $F\xrightarrow{\pi} M$ be a holomorphic line bundle on M. Let K_M be the canonical line bundle on M. We set a Kähler metric ds^2 on M

(2.3) $$ds^2=\sum_{\alpha,\beta=1}^{n} g_{i,\alpha\bar\beta}\, dz_i^\alpha d\bar z_i^\beta$$

and a hermitian metric of F and its curvature tensor

(2.4) $$\{a_i\}$$ $$\theta_{i,\alpha\bar\beta}=\partial_\alpha\partial_{\bar\beta}\log a_i\,,$$

Then, from (1.21),

(2.5) $$\{c_i=a_i\cdot g_i\} \quad \text{where} \quad g_i=\det(g_{i,\alpha\bar\beta})$$

is a hermitian metric of $F\otimes K_M$. With respect to (2.3) and (2.4), an inner product (,) and a norm ‖ , ‖ are defined as in Section 1.1. Since the curvature tensor of $\{c_i\}$ is the sum of $\theta_i=(\theta_{i,\alpha\bar{\beta}})$ and the Ricci curvature with respect to (2.3), we have from Proposition 1.5

$$\|\bar{\partial}\varphi\|^2+\|\vartheta\varphi\|^2 = \int_{\partial X}(c_i|\text{grad}\,\Phi|)^{-1}\sum_{\substack{D_{p-1}=(d_1,\dots,d_{p-1})\\ d_1<\dots<d_{p-1}}}\sum_{\beta,\gamma=1}^{n}\partial_\beta\partial_{\bar{\gamma}}\Phi\varphi_{i,}{}^{\beta}{}_{\bar{D}_{p-1}}\overline{\varphi_i^{\gamma D_{p-1}}}dS + \|\bar{\nabla}\varphi\|^2+\int_X c_i^{-1}\sum_{\substack{D_{p-1}=(d_1,\dots,d_{p-1})\\ d_1<\dots<d_{p-1}}}\sum_{\alpha,\gamma=1}^{n}\theta_{i,\bar{\gamma}}^{\bar{\alpha}}\varphi_{i,\bar{\alpha}\bar{D}_{p-1}}\overline{\varphi_i^{\gamma D_{p-1}}}dV \tag{2.6}$$

for any $\varphi\in B^{0,p}(X,F\otimes K_X)$.

Proposition 2.1. *Let X be a (connected) relatively compact domain with the pseudoconvex boundary ∂X on a complex manifold M and let $F\xrightarrow{\pi}M$ be a holomorphic line bundle on M.*

I) *If there exists a Kähler metric $d\sigma^2$ on a neighborhood W of ∂X and $F\xrightarrow{\pi}M$ is q-positive with respect to $d\sigma^2$ on W, then there exist a proper compact subset K_1 of X and a positive constant C_1 such that*

$$\|\varphi\|^2_{X\backslash K_1}\leqq C_1\{\|\bar{\partial}\varphi\|^2+\|\bar{\partial}^*\varphi\|^2+\|\varphi\|^2_{K_1}\} \tag{2.7}$$

for any $\varphi\in D^{0,p}_{\bar{\partial}}\cap D^{0,p}_{\bar{\partial}^*}\subset L^{0,p}(X,F\otimes K_M)$ *and* $p\geqq q$.

II) *If M is provided with a Kähler metric ds^2 and $F\xrightarrow{\pi}M$ is q-semipositive with respect to ds^2 on a neighborhood V of $\bar{X}=X\cup\partial X$ and q-positive with respect to ds^2 on $V\backslash K$, where K is a proper cmpact subset of X, then there exist a proper compact subset K_2 of X with $K\subset\text{Int}\,K_2$ and a positive constant C_2 such that*

$$\|\varphi\|^2_{X\backslash K_2}\leqq C_2\{\|\bar{\partial}\varphi\|^2+\|\bar{\partial}^*\varphi\|^2\} \tag{2.8}$$

for any $\varphi\in D^{0,p}_{\bar{\partial}}\cap D^{0,p}_{\bar{\partial}^*}\subset L^{0,p}(X,F\otimes K_M)$ *and* $p\geqq q$.

Proof. We prove II) first. Let $\{a_i\}$ be the hermitian metric of F with respect to the covering $\{U_i\}_{i\in I}$ of M corresponding to the assumption. To obtain the required estimate, by Proposition 1.6 iii), it suffices to show that the estimate holds for any element of $B^{0,p}(\bar{X},F\otimes K_M)$. From Remark 2.1 II), there exists a defining C^∞-function Φ for ∂X

such that the complex Hessian of Φ is positive semi-definite when restricted to the complex tangent space of ∂X. Hence, by using notations (2.3), (2.4) and (2.5), it follows that the first term of the right-hand side of (2.6) is non-negative. Hence we have

$$\int_X c_i^{-1} \sum_{\substack{D_{p-1}=(d_1,\cdots,d_{p-1})\\ d_1<\cdots<d_{p-1}}} \sum_{\alpha,\gamma=1}^{n} \Theta_{i,\bar\gamma}^{\bar\alpha} \varphi_{i,\bar\alpha\bar D_{p-1}} \overline{\varphi_i^{\gamma D_{p-1}}} dV \tag{2.9}$$

$$\leqq \|\bar\partial\varphi\|^2 + \|\bar\partial^*\varphi\|^2 \qquad \text{for any} \quad \varphi \in B^{0,p}(\overline{X}, F\otimes K_M).$$

Let $\{\varepsilon_\alpha\}_{1\leqq\alpha\leqq n}$ be the eigenvalues of $G_i^{-1}\Theta_i$, where $G_i=(g_{i,\alpha\bar\beta})$, and choose a system of local coordinates $(z_i^1, \cdots, z_i^n)$ around $x_0 \in X$ as follows:

$$G_i(x_0) = (\delta_{\alpha\beta}) \quad \text{and} \quad \Theta_i(x_0) = (\varepsilon_\alpha(x_0)\delta_{\alpha\beta}). \tag{2.10}$$

By the assumption, for a suitable proper compact subset K_2 of X satisfying $K \subset \operatorname{Int} K_2$, there exists a positive constant κ, independent of the choice of $x_0 \in X \backslash K_2$, satisfying

$$\varepsilon_{n-q+1}(x_0) + n \inf(0, \varepsilon_n(x_0)) \geqq \kappa > 0. \tag{2.11}$$

We apply (2.10) and (2.11) to (2.9), then at $x_0 \in X\backslash K_2$

$$\sum_{\substack{D_{p-1}=(d_1,\cdots,d_{p-1})\\ d_1<\cdots<d_{p-1}}} \sum_{\alpha,\gamma=1}^{n} \theta_{i,\bar\gamma}^{\bar\alpha} \varphi_{i,\bar\alpha\bar D_{p-1}} \overline{\varphi_i^{\gamma D_{p-1}}} \tag{2.12}$$

$$= \sum_{d_1<\cdots<d_{p-1}} \sum_{\alpha=1}^{n} \varepsilon_\alpha(x_0) |\varphi_{i,\bar\alpha\bar D_{p-1}}|^2$$

$$\geqq \varepsilon_{n-q+1}(x^0) \sum_{d_1<\cdots<d_{p-1}} \sum_{\alpha=1}^{n-q+1} |\varphi_{i,\bar\alpha\bar D_{p-1}}|^2$$

$$+ \inf(0, \varepsilon_n(x_0)) \sum_{d_1<\cdots<d_{p-1}} \sum_{\alpha=n-q+2}^{n} |\varphi_{i,\bar\alpha\bar D_{p-1}}|^2 .$$

If $p \geqq q$, then $p+n-q+1 \geqq n+1$, thus any block D_p of p-indices taken from $\{1, 2, \cdots, n\}$ must contain one of the indices $\{1, 2, \cdots, n-q+1\}$ i.e., one of the indices corresponding to the positive eigenvalues $\varepsilon_1(x_0), \cdots, \varepsilon_{n-q+1}(x_0)$. Then we have

$$\sum_{d_1<\cdots<d_{p-1}} \sum_{\alpha=1}^{n-q+1} |\varphi_{i,\bar\alpha\bar D_{p-1}}|^2 \geqq \sum_{d_1<\cdots<d_p} |\varphi_{i,\bar D_p}|^2$$

(2.13) and

$$n \sum_{d_1<\cdots<d_p} |\varphi_{i,\bar D_p}|^2 \geqq \sum_{d_1<\cdots<d_{p-1}} \sum_{\alpha=n-q+2}^{n} |\varphi_{i,\bar\alpha\bar D_{p-1}}|^2 .$$

Hence, from (2.11), (2.12) and (2.13), we have

$$(2.14)\qquad \sum_{d_1<\cdots<d_{p-1}} \sum_{\alpha,\gamma=1}^{n} \theta_{i,\bar{\gamma}}{}^{\bar{\alpha}} \varphi_{i,\bar{\alpha}\bar{D}_{p-1}} \overline{\varphi_i{}^{\gamma D_{p-1}}}$$
$$\geqq \kappa \sum_{d_1<\cdots<d_p} |\varphi_{i,\bar{D}_p}|^2 \qquad \text{at} \quad x_0 \in X \backslash K_2.$$

Since $\varepsilon_{n-q+1} + n \inf(0, \varepsilon_n) \geqq 0$ on K_2, from (2.9) and (2.14), if we take $C_2 = 1/\kappa$, we obtain

$$\|\varphi\|^2_{X\backslash K_2} \leqq C_2 \{\|\bar{\partial}\varphi\|^2 + \|\bar{\partial}^*\varphi\|^2\}$$

for any $\varphi \in B^{0,p}(\overline{X}, F\otimes K_M)$ with $p \geqq q$. This completes the proof of II).

Next we prove I). From the assumption and Remark 2.1 II), we may assume that there exists a neighborhood W of ∂X such that i) the gradient of the defining function Φ nowhere vanishes on W, ii) there exists a Kähler metric $d\sigma^2$ on W, iii) there exist a coordinate cover $\{U_i\}_{i\in I}$ of M, for which $\pi^{-1}(U_i)$ are trivial and $U_i \subsetneqq W$ if $U_i \cap \partial X \neq \emptyset$, and a hermitian metric $\{a_i\}$ of F satisfying the q-positivity with respect to $d\sigma^2$ on W. Hence by the same reason as in the proof of II), we have only to show the required estimate for elements of $B^{0,p}(\overline{X}, F\otimes K_M)$. Take neighborhoods W_i $(i=1, 2, 3)$ of ∂X with $\partial X \subset W_3 \Subset W_2 \Subset W_1 \Subset W$. We may assume that $M = W_1 \cup X$. Then we take a hermitian metric ds^2

$$(2.15)\qquad ds^2 = \sum_{\alpha,\beta=1}^{n} g_{i,\alpha\bar{\beta}} dz_i^\alpha d\bar{z}_i^\beta$$

on M. such that $ds^2 = d\sigma^2$ on W_1.

