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On Quasi-equivalence of Quasifree States of the Canonical Commutation Relations

By

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Abstract

A necessary and sufficient condition for two quasifree states of CCR's (the canonical commutation relations) to yield quasi-equivalent representations is obtained, in the most general setting where states are non gauge-invariant in general and the symplectic form defining CCR might be degenerate (either from the start or after completion relative to the topology induced by states). The criterion consists of the following two conditions: (1) The induced topologies on the test function space are equivalent (2) operators $\tilde{S}$ and $\tilde{S}'$ on the completed test function space (completion relative to the induced topology, after taking quotient by the kernel of the representation) giving the two point functions S and S' (relative to any common inner product giving rise to the induced topology) are such that $\tilde{S}^{1/2}-(\tilde{S}')^{1/2}$ is in the Hilbert Schmidt class.

§ 1. Main Result

The necessary and sufficient condition for two quasifree states of CCR's to have quasi-equivalent representations has been obtained by several authors ([1], [6], [10]). All these results are restricted to the cases where the symplectic form defining the canonical commutation relations stays non-degenerate after the completion of the test function space by the topology induced by quasifree states. The corresponding problem in commutative case (i.e. the extreme case where the symplectic form completely degenerates to zero) has been known in the probability theory as the condition for equivalence of two Gaussian measures. We shall obtain a general result which contains these results as special cases, using the result in the measure theoretical case directly and applying the methods in [1].

To set up the problem, we use the notation for CCR's developed

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by Araki in [3]. Let K be a complex vector space (serving as the direct sum of test function spaces for creation and annihilation operators) with an indefinite inner product $\gamma(x, y)$ (giving rise to CCR's) and a conjugate linear involution Γ (serving as the combination of the complex conjugation and the interchange of the test functions for creation and annihilation operators), satisfying $\gamma(\Gamma x, \Gamma y) = -\gamma(y, x)$.

The self-dual CCR algebra $\mathfrak{A}(K, \Gamma, \gamma)$ over (K, Γ, γ) is the quotient of the free involutive algebra over K (with the identity element 1 adjoined and involution denoted by *) by the two sided ideal generated by $x^*y - yx^* - \gamma(x, y)1$ and $x^* - \Gamma x$ $(x, y \in K)$. A state φ on this involutive algebra is called quasifree if

$$(1.1)\qquad \begin{cases} \varphi(x_1 \cdots x_{2n+1}) = 0 \\ \varphi(x_1 \cdots x_{2n}) = \sum_{\sigma} \prod_{j=1}^{n} \varphi(x_{\sigma(j)} x_{\sigma(j+n)}) \end{cases}$$

where the summation is taken over all permutation σ satisfyings $\sigma(1) < \cdots < \sigma(n)$ and $\sigma(j) < \sigma(j+n)$ $(j = 1, \cdots, n)$. Any quasifree state φ is obviously specified by the two-point function

$$(1.2)\qquad S(x, y) = \varphi(x^* y)$$

and hence we write $\varphi = \varphi_S$. The hermitian form S over K, associated with quasifree state φ in the above manner, can be completely characterized by the following properties:

$$(1.3)\qquad \begin{cases} S(x, x) \geqq 0 \\ S(x, y) - S(\Gamma y, \Gamma x) = \gamma(x, y). \end{cases}$$

The positive (semi-definite) inner product

$$(1.4)\qquad (x, y)_S \equiv S(x, y) + S(\Gamma y, \Gamma x)$$

defines an induced topology τ_S on K. This τ_S is in general non-Hausdorff. Therefore the completion $\overline{K}$ of K by the topology $\tau = \tau_S$ is defined to be the completion of the quotient space $K/\ker S$ ($\ker S \equiv \{x \in K;\ (x, x)_S = 0\}$) by the positive definite inner product on $K/\ker S$ induced by $(\ ,\ )_S$. The Hilbert space topology $\overline{\tau}$ of $\overline{K}$ obtained in this manner depends only on τ and is independent of the choice of S such that $\tau = \tau_S$. Our main result is as follows.

Theorem. *Two quasifree states* φ_S *and* $\varphi_{S'}$ *have quasi-equivalent GNS representations* π_S *and* $\pi_{S'}$ *if and only if the following two conditions hold*:

(1) $\tau_S=\tau_{S'}$ $(\equiv\tau)$.

(2) *Let* $\bar{K}$ *be the completion of* K *by the topology* τ *with an inner product* (x, y) *inducing the topology* $\bar{\tau}$ *and let*

$$S(x, y) = (x, \tilde{S}y), \quad S'(x, y) = (x, \tilde{S}'y).$$

Then $\tilde{S}^{1/2}-\tilde{S}'^{1/2}$ *is in the Hilbert-Schmidt class.*

Remark. The condition that $\tilde{S}^{1/2}-\tilde{S}'^{1/2}$ is in the Hilbert-Schmidt class depends only on the forms S and S' and is independent of the inner product $(,)$ provided that it induces the same topology as S and S'. (See Remark 6.4 (ii) and its proof.)

§ 2. Key Points in the Proof

The proof of the theorem is divided into (I) preliminary reduction, (II) sufficiency proof, and (III) necessity proof.

(I) Preliminary Reduction.

We use a few immediate consequences of quasi-equivalence of representations to reduce the problem to a standard form.

First the topology induced on K by the representations must be the same, namely

Proposition 2.1. *If* φ_S *and* $\varphi_{S'}$ *have quasi-equivalent GNS representations, then two inner products* $(,)_S$ *and* $(,)_{S'}$ *on* K *given by* (1.2) *induce the same topology* τ *on* K. (*A proof in* § 3.)

As a consequence we may extend both φ_S and $\varphi_{S'}$ to quasifree states of $\mathfrak{A}(\bar{K}, \bar{\gamma}, \bar{\Gamma})$ for the τ-completion $\bar{K}$ (see § 1) of K and the problem of quasi-equivalence remains the same. (More precisely, φ_S and $\varphi_{S'}$ are well-defined on the quotient of K by $\ker S=\ker S'$ and have unique extensions on $\bar{K}$ by continuity.) Therefore we may denote $\bar{K}$ by K and assume that K is a Hilbert space with an inner product $(\ ,\ )$, equivalent to (extensions of) both $(\ ,\ )_S$ and $(\ ,\ )_{S'}$. (We may take, for example, $(\ ,\ ) = (\ ,\ )_S$.)

Second, we consider the kernel of γ in (the complete space) K, i.e., $\ker\gamma \equiv \{x \in K;\ \gamma(x, y) = 0 \text{ for all } y \in K\}$. If we restrict the states to the commutative algebra $\mathfrak{A}(\ker\gamma)$, then the necessary and sufficient condition for the quasi-equivalence of restrictions of φ_S and $\varphi_{S'}$ to $\mathfrak{A}(\ker\gamma)$ is known to be described in terms of the restrictions S_0 and S_0' of $S(x, y)$ and $S'(x, y)$ to $\ker\gamma$ by the conditions of our theorem. (See § 3 (9)).

If we take $(\ ,\ ) = (\ ,\ )_S$ and consider an S-orthogonal direct sum decomposition $K = \ker\gamma \oplus K_1$, then $\mathfrak{A}(K)$ is the tensor product of $\mathfrak{A}(\ker\gamma)$ and $\mathfrak{A}(K_1)$ as is well-known and $\varphi_S = \varphi_{S_0} \otimes \varphi_{S_1}$ where $S_1(x, y)$ denotes the restriction of $S(x, y)$ to K_1.

Thus if φ_S and $\varphi_{S'}$ are quasi-equivalent, then S_0 and S_0', the restrictions of S and S' to $\ker\gamma$, satisfy the conditions of Theorem (due to Corollary 3.14 to be proved later) and furthermore $\varphi_{S''} \equiv \varphi_{S_0'} \otimes \varphi_{S_1}$ must be quasi-equivalent to φ_S and hence to φ_S. (The condition on S's in Theorem are obviously transitive and are satisfied by $S_0 \oplus S_1$ and $S_0' \oplus S_1$.) Therefore we can proceed to the necessity proof for the pair $\varphi_{S'}$ and $\varphi_{S''}$ with S' and S'' having the same restrictions on $\ker\gamma$.

Conversely, if S and S' satisfy the conditions of Theorem, then S_0 and S_0' also satisfy the conditions of Theorem and hence φ_S and $\varphi_{S''}$ are quasi-equivalent due to the result on the commutative case. Furthermore $S_0' \oplus S_1$ and S satisfy the condition of Theorem and hence the same holds for $S_0' \oplus S_1$ and S'. Therefore the sufficiency proof can also be reduced to the pair $S_0' \oplus S_1$ and S' which have the same restrictions on $\ker\gamma$. Therefore we may assume that the restrictions S_0 and S_0' of S and S' to $\ker\gamma$ are exactly the same for the purpose of both sufficiency and necessity proof without loss of generality.

Hence, in the rest of this section, we assume that all these reductions are already done, namely K is complete, $(\ ,\ )_S$ and $(\ ,\ )_{S'}$ both give the same (non-degenerate) Hilbert space topology of K, and the restrictions of $(\ ,\ )_S$ and $(\ ,\ )_{S'}$ to $\ker\gamma$ coincide.

(II) Sufficiency Proof.

This consists of 3 steps and one formula, which will be used also for necessity proof.

In the first step, we use the doubling $\hat{K}$ of K ($K \subset \hat{K}$) such that both φ_S and $\varphi_{S'}$ are the restrictions of generalized Fock states $\varphi_{\hat{S}}$ and $\varphi_{\hat{S}'}$ of $\mathfrak{A}(\hat{K})$, the GNS representation of $\mathfrak{A}(\hat{K})$ associated with $\varphi_{\hat{S}}$, when restricted to $\mathfrak{A}(K) \subset \mathfrak{A}(\hat{K})$, is quasi-equivalent to the GNS representation of $\mathfrak{A}(K)$ associated with φ_S, and the same holds for S' and $\hat{S}'$. The main part of this step is the construction of a Bogoliubov transformation B in $\hat{K}$ such that the associated Bogoliubov automorphism τ_B of $\mathfrak{A}(\hat{K})$ satisfies $\varphi_{\hat{S}'} = \varphi_{\hat{S}} \circ \tau_B$. If we can implement τ_B by a unitary transformation, then it immediately follows that GNS representations associated with φ_S and $\varphi_{S'}$ are quasi-equivalent.

The second step is to show that τ_B is unitarily implementable if a certain condition is satisfied by $\hat{S}$ and $\hat{S}'$. We make an explicit construction of a unitary operator U implementing B when $B-1$ is of finite rank in Section 5. We then use this result to construct U for the general case as a limit of U for the finite rank case, where we use the formula for $(\Omega_{\hat{S}}, U\Omega_{\hat{S}})$ (for a finite dimensional K) proven in Appendix A.

The sufficiency proof will be completed in Section 6 by showing the equivalence of the conditions for S and S' in Theorem with the conditions for $\hat{S}$ and $\hat{S}'$.

(III) Necessity Proof.

We first reduce the problem to the case where φ_S and $\varphi_{S'}$ are both faithful for the von Neumann algebra generated by the GNS representations (called standard polarizations). This reduction can be done for a general S by finding another S_1 such that the conditions of Theorem is satisfied for S and S_1 (and hence the GNS representations π_S and π_{S_1} for φ_S and φ_{S_1} are quasi-equivalent by sufficiency proof) and φ_{S_1} has the desired property.

The second step is to show that the unitary operator U, constructed earlier for the finite rank case, brings $\Omega_{\hat{S}}$ (the cyclic vector associated with $\varphi_{\hat{S}}$) to a vector in the natural positive cone for $\pi_{\hat{S}}(\mathfrak{A}(\hat{K}))''$ (with the reference vector $\Omega_{\hat{S}}$). It then follows by a general theorem that if the limit of $(\Omega_{\hat{S}}, U\Omega_{\hat{S}})$ is zero as the finite rank tends to infinity, then two representations π_s and $\pi_{S'}$ are disjoint.

§ 3. Preliminaries

In this section, we collect notations and basic facts of CCR in the setting given in [3].

(1) *Phase space.*

We have introduced in Section 1, a complex vector space K, a conjugation Γ in K, and a hermitian form γ on K, which satisfy some relations. In this paper we call such a triplet (K, Γ, γ) a phase space because it can be viewed as the phase space of canonically quantized fields. The CCR algebra associated with (K, Γ, γ) is a *-algebra $\mathfrak{A}(K, \Gamma, \gamma)$ generated by K and the unit $\mathbf{1}$ with the relations:

$$(3.1)\qquad \begin{cases} x^*y - yx^* = \gamma(x, y)\mathbf{1} \\ x^* = \Gamma x \qquad\qquad \text{for} \quad x, y \in K. \end{cases}$$

(All inner products and forms will be conjugate linear in the first variable and linear in the second variable.)

(2) *Polarization.*

Definition 3.1. *A positive hermitian form S on K such that*

$$(3.2)\qquad S(x, y) - S(\Gamma y, \Gamma x) = \gamma(x, y) \quad \textit{for} \quad x, y \in K$$

is called a polarization of γ.

Lemma 3.2. *There exists a $1-1$ correspondence between a polarization S of γ and a positive hermitian form $(\ ,\ )_S$ on K such that*

$$(3.3)\qquad \begin{cases} (\Gamma x, \Gamma y)_S = (y, x)_S \\ |\gamma(x, y)|^2 \leqq (x, x)_S (y, y)_S \quad \text{for} \quad x, y \in K, \end{cases}$$

given by the mutual relations

(3.4)
$$\begin{cases} (x, y)_S = S(x, y) + S(\Gamma y, \Gamma x) \\ S(x, y) = \frac{1}{2}((x, y)_S + \gamma(x, y)). \end{cases}$$

Proof. See Lemma 3.3 in [3].

Consider a state φ on $\mathfrak{A}(K, \Gamma, \gamma)$, namely, a positive normalized linear functional on $\mathfrak{A}(K, \Gamma, \gamma)$.

Lemma 3.3. *A positive hermitian form S on K defined by*

(3.5)
$$S(x, y) = \varphi(x^*y) \quad \text{for} \quad x, y \in K,$$

is a polarization of γ.

Proof.

$$S(x, y) - S(\Gamma y, \Gamma x) = \varphi(x^*y) - \varphi(yx^*) = \varphi([x^*, y]) = \gamma(x, y).$$

Although this association of polarization with state is not one-to-one in general, it is so if we restrict ourselves to the class of quasifree states (see (1.1) for the definition of quasifree state). For given polarization S, the unique quasifree state φ satisfying (3.5) is denoted by φ_S (the existence of φ_S for every S is proved in [3]).

(3) *Homomorphisms and Bogoliubov transformations.*

Let $(K_1, \Gamma_1, \gamma_1)$ and $(K_2, \Gamma_2, \gamma_2)$ be two phase spaces. A homomorphism of $(K_1, \Gamma_1, \gamma_1)$ into $(K_2, \Gamma_2, \gamma_2)$ is a linear map $f: K_1 \to K_2$ which intertwines Γ's and γ's:

(3.6)
$$\begin{cases} f(\Gamma_1 x) = \Gamma_2 f(x) \\ \gamma_2(fx, fy) = \gamma_1(x, y) \end{cases} \quad \text{for} \quad x, y \in K_1$$

A homomorphism f of phase spaces induces a *-homomorphism τ_f (or simply f) of CCR algebras, τ_f: $\mathfrak{A}(K_1, \Gamma_1, \gamma_1) \to \mathfrak{A}(K_2, \Gamma_2, \gamma_2)$. If a homomorphism f of (K, Γ, γ) onto itself is an isomorphism, f is called a Bogoliubov transformation of (K, Γ, γ) and the associated automorphism τ_f is called a Bogoliubov automorphism.

(4) *Representations.*

Proposition 3.4. *Let* $(\mathfrak{H}_S, \Omega_S, \pi_S)$ *be the GNS representation for a quasifree state* φ_S. *Set*

$$D_n \equiv \pi_S(\mathfrak{A}_n)\Omega_S \quad \text{and} \quad D \equiv \bigcup_{n\geqq 0} D_n,$$

where $\mathfrak{A}_n \subset \mathfrak{A}(K, \Gamma, \gamma)$ *is the space of polynomials with at most n-th order. Then the following properties hold*:

(i) *For any* $\xi \in D$ *and any* $x \in \mathrm{Re}\, K = \{y \in K;\ \Gamma y = y\}$,

$$\sum_{n\geqq 0} \frac{1}{n!} \|\pi_S(x)^n \xi\| < \infty.$$

(More precisely, $\xi \in D_k$ *implies* $\|\pi_S(x)\xi\| \leqq 2^{1/2}(k+1)^{1/2}(x, x)_S^{1/2}\|\xi\|$*). The operator* $\pi_S(x)$ *is essentially self-adjoint on* D *and a unitary operator* $W_S(x)$ *can be defined for* $\xi \in D$ *by*

$$W_S(x)\xi = \sum_{n\geqq 0} \frac{1}{n!} i^n \pi_S(x)^n \xi$$

and for a general $\xi \in \mathfrak{H}_S$ *by continuity.*

(ii) $$W_S(x_1) W_S(x_2) = W_S(x_1 + x_2) \exp\left(-\frac{1}{2}\gamma(x_1, x_2)\right)$$

for $x_1, x_2 \in \mathrm{Re}\, K$.

(iii) $$(\Omega_S, W_S(x)\Omega_S) = \exp\left(-\frac{1}{2}S(x, x)\right)$$

for $x \in \mathrm{Re}\, K$.

(iv) $W_S(x)$ *is strongly continuous in* $x \in \mathrm{Re}\, K$, *where the topology of* $\mathrm{Re}\, K$ *is the one induced by* $(\ ,\ )_S$.

These properties are well-known in Fock representations. The present case is obtained by the restriction of a Fock representation of $\mathfrak{A}(\hat{K}, \hat{\Gamma}, \hat{\gamma})$ for a larger space $\hat{K} \supset K$ to $\mathfrak{A}(K, \Gamma, \gamma)$ (for the definition of $(\hat{K}, \hat{\Gamma}, \hat{\gamma})$, see (10) doubling). The self-adjointness in (i) is due to Nelson's theorem. [7]

(5) *Quasi-equivalence.*

The quasi-equivalence of two representations π_{S_1} and π_{S_2} (denoted by $\pi_{S_1}\sim\pi_{S_2}$) means the quasi-equivalence of the associated representations of Weyl algebras: Let R_S be the von Neumann algebra generated by $\{W_S(x)\,;\ x\in\mathrm{Re}\,K\}$. Then $\pi_{S_1}\sim\pi_{S_2}$ if there exists an isomorphism ϕ: $R_{S_1}\to R_{S_2}$ such that $\phi(W_{S_1}(x))=W_{S_2}(x)$ for all $x\in\mathrm{Re}\,K$.

(6) *Kernels of representations.*

If two representations are quasi-equivalent, their kernels (i.e. those elements of the algebra which are represented by zero operators) must coincide.

Lemma 3.5. *$(x, x)_S=0$, if and only if $\pi_S(x)=0$. If q denotes the quotient map from K to $K_q\equiv K/\ker S$ ($\ker S=\{x\in K;\ (x, x)_S=0\}$), then $\pi_{S_q}(qx)\equiv\pi_S(x)$ is equivalent to the GNS representation associated with the quasifree state φ_{S_q} of $\mathfrak{A}(K_q)$ where $S_q(qx, qy)=S(x, y)$. The quasi-equivalence of φ_{S_1} and φ_{S_2} is equivalent to that of $\varphi_{(S_1)_q}$ and $\varphi_{(S_2)_q}$.*

The proof is straightforward and is omitted. By this lemma, we may restrict our attention to the case where $\ker S=0$.

(7) *Topology.*

Let $\overline{K}$ be the completion of K with respect to $(\ ,\ )_S$. Since S and γ are τ_S-continuous by $S(x, x)\leqq(x, x)_S$ and (3.3), Γ, γ, and S can be continuously extended to $\overline{K}$. By the continuity of $W_S(x)$ (Proposition 3.4 (iv)), the quasi-equivalence problem is not altered by the transition to the completed phase space. Therefore we may assume the τ_S-completeness of K without loss of generality in our problem.

