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Positive Cones and L_p-Spaces for von Neumann Algebras

By

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Abstract

The L_p-space $L_p(M, \eta)$ for a von Neumann algebra M with reference to its cyclic and separating vector η in the standard representation Hilbert space H of M is constructed either as a subset of H (for $2 \leqq p \leqq \infty$), or as the completion of H (for $1 \leqq p < 2$) with an explicitly defined L_p-norm. The Banach spaces $L_p(M, \eta)$ for different reference vector η (with the same p) are isomorphic.

Any L_p element has a polar decomposition where the positive part $L_p^+(M, \eta)$ is defined to be either the intersection with the positive cone $V_\eta^{1/(2p)}$ (for $2 \leqq p \leqq \infty$) or the completion of the positive cone $V_\eta^{1/(2p)}$ (for $1 \leqq p < 2$). Any positive element has an interpretation as the $(1/p)^{th}$ power $\omega^{1/p}$ of an $\omega \in M_*^+$ with its L_p-norm given by $\|\omega\|^{1/p}$.

Product of an L_p element and an L_q element is explicitly defined as an L_r element with $r^{-1} = p^{-1} + q^{-1}$ provided that $1 \leqq r$, and the Hölder inequality is proved.

The L_p-space constructed here is isomorphic to those defined by Haagerup, Hilsum, and Kosaki.

As a corollary, any normal state of M is shown to have one and only one vector representative in the positive cone V_η^α for each $\alpha \in [0, 1/4]$.

§ 1. Main Results

The L_p-space $L_p(M, \tau)$ of a semifinite von Neumann algebra M with respect to a normal trace τ is defined as the linear space of those closed operators which are affiliated with M and satisfy the condition $\|x\|_p = \tau(|x|^p)^{1/p} < \infty$. ([20]. Also see [18].) Extension to non semifinite cases have been worked out by Haagerup [11], Hilsum [12], and Kosaki [15], [16]. We shall present another version of such an extension with emphasis on defining them on the Hilbert space where M is acting rather than going over to the crossed product of M with the modular action.

We shall construct the L_p-space $L_p(M, \eta)$ with reference to a cyclic and separating vector η in the standard representation Hilbert space H of

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a general (σ-finite) von Neumann algebra M utilizing the relative modular operator $\Delta_{\phi,\eta}$ of a normal semifinite weight ϕ on M, which is defined as follows:

$$\Delta_{\phi,\eta}=S_{\phi,\eta}^{*}\overline{S_{\phi,\eta}}\,, \tag{1.1}$$

$$S_{\phi,\eta}x\eta=\xi_{\phi}(x^{*}), \quad x\in N_{\phi}^{*} \tag{1.2}$$

where $\overline{S_{\phi,\eta}}$ is the closure of $S_{\phi,\eta}$, N_{ϕ} is the set of $x\in M$ satisfying $\phi(x^{*}x)<\infty$ and $\xi_{\phi}(x)$ is the GNS vector representation of $x\in N_{\phi}$ in $H_{\phi}=\overline{N_{\phi}/\ker\phi}$ based on the weight ϕ. If ϕ is a vector state ω_{ξ} with $\xi\in H$, then $\xi_{\phi}(x)=x\xi$ and we denote $\Delta_{\phi,\eta}$ also as $\Delta_{\xi,\eta}$. The support of $\Delta_{\phi,\eta}$ is the support $s(\phi)$ of ϕ and $\Delta_{\phi,\eta}^{z}$ is defined as the sum of 0 on $(1-s(\phi))H$ and the usual power of positive selfadjoint operator $\Delta_{\phi,\eta}$ on $s(\phi)H$.