We take a C^∞-real valued function χ on M such that i) $\chi \equiv 1$ on $(M\backslash\overline{X}) \cup W_3$, ii) $\operatorname{supp}\chi \subset W_2$. Then, for any $\varphi \in B^{p,q}(\overline{X}, F\otimes K_M)$, $\chi\varphi$ is again contained in $B^{p,q}(\overline{X}, F\otimes K_M)$ and $\operatorname{supp}\chi\varphi$ is contained in W_1. We set

$$K_1 = X \backslash (X \cap W_3).$$

Since the metric ds^2 is Kähler on W_1, from (1.16)′ of Remark 1.1, the equation (2.6) holds for elements $\chi\varphi$, where $\varphi \in B^{0,p}(\overline{X}, F\otimes K_M)$. Hence by using notations in (2.4), (2.5) and (2.15), we obtain the following inequality in the same way as in the proof of II),

$$\int_{X\cap W_1} c_i^{-1} \sum_{\substack{D_{p-1}=(d_1,\cdots,d_{p-1})\\ d_1<\cdots<d_{p-1}}} \sum_{\alpha,\gamma=1}^{n} \theta_{i,\bar{\gamma}}^{\bar{\alpha}}(\chi\varphi)_{i,\bar{\alpha}\bar{D}_{p-1}} \overline{(\chi\varphi)_i^{\gamma D_{p-1}}} dV \tag{2.16}$$

$$\leqq \|\bar{\partial}(\chi\varphi)\|^2 + \|\bar{\partial}^*(\chi\varphi)\|^2 \quad \text{for any} \quad \varphi \in B^{0,p}(\bar{X}, F\otimes K_M).$$

We estimate the both sides of this inequality. By the assumption, we apply (2.11), (2.12), (2.13) and (2.14) in the proof of I) to $\chi\varphi$ and K_1. Then the left-hand side may be estimated in the following way.

If $p \geqq q$, then

$$\|\varphi\|^2_{X\setminus K_1} \leqq \|\chi\varphi\|^2_{X\cap W_1} \tag{2.17}$$

$$\leqq \frac{1}{\kappa} \int_{X\cap W_1} c_i^{-1} \sum_{d_1<\cdots<d_{p-1}} \sum_{\alpha,\gamma=1}^{n} \theta_{i,\bar{\gamma}}^{\bar{\alpha}}(\chi\varphi)_{i,\bar{\alpha}\bar{D}_{p-1}} \overline{(\chi\varphi)_i^{\gamma D_{p-1}}} dV$$

for some positive constant κ as which is taken in (2.11).

As for the right-hand side,

$$\|\bar{\partial}(\chi\varphi)\|^2 + \|\bar{\partial}^*(\chi\varphi)\|^2$$
$$= \|\bar{\partial}\chi\wedge\varphi + \chi\bar{\partial}\varphi\|^2 + \|-*c_i\partial * c_i^{-1}(\chi\varphi)\|^2$$
$$= \|\bar{\partial}\chi\wedge\varphi + \chi\bar{\partial}\varphi\|^2 + \|\chi\bar{\partial}^*\varphi - *(\partial\chi\wedge *\varphi)\|^2$$
$$= \|\bar{\partial}\chi\wedge\varphi\|^2 + 2\mathrm{Re}(\bar{\partial}\chi\wedge\varphi, \chi\bar{\partial}\varphi) + \|\chi\bar{\partial}\varphi\|^2$$
$$+ \|\chi\bar{\partial}^*\varphi\|^2 + 2\mathrm{Re}(\chi\bar{\partial}^*\varphi, -*(\partial\chi\wedge *\varphi)) + \|*(\partial\chi\wedge *\varphi)\|^2$$
$$\leqq 2\{\|\bar{\partial}\chi\wedge\varphi\|^2 + \|\chi\bar{\partial}\varphi\|^2 + \|\partial\chi\wedge *\varphi\|^2 + \|\chi\bar{\partial}^*\varphi\|^2\}$$
$$\leqq 2\left\{\|\bar{\partial}\varphi\|^2 + \|\bar{\partial}^*\varphi\|^2 + 2c_0\left(\int_{K_1} |\mathrm{grad}\,\chi|^2 \langle\varphi, \varphi\rangle dV\right)\right\}.$$

Here we used [45], p. 18, Lemma A and the fact that the star operator $*$ is isometric with respect to the pointwise inner product $\langle\ ,\ \rangle$ and c_0 is a positive constant depending only the dimension of M and $|\mathrm{grad}\,\chi|^2$ is the length of $\mathrm{grad}\,\chi$ with respect to ds^2.

Therefore, if

$$C_1 \geqq \frac{4}{\kappa} \max\{1, c_0 \sup_{x\in M} |\mathrm{grad}\,\chi|^2(x)\}, \tag{2.18}$$

we have, from (2.16), (2.17) and (2.18),

$$\|\varphi\|^2_{X\setminus K_1} \leqq C_1\{\|\bar{\partial}\varphi\|^2 + \|\bar{\partial}^*\varphi\|^2 + \|\varphi\|^2_{K_1}\}$$

for any $\varphi \in B^{0,p}(\bar{X}, F\otimes K_M)$ with $p \geqq q$. This completes the proof of I).

Let X be a relatively compact domain with smooth boundary ∂X and let Φ be a defining function for X on a neighborhood V of ∂X. Then we may assume that the gradient of Φ nowhere vanishes on V. We set

$$(2.19) \qquad X_t=\{x\in V|\Phi(x)<t\}\cup X, \quad \partial X_t=\{x\in V|\Phi(x)=t\}$$

for any t with $0\leqq t<\sup_{x\in V}\Phi(x)$, and in particular

$$X=X_0, \quad \partial X=\partial X_0 .$$

Let $F\xrightarrow{\pi}M$ be a holomorphic line bundle on M. Then, if we fix a hermitian metric on M and a hermitian metric of F, for each $t\geqq 0$, we can consider Hilbert spaces $L^{\cdot,\cdot}(X_t, F\otimes K_M)$, which are the completions of $C_0^{\cdot,\cdot}(X_t, F\otimes K_M)$ with respect to the norm $\|\ ,\ \|_t^2=\int_{X_t}\langle\ ,\ \rangle dV$, and operators $\bar\partial_t, \bar\partial_t^*$ and ϑ_t in $L^{\cdot,\cdot}(X_t, F\otimes K_M)$.

Proposition 2.2. *Let $\{X_t\}_{t\geqq 0}$ and F be as above. If there exist* i) *a positive constant δ such that for any t $(0\leqq t\leqq\delta)$ X_t is a domain with pseudoconvex boundary ∂X_t, and* ii) *a Kähler metric $d\sigma^2$ on a neighborhood V of ∂X such that $F\xrightarrow{\pi}M$ is q-positive with respect to $d\sigma^2$ on V, then there exist a positive constant C_3 independent of t and a proper compact subset K_3 such that for each t $(0\leqq t\leqq\delta)$*

$$(2.20) \qquad \|\varphi\|_{t,X_t\setminus K_3}^2\leqq C_3\{\|\bar\partial_t\varphi\|_t^2+\|\bar\partial_t^*\varphi\|_t^2+\|\varphi\|_{K_3}^2\}$$

for any $\varphi\in D_{\bar\partial_t}^{0,p}\cap D_{\bar\partial_t^*}^{0,p}\subset L^{0,p}(X_t, F\otimes K_M)$ with $p\geqq q$.

The proof is similar to that of Proposition 2.1 I). In fact, C_3 in that proposition depends only on the length of the gradient of C^∞-function on M and the lower bound of $\varepsilon_{n-q+1}+n\inf(0,\varepsilon_n)$ with respect to $G_i^{-1}\Theta_i$ (see (2.11) and (2.18)).

§ 2.3. Weak Finiteness and Vanishing Theorems

Theorem 2.3. *Let X be a (connected) domain with pseudoconvex boundary ∂X on a complex manifold M and let $F\xrightarrow{\pi}M$ be a holomorphic line bundle on M.*

I) *If there exists a Kähler metric $d\sigma^2$ on a neighborhood of ∂X and*

$F\xrightarrow{\pi}M$ *is q-positive with respect to* $d\sigma^2$ *on its neighborhood, then it holds that if* $p\geqq q$,

i) $L^{0,p}(X, F\otimes K_M)\cong R^{0,p}_{\bar\partial\bar\partial^*}\oplus N^{0,p}\oplus R^{0,p}_{\bar\partial^*\bar\partial}$,

ii) $\dim_C N^{0,p}<+\infty$

and so

iii) *the image of the restriction homomorphism*

$$r: H^p(M, \mathcal{O}(F\otimes K_M))\to H^p(X, \mathcal{O}(F\otimes K_M))$$

has finite dimension.

II) *If M is provided with a Kähler metric* ds^2 *and* $F\xrightarrow{\pi}M$ *is q-semi-positive with respect to* ds^2 *on a neighborhood V of* $\bar X$ *and q-semi-positive with respect to* ds^2 *on* $V\setminus K$, *where K is a proper compact subset of X, then if* $p\geqq q$, *for any* $f\in L^{0,p}(X, F\otimes K_M)$ *with* $\bar\partial f=0$, *there exists* $g\in L^{0,p-1}(X, F\otimes K_M)$ *satisfying* $f=\bar\partial g$ *and so the natural homomorphism*

$$r_1:\ H^p(M, \mathcal{O}(F\otimes K_M))\to H^p(X, \mathcal{O}(F\otimes K_M))$$

$$r_2:\ H^p_c(X, \mathcal{O}(F\otimes K_M))\to H^p(X, \mathcal{O}(F\otimes K_M))$$

are zero maps for $p\geqq q$, *where* $H^p_c(X, \mathcal{O}(F\otimes K_M))$ *denotes the p-th cohomology group with compact supports.*

In particular, we obtain

Corollary 2.4. *Let X, M and F be as above.*

I) *If* $F\xrightarrow{\pi}M$ *is positive on a neighborhood of* ∂X, *it holds that if* $p\geqq 1$,

i) $L^{0,p}(X, F\otimes K_M)\cong R^{0,p}_{\bar\partial\bar\partial^*}\oplus N^{0,p}\oplus R^{0,p}_{\bar\partial^*\bar\partial}$,

ii) $\dim_C N^{0,p}<+\infty$

and so

iii) *the image of the restriction homomorphism*

$$r:\ H^p(M, \mathcal{O}(F\otimes K_M))\to H^p(X, \mathcal{O}(F\otimes K_M))$$

has finite dimension.

II) *If M is provided with a Kähler metric* ds^2 *and* $F\xrightarrow{\pi}M$ *is semi-positive on a neighborhood V of* $\bar X$ *and positive on* $V\setminus K$, *where K* *is a proper compact subset of* X, *then if* $p \geqq 1$, *for any* $f \in L^{0,p}(X, F \otimes K_M)$ *with* $\bar{\partial} f = 0$, *there exists* $g \in L^{0,p-1}(X, F \otimes K_M)$ *satisfying* $f = \bar{\partial} g$ *and the natural homomorphisms*

$$r_1 : H^p(M, \mathcal{O}(F \otimes K_M)) \to H^p(X, \mathcal{O}(F \otimes K_M))$$

$$r_2 : H^p_c(X, \mathcal{O}(F \otimes K_M)) \to H^p(X, \mathcal{O}(F \otimes K_M))$$

are zero maps for $p \geqq 1$.