Lemma 3.6. *If $\pi_{S_1}\sim\pi_{S_2}$, the topologies τ_{S_1} and τ_{S_2} induced by $(\ ,\ )_{S_1}$ and $(\ ,\ )_{S_2}$ coincide on K.*

Proof. Let ϕ: $R_{S_1}\to R_{S_2}$ be an isomorphism giving rise to the quasi-equivalence of π_{S_1} and π_{S_2}. By Proposition 3.4 (iii), (iv),

$$\exp\left(-\frac{1}{2}S_2(x,x)\right)=(\Omega_{S_2}, W_{S_2}(x)\Omega_{S_2})$$

$$=(\Omega_{S_2}, \phi(W_{S_1}(x))\Omega_{S_2})$$

is τ_{S_1}-continuous in $x\in\mathrm{Re}\,K$ and therefore S_2 is τ_{s_1}-continuous. Since $(\ ,\ )_{S_1}$ is Γ-invariant, $(\ ,\ )_{S_2}$ is also continuous with respect to $(\ ,\ )_{S_1}$ by (3.4). By symmetry, $(\ ,\ )_{S_1}$ is also τ_{S_2}-continuous. Thus two topologies must be the same.

This lemma shows the necessity of the condition $\tau_{S_1}=\tau_{S_2}$ for the quasi-equivalence. In the rest of discussion, we may assume that $\tau_{S_1}=\tau_{S_2}$.

(8) *The operators* **S** *and* γ_S.

We define the operators **S** and γ_S in K by

$$\begin{cases} S(x,y)=(x,\mathbf{S}y)_S \\ \gamma(x,y)=(x,\gamma_S y)_S \end{cases} \tag{3.7}$$

for $x, y\in K$. It satisfies

$$\mathbf{S}+\Gamma\mathbf{S}\Gamma=\mathbf{1},\quad \mathbf{S}-\Gamma\mathbf{S}\Gamma=\gamma_S \tag{3.8}$$

(9) *Commutative case.*

If $\gamma\equiv 0$, our theorem reduces to the following known equivalence criterion for two Gaussian measures on a Hilbert space:

Proposition 3.7. *Consider a phase space* (K,Γ,γ) *with* $\gamma=0$. *Let* S_1 *and* S_2 *be two polarizations on* K. *Then* $\pi_{S_1}\sim\pi_{S_2}$ *if and only if*

(i) $(\ ,\ )_{S_1}$ *and* $(\ ,\ )_{S_2}$ *give the same topology on* K *and*

(ii) *a positive operator* T: $K\to K$ *defined by* $(x,y)_{S_2}=(x,Ty)_{S_1}$ *satisfies condition that* $T-\mathbf{1}$ *is in the Hilbert-Schmidt class.*

Condition (ii) on $(\ ,\ )_{S_1}$ and $(\ ,\ )_{S_2}$ stated above is equivalent to condition (ii) in Theorem of Section 1 because of the following observation. For commutative case where $\gamma=0$, we have $S(x,y)=(1/2)\,(x,y)_S$ and $S'(x,y)=(1/2)\,(x,y)_{S'}$ by (3.4). Therefore, if we take $(x,y)=(x,y)_S$, then the operators $\tilde{S}$ and $\tilde{S}'$ in condition (ii) of Theorem are 1/2 and $T/2$ with T defined above and hence condition (ii) of Theorem is that $\mathbf{1}-T^{1/2}$ is in the Hilbert-Schmidt class. This is obviously equivalent to condition (ii) of Proposition (for example, by the spectral resolution of T and the equivalence of $\sum|1-\lambda_j^{1/2}|^2<\infty$ and $\sum|1-\lambda_j|^2<\infty$).

This proposition is usually stated as follows:

Proposition 3.7'. *Let H be a real Hilbert space with two inner products $(\ ,\ )_1$ and $(\ ,\ )_2$ giving rise to the Hilbert space topology. Then the following conditions are equivalent:*

(i) *There exist two Gaussian random processes ϕ_1, ϕ_2 over $(\mathfrak{H}, (\ ,\ )_1)$ and $(\mathfrak{H}, (\ ,\ )_2)$, both realizable on a Borel space $(X, \mathfrak{B})$ with mutually absolutely continuous probability measures μ_1 and μ_2.*

(ii) *$T-1$ is in the Hilbert-Schmidt class where T is the positive operator in H defined by $(x, y)_2=(x, Ty)_1$ for $x, y\in H$.*

Proof. See Theorem 3 in [9].

The connection of the above two propositions can be seen as follows: Let S be a polarization of a phase space $(K, \Gamma, \gamma\equiv 0)$ and consider a Gaussian random process (ϕ, X, μ) over $(\mathrm{Re}\,K, (\ ,\ )_S)$. By definition, (X, μ) is a probability space and ϕ is a linear map of $\mathrm{Re}\,K$ into the space of real-valued measurable functions on X satisfying the following:

(i) $\phi(x)$ is a Gaussian random variable for $x\in \mathrm{Re}\,K$,

(ii) $\displaystyle\int_X \phi(x)\phi(y)\,d\mu=\frac{1}{2}(x, y)_S$ for $x, y\in \mathrm{Re}\,K$.

By the universal construction of $\mathfrak{A}(K)$, ϕ is extended to $\mathfrak{A}(K)$ as a homomorphism of algebras, which will be also denoted by ϕ. Then, by the above two conditions, the map $a\mapsto \int\phi(a)\,d\mu$, $a\in\mathfrak{A}(K)$, satisfies the condition of quasifree state φ_S and $(\phi, L^2(X, \mu)$, the constant function 1 on $X)$ is unitarily equivalent to the GNS representation $(\pi_S, \mathfrak{H}_S, \Omega_S)$.

Let (ϕ, X, μ) and (ϕ', X, μ') correspond to S and S' as above. First assume that μ and μ' are equivalent measures. Then $L^\infty(X, \mu)=L^\infty(X,$

μ') and this identification induces the *-isomorphism of $\pi_S(\mathfrak{A}(K))''$ and $\pi_{S'}(\mathfrak{A}(K))''$ through the unitary equivalence quoted above (and hence the unitary equivalence of $\pi_S(\mathfrak{A}(K))''$ with the von Neumann algebra $L^\infty(X,\mu)$ generated by $\exp i\phi(\mathrm{Re}\, K)$). Conversely assume that π_S and $\pi_{S'}$ are quasi-equivalent. Then there exists a *-isomorphism between $L^\infty(X,\mu)$ and $L^\infty(X,\mu')$ preserving $e^{i\phi(f)}, f\in \mathrm{Re}\, K$. Therefore any subset measurable relative to ϕ (whose characteristic function must be preserved by the above isomorphism) must have a non-zero μ-measure if and only if it has a non-zero μ'-measure. Hence μ and μ' are equivalent. This shows that Propositions 3.7 and 3.7′ are equivalent.

(10) *Doubling.*

For later use, we collect here some definitions and properties of doubling operation on phase spaces. Let (K,Γ,γ) be a phase space with a polarization S. First we equip the double-sized space $K\oplus K$ with a conjugation $\Gamma\oplus\Gamma$, a hermitian form $\gamma\oplus-\gamma$, and a positive inner product

$$(3.9)\qquad (x_1\oplus x_2, y_1\oplus y_2)_{\hat{S}}\equiv (x_1,y_1)_S+(x_2,y_2)_S + (x_1,(1-\gamma_S^2)^{1/2}y_2)_S+(x_2,(1-\gamma_S^2)^{1/2}y_1)_S \quad \text{for}\quad x_1\oplus x_2, y_1\oplus y_2\in K\oplus K.$$

Then $(\ ,\ )_{\hat{S}}$ gives a polarization of $(K\oplus K,\Gamma\oplus\Gamma,\gamma\oplus-\gamma)$. The completion of $(K\oplus K,\Gamma\oplus\Gamma,\gamma\oplus-\gamma)$ by $(\ ,\ )_{\hat{S}}$ (after taking the quotient by $\ker\hat{S}$) is called the doubling of (K,Γ,γ) and denoted by $(\hat{K},\hat{\Gamma},\hat{\gamma})$. Note that the map

$$K\oplus K\ni x_1\oplus x_2\mapsto [x_1\oplus x_2]\in\hat{K}$$

is injective if and only if $\ker\gamma=\{x\in K;\gamma(x,y)=0 \text{ for all } y\in K\}$ ($\ker\gamma$ will be sometimes denoted below by K_0) is trivial because $\ker\hat{S}=\{x\oplus -x;\ x\in\ker\gamma\}$.

The original pair of K and S can be imbedded into the doubling in the following manner:

Lemma 3.8.

(i) *The map* ι *from* $x\in K$ *to* $\iota(x)=[x\oplus 0]\in\hat{K}$ *is a monomor-*

phism of phase spaces. (*In particular*, $\iota\Gamma=\hat{\Gamma}\iota$, $\gamma(x, y)=\hat{\gamma}(\iota x, \iota y)$.)

(ii) $S(x, y)=\hat{S}(\iota x, \iota y)$ *for* $x, y\in K$.

(iii) $\ker\hat{\gamma}=\iota(\ker\gamma)$.

Lemma 3.9. *The spectrum of the operator* $\hat{\gamma}_{\hat{S}}$ *with respect to* $(\ ,\ )_{\hat{S}}$ *is in* $\{-1, 0, 1\}$.

A proof of Lemma 3.9 is actually in [3] Lemma 5.8. The new quasifree state $\varphi_{\hat{S}}$ is a Fock state except for the existence of center. To introduce some notations, we sketch the proof taken from [3]: Define linear maps $h_{\pm}$ of $K\oplus K$ into K by

$$\begin{cases} h_{+}(x_1\oplus x_2)=\mathbf{S}^{1/2}x_1+(\mathbb{1}-\mathbf{S})^{1/2}x_2 \\ h_{-}(x_1\oplus x_2)=(\mathbb{1}-\mathbf{S})^{1/2}x_1+\mathbf{S}^{1/2}x_2 . \end{cases} \tag{3.10}$$

Then it is easy to see that $\Gamma\circ h_{\pm}\circ(\Gamma\oplus\Gamma)=h_{\mp}$ and

$$\begin{cases} (x, y)_{\hat{S}}=(h_{+}(x), h_{+}(y))_S+(h_{-}(x), h_{-}(y))_S \\ \hat{\gamma}(x, y)=(h_{+}(x), h_{+}(y))_S-(h_{-}(x), h_{-}(y))_S \end{cases} \tag{3.11}$$

for $x, y\in K\oplus K$. From the first equation, $h_{\pm}$ maps $\ker\hat{S}$ to $\{0\}$ and hence can be considered as maps from $\hat{K}$ to K. Set

$$\begin{cases} K_0=\{x\in K; \gamma(x, y)=0 \quad \text{for all } y\in K\} \\ K_S=\gamma_S K . \end{cases} \tag{3.12}$$

Since $K_S\subset K_0^{\perp}$ (S-orthogonal complement of K_0) and the restriction of γ_S to $K_0^{\perp}$ is one to one, we can define linear maps $k_{\pm}$ of K_S into $\hat{K}$ by

$$\begin{cases} k_{+}(x)=[2^{-1/2}(1+\gamma_S)^{1/2}\gamma_S^{-1}x\oplus -2^{-1/2}(1-\gamma_S)^{1/2}\gamma_S^{-1}x] \\ k_{-}(x)=[-2^{-1/2}(1-\gamma_S)^{1/2}\gamma_S^{-1}x\oplus 2^{-1/2}(1+\gamma_S)^{1/2}\gamma_S^{-1}x] \end{cases} \tag{3.13}$$

$$\text{for} \quad x\in K_S.$$

Finally we define a linear map k_0 of K_0 into $\hat{K}$ by

$$k_0(x)=2^{1/2}\iota(x) \quad \text{for} \quad x\in K_0. \tag{3.14}$$

Then by direct computation, we obtain the relations

$$
\begin{cases} h_+k_+(x_S)=x_S, & h_-k_-(x_S)=x_S\,, \\ h_-k_+(x_S)=0, & h_+k_-(x_S)=0\,, \\ h_\pm k_0(x_0)=x_0 \end{cases} \tag{3.15}
$$

for $x_S\in K_S$ and $x_0\in K_0$.

From these relations, we see that the three subspaces $k_+(K_S)$, $k_-(K_S)$, and $k_0(K_0)$ of $\hat{K}$ are orthogonal to each other, relative to $(\ ,\ )_{\hat{S}}$ as well as relative to $\hat{\gamma}$. Since the sum of these subspaces contains a dense subset $[(K_S+K_0)\oplus(K_S+K_0)]$ of $\hat{K}$, it is clear that the spectral resolution of $\hat{\gamma}_{\hat{S}}$ is given by the orthogonal direct sum

$$
\hat{K}=\overline{k_+(K_S)}\oplus\overline{k_-(K_S)}\oplus\overline{k_0(K_0)}
$$

with spectrum 1, -1, 0.

(11) *Independence of* $\hat{K}$ *and* $\tau_{\hat{S}}$ *on* S *for fixed* τ_S.

Lemma 3.10. *Let* S_1 *and* S_2 *be two polarizations of a phase space* (K,Γ,γ) *(with zero kernels and complete) and suppose that* $\tau_{S_1}=\tau_{S_2}$. *Then we have* $\tau_{\hat{S}_1}=\tau_{\hat{S}_2}$.

Proof. See [3] Lemma 6.1 (6).

(12) *Duality theorem.*

Finally we quote the duality theorem quoted in [1] in the form convenient for our application.

Definition 3.11. *A polarization* S *of a phase space* (K,Γ,γ) *is called a generalized Fock polarization if the spectrum of* γ_S *is in* $\{-1,0,1\}$.

Theorem 3.12. *Suppose that* S *is a generalized Fock polarization of a phase space* $(\hat{K},\hat{\Gamma},\hat{\gamma})$. *Consider a* $\hat{\Gamma}$*-invariant subspace* H *of* $\hat{K}$ *which contains* $\ker\hat{\gamma}$. *Set* $H^0\equiv\{x\in K;\ \hat{\gamma}(x,y)=0$ *for all* $y\in H\}$. *If we denote by* $R_S(H)$ *the von Neumann algebra generated by* $\{W_S(x)\,;\ x\in\operatorname{Re}H\}$, *then the following properties hold*:

(i) $H^{00}=\overline{H}$ *(the closure of* H *in* K*)*.

(ii) $R_S(H)=R_S(\overline{H})$.

(iii) $R_S(H^0)=R_S(H)'$ *(duality)*.

(iv) $R_S(H_1)\vee R_S(H_2)=R_S(H_1\vee H_2)$.

(v) $R_S(H_1)\cap R_S(H_2)=R_S(\overline{H}_1\cap\overline{H}_2)$.

(vi) Ω_S *is cyclic for* $R_S(H)$ *if and only if* PH *is dense in* PK.

(vii) Ω_S *is separating for* $R_S(H)$ *if and only if* PH^0 *is dense in* PK.

Here P *is the spectral projection of* $\hat{\gamma}_S$ *corresponding to the eigenvalue* 1, *and* Ω_S *is the cyclic vector associated with the GNS representation of quasifree state* φ_S.

Corollary 3.13. $R_{\hat{S}}(\iota K_0)$ is the center of $R_{\hat{S}}(\iota K)$ $(K_0=\ker\gamma)$.

This is immediate from (iii) and (v) above.

(13) *A further reduction.*

Corollary 3.14. *Let* (K,Γ,γ) *be a phase space and* S_1, S_2 *be two polarizations. The following statements hold*:

(i) *If* π_{S_1} *and* π_{S_2} *are quasi-equivalent, then* $(\ ,\ )_{S_1|K_0}$ *and* $(\ ,\ )_{S_2|K_0}$ *satisfy conditions* (i) *and* (ii) *in Proposition* 3.7.

(ii) *Conversely if* $(\ ,\ )_{S_1|K_0}$ *and* $(\ ,\ )_{S_2|K_0}$ *satisfy conditions* (i) *and* (ii) *in Proposition* 3.7, *then there exists a polarization* S_2' *such that* π_{S_2} *and* $\pi_{S_2'}$ *are quasi-equivalent, and* $(\ ,\ )_{S_1|K_0}=(\ ,\ )_{S_2'|K_0}$.

Proof. (i) Let ϕ: $R_{S_1}(K)\rightarrow R_{S_2}(K)$ be an isomorphism giving the quasi-equivalence of π_{S_1} and π_{S_2}. ϕ induces the isomorphism $R_{S_1}(K_0)\rightarrow R_{S_2}(K_0)$, because ϕ maps the center of $R_{S_1}(K)$ onto the center of $R_{S_2}(K)$. (See Corollary 3.13.) Then $\pi_{S_1|K_0}$ and $\pi_{S_2|K_0}$ are quasi-equivalent and (i) follows from Proposition 3.7.

(ii) Let K_1 be the S_2-orthogonal complement of K_0 in K. Set $S_2'=S_{1|K_0}\oplus S_{2|K_1}$. Then one sees that

$$R_{S_2}\cong R_{S_2|K_0}\otimes R_{S_2|K_1}$$

$$R_{S_2'}\cong R_{S_1|K_0}\otimes R_{S_2|K_1}$$

where $\cong$ means unitary equivalence. Since $(\ ,\ )_{S_1|K_0}$ and $(\ ,\ )_{S_2|K_0}$ satisfy conditions (i) and (ii) in Proposition 3.7, $R_{S_1|K_0}$ and $R_{S_2|K_0}$ are quasi-equivalent, and therefore π_{S_2} and $\pi_{S_2'}$ are quasi-equivalent. This completes the proof.

§ 4. Construction of Bogoliubov Transformations

For given polarizations S and S' of a phase space (K, Γ, γ), we show in this section the existence of a Bogoliubov transformation in $(\hat{K}, \hat{\Gamma}, \hat{\gamma})$ that connects $\hat{S}$ and $\hat{S}'$ by applying the methods in [1]. We assume that $\tau_S = \tau_{S'}$ and the restrictions of S and S' to $K_0 = \ker \gamma$ coincide.

(1) *Crossing part.*

Let p (resp. p') be the spectral projection of $\hat{\gamma}_{\hat{S}}$ (resp. $\hat{\gamma}_{\hat{S}'}$) corresponding to the eigenvalue 1. By (1.3) and Lemma 3.9 we have

$$(4.1)\qquad \begin{cases} \hat{\gamma}_{\hat{S}} = p - \bar{p} \\ \hat{\gamma}_{\hat{S}'} = p' - \bar{p}' \end{cases}$$

where $\bar{p} = \hat{\Gamma} p \hat{\Gamma}$ and $\bar{p}' = \hat{\Gamma} p' \hat{\Gamma}$. In the following arguments, we shall use the matrix representations of operators in $\hat{K}$ relative to the decomposition $\hat{K} = (\mathbf{1} - p - \bar{p})\hat{K} + (p + \bar{p})\hat{K}$. For example, p and p' can be expressed as

$$(4.2)\qquad p = \begin{pmatrix} 0 & 0 \\ 0 & P \end{pmatrix}, \quad p' = \begin{pmatrix} 0 & q \\ 0 & Q \end{pmatrix}$$

where we have used the fact that $p' = 0$ on $(\mathbf{1} - p - \bar{p})\hat{K} = \ker \hat{\gamma} = (\mathbf{1} - p' - \bar{p}')\hat{K}$. Due to $p'^2 = p'$ and $\bar{p}' p' = 0$, the operators q and Q satisfy the following relations:

$$(4.3)\qquad \begin{cases} qQ = q, \quad Q^2 = Q, \\ \bar{q} Q = 0, \quad \bar{Q} Q = 0. \end{cases}$$

A Bogoliubov transformation B in $\hat{K}$ transforms a positive definite inner product $(\ ,\ )$ on $\hat{K}$ which gives rise to a polarization of $\hat{\gamma}$ (see Lemma 3.2) to another such inner product by

$$(x, y)_B = (Bx, By) \quad \text{for} \quad x, y \in \hat{K}.$$

If the polarization $(\ ,\ )$ is a generalized Fock polarization (see Definition 3.11), so is the transform $(\ ,\ )_B$. In particular, taking $(\ ,\ )_{\hat{S}}$ as $(\ ,\ )$, we obtain the formulas:

$$(4.4)\qquad \begin{cases} \hat{\gamma}_B = p_B - \overline{p_B} \\ p_B = B^{-1} p B\,, \end{cases}$$

where the suffix B is used to denote operators associated with the polarization $(\ ,\ )_B$.