For $2\leqq p\leqq\infty$, we define the L_p-space as follows:

$$L_{p}(M,\eta)=\{\zeta\in\bigcap_{\xi\in H}D(\Delta_{\xi,\eta}^{(1/2)-(1/p)}):\|\zeta\|_{p}^{(\eta)}<\infty\} \tag{1.3}$$

$$\|\zeta\|_{p}^{(\eta)}=\sup_{\|\xi\|=1}\|\Delta_{\xi,\eta}^{(1/2)-(1/p)}\zeta\|. \tag{1.4}$$

For $1\leqq p<2$, we define the L_p-space $L_{p}(M,\eta)$ as the completion of H with the following L_p-norm:

$$\|\zeta\|_{p}^{(\eta)}=\inf\{\|\Delta_{\xi,\eta}^{(1/2)-(1/p)}\zeta\|:\|\xi\|=1,\, s^{M}(\xi)\geqq s^{M}(\zeta)\} \tag{1.5}$$

where s^{M} denotes the M-support of a vector (the smallest projection in M leaving the vector invariant), $\|\Delta_{\xi,\eta}^{(1/2)-(1/p)}\zeta\|$ is defined to be $+\infty$ if ζ is not in the domain of $\Delta_{\xi,\eta}^{(1/2)-(1/p)}$ and we prove in Lemma 7.1 (1) that any $\zeta\in H$ is in $D(\Delta_{\zeta,\eta}^{(1/2)-(1/p)})$ if $1\leqq p\leqq 2$.

For any $x\in M$ and $\zeta\in L_{p}(M,\eta)\cap H$, $x\zeta\in L_{p}(M,\eta)$ and $\|x\zeta\|_{p}^{(\eta)}\leqq\|x\|\,\|\zeta\|_{p}^{(\eta)}$. Therefore the multiplication of $x\in M$ can be defined for any $\zeta\in L_{p}(M,\eta)$ by continuous extension.

Theorem 1.

(1) *The formulae* (1.4) *and* (1.5) *define a norm for each* p $(1\leqq p\leqq\infty)$ *and* $L_{p}(M,\eta)$ *is a Banach M-module.*

(2) *Assume that* $p^{-1}+(p')^{-1}=1$, *then the sesquilinear form* (ζ,ζ') *for* $\zeta\in L_{p}(M,\eta)\cap H$, $\zeta'\in L_{p'}(M,\eta)\cap H$ *can be uniquely extended to a continuous sesquilinear form on* $L_{p}(M,\eta)\times L_{p'}(M,\eta)$ *(denoted by*

$\langle\,,\,\rangle_{(\eta)}$), *through which* $L_p(M,\eta)$ *is the dual of* $L_{p'}(M,\eta)$ *if* $1<p\leqq\infty$.

(3) *The norm satisfies*

$$\|\zeta\|_p^{(\eta)}=\sup\{|\langle\zeta,\zeta'\rangle_{(\eta)}|:\zeta'\in L_{p'}(M,\eta),\ \|\zeta'\|_{p'}^{(\eta)}\leqq 1\},\tag{1.6}$$

where $p^{-1}+(p')^{-1}=1$ *and* $1\leqq p\leqq\infty$.

(4) *For* $2\leqq p<\infty$ *and* $\zeta_1,\zeta_2\in L_p(M,\eta)$, *the following Clarkson's inequality holds:*

$$\begin{aligned}&(\|\zeta_1+\zeta_2\|_p^{(\eta)})^p+(\|\zeta_1-\zeta_2\|_p^{(\eta)})^p\\&\qquad\leqq 2^{p-1}\{(\|\zeta_1\|_p^{(\eta)})^p+(\|\zeta_2\|_p^{(\eta)})^p\}.\end{aligned}\tag{1.7}$$

The L_p-spaces for different reference vectors η are related as follows.

Theorem 2. *There exists a family of conjugate linear isometry* $J_p(\eta_2,\eta_1)$ *and linear isometry* $\tau_p(\eta_2,\eta_1)$ *from* $L_p(M,\eta_1)$ *onto* $L_p(M,\eta_2)$ *satisfying the following relations:*

(1) *For* $2\leqq p\leqq\infty$, *and* $\zeta\in L_p(M,\eta_1)$,

$$J_p(\eta_2,\eta_1)\zeta=J_{\eta_2,\eta_1}\Delta_{\eta_2,\eta_1}^{(1/2)-(1/p)}\zeta\quad(\in L_p(M,\eta_2))\tag{1.8}$$

$$\tau_p(\eta_2,\eta_1)=J_p(\eta_2,\eta)J_p(\eta,\eta_1)\tag{1.9}$$

where (1.9) *is independent of a cyclic and separating vector* η *and* J_{η_2,η_1} *is obtained by the polar decomposition* $\bar{S}_{\phi,\eta}=J_{\phi,\eta}\Delta_{\phi,\eta}^{1/2}$ (*see* (1.2)).