Proof of Theorem 2.3. I). i) and ii) of I) follow from Proposition 1.11 I) and Proposition 2.1 I). While, if we let $L^{0,p}_{\mathrm{loc}}(Y, F \otimes K_M)$ denote the set of the locally square integrable $(0, p)$ forms on Y with values in $F \otimes K_M$ for any open subset Y of M, then there is a natural isomorphism

$$(2.21) \quad H^p(Y, \mathcal{O}(F \otimes K_M)) \cong \frac{\{f \in L^{0,p}_{\mathrm{loc}}(Y, F \otimes K_M) : \bar{\partial} f = 0\}}{\{f \in L^{0,p}_{\mathrm{loc}}(Y, F \otimes K_M) : f = \bar{\partial} g \text{ for some } g \in L^{0,p-1}_{\mathrm{loc}}(Y, F \otimes K_M)\}}.$$

Here we consider the operator $\bar{\partial}$ in the sense of distribution. Hence, there is the following factorization of the homomorphism r

$$\begin{array}{ccc} H^p(M, \mathcal{O}(F \otimes K_M)) & & \\ \downarrow & \circlearrowleft \quad \searrow^{r} & \\ N^{0,p}_{\bar{\partial}} / R^{0,p}_{\bar{\partial}} & \longrightarrow & H^p(X, \mathcal{O}(F \otimes K_M)) \end{array}$$

From ii) of I) and $N^{0,p}_{\bar{\partial}} / R^{0,p}_{\bar{\partial}} \cong N^{0,p}$, $\mathrm{Im}\, r$ is finite dimensional.

II). The former assertion follows from Propositions 1.10 i), 1.11 II) and 2.1 II). Combining this with (2.21), the latter one follows immediately. Q.E.D.

Remark 2.3. Such finiteness theorems of weak type as Theorem 2.3, I) and Corollary 2.4, I) were treated in [9], [22], [23], [24] and [28] etc., from the viewpoint of boundary regularity of the $\bar{\partial}$-operator. The basic estimates used in these papers are more precise than ours in the following sense.

Let X be a relatively compact domain with smooth boundary ∂X on a complex manifold M and let the defining function Φ of X be *strongly q pseudoconvex* on a neighborhood V of ∂X. For any holomorphic line bundle F on M and a fixed hermitian metric $\{a_i'\}$ of F, we consider $\{a_i = a_i' \exp(\tilde{\Phi})\}$ as a new hermitian metric of F, where $\tilde{\Phi}$ is a C^∞-function on M and coincides with the original Φ on a neighborhood of ∂X. Then, there exist a hermitian metric ds^2 on M and a positive constant C_* such that

(*)

i) for $\{a_i\}$, F is q-positive with respect to ds^2 on a neighborhood of ∂X.

ii) if $s \geqq 1$ and $t \geqq q$,

$$a_i^{-1} \sum_{A_s, B_{t-1}} \sum_{\alpha, \beta=1}^{n} \partial_\alpha \partial_{\bar\beta} \tilde{\Phi} \varphi_{i,}{}^{\alpha}{}_{A_s \bar{B}_{t-1}} \overline{\varphi_i{}^{\bar{A}_s \beta B_{t-1}}}$$
$$\geqq C_* a_i^{-1} \sum_{A_s, B_t} \varphi_{i, A_s \bar{B}_t} \overline{\varphi_i{}^{\bar{A}_s B_t}} \quad \text{on } \partial X \text{ for every } \varphi \in B^{s,t}(\bar{X}, F).$$

(See [5], § 5. 16, Lemmas 18, 19 and 20 and § 5. 17, Lemma 21.)

In general, the metric ds^2 is not Kähler near the boundary ∂X, nevertheless, we can prove that there exist a positive constant C_{**} and a compact subset K of X such that, if $\varphi \in B^{s,t}(\bar{X}, F)$ and $s \geqq 1$, $t \geqq q$,

(**)

i) $\|\varphi\|^2_{X \setminus K} \leqq C_{**} \{\|\bar\partial \varphi\|^2 + \|\bar\partial^* \varphi\|^2 + \|\varphi\|^2_K\}$

ii) $\|\bar\nabla \varphi\|^2 + \|\varphi\|^2 + \int_{\partial X} \langle \varphi, \varphi \rangle dS \leqq C_{**} \{\|\bar\partial \varphi\|^2 + \|\bar\partial^* \varphi\|^2 + \|\varphi\|^2\}$

where the integral $\|\bar\nabla \varphi\|^2$ is defined as same as in Proposition 1. 5 with respect to the covariant differentiation $\bar\nabla$ of type (0, 1) associated to the connection of ds^2.

The above inequalities are not new and, essentially, due to Hörmander [17], Theorem 3. 2. 5 and Proposition 3. 4. 4, Kohn and Rossi [29], 3. 12 Proposition and 5. 8 Theorem, and Andreotti and Vesentini [6]. In particular, the estimate (**) (ii) is a crucial one. In fact, the estimate (**) (ii) implies that the graph norm $Q(\varphi, \varphi) = (\bar\partial \varphi, \bar\partial \varphi) + (\bar\partial^* \varphi, \bar\partial^* \varphi) + (\varphi, \varphi)$ is completely continuous on the space $B^{s,t}(\bar{X}, F)$ (see [22], 6. 2 Theorem and 6. 16 Proposition). In this connection, the graph norm $Q(\, , \,)$ is always completely continuous on the space $C_0^{s,t}(X, F)$ by the ellipticity of the operator $\square=\bar{\partial}\vartheta+\vartheta\bar{\partial}$. By using this property of the graph norm $Q(\ ,\ )$, Kohn and Nirenberg proved boundary regularity of the operator $\square=\bar{\partial}\vartheta+\vartheta\bar{\partial}$, which is, in their sense, strongly elliptic operator of second order, *i.e. in the situation as above, if* $s\geqq 1$ *and* $t\geqq q$,

i) *any element* φ *in* $N^{s,t}=N^{s,t}_{\bar{\partial}}\cap N^{s,t}_{\bar{\partial}^*}\subset L^{s,t}(X,F)$, *which is the completion of* $C^{s,t}_0(X,F)$ *by the norm associated with the above* $\{a_i\}$ *and* ds^2, *can be taken as an element of* $C^{s,t}(\bar{X},F)$

ii) *any element* $\varphi\in C^{s,t}(\bar{X},F)$ with $\bar{\partial}\varphi=0$ and $\varphi\perp N^{s,t}$, *there exists an element* $\psi\in C^{s,t-1}(\bar{X},F)$ *with* $\bar{\partial}\psi=\varphi$.

(*For detailed descriptions. see* [22] *and* [9] 3.1.11 *Proposition and* 3.1.15 *Proposition.*)

But in our situation, the quadratic form $\sum_{D_{p-1}}\sum_{\beta,\gamma=1}^{n}\partial_\beta\bar{\partial}_\gamma\Phi\varphi_{i,}{}^{\beta}{}_{\bar{D}_{p-1}}\cdot\overline{\varphi_i{}^{\gamma D_{p-1}}}$ is at most non-negative on every point of ∂X (compare with (*) (ii)). This obstruction can not be covered by only the curvature conditon of bundles and actually, if in Proposition 2.1 the boundary $\partial X=\{x\in W|\Phi(x)=0\}$ is Levi-flat, local boundary regularity breaks down (see [27], § 9, Propagation of singularities for $\bar{\partial}$).

While, vanishing theorems of weak type as Theorem 2.3, II) and Corollary 2.4, II) were treated in [3], [4], [5] and [36] etc. In these works, the completeness with respect to Kähler metrics was an important ground for the proof. In the situation of Theorem 2.3, II), the actual situation is that a complete Kähler metric $d\sigma^2$ exists on X. $d\sigma^2$ is constructed as follows. Let $d(x)$ be the distance from $x\in X$ to ∂X with respect to the given Kähler metric ds^2 on M and d is a C^∞-function near ∂X. Then $\inf_{\|\lambda\|=1}\sum_{\alpha,\beta=1}^{n}\frac{\partial^2(-\log d)}{\partial z^\alpha\partial\bar{z}^\beta}\lambda_\alpha\bar{\lambda}_\beta$ is bounded from below with some constant, which may not be non-negative generally, near ∂X uniformly (see [8], Principal lemma). We extend d to a C^∞-function $\tilde{d}$ on X without changing the original near ∂X in a suitable manner. If we take a positive constant κ large enough, $d\sigma^2=\sum_{\alpha,\beta=1}^{n}\frac{\partial^2(-\log\tilde{d})}{\partial z^\alpha\partial\bar{z}^\beta}dz^\alpha d\bar{z}^\beta+\kappa ds^2$ is, again, a Kähler metric on X and since $-\log\tilde{d}$ is an exhaustion function on X and the gradient of $\tilde{d}$ nowhere vanishes near ∂X, $d\sigma^2$ is a complete Kähler metric on X (see [31], Proposition 1).

Therefore, if ∂X is pseudoconvex and there exists a Kähler metric on a neighborhood of ∂X, then there exists a complete hermitian metric on X which is Kähler near ∂X. But, in our situation, the existence of these complete metrics does not seem to be so useful. Because, the boundedness from below of the eigenvalues of the curvature of F with respect to these complete metrics can not be easily verified. But pseudoconvexity and the existence of complete Kähler metrics have a deep relation. (See [36], [47], [49].)

In these points, our estimates in Section 2.2 by using the formula (1.30) seem to be most appropriate.

§ 2.4. A Stability Theorem for Spaces of Harmonic Forms

Through this section, we use notations in (2.19) and Proposition 2.2, and set ourselves in the following situation:

i) *There exists a positive constant* δ *such that for any* t $(0 \leqq t \leqq \delta)$ X_t *is a domain with pseudoconvex boundary* ∂X_t.
ii) *There exist a Kähler metric* $d\sigma^2$ *on a neighborhood* V *of* ∂X *and a holomorphic line bundle* $F \xrightarrow{\pi} M$ *such that* F *is q-positive with respect to* $d\sigma^2$ *on* V.

Then, since we may assume that $\partial X_\delta \subsetneqq V$, from Proposition 2.2, weak finite theorem holds for each t $(0 \leqq t \leqq \delta)$, i.e., *if* t $(0 \leqq t \leqq \delta)$ *and* $p \geqq q$, then

i) $L^{0,p}(X_t, F \otimes K_M) \cong R^{0,p}_{\bar\partial_t \bar\partial_t^*} \oplus N^{0,p}_t \oplus R^{0,p}_{\bar\partial_t^* \bar\partial_t}$ *where* $N^{0,p}_t = N^{0,p}_{\bar\partial_t} \cap N^{0,p}_{\bar\partial_t^*}$,

ii) $\dim_C N^{0,p}_t < +\infty$.