Lemma 4.1. *Let B_1 be the operator in $\hat{K}$ defined by*

$$(4.5)\qquad B_1 = \begin{pmatrix} \mathbf{1} & -(q+\bar{q}) \\ 0 & \mathbf{1} \end{pmatrix}.$$

Then B_1 is a Bogoliubov transformation and the transform of p' by B_1^{-1} is given by

$$(4.6)\qquad p'_{B_1^{-1}} = B_1 p' B_1^{-1} = \begin{pmatrix} 0 & 0 \\ 0 & Q \end{pmatrix}.$$

Proof. B_1 is a Bogoliubov transformation because $[B_1, \hat{\Gamma}] = 0$ and $\hat{\gamma}(B_1 x, B_1 y) = \hat{\gamma}(x, y)$ hold trivially. Then (4.6) follows from (4.3).

This B_1 transforms away the crossing part that connects the degenerate part and the non-degenerate part.

(2) *Non-degenerate part.*

Lemma 4.2.

(i) $P + \bar{P} = Q + \bar{Q}$.

(ii) $Q^* = (P - \bar{P}) Q (P - \bar{P})$ *(adjoint is relative to $(\ ,\ )_{\hat{S}}$).*

(iii) $(P-Q)^2 = P\bar{Q}P + \bar{P}Q\bar{P}$.

(iv) $(P-Q)^2$ *is $\hat{S}$-negative, namely*

$$(x, (P-Q)^2 y)_{\hat{S}} \leqq 0 \quad for \quad x, y \in \hat{K}.$$

Proof. (i) If $(Q+\bar{Q})y = 0$ for $y \in (P+\bar{P})\hat{K}$, then $qy = q(Q+\bar{Q})y$

$=0$ by (4.3), $\bar{q}y=0$ due to the same reason and hence $(p'+\bar{p}')y=0$. Since $\ker(p'+\bar{p}')=\ker(p+\bar{p})$ $(=\ker\hat{\gamma})$, this implies $y=0$. Therefore, for $y\in(P+\bar{P})\hat{K}$, $(Q+\bar{Q})y=0$ if and only if $y=0$, namely $Q+\bar{Q}$ is one to one on $(P+\bar{P})\hat{K}$. On the other hand, by (4.3), $Q+\bar{Q}$ is a projection with its range contained in $(P+\bar{P})\hat{K}$. Thus we conclude that $Q+\bar{Q}$ is the identity operator on $(P+\bar{P})\hat{K}$ and hence $P+\bar{P}=Q+\bar{Q}$.

(ii) by the following computation: For $x, y\in\hat{K}$,

$$
\begin{aligned}
(x, Qy)_{\hat{S}} &= \hat{\gamma}(x, (P-\bar{P})Qy) && \text{by (4.1)}\\
&= \hat{\gamma}((P-\bar{P})x, p'y) && \text{by (4.2)}\\
&= \hat{\gamma}(p'(P-\bar{P})x, y) &&\\
&= (p'(P-\bar{P})x, (P-\bar{P})y)_{\hat{S}} && \text{by (4.1)}\\
&= ((P-\bar{P})p'(P-\bar{P})x, y)_{\hat{S}} &&\\
&= ((P-\bar{P})Q(P-\bar{P})x, y)_{\hat{S}} && \text{by (4.2)}.
\end{aligned}
$$

(iii) by the following computation:

$$
\begin{aligned}
(P-Q)^2 &= P(\mathbf{1}-Q)P+(\mathbf{1}-P)Q(\mathbf{1}-P)\\
&= P(P+\bar{P}-Q)P+(P+\bar{P}-P)Q(P+\bar{P}-P)\\
&= P(Q+\bar{Q}-Q)P+\bar{P}Q\bar{P} && \text{by (i)}\\
&= P\bar{Q}P+\bar{P}Q\bar{P}.
\end{aligned}
$$

(iv) First note that if $x\in P\hat{K}$, then $(x, y)_{\hat{S}}=\hat{\gamma}(x, y)$ for all $y\in\hat{K}$. Using this, we have

$$
\begin{aligned}
(x, P\bar{Q}Px)_{\hat{S}} &= (Px, \bar{Q}Px)_{\hat{S}}\\
&= \hat{\gamma}(Px, \bar{Q}Px)\\
&= (Px, (p'-\bar{p}')\bar{Q}Px)_{\hat{S}'}\\
&= -(Px, \bar{p}'Px)_{\hat{S}'}\leqq 0
\end{aligned}
$$

for all $x\in\hat{K}$. (Here we have used the relations $p'\bar{Q}=0$ and $\bar{p}'\bar{Q}=\bar{p}'$, see (4.3).) By replacing x by $\hat{\Gamma}x$, we also have $(x, \bar{P}Q\bar{P}x)\leqq 0$ for all $x\in\hat{K}$. Now (iv) follows from (iii).

By (iv) of this lemma, we can define the unique $\hat{S}$-positive operator

α in $\hat{K}$ such that

$$\sinh^2 \alpha = -(P-Q)^2. \tag{4.7}$$

Lemma 4.3.

(i) *The operator α commutes with P, Q, and $\hat{\Gamma}$.*

(ii) $[(\cosh \alpha \ \sinh \alpha)^{-1} PQ\bar{P}]$ $(\bar{P} \cosh \alpha \ \sinh \alpha)$ *gives the polar decomposition (relative to $(\ ,\ )_{\hat{S}}$) of $PQ\bar{P}$.*

Proof. (i) Lemma 4.2 (iii) implies $[\alpha, \hat{\Gamma}]=0$ and $[\alpha, P]=0$. Since $[\alpha, P-Q]=0$, we also have $[\alpha, Q]=0$.

(ii) We compute $(PQ\bar{P})^*(PQ\bar{P})$ as follows:

$$\begin{aligned}(PQ\bar{P})^*(PQ\bar{P}) &= \bar{P}Q^*PQ\bar{P}\\ &= -\bar{P}QPQ\bar{P} \qquad \text{(by Lemma 4.2 (ii))}\\ &= -\bar{P}Q\bar{P} + \bar{P}Q\bar{P}Q\bar{P} \qquad \text{(by } (P+\bar{P})Q=Q)\\ &= \bar{P}[(P-Q)^4 - (P-Q)^2]\bar{P} \quad \text{(by Lemma 4.2 (iii))}\\ &= \bar{P} \ (\cosh \alpha \ \sinh \alpha)^2.\end{aligned}$$

Therefore there exists a partial isometry u satisfying $PQ\bar{P} = u \cosh \alpha \sinh \alpha$ with $u^*u = \bar{P}P_0$ where P_0 is the $\hat{S}$-orthogonal projection onto $(\ker \alpha)^\perp$ ($\hat{S}$-orthogonal complement of $\ker \alpha$). Since α commutes with P, $\bar{P}$, and Q, we may write $u = (\cosh \alpha \ \sinh \alpha)^{-1} PQ\bar{P}$ where the inverse is well-defined on $(\ker \alpha)^\perp \supset \text{range } (PQ\bar{P})$.

We set

$$H = (u+u^*)\alpha. \tag{4.8}$$

Lemma 4.4. *Adjoint * is relative to $(\ ,\ )_{\hat{S}}$.*

(i) $\bar{u} = u^*$, $u^2 = 0$, *and* $(u+u^*) \cosh \alpha \sinh \alpha = [P, Q]$.

(ii) $H^* = H$ *and* $\bar{H} = H$.

(iii) $\hat{\gamma}(Hx, y) + \hat{\gamma}(x, Hy) = 0$ *for* $x, y \in \hat{K}$.

(iv) $H^2 = \alpha^2$.

Proof. (i) $\bar{u} = u^*$ follows from Lemma 4.3 (i), Lemma 4.2 (i) and (ii). $u^2=0$ follows from $\bar{P}P=0$. The last equality holds from $PQ\bar{P}+\bar{P}Q^*P=PQ(1-P)-(1-P)QP=[P,Q]$. (ii) follows from (i) and Lemma 4.3 (i). (iii) is equivalent to the relation $H(P-\bar{P})+(P-\bar{P})H=0$, which follows from the same equality with H replaced by $PQ\bar{P}$. (iv) Since $uu^*=PP_0$ and $u^*u=\bar{P}P_0$ are orthogonal, $u+u^*$ is a hermitian partially isometric operator with $(u+u^*)^2=P_0$. Thus we have $H^2=(u+u^*)^2\alpha^2=\alpha^2$ (note that α commutes with u and u^*).

Corollary 4.5.

(i) *The operator* $\exp H$ *is a Bogoliubov transformation in* $\hat{K}$.

(ii) $\exp H=\cosh\alpha+H\alpha^{-1}\sinh\alpha$.

(iii) $\exp(-H)\cdot P\cdot\exp H=Q$.

Proof. (i) follows from $\bar{H}=H$ and Lemma 4.4 (iii). (ii) follows from $H^2=\alpha^2$. (iii) requires some computations:

$$e^{-H}Pe^{H}=(\cosh\alpha-H\alpha^{-1}\sinh\alpha)P(\cosh\alpha+H\alpha^{-1}\sinh\alpha)$$

$$=P\cosh^2\alpha-\alpha^{-2}\sinh^2\alpha HPH+\alpha^{-1}\sinh\alpha\cosh\alpha[P,H].$$

From $u^*u=\bar{P}P_0$ and $uu^*=PP_0$, it follows that $Pu=u\bar{P}=u$, $uP=\bar{P}u=0$. From $H=(u+u^*)\alpha$, $u^*=\bar{u}$ and $\alpha P_0=\alpha$, we obtain

$$e^{-H}Pe^{H}=P\cosh^2\alpha-\bar{P}\sinh^2\alpha+\sinh\alpha\cosh\alpha(u-\bar{u})$$

$$=P-(P-Q)^2(P-\bar{P})+(PQ\bar{P}-\bar{P}\bar{Q}P)$$

(by (4.7) and definition of u)

$$=PQ(P-\bar{P})+Q\bar{P}+(PQ\bar{P}+\bar{P}QP)\quad(\text{by } Q+\bar{Q}=P+\bar{P})$$

$$=Q.$$

Let B be a Bogoliubov transformation on $\hat{K}$ defined by

$$B=B_2B_1,\quad B_2=\exp H. \tag{4.9}$$

(See (4.5) for the definition of B_1.)

Corollary 4.6.

(i) $p' = B^{-1} p B$.

(ii) $(Bx, By)_{\hat{S}} = (x, y)_{\hat{S}'}$ *for* $x, y \in \hat{K}$.

(iii) $\hat{S}(Bx, By) = \hat{S}'(x, y)$ *for* $x, y \in \hat{K}$.

Proof. (i) follows from (4.6) and Corollary 4.5 (iii). (ii) First note that $(\mathbf{1} - p - \bar{p}) B = \mathbf{1} - p' - \bar{p}'$ due to (4.9), $(\mathbf{1} - p - \bar{p}) H = 0$, (4.5), (4.2), Lemma 4.2 (i) and the definition of the matrix notation. For $x, y \in \hat{K}$,

$$(Bx, By)_{\hat{S}} = ((\mathbf{1} - p - \bar{p}) Bx, (\mathbf{1} - p - \bar{p}) By)_{\hat{S}} + (Bx, (p + \bar{p}) By)_{\hat{S}}$$

$$= ((\mathbf{1} - p' - \bar{p}') x, (\mathbf{1} - p' - \bar{p}') y)_{\hat{S}} + \hat{\gamma}(Bx, (p - \bar{p}) By)$$

$$= ((\mathbf{1} - p' - \bar{p}') x, (\mathbf{1} - p' - \bar{p}') y)_{\hat{S}'} + \hat{\gamma}(x, (p' - \bar{p}') y)$$

where the last equality is due to $\hat{S}|_{\ker \hat{\gamma}} = S'|_{\ker \hat{\gamma}}$ and (i). (Note that $[B, \hat{\Gamma}] = 0$.) Hence

$$(Bx, By)_{\hat{S}} = ((\mathbf{1} - p' - p') x, (\mathbf{1} - p' - \bar{p}') y)_{\hat{S}'} + (x, (p' + \bar{p}') y)_{\hat{S}'}$$

$$= (x, y)_{\hat{S}'}.$$

(iii) For $x, y \in \hat{K}$,

$$\hat{S}(Bx, By) = 2^{-1}\{(Bx, By)_{\hat{S}} + \hat{\gamma}(Bx, By)\}$$

$$= 2^{-1}\{(x, y)_{\hat{S}'} + \hat{\gamma}(x, y)\}$$

$$= \hat{S}'(x, y).$$

§ 5. Unitary Implementation of Bogoliubov Transformation

In this section, we construct a unitary transformation (between two GNS representations $\pi_{\hat{S}}$ and $\pi_{\hat{S}'}$) that implement the Bogoliubov transformation B given in Section 4, under the condition that $p - p'$ is in the Hilbert-Schmidt class. The assumption given at the beginning of Section 4 is also valid in this section. First we treat the case in which $p - p'$ is a finite-rank operator. The general case then follows from this special case.

(1) *Unitary implementation of* B_1. *(The case of finite rank.)*

In this paragraph, we assume that $B_1 - \mathbf{1}$ is of finite rank. Then in the matrix representation of B_1,

$$B_1=\begin{pmatrix}\mathbf{1} & v\\ 0 & \mathbf{1}\end{pmatrix}, \tag{5.1}$$

v is a finite rank operator and $\bar{v}=v$ ($v=-(q+\bar{q})$ in the notation of § 4). In the following we regard v as an operator on $\hat{K}$ so that $vx=0$ if $x\in(\mathbf{1}-p-\bar{p})\hat{K}$. The action of v on a vector $x\in\hat{K}$ can be expressed as

$$vx=\sum_j \hat{\gamma}(b_j, x)\,a_j \tag{5.2}$$

where $a_j\in(\mathbf{1}-p-\bar{p})\hat{K}$ chosen as linearly independent vectors spanning the (finite-dimensional) range of v, and $b_j\in(p+\bar{p})\hat{K}$ are then uniquely determined. Set

$$q(v)\equiv\sum_j \pi(a_j)\pi(\hat{\Gamma}b_j) \tag{5.3}$$

where π is the GNS representation associated with φ_S. A different choice of linearly independent vectors spanning (range v), say a'_j, must be related by a nonsingular matrix L_{ij} by $a'_i=\sum_j L_{ij}a_j$ and the corresponding vectors in $(p+\bar{p})\hat{K}$, say b'_j, must then be related to b_j by $b_j=\sum_i b'_i\overline{L_{ij}}$. Hence $q(v)$ does not depend on the choice of the basis a_j in the range of v.

Lemma 5.1.

(i) $[q(v), \pi(x)]=\pi(vx)$ *for* $x\in\hat{K}$.

(ii) $-q(v)\subset q(v)^*$ (*skew hermitian*).

(iii) *Every vector in* D *is analytic for* $q(v)$. (*D is defined in Proposition* 3.4.)

Proof. Let $\xi\in D_n$. Then, using Proposition 3.4 (i), we obtain

$$\sum_j\|\pi(a_j)\pi(b_j)^*\xi\|\leqq C(n+2)^{1/2}(n+1)^{1/2}\|\xi\|$$

where $C=2\sum_j\|a_j\|\,\|b_j\|$. By the iteration of this estimate, we have

$$\|q(v)^k\xi\|\leqq\left[\frac{(n+2k)!}{n!}\right]^{1/2}C^k\|\xi\|$$

and the power series

$$\sum_{k\geq 0} \frac{z^k}{k!} \|q(v)^k \xi\|$$

converges if $|z| < (2C)^{-1}$, proving (iii). (i) is obtained by a straightforward calculation. (ii) can be seen as follows: First note that $\bar{v}x = -\sum_j \hat{\gamma}(\hat{\Gamma} b_j, x)\hat{\Gamma} a_j$. Then

$$q(v)^* \supset \sum_j \pi(b_j)\pi(\hat{\Gamma} a_j) = -q(\bar{v}) = -q(v) \quad (\text{by } \bar{v} = v).$$

Corollary 5.2. *$q(v)$ is essentially skew self-adjoint and the unitary operator $Q(v) \equiv \exp(\overline{q(v)})$ (the bar denoting the closure) satisfies*

(i) $Q(v) W_{\hat{S}}(x) Q(v)^* = W_{\hat{S}}(e^v x)$ *for* $x \in \mathrm{Re}\, \hat{K}$;

(ii) $Q(v_1) Q(v_2) = Q(v_1 + v_2)$ *if* $\mathrm{supp}\, v_1 \perp \mathrm{supp}\, v_2$.

Proof. Lemma 5.1 (i) implies $Q(v)\pi(x) = \pi(e^v x) Q(v)$ by algebraic computation if we consider matrix elements between vectors in D and expand $Q(v)$ in power series, which is possible for small v by Lemma 5.1 (iii). This equality in turn implies (i) by algebraic computation if we consider matrix elements between vectors in D and expand $W(x)$ in power series, which is possible by Proposition 3.4 (i). (i) for large v follows from (i) for small v by repetition. Since $q(v_1)$ and $q(v_2)$ commute on D if $\mathrm{supp}\, v_1 \perp \mathrm{supp}\, v_2$, the proof of (ii) is similar.

Note that $e^v = \mathbf{1} + v$ due to $v^2 = 0$.

(2) *Unitary implementation of B_2.*

Let H be a finite rank operator in $\hat{K}$ such that

$$\begin{cases} \hat{\gamma}(Hx, y) + \hat{\gamma}(x, Hy) = 0 \quad \text{for} \quad x, y \in \hat{K}, \\ \bar{H} = H \text{ and } (p + \bar{p}) H (p + \bar{p}) = H. \end{cases} \tag{5.4}$$

Expand H as

$$Hx = \sum_j \hat{\gamma}(b_j, x) a_j \tag{5.5}$$

where $a_j \in (p + \bar{p})\hat{K}$ and $b_j \in (p + \bar{p})\hat{K}$, and set

$$q(H) \equiv \frac{1}{2} \sum_j \pi(a_j)\pi(\Gamma b_j). \tag{5·6}$$

This does not depend on the decomposition (5. 5) for given H by the same reason as before.

Lemma 5. 3.

(i) $[q(H_1), q(H_2)] = q([H_1, H_2])$,

(ii) $[q(H), \pi(x)] = \pi(Hx)$ *for* $x \in \hat{K}$,

(iii) $-q(H) \subset q(H)^*$,

(iv) *every vector in* D *is analytic for* $q(H)$.

Proof. (i) follows from (ii) and (ii)-(iv) can be proved by a method similar to Lemma 5. 2, with computation more or less copied from the proof of Lemma 4. 4 in [1].

Corollary 5. 4. (i) *$q(H)$ is essentially skew self-adjoint and the unitary operator* $Q(H) \equiv \exp(\overline{q(H)})$ *satisfies*

$$Q(H)\, W_{\hat{S}}(x) Q(H)^* = W_{\hat{S}}(e^H x)$$

for $x \in \mathrm{Re}\, \hat{K}$.

(ii) *If* H_1 *and* H_2 *are finite rank operators satisfying* (5. 4) *and commute, then*

$$Q(H_1) Q(H_2) = Q(H_1 + H_2).$$

Proof. The same as the proof of Lemma 5. 1 in [1].

(3) *Unitary implementation of* B_1 *and* B_2. (*General case.*)

In this paragraph, we suppose that H and v are in the Hilbert-Schmidt class. H is assumed to satisfy (5. 4) in addition to $H = H^*$ relative to $(\ ,\ )_S$. (cf. Lemma 4. 4.)

Lemma 5. 5.

(i) *There exists a* Γ*-invariant* (*i.e.,* $\overline{v_n} = v_n$) *finite rank oper-*

ators $\{v_n\}_{n\geqq 1}$ *which converges to* v *in the H.S. norm and for which* $\mathrm{supp}(v_j - v_i) \perp \mathrm{supp}\, v_i$ *whenever* $j > i$.

(ii) *There exists a mutually commuting* $\hat{S}$*-hermitian operators* H_n $(n \geqq 1)$ *of finite rank on* $\hat{K}$ *which satisfies the property for* H *in* (5.4) *and converges to the given* H *in the H.S. norm.*

Proof. (i) Let e_n be the spectral projection of v^*v for the interval $[n^{-1}, \|v\|^2]$ and set $v_n = ve_n$. By the construction, $\{v_n\}_{n\geqq 1}$ satisfies the desired condition.