(2) *For* $p^{-1}+(p')^{-1}=1$, $\zeta\in L_p(M,\eta_1)$ *and* $\zeta'\in L_{p'}(M,\eta_2)$,

$$\langle J_p(\eta_2,\eta_1)\zeta,\zeta'\rangle_{(\eta_2)}=\langle J_{p'}(\eta_1,\eta_2)\zeta',\zeta\rangle_{(\eta_1)},\tag{1.10}$$

$$\langle\tau_p(\eta_2,\eta_1)\zeta,\zeta'\rangle_{(\eta_2)}=\langle\zeta,\tau_{p'}(\eta_1,\eta_2)\zeta'\rangle_{(\eta_1)},\tag{1.11}$$

and

$$\tau_p(\eta_3,\eta_2)\tau_p(\eta_2,\eta_1)=\tau_p(\eta_3,\eta_1).\tag{1.12}$$

The cones

$$V_\eta^\alpha=\text{the closure of }\Delta_\eta^\alpha M_+\eta\quad(0\leqq\alpha\leqq 1/2)\tag{1.13}$$

defined in [2] can be used to define the positive part $L_p^+(M,\eta)$ as follows:

$$L_p^+(M,\eta)=L_p(M,\eta)\cap V_\eta^{1/(2p)}\quad\text{for}\quad 2\leqq p\leqq\infty\tag{1.14}$$

$$L_p^+(M,\eta)=L_p\text{-closure of }V_\eta^{1/(2p)}\quad\text{for}\quad 1\leqq p<2.\tag{1.15}$$

Then we have the following polar decomposition theorem.

Theorem 3.

(1) *Any* $\zeta \in L_p(M, \eta)$ *has the unique polar decomposition* $\zeta = u|\zeta|_p$, *where* u *is a partial isometry in* M *satisfying* $uu^* = s^M(\zeta)$ *(equivalently,* $u^*u = s^M(|\zeta|_p)$*) and*

$$|\zeta|_p = u^*\zeta \in L_p^+(M, \eta). \tag{1.16}$$

Here, $s^M(\zeta)$ *is the* M*-support of* ζ, *namely the smallest projection* $P \in M$ *such that* $P\zeta = \zeta$.

(2) *Under the identification of* $L_p(M, \eta_1)$ *and* $L_p(M, \eta_2)$ *by* $\tau_p(\eta_2, \eta_1)$, *the above polar decomposition is independent of* η.

(3) $\|\zeta\|_p^{(\eta)} = \||\zeta|_p\|_p^{(\eta)}$

(4) *If* $\zeta \in L_p^+(M, \eta)$, *there exists a unique* $\phi \in M_*^+$ *such that*

$$\zeta = \Delta_{\phi,\eta}^{1/p}\eta \tag{1.17}$$

if $2 \leqq p < \infty$, *and*

$$\langle \zeta, \zeta' \rangle_{(\eta)} = (\Delta_{\phi,\eta}^{1/2}\eta, \Delta_{\phi,\eta}^{(1/p)-(1/2)}\zeta') \tag{1.18}$$

for all $\zeta' \in L_{p'}(M, \eta)$, $p^{-1} + (p')^{-1} = 1$ *if* $1 \leqq p \leqq 2$. *For such unique* ϕ, $\|\zeta\|_p^{(\eta)} = \phi(1)^{1/p}$. *If* $\zeta \in L_\infty^+(M, \eta)$, *there exists a unique* $x \in M^+$ *such that* $\zeta = x\eta$. *For such* x, $\|\zeta\|_\infty^{(\eta)} = \|x\|$.

We may symbolically write

$$\zeta = u\phi^{1/p} \tag{1.19}$$

if $|\zeta|_p$ is given either by (1.17) or (1.18).

Special cases $p = \infty$ and $p = 1$ reduce to well-known objects.

Theorem 4.

(1) *The map* $x \in M \mapsto x\eta \in H$ *is an isometric isomorphism from* M *onto* $L_\infty(M, \eta)$.