In this way, we obtain a family $\{N^{0,p}_t\}_{0 \leqq t \leqq \delta}$ of spaces of harmonic forms parametrized by the defining function Φ. By composing the following natural holomorphisms,

$$\text{restriction map} \quad r^{t_1}_{t_2} \colon L^{0,p}(X_{t_1}, F \otimes K_M) \to L^{0,p}(X_{t_2}, F \otimes K_M) \quad (0 \leqq t_2 < t_1 \leqq \delta)$$

$$\text{orthogonal projection} \quad H_t \colon L^{0,p}(X_t, F \otimes K_M) \to N^{0,p}_t$$

$$(0 \leqq t \leqq \delta),$$

we obtain a homomorphism

$$\rho_{t_2}^{t_1}: N_{t_1}^{0,p} \to N_{t_2}^{0,p} \quad (0 \leqq t_2 < t_1 \leqq \delta).$$

We denote other two orthogonal projections by

$$R_t: L^{0,p}(X_t, F \otimes K_M) \to R_{\bar{\partial}_t \bar{\partial}_t^*}^{0,p}$$

$$R_t^*: L^{0,p}(X_t, F \otimes K_M) \to R_{\bar{\partial}_t^* \bar{\partial}}^{0,p} \quad (p \geqq q \text{ and } 0 \leqq t \leqq \delta).$$

The main result of this section is the following stability theorem.

Theorem 2.5. *In the above situation, there exists a postive constant* δ_* $(\delta_* \leqq \delta)$ *such that for any* t $(0 < t \leqq \delta_*)$ *and* $p \geqq q$, *the homomorphism* $\rho_0^t: N_t^{0,p} \to N_0^{0,p}$ *is an isomorphism.*

Remark 2.4. If Φ is *plurisubharmonic*, a stronger result hold (see Chapter III, Theorem 3.5). Our method, which will be used to prove Theorem 2.5, seems to be an interesting one in the theory of $\bar{\partial}$-operator with boundary condition.

The remainder of this section is devoted to the proof of Theorem 2.5. In the first place, we prove the following.

Proposition 2.6. *Under the circumstance mentioned at the beginning of this section, there exist positive constants* δ_0 $(\delta_0 \leqq \delta)$ *and* C_4 *such that, for any* t $(0 \leqq t \leqq \delta_0)$,

$$\|\varphi\|_t^2 \leqq C_4 \{ \|\bar{\partial}_t \varphi\|_t^2 + \|\bar{\partial}_t^* \varphi\|_t^2 \} \tag{2.22}$$

if $\varphi \in D_{\bar{\partial}_t}^{0,p} \cap D \subset L_{\bar{\partial}_t^*}^{0,p}(X_t, F \otimes K_M)$ *and* $(r_0^t(\varphi), h)_0 = 0$ *for all* $h \in N_0^{0,p} \subset L^{0,p}(X, F \otimes K_M)$.

Proof. Assume that the assertion is false. For any C, we can find arbitrarily small t for which (2.20) does not hold. If necessary, take a subsequence and we may assume that there exist sequences $\{t_n\}_{n \geqq 1}$ and $\{\varphi_n\}_{n \geqq 1}$ such that

i) $t_n > 0$, $t_n > t_{n+1}$ and $t_n \to 0$ as $n \to +\infty$,

$$\text{ii)}\quad \varphi_n\in D^{0,p}_{\bar\partial t_n}\cap D^{0,p}_{\bar\partial^*_{t_n}}\subset L^{0,p}(X_{t_n}, F\otimes K_M)$$

$$(2.23)\qquad \|\varphi_n\|^2_{t_n}=1 \quad\text{and}\quad \|\bar\partial_{t_n}\varphi_n\|^2_{t_n},\ \|\bar\partial^*_{t_n}\varphi_n\|^2_{t_n}\to 0 \ \text{as}\ n\to+\infty,$$

$$\text{iii)}\quad (\varphi_n, h)_0=0 \quad\text{for all}\quad h\in N^{0,p}_0\subset L^{0,p}(X, F\otimes K_M).$$

From (2.23) ii), we may assume that $\{\varphi_n\}$ is weakly convergent to some φ in $L^{0,p}(X, F\otimes K_M)$. On the other hand, if $\psi\in C^{0,p-1}(\bar X, F\otimes K_M)$, we have

$$\begin{aligned}|(\bar\partial\psi,\varphi)_0| &= \lim_{n\to+\infty}|(\bar\partial\psi,\varphi_n)_0| = \lim_{n\to+\infty}|(\bar\partial_{t_n}\psi,\varphi_n)_{t_n}-(\bar\partial_{t_n}\psi,\varphi_n)_{X_{t_n}\setminus X}|\\ &\leqq \lim_{n\to+\infty}|(\psi,\bar\partial^*_{t_n}\varphi_n)_{t_n}|+\lim_{n\to+\infty}|(\bar\partial_{t_n}\psi,\varphi_n)_{X_{t_n}\setminus X}|\\ &\leqq \lim_{n\to+\infty}\|\psi\|_{t_n}\|\bar\partial^*_{t_n}\varphi_n\|_{t_n}+\lim_{n\to+\infty}\|\bar\partial_{t_n}\psi\|_{X_{t_n}\setminus X}=0\end{aligned}$$

and

$$\lim_{n\to+\infty}\|\bar\partial\varphi_n\|^2_0\leqq\lim_{n\to+\infty}\|\bar\partial_{t_n}\varphi_n\|^2_{t_n}=0\,.$$

Hence, by Proposition 1.6 ii), we have that $\varphi\in D^{0,p}_{\bar\partial^*}\subset L^{0,p}(X, F\otimes K_M)$ and $\bar\partial^*\varphi=0$ in $L^{0,p-1}(X, F\otimes K_M)$ and $\bar\partial\varphi=0$ in $L^{0,p+1}(X, F\otimes K_M)$. So φ is contained in $N^{0,p}_0$. But each φ_n is orthogonal to $N^{0,p}_0$. Therefore we obtain $\varphi=0$. While, by the same argument in the proof of Proposition 1.11 I), we may assume that $\{\varphi_n\}_{n\geqq 1}$ is strongly convergent to zero on K_3. From (2.20) and (2.23) ii), we have

$$1=\|\varphi_n\|^2_{t_n}\leqq C_3\{\|\bar\partial_{t_n}\varphi\|^2_{t_n}+\|\bar\partial^*_{t_n}\varphi_n\|^2_{t_n}\}+(C_3+1)\|\varphi_n\|^2_{K_3}\to 0$$

as $n\to+\infty$. This is a contradiction. Q.E.D.

Proof of Theorem 2.5. *Step I. If* $p\geqq q$ *and* $0<t\leqq\delta_0$, ρ^t_0: $N^{0,p}_t\to N^{0,p}_0$ *is injective. If* $\varphi\in N^{0,p}_t$ and $\rho^t_0(\varphi)=0$, then $r^t_0(\varphi)\in R^{0,p}_{\bar\partial\bar\partial^*}$. Hence $r^t_0(\varphi)$ is orthogonal to $N^{0,p}_0$. Therefore, from Proposition 2.6 (2.22), we have $\|\varphi\|^2_t\leqq C_4\{\|\bar\partial_t\varphi\|^2_t+\|\bar\partial^*_t\varphi\|^2_t\}=0$. Hence $\varphi=0$.

Step II. There exists a positive constant δ_* ($\delta_*\leqq\delta_0$) *such that if* $p\geqq q$ *and* $0<t\leqq\delta_*$, ρ^t_0: $N^{0,p}_t\to N^{0,p}_0$ *is surjective.*

We remark that the following diagram is commutative: if $t_1>t_2>0$,

$$\begin{array}{ccc} N^{0,p}_{t_1} & \xrightarrow{\rho^{t_1}_0} & N^{0,p}_0 \\ & \circlearrowleft & \\ \rho^{t_1}_{t_2} \searrow & & \nearrow \rho^{t_2}_0 \\ & N^{0,p}_{t_2} & \end{array} \qquad \text{i.e.} \quad \rho^{t_1}_0 = \rho^{t_2}_0 \circ \rho^{t_1}_{t_2} .$$

To see this, it suffices to show that $H_0 \circ r^{t_2}_0 = H_0 \circ r^{t_2}_0 \circ H_{t_2}$ i.e. $\rho^{t_2}_0 = \rho^{t_2}_0 \circ H_{t_2}$ on $N^{0,p}_{\bar{\partial} t_2}$. Take $\varphi \in N^{0,p}_{\bar{\partial} t_2}$, then $r^{t_2}_0(\varphi) = R_0 \circ r^{t_2}_0(\varphi) + H_0 \circ r^{t_2}_0(\varphi) = R_0 \circ r^{t_2}_0(\varphi) + H_0(r^{t_2}_0 \circ R_{t_2}(\varphi) + r^{t_2}_0 \circ H_{t_2}(\varphi)) = R_0 \circ r^{t_2}_0(\varphi) + H_0 \circ r^{t_2}_0 \circ H_{t_2}(\varphi)$. Hence $\rho^{t_2}_0 = \rho^{t_2}_0 \circ H_{t_2}$ on $N^{0,p}_{\bar{\partial} t_2}$. We come back to the beginning. Assume that the assertion is false. From the injectivity of ρ^t_0 and the above remark, it holds that

i) if $t < t'$, $\rho^t_0(N^{0,p}_t)$ is contained in $\rho^{t'}_0(N^{0,p}_{t'})$ as a closed subspace,

ii) $\dim_{\boldsymbol{C}} \rho^t_0(N^{0,p}_t) < \dim_{\boldsymbol{C}} N^{0,p}_0$ for $0 < t \leqq \delta_0$.