(ii) Let E_n be the spectral projection of H^2 for the interval $[n^{-1}, \|H^2\|]$ and set $H_n = HE_n$. By this definition, $\{H_n\}_{n\geqq 1}$ is a sequence of mutually commuting $\hat{S}$-hermitian operators of finite rank and converges to H in the H.S. norm. Now we must cheque the equation (5.4) for H_n. Since the Γ-invariance of H_n follows from the commutativity of H and Γ, we have only to show that

$$\hat{\gamma}(H_n x, y) + \hat{\gamma}(x, H_n y) = 0 \tag{5.7}$$

for $x, y \in \hat{K}$. Since H is $\hat{S}$-hermitian and satisfies (5.4), we have $(p - \bar{p})H + H(p - \bar{p}) = 0$. It then follows that $p - \bar{p}$ commutes with H^2 and hence with E_n. This implies (5.7) because H anticommutes with $p - \bar{p} = \gamma_{\hat{S}}$.

Proposition 5.6. *There exist unitary operators* $Q(H)$ *and* $Q(v)$ *on* $\mathfrak{H}_{\hat{S}}$ *such that*

(i) $Q(v) W_{\hat{S}}(x) Q(v)^* = W_{\hat{S}}(e^v x)$,

(ii) $Q(H) W_{\hat{S}}(x) Q(H)^* = W_{\hat{S}}(e^H x)$,

for $x \in \mathrm{Re}\, \hat{K}$.

Proof. (i) Take a sequence of Γ-invariant finite rank operators which converges to v in H.S. norm and for which

$$\mathrm{supp}\,(v_j - v_i) \perp \mathrm{supp}\, v_i \quad \text{whenever} \quad j > i. \tag{5.8}$$

This is possible by Lemma 5.5 (i). If we show that $\{Q(v_i)\}$ and $\{Q(v_i)^*\}$ are Cauchy sequences in the strong operator topology with limits $Q(v)$ and $Q(v)^*$, then $Q(v)$ is unitary and satisfies (i) due to the same formula for $Q(v_i)$ given by Corollary 5.2 (i). This is further reduced to the proof that $\{Q(v_i)\Omega_{\hat{S}}\}$ and $\{Q(v_i)^*\Omega_{\hat{S}}\}$ are Cauchy sequences because $Q(v_i)W_{\hat{S}}(x)\Omega_{\hat{S}}=W_{\hat{S}}(e^{v_i}x)Q(v_i)\Omega_{\hat{S}}$ where unitary operator $W_{\hat{S}}(e^{v_i}x)$ strongly converges to $W_{\hat{S}}(e^{v}x)$ and similarly for $Q(v_i)^*W_{\hat{S}}(x)\Omega=W_{\hat{S}}(e^{-v_i}x)Q(v_i)^*\Omega_{\hat{S}}$. By (5.8) and Corollary 5.2 (ii), we have

$$\|Q(v_i)\Omega_{\hat{S}}-Q(v_j)\Omega_{\hat{S}}\|^2 =2-(\Omega_{\hat{S}},Q(v_i-v_j)\Omega_{\hat{S}})-(\Omega_{\hat{S}},Q(v_j-v_i)\Omega_{\hat{S}}). \tag{5.9}$$

using the formula

$$(\Omega_{\hat{S}},Q(v_i-v_j)\Omega_{\hat{S}})=\det\left(1+\frac{1}{4}(v_i-v_j)(v_i-v_j)^*\right)^{-1/2} \tag{5.10}$$

(see (A.1)) one sees that (5.9) converges to zero as $i,j\to\infty$ because v_i-v_j converges to zero in the H.S. topology.

(ii) This can be shown by the same method as the existence of $Q(v)$. Take a sequence of finite rank operators $\{H_n\}_{n\geq 1}$, satisfying the condition of Lemma 5.5 (ii). By the same argument as above for $Q(v)$, we see the strong convergence of $Q(H_n)$ and $Q(H_n)^*$ using Corollary 5.4 (ii) and the formula $(\Omega_{\hat{S}},Q(H_i-H_j)\Omega_{\hat{S}})=(\det\cosh|H_i-H_j|)^{-1/4}$ given by (A.1). The limit $Q(H)$ of $Q(H_n)$ then is unitary and satisfies (ii) due to Corollary 5.4 (i).

Lemma 5.7. *The following two conditions are equivalent*:

(i) *H and v are in the Hilbert-Schmidt class.*

(ii) *$p-p'$ is in the Hilbert-Schmidt class.*

Proof. We rewrite condition (i). H is in the H.S. class if and only if α is in the H.S. class (by Lemma 4.4 (iv)) and this is equivalent to the condition that $P-Q$ is in the H.S. class by (4.7) (if $P-Q$ is in the H.S. class, so is α) and Corollary 4.5 (iii) (if H is in the H.S. class, so is $P-Q$). On the other hand, $v=-(q+\bar{q})$ is in the H.S. class if and only if q is in the H.S. class. (By (4.3), $q=-vQ$.) Then the equivalence of (i) and (ii) follows from (4.2).

Proposition 5.8. *Assume that $\tau_S=\tau_{S'}$ and $S|\ker\gamma=S'|\ker\gamma$.*

Suppose that $p-p'$ *be in the Hilbert-Schmidt class. Then* $\varphi_{S'}$ *is given by the vector state for* $R_{\hat{S}}(\iota K)$ *with a representative vector* $Q(v)^* Q(H)^* \Omega_{\hat{S}}$, *where* $\Omega_{\hat{S}}$ *is the cyclic vector in the GNS representation associated with* $\varphi_{\hat{S}}$ *and* ι *is the inclusion map of* K *into* $\hat{K}$ *in Lemma* 3.8.

Proof. By Lemma 5.7, H and v are in the H.S. class and we have unitary operators $Q(H)$ and $Q(v)$ of Proposition 5.6. For $x \in \iota \operatorname{Re} K \subset \operatorname{Re} \hat{K}$, we have the following relations

$$(5.11) \qquad \varphi_{S'}(W_{S'}(x)) = \exp\left(-\frac{1}{2}\hat{S}'(\iota x, \iota x)\right)$$

$$= \exp\left(-\frac{1}{2}\hat{S}(B\iota x, B\iota x)\right) \quad \text{(by Corollary (4.6 iii))}$$

$$= (\Omega_{\hat{S}}, W_{\hat{S}}(B\iota x)\Omega_{\hat{S}})$$

$$= (Q(v)^* Q(H)^* \Omega_{\hat{S}}, W_{\hat{S}}(\iota x) Q(v)^* Q(H)^* \Omega_{\hat{S}})$$

(by Proposition 5.6).

§ 6. Relation between Conditions on S and on $\hat{S}$

In this section, we complete the proof of the sufficiency (stated as Corollary 6.8).

Notation 6.1. *Let* A *and* B *be two bounded linear operators on a Hilbert space. We write* $A \overset{\text{H.S.}}{\sim} B$ *if* $A-B$ *is in the Hilbert-Schmidt class.*

By Proposition 5.8 and Corollary 3.14, two quasifree representations π_S and $\pi_{S'}$ are quasi-equivalent if $\tau_S = \tau_{S'}$, $S|_{\ker \tau} \overset{\text{H.S.}}{\sim} S'|_{\ker \tau}$, and $p \overset{\text{H.S.}}{\sim} p'$. The sufficiency proof will be achieved if we prove the equivalence of this condition with the condition for S and S' given in Theorem. This is formulated as Proposition 6.6 below.

Remark 6.2. Note that $\overset{\text{H.S.}}{\sim}$ is an equivalence relation and does not depend on the choice of inner product as long as the inner product induces the same Hilbert space topology.

Definition 6.3. *Given two polarizations S_1 and S_2 of a phase space (K, Γ, γ), we introduce two equivalence relations on the set of polarizations: The first one is denoted by $S_1 \overset{\text{H.S.}}{\sim} S_2$ and it holds if S_1 and S_2 satisfy the following conditions:*

(1) *There exist constants M_1, M_2 such that*

$$(x, x)_{S_2} \leqq M_2 (x, x)_{S_1} \text{ and } (x, x)_{S_1} \leqq M_1 (x, x)_{S_2}$$

for all $x \in K$. (The equivalence of τ_{S_1} and τ_{S_2}.)

(2) *Set $N \equiv \{x \in K;\ (x, x)_{S_1} = 0\}$ (this equals to $\{x \in K;\ (x, x)_{S_2} = 0\}$ by the above condition) and let $q: K \to K/N$ be the quotient map. Take any positive definite inner product $(\ ,\ )$ on K/N which is equivalent to both $(\ ,\ )_{S_1}$ and $(\ ,\ )_{S_2}$. Then $\tilde{S}_1 - \tilde{S}_2$ is in the Hilbert-Schmidt class, where $\tilde{S}_1$ and $\tilde{S}_2$ are positive (relative to $(\ ,\ )$) operators in the completion $\overline{K/N}$ of K/N satisfying*

$$S_i(x, y) = (qx, \tilde{S}_i qy) \qquad (i = 1, 2) \tag{6.1}$$

for $x, y \in K$.

The second one is denoted by $S_1^{1/2} \overset{\text{H.S.}}{\sim} S_2^{1/2}$ and it holds if S_1 and S_2 satisfy the following conditions:

(1) *S_1 and S_2 induce an equivalent topology (the same as* (1) *above).*

(2′) *$(\tilde{S}_1)^{1/2} - (\tilde{S}_2)^{1/2}$ is in the Hilbert-Schmidt class where operators $\tilde{S}_1$ and $\tilde{S}_2$ are defined by* (6.1).

Remark 6.4. (i) The relation $S_1 \overset{\text{H.S.}}{\sim} S_2$ does not depend on the choice of the inner product $(\ ,\ )$ in (2), as is easily seen.

(ii) The relation $S_1^{1/2} \overset{\text{H.S.}}{\sim} S_2^{1/2}$ also does not depend on the choice of the inner product, relative to which $\tilde{S}_1$ and $\tilde{S}_2$ are defined. However this is not at all trivial and will be proved below.

We shall use the following result in [4], which holds for any bounded linear operators A and B.

$$\| |A| - |B| \|_{\text{H.S.}} \leqq 2^{1/2} \| A - B \|_{\text{H.S.}}\,. \tag{6.2}$$

Corollary 6.5. *Let A, B, and T be positive operators on a Hilbert space. Suppose that T has a bounded inverse.*

(i) $(TAT)^{1/2} \overset{\text{H.S.}}{\sim} A^{1/2}$ *if* $T \overset{\text{H.S.}}{\sim} 1$.

(ii) $(TAT)^{1/2} \overset{\text{H.S.}}{\sim} (TBT)^{1/2}$ *if* $A^{1/2} \overset{\text{H.S.}}{\sim} B^{1/2}$.

Proof: (i) follows from

$$\begin{aligned} \|(TAT)^{1/2} - A^{1/2}\|_{\text{H.S.}} &= \||A^{1/2}T| - A^{1/2}\|_{\text{H.S.}} \\ &\leqq 2^{1/2}\|A^{1/2}T - A^{1/2}\|_{\text{H.S.}} \\ &\leqq 2^{1/2}\|A^{1/2}\| \ \|T-1\|_{\text{H.S.}}. \end{aligned} \tag{6.3}$$

Similarly for (ii).

Proof of Remark 6.4 (ii). Let $(\ ,\ )'$ be another inner product and T be a positive invertible operator such that $(Tx, Ty)' = (x, y)$. If $(\ ,\ )'$ and $(\ ,\ )$ induce the same topology, then T and T^{-1} are bounded. Operators $\tilde{S}_1'$ and $\tilde{S}_2'$, which represent S_1 and S_2 relative to $(\ ,\ )'$ are expressed as

$$\tilde{S}_i' = T^2\tilde{S}_i \quad (i=1, 2). \tag{6.4}$$

The positive square root of a $(\ ,\ )'$-positive operator T^2A relative to $(\ ,\ )'$ (also denoted as $(T^2A)^{1/2}$ below) can be expressed as $T(TAT)^{1/2}T^{-1}$ in terms of the $(\ ,\ )$-positive square root $(TAT)^{1/2}$ of the $(\ ,\ )$-positive operator TAT. Hence

$$\begin{aligned} \|(\tilde{S}_1')^{1/2} - (\tilde{S}_2')^{1/2}\|_{\text{H.S.}} &= \|T\{(T\tilde{S}_1T)^{1/2} - (T\tilde{S}_2T)^{1/2}\}T^{-1}\|_{\text{H.S.}} \\ &\leqq \|T\| \ \|T^{-1}\| \ \||\tilde{S}_1^{1/2}T| - |\tilde{S}_2^{1/2}T|\|_{\text{H.S.}} \\ &\leqq 2^{1/2}\|T\| \ \|T^{-1}\| \ \|\tilde{S}_1^{1/2}T - \tilde{S}_2^{1/2}T\|_{\text{H.S.}} \\ &\leqq 2^{1/2}\|T\|^2 \ \|T^{-1}\| \ \|\tilde{S}_1^{1/2} - \tilde{S}_2^{1/2}\|_{\text{H.S.}}. \end{aligned}$$

Thus $\tilde{S}_1^{1/2} \overset{\text{H.S.}}{\sim} \tilde{S}_2^{1/2}$ implies $(\tilde{S}_1')^{1/2} \overset{\text{H.S.}}{\sim} (\tilde{S}_2')^{1/2}$ and vice versa by the symmetry. This shows the independence of the relation $S_1^{1/2} \overset{\text{H.S.}}{\sim} S_2^{1/2}$ from the reference inner product.

Proposition 6.6. *Let S and S′ be two polarizations of a phase space* (K, Γ, γ) *(here we do not assume* $S|_{\ker\gamma} = S'|_{\ker\gamma}$*) such that they* *satisfy condition* (1) *of Definition* 6.3. *Then the following conditions are equivalent where* $\hat{S}$ *and* $\hat{S}'$ *are defined in Section* 3 (10), p *and* p' *are defined by* (4.1), *the bold face* $\mathbf{S}$*'s are operators defined from the corresponding* S*'s by* (3.7) *and the equivalence notation is as described at the beginning of this section*:

(i) $p \overset{\text{H.S.}}{\sim} p'$ *and* $\hat{S}|_{\ker\hat{\gamma}} \overset{\text{H.S.}}{\sim} \hat{S}'|_{\ker\hat{\gamma}}$.

(ii) $\hat{\mathbf{S}} \overset{\text{H.S.}}{\sim} \hat{\mathbf{S}}'$ *and* $\hat{S}|_{\ker\hat{\gamma}} \overset{\text{H.S.}}{\sim} \hat{S}'|_{\ker\hat{\gamma}}$.

(iii) $\hat{S} \overset{\text{H.S.}}{\sim} \hat{S}'$.

(iv) $S \overset{\text{H.S.}}{\sim} S'$ *and* $\mathbf{S}^{1/2} \overset{\text{H.S.}}{\sim} \mathbf{S}'^{1/2}$.

(v) $S^{1/2} \overset{\text{H.S.}}{\sim} S'^{1/2}$.

Corollary 6.7. *If* $S^{1/2} \overset{\text{H.S.}}{\sim} S'^{1/2}$, *then* π_S *and* $\pi_{S'}$ *are quasi-equivalent.*

Proof of Proposition 6.6. (ii) $\Rightarrow$ (i): The relations in Section 4 (1) and Lemma 4.2 hold in the present case as their proofs do not use the coincidence of S and S' on $\ker\gamma$ (assumed in Section 4 and not assumed here). Since

$$\hat{\mathbf{S}}-\hat{\mathbf{S}}'=\frac{1}{2}((p-p')-\overline{(p-p')}) \qquad \text{(by (3.8) and (4.1))}$$

$$=\frac{1}{2}\begin{pmatrix}0 & \bar{q}-q\\ 0 & P-\bar{P}+\bar{Q}-Q\end{pmatrix} \qquad \text{(by (4.2))}$$

is in the H.S. class, $q=(q-\bar{q})Q$ is in the H.S. class (the equality due to (4.3)). Therefore, by Lemma 4.2,

$$(p-p')+\overline{(p-p')}=\begin{pmatrix}0 & -(q+\bar{q})\\ 0 & 0\end{pmatrix}$$

is in the H.S. class. Since $(p-p')-\overline{(p-p')}$ is in the H.S. class as above, we see that $p-p'$ is in the H.S. class.

(i) $\Rightarrow$ (iii): First let T_0: $\ker\hat{\gamma}\rightarrow\ker\hat{\gamma}$ be the operator defined by $(x,y)_{\hat{S}'}=(x,T_0y)_{\hat{S}}$ for $x,y\in\ker\hat{\gamma}$. Then we have

$$(x,y)_{\hat{S}'}=(x,(p-\bar{p})(p'-\bar{p}')y)_{\hat{S}} + ((\mathbf{1}-p'-\bar{p}')x, T_0(\mathbf{1}-p'-\bar{p}')y)_{\hat{S}}$$

for $x,y\in\hat{K}$. Since $p' \overset{\text{H.S.}}{\sim} p$ and $T_0 \overset{\text{H.S.}}{\sim} (\mathbf{1}-p-\bar{p})$ (by $\hat{S}|_{\ker\hat{\gamma}} \overset{\text{H.S.}}{\sim} \hat{S}'|_{\ker\hat{\gamma}}$), we have

$$(p-\bar{p})(p'-\bar{p}')+(\mathbb{1}-p'-\bar{p}')^{*}T_0(\mathbb{1}-p'-\bar{p}')\overset{\text{H.S.}}{\sim}\mathbb{1}.$$

This shows that $\hat{S}\overset{\text{H.S.}}{\sim}\hat{S}'$.

(iii) $\Rightarrow$ (ii) : $\hat{S}\overset{\text{H.S.}}{\sim}\hat{S}'$ means that $(\hat{T}x, \hat{T}y)_{\hat{S}'}=(x, y)_{\hat{S}}$, for $x, y\in\hat{K}$, with $\hat{T}$ a positive invertible operator such that $\hat{T}^2\overset{\text{H.S.}}{\sim}\mathbb{1}$ which is equivalent to $\hat{T}\overset{\text{H.S.}}{\sim}\mathbb{1}$. Then $\hat{\gamma}_{\hat{S}}\overset{\text{H.S.}}{\sim}\hat{\gamma}_{\hat{S}'}$ (because $\hat{\gamma}_{\hat{S}'}=\hat{T}^2\hat{\gamma}_{\hat{S}}\overset{\text{H.S.}}{\sim}\hat{\gamma}_{\hat{S}}$) and hence $\hat{\mathbf{S}}-\hat{\mathbf{S}}'=\frac{1}{2}(\hat{\gamma}_{\hat{S}}-\hat{\gamma}_{\hat{S}'})$ is in the H.S. class. The second condition of (ii) is a restriction of (iii).

(iv) $\Rightarrow$ (v) : Let T be an S'-positive invertible operator on K defined by $(Tx, Ty)_{S'}=(x, y)_S$ for $x, y\in K$. Then T is also S-positive. By $S\overset{\text{H.S.}}{\sim}S'$, $T^2-\mathbb{1}$ is in the H.S. class in $(\ ,\ )_{S'}$, which is equivalent to $T-\mathbb{1}$ in the H.S. class. S and S' are represented by operators $T^2\mathbf{S}$ and $\mathbf{S}'$ relative to $(\ ,\ )_{S'}$. Now we have

$$\begin{aligned}(T^2\mathbf{S})^{1/2}-(\mathbf{S}')^{1/2}&=T(T\mathbf{S}T)^{1/2}T^{-1}-(\mathbf{S}')^{1/2}\\&\overset{\text{H.S.}}{\sim}(T\mathbf{S}T)^{1/2}-(\mathbf{S}')^{1/2}\\&\overset{\text{H.S.}}{\sim}(\mathbf{S})^{1/2}-(\mathbf{S}')^{1/2}\end{aligned}$$

(by Corollary 6.5 (i)). Here the square roots for $T^2\mathbf{S}$ and $\mathbf{S}'$ are relative to $(\ ,\ )_{S'}$ while those for $T\mathbf{S}T$ and $\mathbf{S}$ are relative to $(\ ,\ )_S$. Therefore (v) follows from (iv).