(2) *The map from* $\zeta \in L_1(M, \eta)$ *to*

$$\phi(x) = \langle \zeta, x^*\eta \rangle_{(\eta)} \quad (x \in M) \tag{1.20}$$

is an isometric isomorphism from $L_1(M, \eta)$ *onto* M_*, *where the inner product in* (1.20) *is the one given by Theorem* 1 (2) *for* $p = 1$.

From definition, $L_2(M, \eta)$ is H, independent of η.

For the definition of the product, we use the following Lemma.

Lemma A. *For $x_0, \cdots, x_n \in M$, $\phi_1, \cdots, \phi_n \in M_*^+$ and complex numbers $z=(z_1, \cdots, z_n)$ in the tube domain*

$$(1.21) \qquad I_1^{(n)} = \{z \in \mathbf{C}^n : \mathrm{Re}\, z_j \geqq 0 \ j=1, \cdots, n, \ \sum_{j=1}^{n} \mathrm{Re}\, z_j \leqq 1\},$$

the expression

$$(1.22) \qquad F(z) = (\Delta_{\phi_j,\eta}^{z'_j} x_j \Delta_{\phi_{j+1},\eta}^{z_{j+1}} x_{j+1} \cdots x_n \eta, \ \Delta_{\phi_j,\eta}^{\bar{z}''_j} x_{j-1}^* \Delta_{\phi_{j-1},\eta}^{\bar{z}_{j-1}} x_{j-2}^* \cdots x_0^* \eta)$$

is well-defined and independent of the division $z_j = z'_j + z''_j$ if

$$(1.23) \qquad \mathrm{Re}\, z_1 + \cdots + \mathrm{Re}\, z_{j-1} + \mathrm{Re}\, z''_j \leqq 1/2, \ \mathrm{Re}\, z''_j \geqq 0\,,$$

$$(1.24) \qquad \mathrm{Re}\, z_n + \cdots + \mathrm{Re}\, z_{j+1} + \mathrm{Re}\, z'_j \leqq 1/2, \ \mathrm{Re}\, z'_j \geqq 0\,.$$

It defines a function of $z=(z_1, \cdots, z_n)$ which is

(i) *holomorphic in the interior of $I_1^{(n)}$,*

(ii) *continuous on $I_1^{(n)}$, and*

(iii) *bounded on $I_1^{(n)}$ by*

$$(1.25) \qquad |F(z)| \leqq \left(\prod_{j=0}^{n} \|x_j\|\right) \omega_\eta(1)^{z_0} \left(\prod_{j=1}^{n} \phi_j(1)^{\mathrm{Re}\, z_j}\right)$$

where $z_0 = 1 - \sum_{j=1}^{n} \mathrm{Re}\, z_j$.

(iv) *Denote*

$$(1.26) \qquad F(z) = \omega_\eta(x_0 \Delta_{\phi_1,\eta}^{z_1} x_1 \cdots \Delta_{\phi_n,\eta}^{z_n} x_n)\,.$$

If $x_j = x'_j x''_j$ with $x'_j, x''_j \in M$ and $z_0 = 1 - \sum_{l=1}^{n} z_l$, then

$$(1.27) \qquad \omega_\eta(x_0 \Delta_{\phi_1,\eta}^{z_1} x_1 \cdots \Delta_{\phi_n,\eta}^{z_n} x_n) = \omega_\eta(x''_j \Delta_{\phi_{j+1},\eta}^{z_{j+1}} x_{j+1} \cdots \Delta_{\phi_n}^{z_n} x_n \Delta_\eta^{z_0} x_0 \Delta_{\phi_1,\eta}^{z_1} x_1 \cdots \Delta_{\phi_j,\eta}^{z_j} x'_j)\,.$$

(Multiple-time KMS condition.)

(v) *$F(z)$ is multilinear in $x_0, \cdots, x_n$.*

(vi) *$F(z)$ is continuous in $(x_0, \cdots, x_n, \phi_1, \cdots, \phi_n, \eta)$ relative to *-strong topology of x's and norm topology of ϕ's and η, provided that*

x's are restricted to a bounded set. The continuity is uniform in z's provided that z's are restricted to a compact set.