These imply that there exists a positive constant δ_1 such that if $0 < t \leqq \delta_1$, $\rho^t_0(N^{0,p}_t) = \rho^{\delta_1}_0(N^{0,p}_{\delta_1})$ and $\dim_{\boldsymbol{C}} \rho^{\delta_1}_0(N^{0,p}_{\delta_1}) < \dim_{\boldsymbol{C}} N^{0,p}_0$. Hence there exists an element $u \neq 0$ of $N^{0,p}_0$ such that if $0 < t \leqq \delta_1$,

$$(2.24) \qquad (u, h)_0 = 0 \quad \text{for all} \quad h \in N^{0,p}_t.$$

We shall show that

$$(u, g)_0 = 0 \quad \text{for all} \quad g \in N^{0,p}_0$$

in this situation. This means $u = 0$ and we arrive at a contradiction. Take a sequence $\{t_n\}_{n \geqq 1}$ such that $t_n \leqq \delta_1$, $t_n > t_{n+1}$ and $t_n \to 0$ as $n \to +\infty$ and extend the definition of u by setting $u = 0$ on $M \backslash X$. We denote it by u'. Then, from (2.24), we have for any element φ of $N^{0,p}_{\bar{\partial} t_n}$

$$\begin{aligned} (u', \varphi)_{t_n} &= (u, r^{t_n}_0(\varphi))_0 = (u, R_0 \circ r^{t_n}_0(\varphi) + H_0 \circ r^{t_n}_0(\varphi))_0 \\ &= (u, R_0 \circ r^{t_n}_0(\varphi))_0 + (u, H_0 \circ r^{t_n}_0(\varphi))_0 \\ &= (u, \rho^{t_n}_0(\varphi))_0 = 0 . \end{aligned}$$

Hence u' is orthogonal to $N^{0,p}_{\bar{\partial} t_n}$ $(n \geqq 1)$ and so $u' \in R^{0,p}_{\bar{\partial}^*_{t_n} \bar{\partial}}$ $(n \geqq 1)$. From Proposition 1.10 ii) and Proposition 2.6, there exists an element $v_n \in L^{0,p+1}(X_{t_n}, F \otimes K_M)$ such that $\bar{\partial}^*_{t_n} v_n = u'$ and $\|v_n\|_{t_n} \leqq C^{1/2}_4 \|u\|_0$ for any $n \geqq 1$. Hence we may asume that $\{v_n\}_{n \geqq 1}$ is weakly convergent to some $v \in L^{0,p+1}(X, F \otimes K_M)$. If $\psi \in C^{0,p}(\overline{X}, F \otimes K_M)$, we have

$$|(\bar{\partial}\psi, v)_0| = \lim_{n \to +\infty} |(\bar{\partial}\psi, v_n)_0|$$

$$= \lim_{n\to+\infty} |(\bar{\partial}_{t_n}\psi, v_n)_{t_n} - (\bar{\partial}_{t_n}\psi, v_n)_{X_{t_n}\setminus X}|$$

$$\leqq \lim_{n\to+\infty} |(\psi, \bar{\partial}^*_{t_n} v_n)_{t_n}| + \lim_{n\to+\infty} |(\bar{\partial}_{t_n}\psi, v_n)_{X_{t_n}\setminus X}|$$

$$\leqq \|\varphi\|_0 \|u\|_0 + C_4^{1/2} \|u\|_0 (\lim_{n\to+\infty} \|\bar{\partial}_{t_n}\psi\|_{X_{t_n}\setminus X})$$

$$= \|\psi\|_0 \|u\|_0 .$$

Hence $v \in D^{0,p+1}_{\bar{\partial}^*} \subset L^{0,p+1}(X, F\otimes K_M)$. While, for any $\varphi \in C_0^{0,p}(X_0, F\otimes K_M)$, we have

$$(\varphi, u)_0 = \lim_{n\to+\infty} (\varphi, u)_{t_n} = \lim_{n\to+\infty} (\varphi, \bar{\partial}^*_{t_n} v_n)_{t_n}$$

$$= \lim_{n\to+\infty} (\bar{\partial}_{t_n}\varphi, v_n)_{t_n} = (\bar{\partial}\varphi, v)_0 .$$

Hence $\vartheta v = u$. Since $v \in D^{0,p+1}_{\bar{\partial}^*}$, we have $\bar{\partial}^* v = u$ (see Proposition 1.7 iii)). Therefore, if $g \in N^{0,p}_0$, we have $(u, g)_0 = (\bar{\partial}^* v, g)_0 = (v, \bar{\partial} g)_0 = 0$. This completes the proof of Theorem 2.5.

Remark 2.5. The author does not know whether we can derive the finite dimensionality of $H^p(X_t, \mathcal{O}(F\otimes K_M))$ for $p \geqq q$ and some $t > 0$ in the situation of this section. To see this, at least, we need some approximation theorem of the Runge type (for example, see Proposition 3.3 and the proof of Theorem 3.6 in Chapter III). But, at present, the author can only prove the following approximation theorem of weak type.

Under the situation of this section, if $p \geqq q-1$, the closure of the space $\bigcup_{0<t\leqq\delta_0} r_0^t(N^{0,p}_{\bar{\partial}_t})$ in $L^{0,p}(X, F\otimes K_M)$ coincides with the closed subspace $N^{0,p}_{\bar{\partial}}$ in $L^{0,p}(X, F\otimes K_M)$, where δ_0 is the positive constant taken in Propositition 2.6, i.e., for any $\varepsilon > 0$ and $\varphi \in N^{0,p}_{\bar{\partial}} \subset L^{0,p}(X, F\otimes K_M)$, there exist a positive constant δ ($\delta \leqq \delta_0$) depending on φ and ε, and $\tilde{\varphi} \in N^{0,p}_{\bar{\partial}_\delta} \subset L^{0,p}(X_\delta, F\otimes K_M)$ satisfying $\|\tilde{\varphi}|_X - \varphi\|^2 < \varepsilon$.

Proof. If there exists a non-zero element $u \in N^{0,p}_{\bar{\partial}} \subset L^{0,p}(X, F\otimes K_M)$ such that, for every t $(0 < t \leqq \delta_0)$,

$$(u, f)_0 = 0 \qquad \text{for any} \quad f \in N^{0,p}_{\bar{\partial}t} \subset L^{0,p}(X_t, F\otimes K_M),$$

then we shall show that

$$(u, g)_0 = 0 \qquad \text{for any} \quad g \in N_{\bar\partial}^{0,p} \subset L^{0,p}(X, F \otimes K_M).$$

This leads us to a contradiction. We set $u' = u$ on X and 0 on $M \setminus X$. Then u' is contained in $L^{0,p}(X_t, F \otimes K_M)$ and orthogonal to $N_{\bar\partial_t}^{0,p}$ for every $0 < t \leqq \delta_0$. Hence $u' \in \overline{R_{\bar\partial_t^* \bar\partial_t}^{0,p}} = R_{\bar\partial_t^* \bar\partial_t}^{0,p}$ (in particular, when $p = q-1$, this equality is secured by the closedness of the range $R_{\bar\partial_t \bar\partial_t^*}^{0,q}$). Take a sequence $\{t_n\}$ such that $t_n \leqq \delta_0$ $(n \geqq 1)$, $t_n > t_{n-1}$ and $t_n \to 0$ as $n \to +\infty$. Then, by the same reason in Step II of the proof of Theorem 2.5, there exists a sequence $\{v_n\}_{n \geqq 1}$ such that $v_n \in L^{0,p+1}(X_{t_n}, F \otimes K_M)$, $\bar\partial_{t_n}^* v_n = u'$ and $\|v_n\|_{t_n} \leqq C_4^{1/2} \|u\|_0$ for every $n \geqq 1$. Hence, by the similar manner in Step II of the proof of Theorem 2.5, there exists an element $v \in L^{0,p+1}(X, F \otimes K_M)$ satisfying $\bar\partial^* v = u$ in $L^{0,p}(X, F \otimes K_M)$. Therefore, if $g \in N_{\bar\partial}^{0,p} \subset L^{0,p}(X, F \otimes K_M)$, we have $(u, g)_0 = 0$. Q.E.D.

Chapter III. Cohomology Groups on Weakly 1-Complete Manifolds

§ 3.1. Definitions and the Basic Estimate

Let X be a connected complex manifold of dimension n.

Definition 3.1. A complex manifold X is said to be *weakly* 1-*complete* if there exists a C^∞-exhausting plurisubharmonic function Φ on X. Φ is called an *exhaustion function* on X.

Remark 3.1. If c is a non-critical value of Φ, plurisubharmonicity of Φ implies that $X_c = \{x \in X \mid \Phi(x) < c\}$ is *a domain with pseudoconvex boundary* $\partial X_c = \{x \in X \mid \Phi(x) = c\}$.

In Sections 3.1 and 3.2, we set the following situation.

(3.1) i) *X is a weakly* 1-*complete manifold with respect to* Φ.

ii) *There exists a Kähler metric $d\sigma^2$ on $X \setminus K_4$.*

iii) *$F \xrightarrow{\pi} X$ is a holomorphic line bundle on X which is q-positive with respect to $d\sigma^2$ on $X \setminus K_4$, where K_4 is a proper compact subset* *of* X.

We choose compact subsets K'', K' and K of X such that $K_4 \subset \mathrm{Int}\, K''$, $K'' \subset \mathrm{Int}\, K'$ and $K' \subset \mathrm{Int}\, K$. We set

(3.2) $$\{a_i\}$$

the hermitian metric of F with respect to the coordinate cover $\{U_i\}_{i\in I}$ corresponding to the assumption. Let K_X be the canonical line bundle of X. Then we define a hermitian metric ds^2 on X and a hermitian metric of $F\otimes K_X$ as follows:

(3.3) i) $ds^2 = \sum_{\alpha,\beta=1}^{n} g_{i,\alpha\bar\beta} dz_i^\alpha d\bar z_i^\beta$,

ii) $ds^2 = d\sigma^2$ on $X\backslash K'$,

iii) $c_i = a_i g_i$ where $g_i = \det(g_{i,\alpha\bar\beta})$.

We set

i) $\theta_i = (\theta_{i,\alpha\bar\beta})$ where $\theta_{i,\alpha\bar\beta} = \partial_\alpha \bar\partial_{\bar\beta} \log a_i$,

ii) $G_i = (g_{i,\alpha\bar\beta})$.

For any open subset Y of X, let $L^{0,p}(Y, F\otimes K_X)$ be the completion of $C_0^{0,p}(Y, F\otimes K_X)$ by the norm $\|\ ,\ \|_Y$ defined by (3.3). We sometimes omit the symbol Y if it is clear. Linear operators $\bar\partial, \bar\partial^*$ (resp. $\bar\partial_\Psi^*$) and ϑ (resp. ϑ_Ψ) are defined in the same way as in Chapter I, Section 1.2. We fix non-critical values c_0 and c_1 of Φ such that $c_1 > c_0$ and $c_0 > \sup_{x\in K} \Phi(x)$. We set

$$X_i = \{x \in X \mid \Phi(x) < c_i\} \text{ and } \partial X_i = \{x\in X \mid \Phi(x) = c_i\} \quad \text{for } i = 0, 1.$$

We take a C^∞-increasing convex function λ on $(-\infty, \infty)$ such that

(3.4) $$\lambda(t) \begin{cases} = 0 & \text{if } t \leqq c_0 \\ > 0 & \text{if } t > c_0 . \end{cases}$$

We define hermitian metrics of F and $F\otimes K_X$ respectively as follows:

(3.5) i) $a_{m,i} = a_i \exp(m\lambda(\Phi))$

ii) $c_{m,i} = a_{m,i} g_i$

for every non-negative integer m.

We set

$$\Theta_{m,i}=(\theta_{m,i,\alpha\bar{\beta}}) \quad \text{where} \quad \theta_{m,i,\alpha\bar{\beta}}=\partial_\alpha\bar{\partial}_\beta \log a_{m,i} \quad (m\geqq 1).$$

With respect to (3.3) i) and (3.5), we set

$$(\varphi,\psi)_m=\int_{X_1}\langle\varphi,\psi\rangle_m dV$$

$$\|\varphi\|_m^2=(\varphi,\varphi)_m \qquad \text{for any} \quad m\geqq 0$$

where $\langle\varphi,\psi\rangle_m=\langle\varphi,\psi\rangle e^{-m\lambda(\Phi)}$.