(v) $\Rightarrow$ (iv) : Let $\tilde{S}_i$ $(i=1,2)$ be an operator representation of S_i $(i=1,2)$ relative to a fixed reference inner product (see (6.1)). Since $2(\tilde{S}_1-\tilde{S}_2)=((\tilde{S}_1)^{1/2}+(\tilde{S}_2)^{1/2})((\tilde{S}_1)^{1/2}-(\tilde{S}_2)^{1/2})+((\tilde{S}_1)^{1/2}-(\tilde{S}_2)^{1/2})((\tilde{S}_1)^{1/2}+(\tilde{S}_2)^{1/2})$, $S_1^{1/2}\overset{\text{H.S.}}{\sim}S_2^{1/2}$ implies $S_1\overset{\text{H.S.}}{\sim}S_2$. Now taking $(\ ,\ )_{S'}$ as a reference inner product, we have $\tilde{S}=T^2\mathbf{S}$ and $\tilde{S}'=\mathbf{S}'$ as in the proof of (iv) $\Rightarrow$ (v). Note that $T-\mathbb{1}$ is in the H. S. class (by $S\overset{\text{H.S.}}{\sim}S'$), and hence $(T\mathbf{S}T)^{1/2}\overset{\text{H.S.}}{\sim}\mathbf{S}^{1/2}$ (Corollary 6.6 (i)). Using this, we have

$$\begin{aligned}\mathbf{S}^{1/2}-\mathbf{S}'^{1/2}&\overset{\text{H.S.}}{\sim}(T\mathbf{S}T)^{1/2}-(\mathbf{S}')^{1/2}\\&\overset{\text{H.S.}}{\sim}T(T\mathbf{S}T)^{1/2}T^{-1}-(\mathbf{S}')^{1/2}\\&=(T^2\mathbf{S})^{1/2}-(\mathbf{S}')^{1/2}\\&=(\tilde{S})^{1/2}-(\tilde{S}')^{1/2}.\end{aligned}$$

By this relation, one sees that (iv) holds if (v) is satisfied.

We are now left with the equivalence of (iii) and (iv).

(iii) $\Rightarrow$ (iv) : In terms of the imbedding ι given by Lemma 3.8, $(\ ,\ )_S$ and $(\ ,\ )_{S'}$ are unitarily equivalent to the restrictions of $(\ ,\ )_{\hat{S}}$ and $(\ ,\ )_{\hat{S}}$ to the subspace ιK of $\hat{K}$. Therefore $\tilde{S} \overset{\text{H.S.}}{\sim} \tilde{S}'$ trivially implies $S \overset{\text{H.S.}}{\sim} S'$. If $(x, y)_S = (Tx, Ty)_{S'}$, then $S' = T^2 S$ and hence $S \overset{\text{H.S.}}{\sim} S'$ implies that $S - S'$ is in the H.S. class.

By earlier proof, $\hat{S} \overset{\text{H.S.}}{\sim} \hat{S}'$ implies $p \overset{\text{H.S.}}{\sim} p'$. Let $\iota' x = [0 \oplus x] \in \hat{K}$ for $x \in K$. Then $(\iota' x, \iota' y)_{\hat{S}} = (x, y)_S$ and $(\iota' x, \iota' y)_{\hat{S}'} = (x, y)_{S'}$. Let $(\iota')^*_S$ be the adjoint of the isometry ι' from $(K, (\ ,\ )_S)$ to $(\hat{K}, (\ ,\ )_{\hat{S}})$ and $(\iota')^*_{S'}$ be that of ι' from $(K, (\ ,\ )_{S'})$ to $(\hat{K}, (\ ,\ )_{\hat{S}'})$. If $(\xi, \eta)_{\hat{S}} = (\hat{T}\xi, \hat{T}\eta)_{\hat{S}'}$ and $(x, y)_S = (Tx, Ty)_{S'}$, then

$$(6.5) \qquad (\iota')^*_{S'} = T^2 (\iota')^*_S \hat{T}^{-2}.$$

Since $\hat{S} \overset{\text{H.S.}}{\sim} \hat{S}'$ and $S \overset{\text{H.S.}}{\sim} S'$, $T^2 - 1$ and $\hat{T}^{-2} - 1$ are in the H.S. class and hence

$$(\iota')^*_{S'} \overset{\text{H.S.}}{\sim} (\iota')^*_S .$$

From (3.9) and the relation $(\mathbf{1} - \gamma_S^2)^{1/2} = 2S^{1/2}(\mathbf{1} - S)^{1/2}$, we have

$$(\iota')^*_S \iota = 2S^{1/2}(\mathbf{1} - S)^{1/2}, \quad (\iota')^*_{S'} \iota = 2(S')^{1/2}(\mathbf{1} - S')^{1/2}.$$

As an isometry, ι is bounded and hence we obtain

$$(6.6) \qquad S^{1/2}(\mathbf{1} - S)^{1/2} \overset{\text{H.S.}}{\sim} (S')^{1/2}(\mathbf{1} - S')^{1/2}.$$

Let $A = S^{1/2} + (\mathbf{1} - S)^{1/2}$ and $A' = (S')^{1/2} + (\mathbf{1} - S')^{1/2}$. By (6.6), we have $A^2 \overset{\text{H.S.}}{\sim} (A')^2$. Let $(1-t)^{1/2} = \sum c_n t^n \ (|t| < 1)$. Since $1 \leqq A^2 \leqq 2$,

$$A = 2^{1/2}(1 - [1 - 2^{-1}A^2])^{1/2} = 2^{1/2} \sum c_n (1 - 2^{-1}A^2)^n$$

is absolutely convergent. Using the same formula for A' (convergent in S'-topology and hence in equivalent S-topology), we obtain

$$(6.7) \qquad \|A - A'\|_{\text{H.S.}} \leqq 2^{1/2} \|T\| \ \|T^{-1}\| \ \|A^2 - (A')^2\|_{\text{H.S.}} \sum n |c_n| 2^{-n}$$

where $\|B\|_S \leqq \|T\| \ \|B\|_{S'} \|T^{-1}\| \leqq 2^{-k} \|T\| \ \|T^{-1}\|$ for $B = (\mathbf{1} - 2^{-1}A'^2)^k$. Hence

$$A(S^{1/2} - (S')^{1/2}) = AS^{1/2} - A'(S')^{1/2} + (A' - A)(S')^{1/2}$$
$$= (S - S') + (S^{1/2}(\mathbf{1} - S)^{1/2} - (S')^{1/2}(\mathbf{1} - S')^{1/2}) + (A' - A)(S')^{1/2}$$

is in the H.S. class. Since A^{-1} is bounded, we conclude that $S^{1/2}-(S')^{1/2}$ is in the H.S. class. Namely (iii) implies (iv).

(iv) $\Rightarrow$ (iii): Let $k_\varepsilon(\varepsilon=\pm)$ be the isometric maps from $(K_S, (\ ,\ )_S)$ into $(\hat{K}, (\ ,\ )_{\hat{S}})$ given by (3.13) (isometry due to (3.11) and (3.15)) and $\iota_0=2^{-1/2}k_0$ be the isometric map from $(\ker\gamma, (\ ,\ )_S)$ onto $(\ker\hat{\gamma}, (\ ,\ )_{\hat{S}})$ with k_0 given by (3.14). We have the following $\hat{S}$-orthogonal sum decomposition.

$$\hat{K}=\overline{k_+K_S}\oplus\overline{k_-K_S}\oplus\iota_0(\ker\gamma). \tag{6.8}$$

For $\varepsilon=\pm$ and $\eta=\pm$, let A's be defined by

$$(k_\varepsilon x, k_\eta y)_{\hat{S}'}-(k_\varepsilon x, k_\eta y)_{\hat{S}}=(x, A_{\varepsilon,\eta}y)_S,$$

$$(\iota_0 z, k_\varepsilon x)_{\hat{S}'}-(\iota_0 z, k_\varepsilon x)_{\hat{S}}=(z, A_\varepsilon x)_S,$$

$$(\iota_0 z, \iota_0 w)_{\hat{S}'}-(\iota_0 z, \iota_0 w)_{\hat{S}}=(z, A_0 w)_S,$$

where $x, y\in K_S$ and $z, w\in\ker\gamma$. For an orthonormal basis ξ_j of $\hat{K}$, $\sum|(\xi_i,\xi_j)_{\hat{S}'}-(\xi_i,\xi_j)_{\hat{S}}|^2<\infty$ is equivalent to $\hat{S}'\overset{\text{H.S.}}{\sim}\hat{S}$ and hence we obtain (iii) if we prove that all A's are in the H.S. class.

Let $(x,y)_{S'}=(Tx,Ty)_S$ with S- (and S'-) positive T. Let $h_\varepsilon(\varepsilon=\pm)$ be given by (3.10) and h'_ε be the same for S'. By (3.11) and (3.15), we obtain

$$A_{\varepsilon\eta}=\sum_{\sigma=\pm}\{(Th'_\sigma k_\varepsilon)^*(Th'_\sigma k_\eta)-(h_\sigma k_\varepsilon)^*(h_\sigma k_\eta)\},$$

$$2^{1/2}A_\varepsilon=\sum_{\sigma=\pm}(\mathbf{1}-p-\bar{p})T^2h'_\sigma k_\varepsilon,$$

$$A_0=(\mathbf{1}-p-\bar{p})(T^2-\mathbf{1})(\mathbf{1}-p-\bar{p}).$$

Since $h_\sigma k_\varepsilon=\delta_{\sigma\varepsilon}k_\varepsilon^*k_\varepsilon$ is bounded and $T-\mathbf{1}$ is in the H.S. class by (iv), it is enough to show that $(h'_\sigma-h_\sigma)k_\eta$ is in the H.S. class in $(K, (\ ,\ )_S)$. Since $\Gamma(h'_\sigma-h_\sigma)k_\eta\Gamma=(h'_{-\sigma}-h_{-\sigma})k_{-\eta}$, it is enough to prove this for $\sigma=+$.

By using definitions of h's and k's, we obtain

$$\begin{aligned}2(h'_+-h_+)k_+&=2\{((S')^{1/2}-S^{1/2})S^{1/2}-((\mathbf{1}-S')^{1/2}\\&\quad-(\mathbf{1}-S)^{1/2})(\mathbf{1}-S)^{1/2}\}(2S-\mathbf{1})^{-1}\\&=(A'-A)A^{-1}+\{(A')^{-1}(2S'-\mathbf{1})\\&\quad-A^{-1}(2S-\mathbf{1})\}(2S-\mathbf{1})^{-1}A\end{aligned}$$

where we define

$$A=S^{1/2}+(\mathbf{1}-S)^{1/2}, \quad A'=(S')^{1/2}+(\mathbf{1}-S')^{1/2},$$

and we have used the formulae

$$S^{1/2}-(\mathbf{1}-S)^{1/2}=A^{-1}(2S-\mathbf{1}),$$

$$2(a\alpha-b\beta)=(a+b)(\alpha-\beta)+(a-b)(\alpha+\beta).$$

Since $\gamma_{S'}=T^{-2}\gamma_S$, we have $2S'-\mathbf{1}=T^{-2}(2S-\mathbf{1})$. Furthermore $\|A^{-1}\| \leqq 1$, $\|A'^{-1}\|_S \leqq \|T\| \ \|T^{-1}\| \ \|A'^{-1}\|_{S'} \leqq \|T\| \ \|T^{-1}\|$, $\|A\| \leqq 2^{1/2}$. Hence

$$2\|(h'_+-h_+)k_+\|_{\text{H.S.}} \leqq (1+c)\|A'-A\|_{\text{H.S.}}+c\|T^2-\mathbf{1}\|_{\text{H.S.}}$$

where $c=2^{1/2}\|T\| \ \|T^{-1}\|$. Since

$$\|(\mathbf{1}-S')^{1/2}-(\mathbf{1}-S)^{1/2}\|_{\text{H.S.}}=\|\Gamma((S')^{1/2}-S^{1/2})\Gamma\|_{\text{H.S.}}$$
$$=\|(S')^{1/2}-S^{1/2}\|_{\text{H.S.}},$$

(iv) implies that $(h'_+-h_+)k_+$ is in the H. S. class.

Similarly, we have

$$\begin{aligned} 2(h'_+-h_+)k_- &= -2\{((S')^{1/2}-S^{1/2})(\mathbf{1}-S)^{1/2}-((\mathbf{1}-S')^{1/2}-(\mathbf{1}-S)^{1/2})S^{1/2}\} \\ &\qquad \times(2S-\mathbf{1})^{-1} \\ &= (A'-A)A^{-1}-\{(A')^{-1}(2S'-\mathbf{1})-A^{-1}(2S-\mathbf{1})\}(2S-\mathbf{1})^{-1}A, \end{aligned}$$

which is in the H.S. class.

§ 7. Standard Polarization

In this section we assume that K is separable (not necessarily the same as K of our theorem). In our application, this condition will be satisfied.

Definition 7.1. *A polarization S of (K, Γ, γ) is called standard if γ_S does not have eigenvalue* 1 *with respect to* $(\ ,\ )_S$.

Lemma 7.2. *S is standard if and only if Ω_S (the cyclic vector in the GNS representation associated with quasifree state φ_S) is cyclic* *and separating for* $R_S(K)$.

Proof. By Theorem 3.12. Details are the same as in Lemma 2.3 of [1].

Lemma 7.3. *Let* S_1 *be a polarization of* (K, Γ, γ). *Then there exists a polarization* S_2 *of* (K, Γ, γ) *such that*

(i) S_2 *is standard,*

(ii) $S_1^{1/2} \overset{\text{H.S.}}{\sim} S_2^{1/2}$.

Proof. Let e be the eigenprojection of γ_{S_1} corresponding to the eigenvalue 1. Take a positive and invertible trace class operator χ in eK. Such an operator always exists because K is assumed to be separable. Set

$$T = \mathbb{1} + \chi + \bar{\chi}$$

$$(x, y)_{S_2} = (x, Ty)_{S_1}.$$

Since the inner product $(\ ,\ )_{S_2}$ satisfies the condition in Lemma 3.2, it defines a polarization (note that $T \geqq \mathbb{1}$ and $\Gamma T \Gamma = T$). On the other hand, the matrix representation of γ_{S_2} with respect to the decomposition $K = eK + \bar{e}K + (\mathbb{1} - e - \bar{e})K$ is given by

$$\begin{bmatrix} (\mathbb{1}+\chi)^{-1} & & \\ & -(\mathbb{1}+\bar{\chi})^{-1} & \\ & & (\mathbb{1}-e-\bar{e})\gamma_{S_1} \end{bmatrix}.$$

From this expression, γ_{S_2} does not have eigenvalue 1 and the polarization S_2 is standard. Condition (ii) is satisfied by the construction.

By this lemma, we may assume that both S and S' are standard for the necessity proof, which we shall do. Let $R = \{\exp i\pi_S(x)\,;\, x \in \mathrm{Re}\, K\}''$ be the von Neumann algebra for the GNS representation $(\mathfrak{H}_S, \pi_S, \Omega_S)$ associated with the quasifree state φ_S of $\mathfrak{A}(K, \Gamma, \gamma)$. Furthermore we may identify $(\mathfrak{H}_S, \pi_S, \Omega_S)$ with $(\mathfrak{H}_{\hat{S}}, \pi_{\hat{S}}|\iota K, \Omega_{\hat{S}})$ under the identifying map ι from K into $\hat{K}$ given by $\iota x = [x \oplus 0]$, (see Lemma 3.8 (i)). The main objective of this section is to give an estimation for $\|\varphi_S - \varphi_{S'}\|$. For this purpose we need an explicit form of the modular conjugation for R associated with $\Omega_{\hat{S}}$.

Let $j: \hat{K} \to \hat{K}$ be the conjugate linear Bogoliubov transformation defined by

$$j(x_1 \oplus x_2) = \Gamma x_2 \oplus \Gamma x_1 \quad \text{for} \quad x_1, x_2 \in K. \tag{7.1}$$

Since $[j, \hat{\Gamma}] = 0$ and $\hat{\gamma}(jx, jy) = \overline{\hat{\gamma}(x, y)}$, j induces a conjugate linear automorphism ζ of $\mathfrak{A}(\hat{K})$. Since $\hat{S}(x, y) = \hat{S}(jy, jx)$, we can define the unitary conjugation J in $\mathfrak{H}_{\hat{S}}$ by the relation

$$J(\pi(a)\Omega_{\hat{S}}) = \pi(\zeta a)\Omega_{\hat{S}} \quad \text{for} \quad a \in \mathfrak{A}(\hat{K}). \tag{7.2}$$

Lemma 7.4. *J gives the modular conjugation for $\Omega_{\hat{S}}$.*

Proof. In [1], this is effectively proved by Corollary 3.4, (3.2) and (3.3) where $T_{\Pi_S}(\omega)$ is the present J.

Lemma 7.5. *Suppose that* $\dim(K/K_0)$ *is finite. Then we have*

(i) $j|_{\ker \hat{\gamma}} = \hat{\Gamma}|_{\ker \hat{\gamma}}$,

(ii) $[v, j] = 0$, $[H, j] = 0$,

(iii) $[Q(v), J] = 0$, $[Q(H), J] = 0$.

Proof. (i) follows from the definition and (iii) is a consequence of (ii). (ii): Since $\hat{\gamma}(jx, jy) = \hat{\gamma}(y, x)$, $(jx, jy)_{\hat{S}} = (y, x)_{\hat{S}}$, and $(jx, jy)_{\hat{S}'} = (y, x)_{\hat{S}'}$, we have $[\hat{S}, j] = 0$ and $[\hat{S}', j] = 0$. Hence $[p, j] = 0$ and $[p', j] = 0$. Then (ii) follows from the definition of H. (4.8) and the fact that $v = (\mathbf{1} - p' - \bar{p}')(p + \bar{p})$.

For a cyclic and separating vector $\xi \in \mathfrak{H}_{\hat{S}}$ for R, we denote the natural positive cone associated with ξ by V_ξ. It is known that $xJxJ\Psi \subset V_\xi$ and $J\Psi = \Psi$ for any $\Psi \in V_\xi$ and $x \in R$. (Theorem 4 of [2].)

Lemma 7.6. *Suppose that* $\dim(K/K_0)$ *is finite and let $\Omega \in \mathfrak{H}$ be a cyclic and separating vector for R. If $\xi \in V_\Omega$ is cyclic for R, then $Q(v)^*\xi$ is cyclic for R and $Q(v)^*\xi \in V_\Omega$.*

Proof. Through the map ι, we can identify $\ker\hat{\gamma}$ with $\ker\gamma$. Since $v(\ker\hat{\gamma})=0$, $\dim(\hat{K}/\ker\hat{\gamma})<\infty$, and $\hat{\gamma}$ is non-degenerate on $\hat{K}/\ker\hat{\gamma}$, we can expand the restriction of v to $K\subset\hat{K}$ as follows;

$$v(x)=\sum_{i=1}^{N}\hat{\gamma}(b_i,x)a_i \quad \text{for} \quad x\in K$$

where $\{a_i\}_{i=1}^{N}\subset\ker\gamma$ and $\{b_i\}_{i=1}^{N}\subset K\ominus\ker\gamma$ (S-orthogonal complement of $\ker\gamma$). Furthermore, by $\bar{v}=v$, we may assume that

$$\Gamma a_i=a_i, \quad \text{and} \quad \Gamma b_i=-b_i \quad (i=1,\cdots,N). \tag{7.3}$$

Then, using $[v,j]=0$; we have

$$\begin{aligned} v(x+jx') &= v(x)+jv(x') \\ &= \sum_i \gamma(b_i,x)a_i+\sum_i \overline{\gamma(b_i,x')}\,ja_i \\ &= \sum_i \hat{\gamma}(b_i+jb_i,x+jx')a_i \end{aligned}$$

for $x,x'\in K$ due to $ja_i=\Gamma a_i=a_i$, $\overline{\gamma(b_i,x')}=\hat{\gamma}(jb_i,jx')$ and the vanishing of each cross term. This implies

$$v(\hat{x})=\sum_i \hat{\gamma}(b_i+jb_i,\hat{x})a_i \quad \text{for} \quad \hat{x}\in\hat{K}$$

because $K+jK=\hat{K}$. Using this expansion of v, we have

$$q(v)=r(v)+Jr(v)J$$

where $r(v)=\sum_i \pi_S(a_i)\pi_S(b_i)^*$. Note that $r(v)\in\pi_S(\mathfrak{A}(K))$, and $r(v)^*\supset -r(v)$ by (7.3). Furthermore the same estimate as the proof of Lemma 5.1 (iii) shows that D is a dense set of analytic vectors for $r(v)$ and hence $r(v)$ is essentially skew self-adjoint. Any operator A in R' satisfies $(A\Psi,B\Phi)=(B^*\Psi,A^*\Phi)$ for $\Psi,\Phi\in D$ (Lemma 3.4) and $B=\pi_S(x)$, $x\in\operatorname{Re}K$ and hence for $B=r(v)\in\pi_S(\mathfrak{A}(K))$. Since D is the core of $\overline{r(v)}$, $e^{-\overline{r(v)}}$ commutes with $A\in R'$ and $e^{-\overline{r(v)}}\in R$. Thus $r(v)$ and $Jr(v)J$ are affiliated with R and R', respectively, and we have

$$Q(v)^*=e^{-\overline{q(v)}}=e^{-\overline{r(v)}}Je^{-\overline{r(v)}}J\,.$$

Hence $Q(v)^*\xi\in V_\Omega$ if $\xi\in V_\Omega$. The cyclicity of $Q(v)^*\xi$ is immediate from the cyclicity of ξ because $e^{-\overline{r(v)}}$ is unitary.