(vii) *If $\sum_{j=1}^{n} z_j=1$, then $F(z)$ is independent of η.*

If $r^{-1}=\sum_{j=1}^{n}(p_j)^{-1}$, $r^{-1}+(r')^{-1}=1$, $\zeta_j\in L_{p_j}(M,\eta)$, $x_j\in M$ $(j=0,\cdots,n)$, and $\zeta_j=u_j\phi_j^{1/p_j}$ $(j=1,\cdots,n)$ is the polar decomposition, then the product

$$\zeta=x_0\zeta_1x_1\zeta_2\cdots\zeta_nx_n\in L_r(M,\eta)\quad(=L_{r'}(M,\eta)^*) \tag{1.28}$$

is defined by

$$\langle\zeta,\zeta'\rangle_{(\eta)}=\omega_\eta(\Delta_{\phi',\eta}^{1/r'}u'^*x_0u_1\Delta_{\phi_1,\eta}^{1/p_1}x_1\cdots u_n\Delta_{\phi_n,\eta}^{1/p_n}x_n) \tag{1.29}$$

where $\zeta'\in L_{r'}(M,\eta)$ and $\zeta'=u'\phi'^{1/r'}$ is its polar decomposition.

Theorem 5. *The product* (1.28) *is multilinear and satisfies*

$$\|\zeta\|_r^{(\eta)}\leqq(\prod_{j=0}^{n}\|x_j\|)\prod_{j=1}^{n}\|\zeta_j\|_{p_j}^{(\eta)}. \tag{1.30}$$

A polar decomposition different from Theorem 3 is given by the following.

Theorem 6. *Any $\zeta\in L_p(M,\eta)$ has the unique polar decomposition*

$$\zeta=\zeta_1-\zeta_2+i(\zeta_3-\zeta_4) \tag{1.31}$$

where $\zeta_j\in L_p^+(M,\eta)$, $s^M(\zeta_1)\perp s^M(\zeta_2)$ and $s^M(\zeta_3)\perp s^M(\zeta_4)$. Here $s^M(\zeta)$ is the M-support of ζ, i.e. the smallest projection $P\in M$ such that $P\zeta=\zeta$.

The polar decompositions have versions appropriate for the positive cone V_η^α itself.

Theorem 7.

(1) *Any ζ in the domain of $\Delta_\eta^{(1/2)-2\alpha}$ $(0\leqq\alpha\leqq1/2)$ has the polar decomposition $\zeta=u|\zeta|_\alpha$ where u is a partial isometry in M satisfying $uu^*=s^M(\zeta)$ (or equivalently $u^*u=s^M(|\zeta|_\alpha)$), $|\zeta|_\alpha\in V_\eta^\alpha$ and $|\zeta|_\alpha=\Delta_{\phi,\eta}^{2\alpha}\eta$ for*

some $\phi \in M_*$ *if* $0<\alpha \leqq 1/2$ *and* $|\zeta|_0 = T\eta$ *for a positive selfadjoint operator* T *affiliated with* M. *Such* ζ *can be written also as* $\zeta = u'|\zeta|'_\alpha$ *where* u' *is a partial isometry in* M', $u'u'^* = s^{M'}(\zeta)$ *(or equivalently* $u'^*u' = s^{M'}(|\zeta|'_\alpha)$*) and* $|\zeta|'_\alpha \in V_\eta^\alpha$.

(2) *For* $1/4 \leqq \alpha \leqq 1/2$, *any* ζ *in* H *has the unique polar decomposition* $\zeta = u|\zeta|_\alpha$ *where* u *is a partial isometry in* M *satisfying* $uu^* = s^M(\zeta)$ *(or equivalently* $u^*u = s^M(|\zeta|_\alpha)$*) and* $|\zeta|_\alpha \in V_\eta^\alpha$. *For* $1/4 \geqq \alpha' \geqq 0$, *any* ζ *in* H *has the unique decomposition* $\zeta = u'|\zeta|'_{\alpha'}$ *where* u' *is a partial isometry in* M' *satisfying* $u'u'^* = s^{M'}(\zeta)$ *(or equivalently* $u'^*u' = s^{M'}(|\zeta|'_{\alpha'})$*) and* $|\zeta|'_{\alpha'} \in V_\eta^{\alpha'}$.