For any m, the completion of $C_0^{0,p}(X_1, F\otimes K_X)$ by the norm $\|\ ,\ \|_m$ coincides with $L^{0,p}(X_1, F\otimes K_X)$. We denote the adjoint operator of the maximal closed extension $\bar{\partial}$ in $L^{\cdot\cdot}(X_1, F\otimes K_X)$ with respect to the inner product $(\ ,\ )_m$ by $\bar{\partial}_m^*$. Then from Proposition 1.7 iv), we have $D_{\bar{\partial}^*}^{0,p}=D_{\bar{\partial}_m^*}^{0,p}$ in $L^{0,p}(X_1, F\otimes K_X)$ for any $m\geqq 1$ and $p\geqq 1$.

Proposition 3.1. *There is a positive constant C_4 such that*

$$\text{(3.6)} \qquad \|\varphi\|_{m,X_1\backslash K}^2\leqq C_4\{\|\bar{\partial}\varphi\|_m^2+\|\bar{\partial}_m^*\varphi\|_m^2+\|\varphi\|_K^2\}$$

if $m\geqq 0$, $\varphi\in D_{\bar{\partial}}^{0,p}\cap D_{\bar{\partial}^*}^{0,p}$ and $p\geqq q$.

Proof. From (1.34) and Proposition 1.6 iii), it sufficies to show that the estimate holds for any element of $B^{0,p}(\bar{X}_1, F\otimes K_X)$. Let χ be a C^∞-function on X such that $\chi=1$ on $X\backslash K$ and $\operatorname{supp}\chi\subsetneqq X\backslash K'$. From Remark 3.1, we obtain the following inequality in the same way as in the proof of Proposition 2.1 I).

$$\text{(3.7)} \qquad \int_{X\backslash K'} c_{m,i}^{-1} \sum_{\substack{D_{p-1}=(d_1,\cdots,d_{p-1})\\ d_1<\cdots<d_{p-1}}} \sum_{\alpha,\gamma=1}^{n} \theta_{m,i,\bar{\gamma}}^{\ \bar{\alpha}}(\chi\varphi)_{i,\bar{\alpha}\bar{D}_{p-1}}\overline{(\chi\varphi)_i^{\ \gamma D_{p-1}}}dV$$

$$\leqq\|\bar{\partial}(\chi\varphi)\|_m^2+\|\bar{\partial}_m^*(\chi\varphi)\|_m^2$$

for $\varphi\in B^{0,p}(\bar{X}_1, F\otimes K_X)$.

In the first place, we estimate the left hand side. Let $\{\varepsilon_{m,\alpha}\}_{1\leqq\alpha\leqq n}$ (resp. $\{\varepsilon_\alpha\}_{1\leqq\alpha\leqq n}$) be the eigenvalues of the matrix $G_i^{-1}\Theta_{m,i}$ (resp. $G_i^{-1}\Theta_i$). Since $G_i^{-1}\Theta_i\leqq G_i^{-1}\Theta_{m,i}$, by the minimum-maximum principle for the eigenvalues, we have $\varepsilon_{m,n-q+1}\geqq\varepsilon_{n-q+1}$ and $\varepsilon_{m,n}\geqq\varepsilon_n$ on $X\backslash K''$. Hence at any point $x_0\in X\backslash K'$, we have the following similarly to the proof of Proposition 2.1 II): If $p \geqq q$, then

$$\sum_{D_{p-1}} \sum_{\alpha,\gamma=1}^{n} \theta_{m,i\bar{\gamma}}^{\bar{\alpha}} (\chi\varphi)_{i,\bar{\alpha}\bar{D}_{p-1}} \overline{(\chi\varphi)_i^{\gamma D_{p-1}}}$$

$$\geqq (\varepsilon_{m,n-q+1}(x_0) + n \inf(0, \varepsilon_{m,n}(x_0))) \sum_{D_p} |(\chi\varphi)_{i,\bar{D}_p}|^2$$

$$\geqq (\varepsilon_{n-q+1}(x_0) + n \inf(0, \varepsilon_n(x_0)) \sum_{D_p} |(\chi\varphi)_{i,\bar{D}_p}|^2$$

$$\geqq \kappa \sum_{D_p} |(\chi\varphi)_{i,\bar{D}_p}|^2,$$

where $\kappa = \inf_{x \in \bar{X}_1 \backslash \mathrm{Int}\, K''} (\varepsilon_{n-q-1}(x) + n \inf(0, \varepsilon_n(x))) > 0$.

Hence we have

$$(3.8) \qquad \|\varphi\|^2_{m,X_1\backslash K} \leqq \|\chi\varphi\|^2_{m,X_1\backslash K'}$$

$$\leqq 1/\kappa \int_{X_1\backslash K'} c_{m,i}^{-1} \sum_{D_{p-1}} \sum_{\alpha,\gamma=1}^{n} \theta_{m,i,\bar{\gamma}}^{\bar{\alpha}} (\chi\varphi)_{i,\bar{\alpha}\bar{D}_{p-1}} \overline{(\chi\varphi)_i^{\gamma D_{p-1}}} dV.$$

Next, repeating the discussion of (2.17) in the proof of Proposition 2.1 I), we can estimate the right hand side of (3.7) in the following way:

$$(3.9) \qquad 1/\kappa \{\|\bar{\partial}(\chi\varphi)\|^2_m + \|\bar{\partial}^*_m(\chi\varphi)\|^2_m\} \leqq C_4 \{\|\bar{\partial}\varphi\|^2_m + \|\bar{\partial}^*_m\varphi\|^2_m + \|\varphi\|^2_K\},$$

where C_4 is a positive constant and independent of m (see (2.18)).

Still, in (3.9), we used $\| \ , \ \|_{m,K} = \| \ , \ \|_K$ for any $m \geqq 1$ (see (3.4)).

From (3.7) (3.8) and (3.9), we obtain the required estimate (3.6) for elements of $B^{0,p}(\bar{X}_1, F \otimes K_X)$. Q.E.D.

§ 3.2. Finiteness, Isomorphic and Representation Theorems

Let our situation be the same as in Section 3.1. From (3.1) and Remark 3.1, for any non-critical value c of Φ, $X_c = \{x \in X | \Phi(x) < c\}$ and $F \xrightarrow{\pi} X$ satisfy the conditions of Theorem 2.3 I) and so we have

$$(3.10) \qquad L^{0,p}(X_c, F \otimes K_X) \simeq R^{0,p}_{\bar{\partial}\bar{\partial}^*} \oplus N^{0,p}_c \oplus R^{0,p}_{\bar{\partial}^*\bar{\partial}}$$

and

$$\dim_C N^{0,p}_c < +\infty, \quad \text{for} \quad p \geqq q.$$

Proposition 3.2. *There exist a positive constant* C_5 *and an* *integer* m_0 *such that for any* $m \geqq m_0$ *and* $p \geqq q$

(3.11) $$\|\varphi\|_m^2 \leqq C_3\{\|\bar{\partial}\varphi\|_m^2 + \|\bar{\partial}_m^*\varphi\|_m^2\}$$

for $\varphi \in D_{\bar{\partial}}^{0,p} \cap D_{\bar{\partial}^*}^{0,p} \subset L^{0,p}(X_1, F \otimes K_X)$, *and*

$$(\varphi, h)_{X_0} = 0 \quad \textit{for all} \quad h \in N_{c_0}^{0,p} \subset L^{0,p}(X_0, F \otimes K_X).$$

Proof. Assume that the assertion were false. There would be a sequence $\{\varphi_m\}$ such that

(3.12) i) $\varphi_m \in D_{\bar{\partial}}^{0,p} \cap D_{\bar{\partial}^*}^{0,p}$,

ii) $\|\varphi_m\|_m^2 = 1$ and $\|\bar{\partial}\varphi_m\|_m^2$, $\|\bar{\partial}_m^*\varphi_m\|_m^2 \to 0$ as $\to +\infty$

and

iii) $(\varphi_m, h)_{X_0} = 0$ for all $h \in N_{c_0}^{0,p} \subset L^{0,p}(X_0, F \otimes K_X)$.

Let $g_m = e^{-m\lambda(\Phi)}\varphi_m$, then we have

(3.13) i) $\vartheta g_m = e^{-m\lambda(\Phi)}\bar{\partial}_m^*\varphi_m$,

ii) $\|\vartheta g_m\|_{-m} = \|\bar{\partial}_m^*\varphi_m\|_m$.

By (3.12), we have

$$\|g_m\| \leqq \|g_m\|_{-m} = \|\varphi_m\|_m = 1.$$

Therefore, choosing a subsequence if necessary, we may assume that $\{g_m\}$ has a weak limit g in $L^{0,p}(X_1, F \otimes K_X)$. From (3.12) ii) and (3.13) ii), we have

(3.14) $$\lim_{m \to +\infty} \|\vartheta g_m\| \leqq \lim_{m \to +\infty} \|\vartheta g_m\|_{-m} = \lim_{m \to +\infty} \|\bar{\partial}_m^*\varphi_m\|_m = 0.$$

On the other hand, for every $\varepsilon > 0$,

$$\int_{\{x \in X \mid \Phi(x) > c_0 + \varepsilon\}} e^{m\lambda(\Phi)} \langle g_m, g_m \rangle dV \leqq 1,$$

and so we have

$$e^{m\lambda(c_0+\varepsilon)} \int_{\{x \in X \mid \Phi(x) \geqq c_0 + \varepsilon\}} \langle g_m, g_m \rangle dV \leqq 1.$$

It follows that $\int_{\{x \in X \mid \Phi(x) \geqq c_0 + \varepsilon\}} \langle g_m, g_m \rangle dV$ tends to zero and hence $g_m \to 0$ almost everywhere in $\{x \in X \mid \Phi(x) \geqq c_0 + \varepsilon\}$. Hence $g = 0$ on $\{x \in X \mid \Phi(x) \geqq c_0 + \varepsilon\}$ for every $\varepsilon > 0$. Combining this with (3.14), we have

(3.15) $$\vartheta g=0 \quad \text{and} \quad \operatorname{supp} g \subseteqq \overline{X}_0 .$$

By Proposition 1.6 i), we have

(3.16) $$\bar{\partial}^* (g|_{X_0}) = 0 .$$

By (3.12) ii), we have

(3.17) $$\bar{\partial} (g|_{X_0}) = 0 .$$

From (3.12) iii), (3.16) and (3.17), we have

(3.18) $$g = 0 .$$

Since $\bar{\partial}^* = \bar{\partial}^*_m$ on X_0 for every m, by the same argument as in the proof of Proposition 1.11 I), we may assume that $\{g_m\}$ converges strongly on K. Since $g_m = \varphi_m$ on K, from (3.6), (3.12) ii) and (3.18), we have

$$1 = \|\varphi_m\|_m^2 \leqq C_4 \{\|\bar{\partial}\varphi_m\|_m^2 + \|\bar{\partial}^*_m \varphi_m\|_m^2\} + (C_4+1)\|\varphi_m\|_K^2 \to 0$$

as $m \to +\infty$. This is a contradiction. Q.E.D.