Lemma 7.7. *Consider a von Neumann algebra F on a Hilbert* *space* $\mathfrak{H}_F$ *and two cyclic and separating vectors* $\xi_1, \xi_2 \in \mathfrak{H}_F$ *for* F. *Suppose that* ξ_1 *and* ξ_2 *have a common modular conjugation operator, i.e., there is an antiunitary involution* J *in* $\mathfrak{H}_F$ *such that*

(i) $J\xi_i = \xi_i \quad (i=1, 2)$,

(ii) $JFJ = F'$,

(iii) $(\xi_i, AJA\xi_i) \geqq 0$ *for all* $A \in F \quad (i=1, 2)$.

Then there is a self-adjoint unitary operator u *in the center of* F *such that*

$$u\xi_2 \in V_{\xi_1}.$$

Proof. It is known that there exists a unitary operator $\theta(\xi_2, \xi_1)$ in F' such that

$$J_{\xi_2} = \theta(\xi_2, \xi_1) J_{\xi_1} \theta(\xi_2, \xi_1)^*,$$

$$\theta(U'\xi_2, \xi_1) = U'\theta(\xi_2, \xi_1)$$

if U' is a unitary operator in F', and $\xi_2 \in V_{\xi_1}$ if and only if $\theta(\xi_2, \xi_1) = \mathbf{1}$. (For example, Lemma 2. 5. 35~37 in [5].) In the above case, we have $J_{\xi_2} = J_{\xi_1} = J$, and hence $J\theta(\xi_2, \xi_1)J = \theta(\xi_2, \xi_1)$. Thus $u \equiv \theta(\xi_2, \xi_1) \in F \cap F'$. By Lemma 3 of [4], for example, $u^* = JuJ$ and hence $u^* = u$. Since $\theta(u\xi_2, \xi_1) = u\theta(\xi_2, \xi_1) = \mathbf{1}$, $u\xi_2$ is a vector in V_{ξ_1}.

We note that $u = \pm \mathbf{1}$ if F is a factor.

To apply this lemma, let $R = A \otimes F$ and $\mathfrak{H}_{\hat{S}} = \mathfrak{H}_A \otimes \mathfrak{H}_F$ be the tensor product factorization of $(R, \mathfrak{H}_{\hat{S}})$ corresponding to the $\hat{S}$-orthogonal decomposition of phase space $\hat{K} = \ker \hat{\gamma} \oplus (\hat{K} \ominus \ker \hat{\gamma})$ with $\Omega_{\hat{S}} = \Omega_A \otimes \Omega_F$, where $(\mathfrak{H}_A, \pi_A, \Omega_A)$ is obtained by the GNS-construction from the state $\varphi_{(S|\ker \gamma)}$ of $\mathfrak{A}(\ker \gamma, \Gamma|\ker \gamma, O)$ with $A = \{\exp i\pi(x);\ x \in \mathrm{Re}\ (\ker \gamma)\}''$ and similarly for $\mathfrak{H}_F$, Ω_F and F. Since $Q(H) = \mathbf{1} \otimes Q$ (we denote Q again by $Q(H)$ below), we apply the above lemma for $\xi_1 = \Omega_F$ and $\xi_2 = Q\Omega_F$ in Corollary 7. 9 below. Once we have $QV_{\Omega_F} \subset V_{\Omega_F}$, then $(\mathbf{1} \otimes Q) V_{\Omega_{\hat{S}}} \subset V_{\Omega_{\hat{S}}}$. The condition of the above Lemma is satisfied for Ω_F and $Q\Omega_F$ due to the following:

Lemma 7.8. *If* $\dim(K/\ker \gamma)$ *is finite, the following assertions* *hold.*

(i) Ω_F *and* $Q(H)^*\Omega_F \in \mathfrak{H}_F$ *are cyclic and separating for* F.

(ii) $JFJ = F'$.

(iii) $J\Omega_F = \Omega_F$, $JQ(H)^*\Omega_F = Q(H)^*\Omega_F$.

(iv) $(\Omega_F, AJAJ\Omega_F) \geqq 0$ *and* $(Q(H)^*\Omega_F, AJAJQ(H)^*\Omega_F) \geqq 0$

for all $A \in F$. *Here* J *is defined in the same way as* (7.2) *on* $\mathfrak{H}_F$.

Proof. Since S and S' are standard, (i) follows from Lemma 7.2. By the same reason as Lemma 7.4, J is the modular conjugation and hence (ii), the first equation of (iii) and the first inequality of (iv) follow. The second equation of (iii) follows from the first by Lemma 7.5 (iii). If we identify $Q(H)^*\Omega_F$ with the corresponding vector for S', J defined for S' coincide with the present J due to the second equation of (iii). (Note that ζ is common.) Therefore the second inequality of (iv) follows.

Corollary 7.9. *Suppose that* $\dim(K/\ker\gamma)$ *is finite. Then we have* $Q(H)^*\Omega_{\hat{S}} \in V_{\Omega_{\hat{S}}}$.

Proof. Applying Lemma 7.7 to the system $(F, \mathfrak{H}_F, \Omega_F, Q(H)^*\Omega_F)$, we have

$$cQ(H)^*\Omega_{\hat{S}} \in V_{\Omega_{\hat{S}}}.$$

By (A.1), $(\Omega_{\hat{S}}, Q(H)^*\Omega_{\hat{S}}) = (\det\cosh H)^{-1/4} \geqq 0$. If $\xi \in V_{\Omega_{\hat{S}}}$, then $(\Omega_{\hat{S}}, \xi) \geqq 0$. Hence $c = 1$.

Proposition 7.10. *Suppose that* $\dim(K/\ker\gamma)$ *is finite and that* $S|_{\ker\gamma} = S'|_{\ker\gamma}$. *Regard* φ_S *and* $\varphi_{S'}$ *as states of* R. *Then*

$$\|\varphi_S - \varphi_{S'}\| \geqq \|\Omega_{\hat{S}} - Q(v)^*Q(H)^*\Omega_{\hat{S}}\|^2$$
$$= 2(1 - (\Omega_{\hat{S}}, Q(H)Q(v)\Omega_{\hat{S}})).$$

Proof. By (5.11), φ_S and $\varphi_{S'}$ are given by vectors $\Omega_{\hat{S}}$ and $Q(v)^* \times Q(H)^*\Omega_{\hat{S}}$. By Lemma 7.6 and Corollary 7.9, $\Omega_{\hat{S}}$ and $Q(v)^*Q(H)^*\Omega_{\hat{S}}$ are in the same natural positive cone $V_{\Omega_{\hat{S}}}$. Now the inequality is the von-Neumann algebra version of the Powers-Størmer inequality. (Theorem 4 (8) of [2]).

§ 8. Necessity Proof

In this section, we give the proof of necessity. First we consider the case of standard polarizations. The general case will be reduced to this special case by the result of Section 7.

Lemma 8. 1. *Let R be a W^*-algebra and $\tilde{\varphi}_1, \tilde{\varphi}_2$ be two faithful normal states of R. Suppose that $\tilde{\varphi}_1$ and $\tilde{\varphi}_2$ are quasi-equivalent. Then we have*

$$\|\tilde{\varphi}_1-\tilde{\varphi}_2\|<2\,. \tag{8.1}$$

Proof. By the proof of (6. 15) in [1].

Lemma 8. 2. *Let S and S' be two polarizations such that the associated inner products $(\,,\,)_S$ and $(\,,\,)_{S'}$ induce the same topology, with respect to which K is separated and complete. Let p, p' and $\mathbf{S}$, $\mathbf{S}'$ be operators on $\hat{K}$ and K, respectively, defined from S and S' as in* Section 3 (10) *and* (8). *Let T be defined on K by*

$$(x, y)_{S'} = (x, Ty)_S$$

for all $x, y\in K$. Then T is a bounded operator with bounded inverse and

$$4\|p-p'\|^2_{\mathrm{H.S.}}=\|\beta(\mathbf{1}-(\beta')^{-2}T^{-1}\beta^2)E\beta^{-1}\|^2_{\mathrm{H.S.}}+\|\beta(E-E')\beta^{-1}\|^2_{\mathrm{H.S.}} + \|\beta E(\mathbf{1}-\beta^{-2}TE'(\beta')^2)\beta^{-1}\|^2_{\mathrm{H.S.}} \tag{8.2}$$

where the Hilbert Schmidt norm is relative to $(\,,\,)_{\hat{S}}$ on the left hand side, relative to $(\,,\,)_S$ on the right hand side, $\mathbf{1}-E$ and $\mathbf{1}-E'$ are orthogonal projections on $\ker\gamma$, *orthogonality being with respect to $(\,,\,)_S$ and $(\,,\,)_{S'}$, respectively, and*

$$\beta=\mathbf{S}^{1/2}+(\mathbf{1}-\mathbf{S})^{1/2},\quad \beta'=(\mathbf{S}')^{1/2}+(\mathbf{1}-\mathbf{S}')^{1/2}. \tag{8.3}$$

Remark 8. 3. If $\ker\gamma=0$, (8. 2) is the same as (6. 9) in [1].

Proof. The assumption that $(\ ,\ )_S$ and $(\ ,\ )_{S'}$ induce the same topology on K is equivalent to the conclusion that T is a bounded operator with bounded inverse.

Denote K with the inner product $(x, y)_S$ by K_S, the completion of $K/\ker\gamma$ with the inner product $(\gamma_S x, \gamma_S y)_S = (x, y)^{\#}_S$ by $K^{\#}_S$ and $\langle K\rangle_S = K_S \oplus K^{\#}_S$, with its inner product denoted as $\langle\ ,\ \rangle_S$. Let α be the mapping from $\hat{K}$ into $\langle K\rangle_S$ uniquely determined by

$$\alpha(x\oplus y) = (x+y)\oplus(x-y). \tag{8.4}$$

We note that

$$2(\xi_1, \xi_2)_{\hat{S}} = \langle(\beta\oplus\beta^{-1})\alpha\xi_1,\ (\beta\oplus\beta^{-1})\alpha\xi_2\rangle_S, \tag{8.5}$$

β is a bounded operator with a bounded inverse ($\mathbf{1}\leqq\beta\leqq 2^{1/2}$) and the range of α is $K\oplus K/\ker\gamma$ dense in $\langle K\rangle_S$. Hence α is a bounded map onto $\langle K\rangle_S$ with a bounded inverse.

From (3.13), we have

$$\begin{aligned} \alpha k_+(x) &= \gamma_S^{-1}(S^{1/2}-(\mathbf{1}-S)^{1/2})x\oplus(S^{1/2}+(\mathbf{1}-S)^{1/2})\gamma_S^{-1}x \\ &= \beta^{-1}x\oplus\beta\gamma_S^{-1}x, \end{aligned} \tag{8.6}$$

$$\alpha k_-(x) = \beta^{-1}x\oplus-\beta\gamma_S^{-1}x \tag{8.7}$$

where the equation holds for $x\in EK\cap D(\gamma_S^{-1})$ and hence by continuity for $x\in EK$ for which $\gamma_S^{-1}x\in K^{\#}_S$. Furthermore, if $x, y\in EK$, then

$$2(x\oplus y) = \alpha k_+(\beta x+\beta^{-1}\gamma_S y) + \alpha k_-(\beta x-\beta^{-1}\gamma_S y). \tag{8.8}$$

Hence

$$2\alpha p\alpha^{-1}(x\oplus y) = (Ex+\beta^{-2}\gamma_S y)\oplus(\beta^2\gamma_S^{-1}Ex+Ey). \tag{8.9}$$

The inner product $(\ ,\ )^{\#}_S$ is γ-dual to $(\ ,\ )_S$. (Namely $(x, x)^{\#1/2}_S$ is the supremum of $|\gamma(x, y)|$ for $(y, y)_S\leqq 1$.) Therefore if $(\ ,\ )_S$ and $(\ ,\ )_{S'}$ give the same topology on K, then $(\ ,\ )^{\#}_S$ and $(\ ,\ )^{\#}_{S'}$ give the same topology on $K/(\ker\gamma)$ and hence we have a natural identification of $K^{\#}_S$ with $K^{\#}_{S'}$. Then

$$2\alpha p'\alpha^{-1}(x\oplus y) = (E'x+\beta'^{-2}\gamma_{S'}y)\oplus(\beta'^2\gamma_{S'}^{-1}E'x+E'y). \tag{8.10}$$

From

$$(x, TE'y)_S = (x, E'y)_{S'} = \gamma(x, \gamma_{S'}^{-1}E'y) = (x, \gamma_S\gamma_{S'}^{-1}E'y)_S \tag{8.11}$$

we obtain

$$TE'=\gamma_S\gamma_{S'}^{-1}E', \quad \text{i.e.} \quad \gamma_S^{-1}ETE'=E\gamma_{S'}^{-1}E'. \tag{8.12}$$

Similarly

$$T^{-1}E=\gamma_{S'}\gamma_S^{-1}E, \quad \text{i.e.} \quad T^{-1}\gamma_S=\gamma_{S'}E. \tag{8.13}$$

From (8.9), (8.10), (8.12) and (8.13), we obtain

$$\begin{aligned} &2\{(\beta\oplus\beta^{-1})\alpha(p-p')\alpha^{-1}(\beta^{-1}\oplus\beta)\}(x\oplus y) \\ &=\{\beta(E-E')\beta^{-1}x+\beta(\beta^{-2}-(\beta')^{-2}T^{-1})\beta\gamma_S y\} \\ &\oplus\{\gamma_S^{-1}E\beta^{-1}(\beta^2-TE'(\beta')^2)\beta^{-1}x+\beta^{-1}(E-E')\beta y\} \end{aligned} \tag{8.14}$$

where we have used $z=Ez$ in $K_S^\#$.

Since $(E-E')z=(\mathbb{1}-E')z-(\mathbb{1}-E)z\in\ker\gamma$, we have $\gamma_S(E-E')z=0$. Since β and γ_S commute as functions of S, the second term $\beta^{-1}(E-E')\beta y$ in the second summand on the right hand side is 0 in $K_S^\#$. If y_i is an orthonormal basis in $K_S^\#$, then $\gamma_S y_i$ is an orthonormal basis of EK. Since $2^{-1/2}(\beta\oplus\beta^{-1})\alpha$ is unitary due to (8.5), we have

$$\begin{aligned} &4\|p-p'\|_{\text{H.S.}}^2 \\ &=\sum_k\|\{(\beta\oplus\beta^{-1})\alpha(p-p')\alpha^{-1}(\beta^{-1}\oplus\beta)\}(x_k\oplus 0)\|_{\langle,\rangle_S}^2 \\ &\quad+\sum_j\|\{(\beta+\beta^{-1})\alpha(p-p')\alpha^{-1}(\beta^{-1}\oplus\beta)\}(0\oplus y_j)\|_{\langle,\rangle_S}^2 \\ &=\|\beta(E-E')\beta^{-1}\|_{\text{H.S.}}^2+\|E\beta^{-1}(\beta^2-TE'(\beta')^2)\beta^{-1}\|_{\text{H.S.}}^2 \\ &\quad+\|\beta(\beta^{-2}-(\beta')^{-2}T^{-1})\beta E\|_{\text{H.S.}}^2. \end{aligned} \tag{8.15}$$

Lemma 8.4. *Assume $(x,y)_S=(x,y)_{S'}$ for all $x,y\in\ker\gamma$ and $k^{-1}\|x\|_S^2\leqq\|x\|_{S'}^2\leqq k\|x\|_S^2$ for all $x\in K$. Then $\|p-p'\|_{\text{H.S.}}\leqq G$ implies*

$$\|S^{1/2}-(S')^{1/2}\|_{\text{H.S.}}\leqq 2.5kG\cosh(k\pi/2), \tag{8.16}$$

$$\|\mathbb{1}-T\|_{\text{H.S.}}\leqq(8+10\sinh k\pi/2)k^2G. \tag{8.17}$$

Proof. By (8.2), we obtain

$$\begin{aligned} &\|\beta^{-2}(2S-\mathbb{1})-(\beta')^{-2}(2S'-\mathbb{1})\|_{\text{H.S.}} \\ &=\|\beta^{-2}(2S-\mathbb{1})-(\beta')^{-2}(2S'-\mathbb{1})E\|_{\text{H.S.}} \\ &\leqq\|\beta^{-1}\|^2\|2S-\mathbb{1}\|\,\|\beta(\mathbb{1}-(\beta')^{-2}T^{-1}\beta^2)E\beta^{-1}\|_{\text{H.S.}}\leqq 2G, \end{aligned} \tag{8.18}$$

where we have used $(2S'-\mathbb{1})(\mathbb{1}-E)=0$ (due to $(\mathbb{1}-E)K=\ker\gamma$) and (8.13).

Let θ and θ' be operators with spectrum in the interval $[-\pi/4, \pi/4]$ satisfying

$$S=\sin^2(\theta+\pi/4),\quad S'=\sin^2(\theta'+\pi/4). \tag{8.19}$$

Then the left hand side of (8.18) is $\|\tan\theta-\tan\theta'\|_{\text{H.S.}}$ and

$$\begin{aligned}\|\mathrm{Tan}^{-1}A-\mathrm{Tan}^{-1}A'\|_{\text{H.S.}} &= \left\|\int_0^1\{(\mathbb{1}+A^2x^2)^{-1}A-(\mathbb{1}+A'^2x^2)^{-1}A'\}dx\right\|_{\text{H.S.}}\\ &\leqq \int_0^1 dx\|(\mathbb{1}+A^2x^2)^{-1}\{A(\mathbb{1}+A'^2x^2)\\ &\quad -(\mathbb{1}+A^2x^2)A'\}(\mathbb{1}+A'^2x^2)^{-1}\|_{\text{H.S.}}\\ &\leqq \sup_x(\|(\mathbb{1}+A^2x^2)^{-1}\{(A-A')\\ &\quad +(Ax)(A'-A)(A'x)\}(\mathbb{1}+A'^2x^2)^{-1}\|_{\text{H.S.}}).\end{aligned} \tag{8.20}$$

The S-norm of $(\mathbb{1}+A^2x^2)^{-1}$ and $(\mathbb{1}+A^2x^2)^{-1}(Ax)$ are bounded by $\mathbb{1}$ and $(\mathbb{1}/2)$. Since

$$\|B\|_S\leqq\|T^{-1/2}BT^{1/2}\|_{S'}\leqq\|T^{-1/2}\|_{S'}\|T^{1/2}\|_{S'}\|B\|_{S'}\leqq k\|B\|_{S'}, \tag{8.21}$$

the S-norm of $(\mathbb{1}+A'_2x^2)^{-1}$ and $(\mathbb{1}+A'^2x^2)^{-1}(A'x)$ are bounded by k and $k/2$. Together with (8.18) and (8.20), we have

$$\|\theta-\theta'\|_{\text{H.S.}}\leqq 5kG/2\,. \tag{8.22}$$

Hence by Taylor expansion

$$\begin{aligned}\|S^{1/2}-(S')^{1/2}\|_{\text{H.S.}} &\leqq\|\theta-\theta'\|_{\text{H.S.}}\textstyle\sum(2n)!^{-1}(k\pi/2)^{2n}\\ &\leqq 2.5kG\cosh(k\pi/2)\ .\end{aligned}$$

where we have used $\|\theta\|_S\leqq\pi/4$, $\|\theta'\|_S\leqq k\|\theta'\|_{S'}\leqq k\pi/4$

Since $\|\beta^2-\beta'^2\|_{\text{H.S.}}=\|\cos 2\theta-\cos 2\theta'\|_{\text{H.S.}}\leqq 5kG\sinh(k\pi/2)$, we obtain

$$\begin{aligned}\|(\mathbb{1}-T^{-1})E\|_{\text{H.S.}} &=\|\beta'^2\beta^{-1}\{\beta(\mathbb{1}-(\beta')^{-2}T^{-1}\beta^2)E\beta^{-1}\}\beta^{-1}\\ &\quad -(\beta'^2-\beta^2)\beta^{-2}E\|_{\text{H.S.}}\\ &\leqq(2k)(2G)+5kG\sinh(k\pi/2)\end{aligned} \tag{8.23}$$

where we have used $\|\beta'^2\|_S\leqq k\|\beta'^2\|_{S'}\leqq 2k$. Since $(\mathbb{1}-E)(\mathbb{1}-T^{-1})(\mathbb{1}-E)$

$=0$, we have $(\mathbf{1}-T^{-1})(\mathbf{1}-E)=((\mathbf{1}-E)(\mathbf{1}-T^{-1})E)^*$ and hence

$$\begin{aligned}(8.24)\qquad \|\mathbf{1}-T^{-1}\|_{\mathrm{H.S.}} &\leqq \|(\mathbf{1}-T^{-1})E\|_{\mathrm{H.S.}}+\|(\mathbf{1}-T^{-1})(\mathbf{1}-E)\|_{\mathrm{H.S.}} \\ &\leqq 2\|(\mathbf{1}-T^{-1})E\|_{\mathrm{H.S.}} \\ &\leqq (8+10\sinh(k\pi/2))kG\,.\end{aligned}$$

Hence

$$\|T-\mathbf{1}\|_{\mathrm{H.S.}}\leqq\|T\|\ \|\mathbf{1}-T^{-1}\|_{\mathrm{H.S.}}\leqq(8+10\sinh(k\pi/2))k^2G\,.$$

With these lemmas as a preparation, we proceed to the main assertion of this section.