(3) *Any* $\zeta \in V_\eta^\alpha$ *for* $1/4 \leqq \alpha \leqq 1/2$ *has the form* $\zeta = \Delta_{\phi,\eta}^{2\alpha}\eta$ *where* $\phi \in M_*^+$, $\eta \in D(\Delta_{\phi,\eta}^{2\alpha})$, *and* ϕ *is uniquely determined by* ζ.

(4) *Any* ζ *in the domain of* $\Delta_\eta^{(1/2)-2\alpha}$ $(0 \leqq \alpha \leqq 1/2)$ *has the unique decomposition,*

$$\zeta = \zeta_1 - \zeta_2 + i(\zeta_3 - \zeta_4) \tag{1.32}$$

with $\zeta_1, \cdots, \zeta_4 \in V_\eta^\alpha$ *and*

$$s^M(\zeta_1) \perp s^M(\zeta_2),\ s^M(\zeta_3) \perp s^M(\zeta_4) \quad \text{for} \quad 1/4 \leqq \alpha \leqq 1/2, \tag{1.33a}$$

$$s^{M'}(\zeta_1) \perp s^{M'}(\zeta_2),\ s^{M'}(\zeta_3) \perp s^{M'}(\zeta_4) \quad \text{for} \quad 0 \leqq \alpha \leqq 1/4. \tag{1.33b}$$

If $\alpha = 1/4$, *the two decompositions coincide.*

Corollary. *Any* $\phi \in M_*^+$ *has a unique vector representative* $\xi_\eta^\alpha(\phi) \in V_\eta^\alpha$ *for each* $\alpha \in [0, 1/4]$, *i.e.*

$$(x\xi_\eta^\alpha(\phi), \xi_\eta^\alpha(\phi)) = \phi(x) \quad (x \in M). \tag{1.34}$$

Our strategy for proof of the above main results is first to show that $L_p(M,\eta)$, $2 \leqq p < \infty$, which is a subset of H in our approach, is a uniformly convex (hence reflexive) M-module and $\zeta \in L_p(M,\eta)$ has a unique polar decomposition $\zeta = u|\zeta|_p^{(\eta)}$ with $|\zeta|_p^{(\eta)} = \Delta_{\phi,\eta}^{1/p}\eta \in V_\eta^{1/(2p)}$, $\phi \in M_*^+$ and $\|\zeta\|_p^{(\eta)} = \phi(1)^{1/p}$. Then $L_p(M,\eta)$ for $2 \geqq p > 1$ can easily be identified with the dual $L_{p'}(M,\eta)^*$ where $(p')^{-1} + p^{-1} = 1$, $L_p^+(M,\eta)$ being exactly the polar of $L_{p'}^+(M,\eta)$ and the polar decomposition $\zeta = u|\zeta|_p^{(\eta)}$ of $\zeta \in L_p(M,\eta)$ being derived from that of $\zeta' \in L_{p'}(M,\eta)$ achieving "maximum" inner product with ζ.

Our main tool is the relative modular operator (defined by (1.1) and (1.2)) which has been used previously in [6], [7]. (Also see [5], [10].) In Appendix C, we collect its properties relevant to our application and provide a brief outline of their proof.

Main lemma providing a control over the unbounded relative modular operator is its domain properties and Hölder type inequality given by Lemma A (stated in Section 1 and proved in Appendix A). This lemma originates in the multiple time KMS condition first found in [1], where it is formulated in terms of boundary values of time correlation functions (rather than modular operators). The present form is a straightforward generalization of Theorem 3.1 and Theorem 3.2 in [3]. (Also see [13].)