Proposition 3.3. *If $\psi \in L^{0,p}((X_0, F\otimes K_X)$ with $\bar{\partial}\psi = 0$ and $p \geqq q-1$, then for any $\varepsilon > 0$, there exists an element $\tilde{\psi}$ of $L^{0,p}(X_1, F\otimes K_X)$ such that $\|\tilde{\psi}|_{X_0} - \psi\|_{X_0}^2 < \varepsilon$ and $\bar{\partial}\tilde{\psi} = 0$.*

Proof. It suffices to show that if $u \in L^{0,p}(X_0, F\otimes K_X)$ and

(3.19) $$(f, u)_{X_0} = 0$$

for any $f \in L^{0,p}(X_1, F\otimes K_X)$ with $\bar{\partial} f = 0$, then we have

(3.20) $$(g, u)_{X_0} = 0$$

for $g \in L^{0,p}(X_0, F\otimes K_X)$ with $\bar{\partial} g = 0$.

We extend u by setting $u=0$ on $X \backslash X_0$. We denote it by u'. Then, from (3.19), u' is orthogonal to $N_{\bar{\partial}}^{0,p} \subset L^{0,p}(X_1, F\otimes K_X)$. Hence we have $u' \in \overline{R^{0,p}_{\bar{\partial}^*_m \bar{\partial}}}$ for any $m \geqq 0$. While Proposition 3.1 implies that $R^{0,p}_{\bar{\partial}^*_m \bar{\partial}}$ is closed for every $p \geqq q-1$ and $m \geqq 0$. Hence from Proposition 1.10 ii) and Proposition 3.2, for any $m \geqq m_0$, there exists an element $v_m \in D_{\bar{\partial}^*}^{0,p+1}$ such that

(3.21) $$u' = \bar{\partial}^*_m v_m \quad \text{and} \quad \|v_m\|_m^2 \leqq C_5 \|u\|_{X_0}^2 .$$

We set

$$w_m = e^{-m\lambda(\Phi)} v_m \qquad \text{for} \quad m \geqq m_0 .$$

Then we have

$$\text{(3. 22)} \qquad \text{i)} \quad \|w_m\|^2 \leqq C_5 \|u\|_{X_0}^2, \qquad \text{ii)} \quad \vartheta w_m = u'.$$

Hence $\{w_m\}$ has a subsequence which is weakly convergent to some w in $L^{0,p+1}(X_1, F \otimes K_X)$. From (3. 22), we have

$$\text{(3. 23)} \qquad \vartheta w = u' \quad \text{and} \quad \operatorname{supp} w \subseteqq \overline{X}_0.$$

By Proposition 1. 6 i), we have

$$\text{(3. 24)} \qquad \bar{\partial}^* (w|_{X_0}) = u.$$

Therefore, if $g \in L^{0,p}(X_0, F \otimes K_X)$ with $\bar{\partial} g = 0$, then $(u, g)_{X_0} = (\bar{\partial}^* (w|_{X_0}), g)_{X_0} = (w|_{X_0}, \bar{\partial} g)_{X_0} = 0$. Q.E.D.

For any real number c with $c > \sup_{x \in K} \Phi(x)$, we set

$$H^{0,p}(X_c, F \otimes K_X) = N_{\bar{\partial}}^{0,p} / \overline{R_{\bar{\partial}\bar{\partial}^*}^{0,p}}$$

where $X_c = \{x \in X | \Phi(x) < c\}$ and $L^{0,p}(X_c, F \otimes K_X) \cong \overline{R_{\bar{\partial}\bar{\partial}^*}^{0,p}} \oplus N_c^{0,p} \oplus \overline{R_{\bar{\partial}^*\bar{\partial}}^{0,p}}$.

If c is a non-critical value of Φ, from (3. 10), we have

$$\text{(3. 25)} \qquad H^{0,p}(X_c, F \otimes K_X) = N_{\bar{\partial}}^{0,p} / R_{\bar{\partial}\bar{\partial}^*}^{0,p} \cong N_c^{0,p}$$

and

$$\dim H^{0,p}(X_c, F \otimes K_X) < +\infty \quad \text{for} \quad p \geqq q.$$

Proposition 3. 4. *The restriction homomorphism*

$$r: H^{0,p}(X_1, F \otimes K_X) \to H^{0,p}(X_0, F \otimes K_X)$$

is an isomorphism for $p \geqq q$.

Proof. Step I. The homomorphism r: $H^{0,p}(X_1, F \otimes K_X) \to H^{0,p}(X_0, F \otimes K_X)$ *is injective.*

Take an element f in $L^{0,p}(X_1, F \otimes K_X)$ such that $\bar{\partial} f = 0$ and $f = \bar{\partial} g$ in $L^{0,p}(X_0, F \otimes K_X)$ for some $g \in L^{0,p-1}(X_0, F \otimes K_X)$. Then f satisfies the relation $(f, h)_{X_0} = 0$ for all $h \in N_{c_0}^{0,p}$. Hence, from Proposition 1. 10 i) and Proposition 3. 2, there exists an element $\tilde{g}$ in $L^{0,p-1}(X_1, F \otimes K_X)$ such that $\bar{\partial} \tilde{g} = f$ in $L^{0,p}(X_1, F \otimes K_X)$.

Step II. The homomorphism $r: H^{0,p}(X_1, F\otimes K_X)\to H^{0,p}(X_0, F\otimes K_X)$ *is surjective.*

From Proposition 3.3, $\mathrm{Im}\, r$ is dense in $H^{0,p}(X_0, F\otimes K_X)$. Since r is injective and $H^{0,p}(X_0, F\otimes K_X)$ is a finite dimensional vector space in view of (3.25), r is surjective. Q.E.D.

In particular, we obtain the following theorem from (3.25) and Proposition 3.4.

Theorem 3.5. (*Compare with Theorem* 2.5.) *The homomorphism*

$$\rho_{c_0}^{c_1}: N_{c_1}^{0,p}\to N_{c_0}^{0,p}$$

is an isomorphism $(p\geqq q)$.

Theorem 3.6 (*Finiteness Theorem*). *The restriction homomorphism*

$$r: H^p(X, \mathcal{O}(F\otimes K_X))\to H^{0,p}(X_0, F\otimes K_X)$$

is an isomorphism and so

$$\dim_{\boldsymbol{C}} H^p(X, \mathcal{O}(F\otimes K_X))<+\infty \quad (p\geqq q).$$

Proof. By Sard's theorem, we can choose a sequence $\{c_\nu\}_{\nu\geqq 2}$ of real numbers such that

(3.26) i) $c_2>c_1$,

ii) $c_{\nu+1}>c_\nu$ and $c_\nu\to+\infty$ as $\nu\to+\infty$,

iii) The boundary ∂X_{c_ν} of $\{x\in X|\Phi(x)\leqq c_\nu\}$ is smooth for any $\nu\geqq 2$.

We set

$$X_\nu=\{x\in X|\Phi(x)<c_\nu\} \quad \text{for} \quad \nu\geqq 2.$$

For any pair $(c_{\nu+1}, c_\nu)$ $(\nu\geqq 1)$, we can apply Proposition 3.2 and so Proposition 3.3 holds, i.e., *if* $\nu\geqq 0$ *and* $p\geqq q-1$, *the restriction homomorphism*

(3.27) $r_{c_\nu}^{c_{\nu+1}}: N_{\bar\partial_{\nu+1}}^{0,p} \to N_{\bar\partial_\nu}^{0,p}$ *has a dense image with respect to the norm* $\|\ ,\ \|_{X_\nu}$ *where* $N_{\bar\partial_\nu}^{0,p} \subset L^{0,p}(X_\nu, F\otimes K_X)$.

Moreover, from Proposition 3.4, we have the following: *if* $\nu \geqq 0$ *and* $p \geqq q$, *the restriction homomorphism*

(3.28) $$r_{c_\nu}^{c_{\nu+1}}: H^{0,p}(X_{\nu+1}, F\otimes K_X) \to H^{0,p}(X_\nu, F\otimes K_X)$$

is an isomorphism.

Let $L_{\mathrm{loc}}^{0,p}(X, F\otimes K_X)$ be the set of the locally square integrable $(0, p)$ forms on X with values in $F\otimes K_X$. Then, there is a natural isomorphism such that for any $p\geqq 1$ and open subset Y of X,

(3.29) $$H^p(Y, \mathcal{O}(F\otimes K_X)) \cong \frac{\{f\in L_{\mathrm{loc}}^{0,p}(Y, F\otimes K_X)\,|\,\bar\partial f=0\}}{\{f\in L_{\mathrm{loc}}^{0,p}(Y, F\otimes K_X)\,|\,f=\bar\partial g \text{ for some } g\in L_{\mathrm{loc}}^{0,p-1}(Y, F\otimes K_X)\}}.$$

After these preparations, we come back to the proof of Theorem 3.6.

Step I. The homomorphism $r: H^p(X, \mathcal{O}(F\otimes K_X)) \to H^{0,p}(X_0, F\otimes K_X)$ *is injective* $(p\geqq q)$.

In view of (3.29), it suffices to show that if we take an element $f\in L_{\mathrm{loc}}^{0,p}(X, F\otimes K_X)$ with $\bar\partial f=0$ and $f=\bar\partial g_0'$ for some $g_0'\in L^{0,p-1}(X_0, F\otimes K_X)$, there exists an element $g\in L_{\mathrm{loc}}^{0,p-1}(X, F\otimes K_X)$ such that $\bar\partial g=f$. We set

$$f_\nu = f|_{X_\nu} \quad \text{for every} \quad \nu\geqq 0.$$

Then from (3.28), there exists $g_\nu'\in L^{0,p-1}(X_\nu, F\otimes K_X)$ such that $\bar\partial g_\nu' = f_\nu$ for every $\nu\geqq 1$.

Let us show that we can choose, by induction, a sequence $\{g_\nu\}_{\nu\geqq 0}$ so that

(3.30) i) $g_\nu\in L^{0,p-1}(X_\nu, F\otimes K_X)$,

ii) $\bar\partial g_\nu = f_\nu$,

iii) $\|g_\nu|_{X_{\nu-1}} - g_{\nu-1}\|_{X_{\nu-1}}^2 < \dfrac{1}{2^{\nu+1}}$.