Lemma 8.5. *Let S and S' be two standard polarizations of a separable phase space (K,Γ,γ) such that they coincide on the kernel of γ. Suppose that π_S and $\pi_{S'}$ are quasi-equivalent. Then conditions of Proposition* 6.6 *hold.*

Proof. Since $\pi_S\sim\pi_{S'}$, we can regard φ_S and $\varphi_{S'}$ as states on a von Neumann algebra R associated with the representation π_S $(R=\pi_S(\mathfrak{A}(K))'')$. Since S and S' are standard,

$$(8.25)\qquad \|\varphi_S-\varphi_{S'}\|<2\,,$$

by Lemma 8.1.

Take an increasing sequence $\{K_i\}_{i\geqq 0}$ of Γ-invariant subspaces of K such that

(i) K_0 is the kernel of γ.

(ii) $\dim(K_i/K_0)$ is finite for all i.

(iii) $\bigcup_i K_i$ is dense in K.

Let γ_i, S_i, S_i' be the restrictions of γ, S, S' to K_i. Then we obtain a family of phase spaces (K_i,γ_i) with polarizations S_i and S_i'.

By construction, S_i and S_i' are standard because S and S' are standard and the GNS representation of φ_{S_i} can be identified with $(\pi_S|\mathfrak{A}(K_i),\Omega_S)$. Furthermore, using the identification $\pi_{S'}(\mathfrak{A}(K))''=\pi_S(\mathfrak{A}(K))''$ $(=R)$, we may set $R_i\equiv\pi_{S_i'}(\mathfrak{A}(K_i))''=\pi_{S_i'}(\mathfrak{A}(K_i))''\subset R$ and consider φ_{S_i} and $\varphi_{S_i'}$ as states of R_i. Then φ_{S_i} and $\varphi_{S_i'}$ are restrictions of φ_S and $\varphi_{S'}$ to R_i. Therefore

(8.26) $$\|\varphi_{S_i}-\varphi_{S_i'}\|\leqq\|\varphi_S-\varphi_{S'}\|\,.$$

Applying Proposition 7.10 and inequality (A.3) to the pair (S_i, S_i'). we have

(8.27) $$\|\varphi_{S_i}-\varphi_{S_i'}\| \geqq 2-2\det(\cosh H_i)^{-1/4}\det(\mathbb{1}+2^{-1}v_i(\mathbb{1}-\tanh H_i)v_i^*)^{-1/4}$$

where P_i, P_i', v_i, H_i are operators associated with $(\hat{S}_i, \hat{S}_i')$ by the construction of Section 4. (Note that they do not coincide with the restrictions of P, P', v, H to $\hat{K}_i$ in general.)

By (8.25), (8.26) and (8.27), it is enough to derive

(8.28) $$\overline{\lim_{i\to\infty}}\det(\cosh H_i)\det(\mathbb{1}+2^{-1}v_i(\mathbb{1}-\tanh H_i)v_i^*)=+\infty$$

from the negation of conditions in Proposition 6.6.

We first note that

(8.29) $$\cosh H_i\geqq\mathbb{1}$$

(8.30) $$v_i(\mathbb{1}-\tanh H_i)v_i^*\geqq(1-\tanh\|H_i\|)v_iv_i^*\geqq 0\,.$$

These imply that both determinants in (8.28) is bounded below by 1. It is therefore sufficient to prove the following alternatives:

(I) $\overline{\lim}\operatorname{tr}\sinh^2 H_i=\infty$, or

(II) $\overline{\lim}\operatorname{tr}\sinh^2 H_i=M<\infty$ and

$\overline{\lim}\operatorname{tr} v_iv_i^*=\infty$.

(Note that $\|H_i\|$ is bounded uniformly by $\sinh^{-1}(M^{1/2})$ if the first condition of (II) is satisfied and hence $1-\tanh\|H_i\|$ is bounded below by $1-(1+M)^{-1/2}M^{1/2}>0$.)

We now bound $\|p_i-p_i'\|_{\text{H.S.}}^2=\|q_i\|_{\text{H.S.}}^2+\|P_i-Q_i\|_{\text{H.S.}}^2$ by quantities appearing in (I) and (II). We have $\|q_i\|_{\text{H.S.}}=\|v_iQ_i\|_{\text{H.S.}}\leqq\|v_i\|_{\text{H.S.}}\|Q_i\|$ and $\|Q_i\|\leqq\|p_i'\|\leqq k\|p_i'\|_{S'}=k$. Proof of Lemma 4.2 (iii) shows $P_i(\mathbb{1}-Q_i)P_i=P_i\bar{Q}_iP_i$ and hence $\|P_i(1-Q_i)P_i\|_{\text{H.S.}}^2+\|\bar{P}_iQ_i\bar{P}_i\|_{\text{H.S.}}^2=\|(P_i-Q_i)^2\|_{\text{H.S.}}^2=\operatorname{tr}\sinh^4 H_i$. Further $P_iQ_i\bar{P}_i=u_i\sinh\alpha_i\cosh\alpha_i$ by Lemma 4.3 (ii) and $\bar{P}_iQ_iP_i=-(P_iQ_i\bar{P}_i)^*=-u_i^*\cosh\alpha_i\sinh\alpha_i$ by Lemma 4.2 (ii). Therefore $\|P_iQ_i\bar{P}_i\|_{\text{H.S.}}^2+\|\bar{P}_iQ_iP_i\|_{\text{H.S.}}^2=\operatorname{tr}\sinh^2 H_i\cosh^2 H_i$. Finally $\|\sinh H_i\|\leqq\|P-Q\|\leqq\|P\|+\|Q\|\leqq 1+k$. Therefore

(8.31) $$\|p_i-p_i'\|_{\text{H.S.}}^2\leqq k^2\operatorname{tr} v_iv_i^*+\operatorname{tr}\sinh^2 H_i(\mathbb{1}+2\sinh^2 H_i)$$

$$\leqq k^2 \operatorname{tr} v_i v_i^* + (1+2(1+k)^2) \operatorname{tr} \sinh^2 H_i .$$

Hence (8. 28) follows from

$$\overline{\lim}\, \|p_i - p_i'\|_{\text{H.S.}} = \infty . \tag{8.32}$$

Let E_i be the orthogonal projection on K_i, orthogonality being relative to $(\ ,\ )_S$. Since S_i and S_i' are restrictions of S and S' to K_i, the inner products $(\ ,\ )_{S_i}$ and $(\ ,\ )_{S_s'}$ are restrictions of $(\ ,\ )_S$ and $(\ ,\ )_{S'}$. In particular

$$(x, y)_{S_s'} = (x, T_i y)_{S_i}, \quad T_i = E_i T E_i . \tag{8.33}$$

Therefore

$$\lim \|\mathbf{1} - T_i\|_{\text{H.S.}} = \lim \|E_i(\mathbf{1} - T) E_i\|_{\text{H.S.}} = \|\mathbf{1} - T\|_{\text{H.S.}} . \tag{8.34}$$

We also have $S_i = E_i S E_i$ and hence

$$\lim S_i^{1/2} = (\lim S_i)^{1/2} = S^{1/2}. \tag{8.35}$$

Similarly $T_i S_i' = E_i T S' E_i$, $S' = \lim T^{-1} T_i S_i' = \lim S_i'$ and

$$\lim (S_i')^{1/2} = (S')^{1/2}. \tag{8.36}$$

Therefore

$$\|S^{1/2} - (S')^{1/2}\|_{\text{H.S.}} \leqq \underline{\lim} \|S_i^{1/2} - (S_i')^{1/2}\|_{\text{H.S.}} . \tag{8.37}$$

If condition (iv) of Proposition 6. 6 is violated, then either (a) $\|\mathbf{1} - T\|_{\text{H.S.}} = \infty$ or (b) $\|S^{1/2} - (S')^{1/2}\|_{\text{H.S.}} = \infty$. In the case (a), (8. 34) and (8. 17) imply $\sup \|p_i - p_i'\|_{\text{H.S.}} = \infty$. In the case (b), (8. 37) and (8. 16) imply the same. Therefore if conditions of Proposition 6. 6 are violated, then (8. 32) holds and π_S and $\pi_{S'}$ can not be quasi-equivalent.

Necessity Proof of Theorem. (See § 1 for the statement.) We assume that the reduction of the problem described in Section 2 and Section 3 has been made, namely $(\ ,\ )_S$ and $(\ ,\ )_{S'}$ induce the same Hilbert space topology on K and the restrictions of S and S' to the kernel of γ coincide. A slight modification of the proof of Lemma 6. 9 in [1] shows that there exists a direct sum decomposition of the phase space, $K = K_1 \oplus K_2$, such that

(i) K_1 is separable,

(ii) both S and S' split,

(iii) $S|_{K_2}=S'|_{K_2}$.

Thus the problem is furthermore reduced to the case in which K is separable. In that case, by Lemma 7.3, we can replace S and S' by standard polarizations. Now the necessity of conditions in Theorem is a direct consequence of Lemma 8.5 and Proposition 6.6. This completes the proof of Theorem.

Corollary 8.6. *Let (K, Γ, γ) be a phase space, B a Bogoliubov transformation in K, and S a polarization of γ. Then the Bogoliubov automorphism τ_B of $\mathfrak{A}(K)$ induces the automorphism of $R_S(K)$ if and only if*

(i) *B and B^{-1} are continuous with respect to the inner product $(,)_S$,*

(ii) *$B^*B-\mathbb{1}$ is in the H.S. class ($*$ and the H.S. class refer to $(,)_S$),*

(iii) *$BS^{1/2}B^{-1}-S^{1/2}$ is in the H.S. class.*

Proof. Set $(x, y)_{S'}=(B^{-1}x, B^{-1}y)_S$. Then S' is a polarization of γ and then B induces the automorphism of $R_S(K)$ if and only if π_S and $\pi_{S'}$ are quasi-equivalent. Now condition (i) of this Corollary is equivalent to condition (1) of Theorem. Conditions (ii) and (iii) of this Corollary are equivalent to condition (iv) in Proposition 6.6 and hence condition (2) of Theorem (under the condition (i)).

§ 9. Alternative Conditions

On the subspace $\ker\gamma$, $S=S'=1/2$ and hence we can not expect to find a condition on S and S' alone equivalent to conditions of Proposition 6.6. However, under the side condition that the restrictions of S and S' to $\ker\gamma$ satisfy the condition for $\overset{\text{H.S.}}{\sim}$, we can find a condition on S and S' equivalent to conditions of Proposition 6.6. This will be achieved in Corollary 9.2 below and clarify the relation of our Theorem and results obtained in [1].

As a preparation we derive a condition expressed in terms of the relation $\overset{\text{H.S.}}{\sim}$ between forms. In Proposition 6.6, this is done in the doubled space $\hat{K}$ (condition (iii)) but not in the original space. For this purpose, we recall a definition of the geometric mean discussed in [8]. The geometric mean of two positive hermitian forms $S(x, y)$ and $\bar{S}(x, y) = S(\Gamma y, \Gamma x)$ is denoted by $(S\bar{S})^{1/2}$ and is given by

$$(9.1) \qquad (S\bar{S})^{1/2}(x, y) = (x, \mathbf{S}^{1/2}(1-\mathbf{S})^{1/2}y)_S .$$

A similar relation holds for S'.

Let τ be a Hilbert space topology on K. For two bounded positive hermitian forms S_1 and S_2, we denote

$$S_1 \overset{\tau\,\mathrm{H.S.}}{\sim} S_2$$

if $S_i(x, y) = (x, \tilde{S}_i y)$ $(i=1, 2)$ for an inner product (,) giving rise to the topology τ and $\tilde{S}_1 - \tilde{S}_2$ is in the Hilbert-Schmidt class (relative to (,)).

Proposition 9.1. *Assume that* $(,)_S$ *and* $(,)_{S'}$ *induce the same topology* τ, *are non-degenerate and complete on* K. *Each of the following conditions is equivalent to conditions in Proposition* 6.6.

(i) $S \overset{\mathrm{H.S.}}{\sim} S'$ *and* $(S\bar{S})^{1/2} \overset{\mathrm{H.S.}}{\sim} (S'\bar{S}')^{1/2}$.

(ii) *For any fixed numbers* λ, $\mu > 0$,

$$\lambda(S+\bar{S}) + \mu(S\bar{S})^{1/2} \overset{\tau\,\mathrm{H.S.}}{\sim} \lambda(S'+\bar{S}') + \mu(S'\bar{S}')^{1/2} .$$

Proof. If $\mathbf{S}^{1/2} - (\mathbf{S}')^{1/2}$ is in the Hilbert-Schmidt class, then

$$(\mathbf{1}-\mathbf{S})^{1/2} - (\mathbf{1}-\mathbf{S}')^{1/2} = \Gamma(\mathbf{S}^{1/2} - (\mathbf{S}')^{1/2})\Gamma$$

is in the Hilbert-Schmidt class. Hence condition (iv) of Proposition 6.6 implies condition (i) above. Obviously (ii) follows from (i).

Let us assume (ii). Let

$$(9.2) \qquad (x, y)_{S'} = (x, Ty)_S .$$

Then (ii) is the same as

$$(9.3) \qquad \lambda + \mu\mathbf{S}^{1/2}(\mathbf{1}-\mathbf{S})^{1/2} \overset{\mathrm{H.S.}}{\sim} T(\lambda + \mu(\mathbf{S}')^{1/2}(\mathbf{1}-\mathbf{S}')^{1/2}).$$

Since $T(2\mathbf{S}'-\mathbf{1}) = 2\mathbf{S}-\mathbf{1}$, we obtain

$$(9.4) \qquad f(\mathbf{S}^{1/2}) \overset{\mathrm{H.S.}}{\sim} f((\mathbf{S}')^{1/2})$$

where the function f is defined for $0 \leqq t \leqq 1$ by

$$f(t) = (\lambda + \mu t (1-t^2)^{1/2})^{-1} (2t^2 - 1). \tag{9.5}$$

This function is monotone increasing and its inverse function f^{-1} is given for $-\lambda^{-1} \leqq x \leqq \lambda^{-1}$ by

$$2^{1/2} f^{-1}(x) = (1 + (4 + \mu^2 x^2)^{-1} x \{4\lambda + \mu g(x)^{1/2}\})^{1/2}, \tag{9.6}$$

$$g(x) = 4 + (\mu^2 - 4\lambda^2) x^2. \tag{9.7}$$

Since $g(x)$ is sandwiched by 4 and μ^2/λ^2, we have a convergent expansion

$$g(x)^{1/2} = k^{1/2} \sum c_n \{(g(x)/k) - 1\}^n \tag{9.8}$$

for $k \geqq \max(4, \mu^2/\lambda^2)$ where $(1+\alpha)^{1/2} = \sum c_n \alpha^n$ for $|\alpha| < 1$ and hence

$$\|g(A)^{1/2} - g(A')^{1/2}\|_{\text{H.S.}} \leqq \|g(A) - g(A')\|_{\text{H.S.}} k^{-1/2} \left(|c_1| + \|T\| \, \|T^{-1}\| \sum_{n=2}^{\infty} n |c_n| l^{n-1}\right) \tag{9.9}$$

where $l = 1 - k^{-1} \min(4, \mu^2/\lambda^2)$ (<1) and the sum converges. (The square root is relative to $(\ ,\ )_S$ for $g(A)$ and relative to $(\ ,\ )_{S'}$ for $g(A')$.)

From (9.6), we obtain

$$2^{1/2} f^{-1}(x) = (1 + \lambda x) F(x)^{1/2} G(x)^{-1/2}, \tag{9.10}$$

$$F(x) = 4\lambda(1 - \lambda x) + \mu(g(x)^{1/2} + \mu x), \tag{9.11}$$

$$G(x) = \lambda \mu^2 x^2 + \lambda g(x) + \mu(1 + \lambda^2 x^2) g(x)^{1/2}. \tag{9.12}$$

Since $g(x)^{1/2} + \mu x \geqq 0$ for $|x| \leqq \lambda^{-1}$, $F(x) \geqq a$ for some $a > 0$. We also have $G(x) \geqq \lambda g(x) \geqq \lambda \min(4, \lambda^{-2}\mu^2)$. If $\|A - A'\|_{\text{H.S.}} < \infty$, then $F(A) - F(A')$ and $G(A) - G(A')$ are in the Hilbert-Schmidt class by (9.9) and hence $F(A)^{1/2} - F(A')^{1/2}$ and $G(A)^{1/2} - G(A')^{1/2}$ are in the Hilbert-Schmidt class due to the same reason as (9.9). Therefore $f^{-1}(A) - f^{-1}(A')$ is in the Hilbert-Schmidt class for $A = f(S^{1/2})$ and $A' = f((S')^{1/2})$ due to (9.4). Namely we have

$$S^{1/2} \sim (S')^{1/2}. \tag{9.13}$$

Using this in (9.3), we obtain

$$(\lambda + \mu (S')^{1/2} (\mathbb{1} - S')^{1/2}) \overset{\text{H.S.}}{\sim} T(\lambda + \mu (S')^{1/2} (\mathbb{1} - S')^{1/2}).$$

Since $(\lambda+\mu(S')^{1/2}(\mathbf{1}-S')^{1/2})^{-1}$ is bounded (relative to $(\ ,\ )_{S'}$ by λ^{-1} and hence relative to $(\ ,\ )_S$), we obtain

$$\mathbf{1} \overset{\text{H.S.}}{\sim} T. \tag{9.14}$$

Therefore (ii) implies (9.13) and (9.14), which are condition (iv) of Proposition 6.6.

Corollary 9.2. *Under the same assumption as Proposition* 9.1, *the following two conditions together are equivalent to conditions of Proposition* 6.6.

(a) $(\ ,\ )_S|\ker\gamma \overset{\text{H.S.}}{\sim} (\ ,\ )_{S'}|\ker\gamma$.

(b) $\beta(S)^{-2}((2S-\mathbf{1})(2S'-\mathbf{1})^{-1}E')^{-}\beta(S')^2 \overset{\text{H.S.}}{\sim} E'$,

where $\mathbf{1}-E'$ *is* S'*-orthogonal projection on* $\ker\gamma$, $\beta(x)=x^{1/2}+(1-x)^{1/2}$ *and* $(\ )^{-}$ *denotes the closure of an operator.*

Proof. Setting $\lambda=1$ and $\mu=2$ in the condition (ii) of Proposition 9.1 and writing it in terms of operators relative to $(\ ,\ )_S$, we obtain

$$\mathbf{1}+2S^{1/2}(\mathbf{1}-S)^{1/2} \overset{\text{H.S.}}{\sim} T(\mathbf{1}+2(S')^{1/2}(\mathbf{1}-S')^{1/2}) \tag{9.15}$$

where $(x, y)_{S'}=(x, Ty)_S$.