The set $\mathcal{L}_p^*(M, \eta)$ of certain formal monomials of elements of M and complex powers (with positive real parts) of relative modular operators (p specifying the sum of real parts of powers not to exceed $1-p^{-1} = (p')^{-1}$) and its subset $\mathcal{L}_{p'}(M, \eta)$ are introduced in Section 2. In fact the set $\mathcal{L}_p(M, \eta)$ consists of $A = u\Delta_{\phi,\eta}^{1/p}$, $\phi \in M_*^+$, which will be identified with $A\eta \in L_p(M, \eta)$ for $2 \leqq p < \infty$ and with an element of $L_p(M, \eta)$ with L_p norm $\phi(1)^{1/p}$ and having the "maximal inner product" with $u\Delta_{\phi,\eta}^{1/p'}\eta$ in $L_{p'}(M, \eta)$ for $2 \geqq p > 1$. $\mathcal{L}_p^*(M, \eta)$ is introduced here for the purpose of defining products of elements of $L_p(M, \eta)$ and $L_q(M, \eta)$, which is technically used in the proof of uniform strong differentiability in Section 9 and is fully treated in Section 12. Lemma A enables us to define an "inner product" $\langle A, B \rangle_{(\eta)}$ between $A \in \mathcal{L}_p(M, \eta)$ and $B \in \mathcal{L}_p^*(M, \eta)$, which coincides with $(A\eta, B\eta)$ in H whenever η is in domains of A and B. This leads to an identification of $\mathcal{L}_p^*(M, \eta)$ (modulo an equivalence) with $L_p(M, \eta)^*$ (after $\mathcal{L}_p(M, \eta) = L_p(M, \eta)$ is shown by polar decomposition) (in Section 7) and also to the Hölder inequality for the above mentioned product (in Section 12).

In Section 3, the polar decomposition of a vector ζ in the domain of $\Delta_\eta^{(1/2)-2\alpha}$, in the form $\zeta = u|\zeta|_p^{(\eta)}$, $|\zeta|_p^{(\eta)} = \Delta_{\phi,\eta}^{1/p}\eta$ with a partial isometry u in M and a normal semifinite weight ϕ on M, is derived by an application of Carlson's theorem, a technique used in [4]. Here the Connes characterization of unitary Randon-Nikodym cocycle is used in the form discussed in Appendix B, where we allow non-faithful normal semifinite weights.

As an immediate application of results in Section 3, we obtain existence and uniqueness of polar decomposition of $\zeta \in L_p(M, \eta)$ $(2 \leqq p < \infty)$ as above with $\phi \in M_*^+$ and the formula $\|\zeta\|_p^{(\eta)} = \phi(1)^{1/p}$ in Section 4. At the same time, the set of $\Delta_{\phi,\eta}^{1/p}\eta$ with $\phi \in M_*^+$ is identified with $L_p^+(M, \eta)$ defined by (1.14).

In Section 5, we show that $L_p(M, \eta)$ for $p=1$ and ∞ are canonically identified with M_* and M.

In Section 6, we prove the completeness of $L_p(M, \eta)$ for $2 \leqq p < \infty$ by using an easily provable inequality between $\|\zeta\|_p^{(\eta)}$ and the norm $\|\zeta\|$ in H.

In Section 7, we derive a few technical lemmas related to $\mathcal{L}_p(M, \eta)$ and $\mathcal{L}_p^*(M, \eta)$ introduced in Section 2. They provide useful tools in subsequent two sections, where Clarkson's inequality (and hence the uniform convexity) and uniform strong differentiability of the norm (and hence the uniform convexity of the dual space) are proved for $L_p(M, \eta)$, $2 \leqq p < \infty$.

Once the properties of $L_p(M, \eta)$, $2 \leqq p < \infty$ are established, properties of $L_p(M, \eta)$ for $1 < p < 2$ are easily derived in Section 10.

The isomorphism of $L_p(M, \eta)$ for different reference vectors η are established in Section 11. As mentioned earlier, product is treated in Section 12. Linear polar decomposition theorems for L_p-spaces as well as for $D(\Delta_\eta^\alpha)$ $(|\alpha| \leqq 1/2)$ are then proved in Section 13.

Section 14 provides a summary of proof of Theorems of Section 1 in terms of Lemmas proved in preceding sections.

A brief discussion of the connection with other works is in Section 15.

In Appendix D, operator monotone function is shown to be applicable also for semibounded operators (or positive forms). This result (in a special case of the function x^ν, $0 \leqq \nu \leqq 1$) is used in Appendix C to derive an inequality for powers of relative modular operators.