We set

$$g_0 = g_0' .$$

Suppose $g_0, \cdots, g_{\nu-1}$ are chosen. Then $g_\nu'|_{X_{\nu-1}} - g_{\nu-1} \in D_{\bar\partial}^{0,p-1} \subset L^{0,p-1}(X_{\nu-1}, F\otimes K_X)$ and $\bar\partial(g_\nu'|_{X_{\nu-1}} - g_{\nu-1}) = 0$. Therefore, by (3.27) there exists $g'' \in L^{0,p}(X_\nu, F\otimes K_X)$ such that $\|g_\nu'|_{X_{\nu-1}} - g_{\nu-1} - g''|_{X_{\nu-1}}\|^2_{X_{\nu-1}} < \frac{1}{2^{\nu+1}}$ and $\bar\partial g'' = 0$. We set $g_\nu = g_\nu' - g''$. Then it is clear that g_ν has the required properties (3.30). Hence we have obtained the sequence $\{g_\nu\}_{\nu\geqq 0}$. From (3.30), for any ν, $\{g_\mu\}_{\mu\geqq\nu}$ converges with respect to the norm $\|\ ,\ \|_{X_\nu}$, and clearly the limit is the same as the restriction of $\lim_{\mu\geqq\eta} g_\mu$ for any $\eta \geqq \nu+1$. Thus we can define an element g of $L^{0,p-1}_{\mathrm{loc}}(X, F\otimes K_X)$ by $g = \lim_{\nu\to+\infty} g_\nu$. For every $\nu \geqq 1$,

$$\lim_{\mu\geqq\nu} g_\nu = g \quad \text{in} \quad L^{0,p-1}(X_\nu, F\otimes K_X)$$

$$\lim_{\mu\geqq\nu} \bar\partial g_\mu|_{X_\nu} = f_\nu \quad \text{in} \quad L^{0,p}(X_\nu, F\otimes K_X).$$

Since $\bar\partial$ is a closed operator in $L^{0,p-1}(X_\nu, F\otimes K_X)$ for every $\nu \geqq 1$, we have for any $\nu \geqq 0$

$$\bar\partial g = f_\nu \quad \text{in} \quad L^{0,p}(X_\nu, F\otimes K_X).$$

Hence we have $f = \bar\partial g$ in $L^{0,p}_{\mathrm{loc}}(X, F\otimes K_X)$.

Step II. The homomorphism $r: H^p(X, \mathcal{O}(F\otimes K_X)) \to H^{0,p}(X_0, F\otimes K_X)$ *is surjective.*

In the first place, we prove the following.

(3.31) *The restriction map* $r: \{f \in L^{0,p}_{\mathrm{loc}}(X, F\otimes K_X) \,|\, \bar\partial f = 0\} \to N^{0,p}_{\bar\partial_0}$ *has a dense image* ($p \geqq q-1$).

Take an element $\psi \in N^{0,p}_{\bar\partial_0}$. From (3.27), we can choose a sequence $\{\varphi_\nu\}_{\nu\geqq 0}$ such that for any $\varepsilon > 0$,

(3.32) i) $\varphi_0 = \psi$,

ii) $\varphi_\nu \in N^{0,p}_{\bar\partial_\nu}$,

iii) $\|\varphi_\nu|_{X_{\nu-1}} - \varphi_{\nu-1}\|^2_{X_{\nu-1}} < \frac{\varepsilon}{2^\nu} \quad (\nu \geqq 1)$.

By (3.32) iii), for any ν, $\{\varphi_\nu\}_{\nu\geqq 1}$ converges with respect to the norm $\| \ , \|_{X_\nu}$. Hence, by the same argument as in Step I, we can define an element φ of $L^{0,p}_{\mathrm{loc}}(X, F\otimes K_X)$ with $\bar{\partial}\varphi=0$. Then it is clear that $\|\varphi|_{X_0}-\psi\|^2_{X_0}<\varepsilon$. This completes the proof of (3.31). Next, by (3.29) and (3.31), we see that the image of r is dense in $H^{0,p}(X_0, F\otimes K_X)$. Since r is injective and $H^{0,p}(X_0, F\otimes K_X)$ is finite dimensional vector space, r is surjective. Q.E.D.

Theorem 3.7 (*Isomorphic Theorem*). *The restriction homomorphism*

$$r\colon H^p(X, \mathcal{O}(F\otimes K_X))\to H^p(X_c, \mathcal{O}(F\otimes K_X))$$

is an isomorphism for $p\geqq q$, *where* $c>\sup_{x\in K_4}\Phi(x)$.

Proof. We may assume $c>c_0$. Then we have the following factorization:

$$\begin{array}{ccc} H^p(X, \mathcal{O}(F\otimes K_X)) & \xrightarrow{\ r_1\ } & H^{0,p}(X_0, F\otimes X_X) \\ & {\scriptstyle r_2}\searrow \quad \circlearrowleft \quad \nearrow{\scriptstyle r_3} & \\ & H^p(X_c, \mathcal{O}(F\otimes K_X)) & \end{array}$$

Since, by Theorem 3.6, the homomorphism r_1 is an isomorphism, r_2 is injective. Similarly since the homomorphism r_3 is an isomorphism, r_2 is surjective. Q.E.D.

Theorem 3.8 (*Representation Theorem*). *There is a natural isomorphism*

$$\rho_c\colon H^p(X_c, \mathcal{O}(F\otimes K_X))\to N^{0,p}_c \quad for \quad p\geqq q$$

where c is a non-critical value of Φ *with* $c>\sup_{x\in K_4}\Phi(x)$.

Proof. For any c' with $c'>c$, from Theorems 3.6 and 3.7, we obtain isomorphisms r_1 and r_2

$$\begin{array}{ccc} & H^p(X_{c'}, \mathcal{O}(F\otimes K_X)) & \\ {\scriptstyle r_1}\swarrow & & \searrow{\scriptstyle r_2} \\ H^p(X_c, \mathcal{O}(F\otimes K_X)) & \xrightarrow{\ \rho'_c\ } & H^{0,p}(X_c, F\otimes K_X). \end{array}$$

We define a morphism $\rho_c': H^p(X_c, \mathcal{O}(F\otimes K_X)) \to H^{0,p}(X_c, F\otimes K_X)$ by $\rho_c' = r_2 \circ r_1^{-1}$. It is clear that ρ_c' is well defined and does not depend on the choice of c'. ρ_c' is an isomorphism from $H^p(X_c, \mathcal{O}(F\otimes K_X))$ to $H^{0,p}(X_c, F\otimes K_X)$. The composition of ρ_c' and an isomorphism $H^{0,p}(X_c, F\otimes K_X) \cong N_c^{0,p}$ gives an isomorphism $\rho_c: H^p(X_c, \mathcal{O}(F\otimes K_X)) \to N_c^{0,p}$.

Q.E.D.

In particular for $q=1$, we obtain the following.

Theorem 3.9. *Let X be a weakly 1-complete manifold with reaspect to an exhaustion function Φ and let $F \xrightarrow{\pi} X$ be a holomorphic line bundle which is positive on $X \backslash K$. Then with respect to the hermitian metric $\{a_i\}$ of F corresponding to the assumption and the hermitian metric ds^2 on X induced by the curvature of $\{a_i\}$, it holds that, for any two non-critical value c and c' with $c' > c > \sup_{x\in K} \Phi(x)$ and $p \geqq 1$,*

1) $L^{0,p}(X_c, F\otimes K_X) \cong R^{0,p}_{\bar\partial\bar\partial^*} \oplus N_c^{0,p} \oplus R^{0,p}_{\bar\partial^*\bar\partial}$ *and* $\dim_{\mathbf{C}} N_c^{0,p} < +\infty$,
2) *the homomorphism* $\rho_c^{c'}: N_{c'}^{0,p} \to N_c^{0,p}$ *is isomorphic,*
3) $\dim_{\mathbf{C}} H^p(X, \mathcal{O}(F\otimes K_X)) < +\infty$,
4) *the restriction homomorphism*

$$r: H^p(X, \mathcal{O}(F\otimes K_X)) \to H^p(X_c, \mathcal{O}(F\otimes K_X))$$

is isomorphic,

5) *there is an isomorphism*

$$\rho_c: H^p(X_c, \mathcal{O}(F\otimes K_X)) \to N_c^{0,p}.$$

Remark 3.2. In Theorem 3.9, 3) and 4) were proved by T. Ohsawa in [34]. He reduced these problems to the $\bar\partial$-operator theory without boundary conditions.

§ 3.3. Vanishing Theorems

Theorem 3.10. *Let X be a connected weakly 1-complete Kähler manifold and let $F \xrightarrow{\pi} X$ be a holomorphic line bundle on X which is q-semi-positive on X and q-positive on $X \backslash K$ with respect to the given*

Kähler metric, where K is aproper compact subset. Then

$$H^p(X, \mathcal{O}(F\otimes K_X))=0 \quad for \quad p\geqq q.$$

Proof. For a suitable non-critical value c of the exhaustion function Φ on X, we obtain, from Theorems 3.7 and 3.8, the isomorphism $H^p(X, \mathcal{O}(F\otimes K_X))\cong N_c^{0,p}$ for $p\geqq q$. But, from the assumption, $N_c^{0,p}$ is the null space (see the proof of Proposition 1.11 II)). This completes the proof.

Remark 3.3. Any connected compact complex manifold is weakly 1-complete, any real constant function being an exhaustion function. Therefore we obtain the following.

Let X be a connected compact complex Kähler manifold and let $F\xrightarrow{\pi}X$ *be q-semi-positive on X and q-positive at least one point of X with respect to the given Kähler metric. Then*

$$H^p(X, \mathcal{O}(F\otimes K_X))=0 \quad for \quad p\geqq q.$$

Remark 3.4. With respect to the difference between weakly 1-completeness and pseudoconvexity, Diedrich and Fornaess showed that there are domains with pseudoconvex boundaries Ω on $\mathbf{C}^n$ such that $\bar{\Omega}$ does not have a Stein neighborhood basis (see [48]). In this situation, if the defining function Φ of Ω is plurisubharmonic on a neighborhood W of $\partial\Omega$, domains $\{\Omega_r=\Omega\cup\{x\in W|\Phi(x)<r\}\}_{r>0}$ $(\partial\Omega=\{x\in W|\Phi(x)=0\})$ consist of a Stein basis of $\bar{\Omega}$ since any domain with pseudoconvex boundary on $\mathbf{C}^n$ is Stein.

Note added. As mentioned in Remark 2.3, in the situation of Chapters II and III, the local boundary regularity does not always hold. But recently the author has shown that the global boundary regularity holds in the following sense if according to the degree of the required boundary regularity, we take the tensor product of the line bundle F, which is positive on a neighborhood of ∂X, sufficiently many times, we can solve the $L^2-\bar{\partial}$-Neumann problem satisfying the required boundary regularity. As a consequence of this regularization theorem, $H^p(X_c, \mathcal{O}(F^{\otimes m}\otimes K_X))$ can be represented by the space of harmonic forms being C^k-class $(0\leqq k<\infty)$ up to ∂X_c if m is large enough. See a forthcoming paper "Global regularity and spectra of Laplace-Beltrami operators on pseudoconvex domains".

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