Then (9.15) is equivalent to two relations obtained by multiplying (9.15) by $\mathbf{1}-E'$ and E' from the right. Since the range of $\mathbf{1}-E'$ is $\ker\gamma$, on which $S=S'=1/2$, the multiplication by $\mathbf{1}-E'$ yields the condition (a). Since $\mathbf{1}-E'$ is the eigenprojection of S' for the eigenvalue 1/2, it commutes with S'. Since

$$(x, TE'y)_S=(x, E'y)_{S'}=\gamma(x, (2S'-\mathbf{1})^{-1}E'y)$$
$$=(x, (2S-\mathbf{1})(2S'-\mathbf{1})^{-1}E'y)_S,$$

we have $TE'=((2S-\mathbf{1})(2S'-\mathbf{1})^{-1}E')^{-}$ where $(2S'-\mathbf{1})^{-1}E'$ is well-defined (in terms of spectral decomposition of S' relative to $(\ ,\ )_{S'}$) and y is in its domain. Since $\beta(x)^2=1+2x^{1/2}(1-x)^{1/2}$ is bounded with a bounded inverse for $0\leqq x\leqq 1$, the multiplication of (9.15) by E' yields an equation equivalent to (b).

Remark 9.3. We have

$$\begin{aligned}\beta(x)^{-2}(2x-1) &= (\mathrm{sign}(2x-1))\beta(x)^{-1}|x^{1/2}-(1-x)^{1/2}| \\ &= (\mathrm{sign}(2x-1))\{\beta(x)^2(x^{1/2}-(1-x)^{1/2})^{-2}\}^{-1/2} \\ &= (\mathrm{sign}(2x-1))\exp(-\mathrm{Tanh}^{-1}\{2x^{1/2}(1-x)^{1/2}\}).\end{aligned}$$

If we set $E'=1$ (the assumption of non-degenerate γ) and substitute the above relation in (b), then we obtain the criterion (2) in Lemma 6.5 of [1]. (Condition (3) in Theorem of [1] should be written in the same way as condition (2) in Lemma 6.5 there, namely $\sigma(S)$ and $\sigma(S')$ are missing by misprint.) By inverting the role of S and S', (b) can be replaced by the following condition which corresponds to (3) in Lemma 6.5 of [1].

(b)′ $\beta(\mathbb{S}')^{-2}((2\mathbb{S}'-\mathbb{1})(2\mathbb{S}-\mathbb{1})^{-1}E)^{-}\beta(\mathbb{S})^2 \overset{\text{H.S.}}{\sim} E$.

As already stated, $\mathbb{S}^{1/2} \overset{\text{H.S.}}{\sim} \mathbb{S}'^{1/2}$ does not imply $S \overset{\text{H.S.}}{\sim} S'$. The following example of S and S' shows that $S \overset{\text{H.S.}}{\sim} S'$ does not imply $\mathbb{S}^{1/2} \overset{\text{H.S.}}{\sim} \mathbb{S}'^{1/2}$.

Example 9.4. Let (K, Γ, γ) be a phase space such that there exists a Fock polarization S (Spec $\mathbb{S}=\{0, 1\}$). Let e be the eigenprojection of $\mathbb{S}$ corresponding to the eigenvalue 1. Take a positive H.S. class operator χ in eK which is not in the trace class. Set $T=\mathbb{1}+\chi+\bar{\chi}$ and $(x, y)_{S'}=(x, Ty)_S$. Then it can be easily checked that S' is a polarization and that $S \overset{\text{H.S.}}{\sim} S'$. According to the decomposition $K=eK \oplus \bar{e}K$, $\mathbb{S}$ and $\mathbb{S}'$ have the following matrix representations;

$$\mathbb{S}=\begin{bmatrix}\mathbb{1} & 0\\ 0 & 0\end{bmatrix}, \quad \mathbb{S}'=\begin{bmatrix}(\mathbb{1}+2^{-1}\chi)(\mathbb{1}+\chi)^{-1} & 0\\ 0 & 2^{-1}\bar{\chi}(\mathbb{1}+\bar{\chi})^{-1}\end{bmatrix}.$$

From this, we see that

$$\mathbb{S}^{1/2}-\mathbb{S}'^{1/2} \overset{\text{H.S.}}{\sim} \begin{bmatrix}0 & 0\\ 0 & 2^{-1/2}\bar{\chi}^{1/2}\end{bmatrix}$$

is not in the H.S. class.

Appendix. A Formula for $(\Omega_{\hat{S}}, Q(H)Q(v)\Omega_{\hat{S}})$

In this section we prove the following formula for finite rank operators H and v on $\hat{K}$ where the notation is as in Section 5. (The formula can be shown to hold for H and v in the Hilbert-Schmidt class as a straightforward extension by continuity from the present case.)

$$(\Omega_{\hat{S}}, Q(H)Q(v)\Omega_{\hat{S}}) = (\det\cosh H)^{-1/4}(\det(\mathbb{1}+c))^{-1/2}, \tag{A.1}$$

$$c = 4^{-1}v(\mathbb{1}-2(\tanh H)\hat{S})v^* . \tag{A.2}$$

Here the first determinant is obviously positive. The second determinant has to be positive due to Lemma 7.6, Corollary 7.9 and $(\xi, \eta) \geqq 0$ for all $\xi, \eta \in V_{\Omega_{\hat{S}}}$ (Theorem 4 (1) of [2]). Hence (A.1) implies

$$\begin{aligned}(\Omega_{\hat{S}}, Q(H)Q(v)\Omega_{\hat{S}}) &= (\det\cosh H)^{-1/4}(\det(\mathbb{1}+c)(\mathbb{1}+c^*))^{-1/4} \\ &\leqq (\det\cosh H)^{-1/4}(\det(\mathbb{1}+2^{-1}v(\mathbb{1}-\tanh H)v^*)^{-1/4}\end{aligned} \tag{A.3}$$

where we have used the inequality

$$\begin{aligned}\det(\mathbb{1}+c+c^*+cc^*) &= \exp \operatorname{tr}\log(\mathbb{1}+c+c^*+cc^*) \\ &\geqq \exp\operatorname{tr}\log(\mathbb{1}+c+c^*) = \det(\mathbb{1}+c+c^*)\end{aligned}$$

(because $\log x$ is operator monotone) and the anticommutativity of H and $\hat{\gamma}_{\hat{S}} = 2\hat{S}-\mathbb{1}$ (Lemma 4.4 (iii)) in the following form:

$$(\tanh H)\hat{S} + \hat{S}(\tanh H) = \tanh H . \tag{A.4}$$

The idea of proof is to decompose the phase space $E\hat{K}$ into a direct sum of 2-dimensional phase spaces. Then the expectation $(\Omega, Q(H)Q(v)\Omega)$ turns out to be the product of the expectation of 2-dimensional components. In this calculation, we use the central decomposition and first make computation with abelian part replaced by a number. We then calculate the expectation of the abelian part by Gaussian integral.

Let $\Lambda = i(p-\bar{p})$ $(= i\hat{\gamma}_{\hat{S}})$ be a partially isometric operator with range $E\hat{K}$ where $E = p+\bar{p}$ as in the main text. (Isometry relative to $(\ ,\ )_{\hat{S}}$.) It commutes with $\hat{\Gamma}$, satisfies $\hat{\gamma}(x, y) = -i(x, \Lambda y)_{\hat{S}}$ and anticommutes with H (Lemma 4.4 (iii)).

By assumption, H and v are of finite rank, both $H = H^*$ and v^*v ($*$ referring to $(\ ,\ )_{\hat{S}}$) annihilate $\ker\hat{\gamma}$, which is invariant under Λ and $\hat{\Gamma}$, $\Lambda^2 = -\mathbb{1}$ on $(\ker\hat{\gamma})^{\perp}$, $\hat{\Gamma}$ commutes with all others and $\hat{\Gamma}^2 = \mathbb{1}$. Hence there exists a finite dimensional subspace L of $\hat{K}$, invariant under

H, v^*v, Λ and $\hat{\Gamma}$, containing ranges of H and v^*v, and $\hat{S}$-orthogonal to $\ker \hat{\gamma}$. Let $\hat{K}=L\oplus L_0$ be an $\hat{S}$-orthogonal decomposition. It then follows that there exists a subspace L_+ of L with the following properties:

(i) $L=L_+\oplus\Lambda L_+$ ($\hat{S}$-orthogonal sum),

(ii) L_+ is $\hat{\Gamma}$-invariant,

(iii) H is reduced by the decomposition (i),

(iv) the restriction of H to L_+ is positive.

(L_+ is taken to be the spectral subspace of $H|L$ for positive eigenvalues plus a half of zero eigenspace which is mapped to the other half by Λ. Such a decomposition of zero eigenspace of H exists because it is $\hat{\Gamma}$- and Λ-invariant, $[\hat{\Gamma},\Lambda]=0$ and hence Λ defines a complex structure on its $\hat{\Gamma}$-real part due to $\Lambda^2=-\mathbb{1}$ on $(\ker\hat{\gamma})^{\perp}$.)

Take a $\hat{\Gamma}$-invariant spectral basis $\{b_j\}_{j\geqq 1}$ of H_+ (the restriction of H to L_+):

$$H_+b_j=\lambda_j b_j\,,\ \lambda_j\geqq 0\,,\ \hat{\Gamma}b_j=b_j\,,\quad \|b_j\|_{\hat{S}}=1 \tag{A. 5}$$

($j=1,2,3,\cdots$). Let L_j be the subspace spanned by $\{b_j,\Lambda b_j\}$. Then we have a finite direct sum decomposition of the phase space, $\hat{K}=\sum_{j\geqq 0}^{\oplus}L_j$.

Using this basis, we expand H and v. Then $Q(H)$ and $Q(v)$ can be written as

$$Q(H)Q(v)=\prod_{j\geqq 1}(e^{q(H_j)}e^{q(v_j)}) \tag{A. 6}$$

where

$$q(H_j)=\frac{1}{2}i\lambda_j(\pi_{\hat{S}}(b_j)\pi_{\hat{S}}(\Lambda b_j)+\pi_{\hat{S}}(\Lambda b_j)\pi_{\hat{S}}(b_j)) \tag{A. 7}$$

$$q(v_j)=i(\pi_{\hat{S}}(vb_j)\pi_{\hat{S}}(\Lambda b_j)-\pi_{\hat{S}}(v\Lambda b_j)\pi_{\hat{S}}(b_j)). \tag{A. 8}$$

Let $R=R_0\otimes(\bigotimes_{j\geqq 1}R_j)$ and $\Omega_{\hat{S}}=\Omega_0\otimes(\bigotimes_{j\geqq 1}\Omega_j)$ be the factorization of the von Neumann algebra $R=R_{\hat{S}}(\hat{K})$ and the cyclic vector $\Omega_{\hat{S}}$ with respect to the decomposition $\hat{K}=L_0\oplus\sum_{j\geqq 1}L_j$.

In (A. 6)-(A. 8), operators connected with L_0 are $\pi_{\hat{S}}(vb_j)$ and $\pi_{\hat{S}}(v\Lambda b_j)$ which are self-adjoint (due to $\hat{\Gamma}vb_j=v\hat{\Gamma}b_j=vb_j$, $\hat{\Gamma}v\Lambda b_j=v\Lambda\hat{\Gamma}b_j=v\Lambda b_j$) and affiliated with the commutative von Neumann subalgebra $R_{\hat{S}}(\ker\hat{\gamma})$ of R_0 because range $v\subset\ker\hat{\gamma}\subset L_0$. By joint spectral decomposition, we have

(A. 9) $$(\Omega_{\hat{S}}, Q(H)Q(v)\Omega_{\hat{S}}) = \int d\mu(z) \prod_{j\geqq 1} (\Omega_{\hat{S}}, e^{q(H_j)}e^{q_j}\Omega_{\hat{S}})$$

where

(A. 10) $$q_j = i(z_{j1}\pi_{\hat{S}}(\Lambda b_j) - z_{j2}\pi_{\hat{S}}(b_j)),$$

and $d\mu$ is a Gaussian measure (in general, allowing δ-functions which specify linear relations among components of z) with mean 0 and covariance given by

(A. 11) $$\int d\mu(z) z_{ja} z_{kb} = (\Omega_{\hat{S}}, \pi_{\hat{S}}(v\Lambda^{\delta(a)}b_j)\pi_{\hat{S}}(v\Lambda^{\delta(b)}b_k)\Omega_{\hat{S}}).$$

Here $\delta(a) = a-1$, for $a=1$, 2 and the same for $\delta(b)$.

Let $\alpha = \pi_{\hat{S}}(\Lambda b_j)$, $\beta = \pi_{\hat{S}}(b_j)$ and

(A. 12) $$f(\lambda) = (\Omega_{\hat{S}}, e^{i\lambda(\alpha\beta+\beta\alpha)}e^{i(z_1\alpha - z_2\beta)}\Omega_{\hat{S}})$$

for a fixed j. Then

(A. 13) $$f(\lambda_j/2) = (\Omega_{\hat{S}}, e^{q(H_j)}e^{q_j}\Omega_{\hat{S}})$$

if $z_a = z_{ja}$ $(a=1, 2)$.

The operators α and β are self-adjoint for which $\Omega_{\hat{S}}$ gives the quasi-free state with

(A. 14) $$[\alpha, \beta] = \hat{\gamma}_S(\Lambda b_j, b_j) = -i \quad (\hat{\Gamma} b_j = b_j),$$

(A. 15a) $$(\Omega_{\hat{S}}, \alpha^2\Omega_{\hat{S}}) = 2^{-1}\{(\hat{\Gamma}\Lambda b_j, \Lambda b_j)_{\hat{S}} + \hat{\gamma}(\hat{\Gamma}\Lambda b_j, \Lambda b_j)\} = 1/2,$$

(A. 15b) $$(\Omega_{\hat{S}}, \beta^2\Omega_{\hat{S}}) = 2^{-1}\{(\hat{\Gamma} b_j, b_j)_{\hat{S}} + \hat{\gamma}(\hat{\Gamma} b_j, b_j)\} = 1/2,$$

(A. 15c) $$(\Omega_{\hat{S}}, \alpha\beta\Omega_{\hat{S}}) = 2^{-1}\{(\hat{\Gamma}\Lambda b_j, b_j)_{\hat{S}} + \hat{\gamma}(\hat{\Gamma}\Lambda b_j, b_j)\} = -i/2,$$

where we have used $\hat{\Gamma} b_j = b_j$, $\hat{\Gamma}\Lambda b_j = \Lambda b_j$, $\hat{\gamma}(x, y) = -i(x, \Lambda y)_{\hat{S}}$, $(b_j, \Lambda b_j)_{\hat{S}} = 0$, $\|b_j\|_{\hat{S}} = \|\Lambda b_j\|_{\hat{S}} = 1$ and $\Lambda^2 = -\mathbb{1}$. From (A. 15), we obtain

(A. 16) $$(\alpha - i\beta)\Omega_{\hat{S}} = 0$$

by computing its norm. On the other hand, by (A. 14), we obtain

(A. 17a) $$\alpha e^{i(z_1\alpha - z_2\beta)} = e^{i(z_1\alpha - z_2\beta)}(\alpha - z_2)$$

(A·17b) $$\beta e^{i(z_1\alpha - z_2\beta)} = e^{i(z_1\alpha - z_2\beta)}(\beta - z_1)$$

(A. 18a) $$e^{i\lambda(\alpha\beta+\beta\alpha)}\alpha = e^{-2\lambda}\alpha e^{i\lambda(\alpha\beta+\beta\alpha)}$$

(A. 18b) $$e^{i\lambda(\alpha\beta+\beta\alpha)}\beta = e^{2\lambda}\beta e^{i\lambda(\alpha\beta+\beta\alpha)}$$

where (A. 17) and (A. 18) for small λ can be obtained through an algebraic computation by series expansion of matrix elements between (analytic) vectors in D (Proposition 3. 4) and by closure, and (A. 18) for large λ by repeated use of the same formula for small λ.

We now derive an equation for f. First we have

(A. 19) $$f'(\lambda)=i(\Omega_{\hat{S}},e^{i\lambda(\alpha\beta+\beta\alpha)}(\alpha\beta+\beta\alpha)e^{i(z_1\alpha-z_2\beta)}\Omega_{\hat{S}}).$$

We then write

(A. 20) $$\alpha\beta+\beta\alpha=(e^{2\lambda}\alpha+ie^{-2\lambda}\beta)A+B(\alpha-i\beta+z_2-iz_1)+C$$

where we may take

(A. 21) $$A=-i(\cosh 2\lambda)^{-1}\{\alpha+(e^{-4\lambda}+1)^{-1}(z_2-iz_1)\},$$

(A. 22) $$B=i(\cosh 2\lambda)^{-1}\{e^{2\lambda}\alpha-e^{-2\lambda}(e^{-4\lambda}+1)^{-1}(z_2-iz_1)\},$$

(A. 23) $$C=i\tanh 2\lambda+i2^{-1}(\cosh 2\lambda)^{-2}(z_2-iz_1)^2.$$

Due to (A. 16)-(A. 18),

(A. 24) $$(\alpha-i\beta+z_2-iz_1)e^{i(z_1\alpha-z_2\beta)}\Omega_{\hat{S}}=0,$$

(A. 25) $$(e^{2\lambda}\alpha+ie^{-2\lambda}\beta)^*e^{-i\lambda(\alpha\beta+\beta\alpha)}\Omega_{\hat{S}}=0.$$

Therefore

(A. 26) $$f'(\lambda)=-\{\tanh 2\lambda+2^{-1}(\cosh 2\lambda)^{-2}(z_2-iz_1)^2\}f(\lambda).$$

Therefore

(A. 27) $$f(\lambda)=(\cosh 2\lambda)^{-1/2}\exp\{-4^{-1}(\tanh 2\lambda)(z_2-iz_1)^2\}f(0).$$

By a similar (and simpler) calculation, we obtain

(A. 28) $$f(0)=(\Omega_{\hat{S}},e^{i(z_1\alpha-z_2\beta)}\Omega_{\hat{S}})=\exp-4^{-1}(z_1^2+z_2^2).$$

Therefore

(A. 29) $$f(\lambda)=(\cosh 2\lambda)^{-1/2}\exp-4^{-1}\{z_1^2+z_2^2+(z_2-iz_1)^2\tanh 2\lambda\}.$$

Substituting (A. 13) and (A. 29) in (A. 9), we obtain

(A. 30) $$(\Omega_{\hat{S}},Q(H)Q(v)\Omega_{\hat{S}})$$
$$=(\prod_j\cosh\lambda_j)^{-1/2}\int d\mu(z)\exp-4^{-1}\sum_j\{z_{j1}^2+z_{j2}^2$$
$$+(z_{j2}-iz_{j1})^2\tanh\lambda_j\}$$

$$= (\det \cosh H)^{-1/4} \int d\mu(z) \exp -4^{-1}(z, (1-\tanh H + i(\tanh H)\Lambda) z)$$

where $z=\sum z_{j1}b_j+\sum z_{j2}\Lambda b_j \in L$. (Note that eigenvalues of H are $\pm\lambda_j$, which accounts for the power $-1/4$ instead of $-1/2$.)

On the other hand, the covariance of the Gaussian measure μ is given by (A. 11), namely

$$\text{(A. 31)} \qquad \int d\mu(z) z_{ja} z_{kb} = \hat{S}(v\Lambda^{\delta(a)}b_j, v\Lambda^{\delta(b)}b_k)$$
$$= 2^{-1}(v\Lambda^{\delta(a)}b_j, v\Lambda^{\delta(b)}b_k)_{\hat{s}} \quad (\text{range } v \subset \ker \hat{\gamma})$$
$$= 2^{-1}(\Lambda^{\delta(a)}b_j, v^* v\Lambda^{\delta(b)}b_k)_{\hat{s}} .$$

Therefore, by a formula of (finite dimensional) Gaussian integral, (A. 30) becomes

$$\text{(A. 32)} \qquad (\Omega_{\hat{s}}, Q(H)Q(v)\Omega_{\hat{s}}) = (\det \cosh H)^{-1/4} \det(\mathbf{1}+c_1)^{-1/2},$$

$$\text{(A. 33)} \qquad c_1 = 4^{-1}v(\mathbf{1}-\tanh H(\mathbf{1}-i\Lambda))v^*$$
$$= 4^{-1}v(\mathbf{1}-2(\tanh H)\hat{\mathbf{S}})v^* .$$

(Due to $H\Lambda=-\Lambda H$, we may also write $(\mathbf{1}-\hat{\mathbf{S}})\tanh H$ instead of $(\tanh)\hat{\mathbf{S}}$.)

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