§ 2. Immediate Consequences of the Multiple-Time KMS Condition

The multiple-time KMS condition has been found to hold for any KMS state in [1], where it is formulated in terms of boundary values (hence in terms of time translation automorphisms) rather than the modular operators: If all ϕ_j coincide with ω_η, then for real $t=(t_1, \cdots, t_n)$,

$$F(it)=\omega_\eta(x_0\sigma^\eta_{s_1}(x_1)\cdots\sigma^\eta_{s_n}(x_n)) \tag{2.1}$$

$$\begin{aligned} F(it_1, \cdots, it_{j-1}, it_j+1, it_{j+1}+1, \cdots, it_n+1) \\ =\omega_\eta(\sigma^\eta_{s_j}(x_j)\cdots\sigma^\eta_{s_n}(x_n)x_0\sigma^\eta_{s_1}(x_1)\cdots\sigma^\eta_{s_{j-1}}(x_{j-1})), \end{aligned} \tag{2.2}$$

where $s_k=t_1+\cdots+t_k$. The proof of Lemma A which is an adaptation of the proof in [3] will be presented in Appendix A for the sake of completeness. In this section, we discuss immediate consequences of Lemma A, which will be used in subsequent proofs of main results. We use the notation of Lemma A.

Corollary 2.1. *If η is in the domains of the two operators.*

$$A=\Delta^{z_j'}_{\phi_j,\eta}x_j\Delta^{z_{j+1}}_{\phi_{j+1},\eta}x_{j+1}\cdots\Delta^{z_n}_{\phi_n,\eta}x_n \tag{2.3}$$

$$B=\Delta^{\overline{z_j''}}_{\phi_j,\eta}x^*_{j-1}\Delta^{\bar{z}_{j-1}}_{\phi_{j-1},\eta}\cdots x^*_1\Delta^{\bar{z}_1}_{\phi_1,\eta}x^*_0 \tag{2.4}$$

where $z\in I_1^{(n)}$ with $z_j=z_j'+z_j''$ ((1.23) and (1.24) of Lemma A are not assumed), then

$$(A\eta, B\eta)=\omega_\eta(x_0\Delta^{z_1}_{\phi_1,\eta}x_1\cdots\Delta^{z_n}_{\phi_n,\eta}x_n). \tag{2.5}$$

Proof. Due to $z\in I_1^{(n)}$, either (1.23) or (1.24) holds. Suppose (1.23) holds. (The case (1.24) is similar.) Then there exists $k>j$, $\mathrm{Re}\, z_k'\geqq 0$, $\mathrm{Re}\, z_k''\geqq 0$, $z_k'+z_k''=z_k$ (or $\mathrm{Re}\, w_j'\geqq 0$, $\mathrm{Re}\, w_j''\geqq 0$, $w_j'+w_j''=z_j$) such that both (1.23) and (1.24) hold if j is replaced by k (or if z_j' and z_j'' are replaced by w_j' and w_j'') and hence $F(z)$ is given by the inner product (1.22) where the same replacement is to be made. The equation (2.5) is then obtained by transposing x_l ($j\leqq l<k$) and appropriate powers of $\Delta_{\phi_l,\eta}$ ($j\leqq l\leqq k$) from one member of the inner product to another.

Lemma 2.2. *If η is in the domains of the operators,*

$$A_1=\Delta^{z_j''}_{\phi_j,\eta}x_j\Delta^{z_{j+1}}_{\phi_{j+1},\eta}x_{j+1}\cdots\Delta^{z_n}_{\phi_n,\eta}x_n \tag{2.6}$$

$$A_2=\Delta^{w''_j}_{\psi_j,\eta}y_j\Delta^{w_{j+1}}_{\psi_{j+1},\eta}y_{j+1}\cdots\Delta^{w_m}_{\psi_m,\eta}y_m \tag{2.7}$$

with Re $z_l\geqq 0$, Re $w_l\geqq 0$ $(l\geqq j+1)$, Re $z''_j\geqq 0$, Re $w''_j\geqq 0$, *and* $\phi_l\in M^+_*$, $x_l\in M$, $\psi_l\in M^+_*$, $y_l\in M$ $(l\geqq j)$, *and if* $A_1\eta=A_2\eta$, *then*

$$\begin{aligned}&\omega_\eta(x_0\Delta^{z_1}_{\phi_1,\eta}x_1\cdots\Delta^{z_n}_{\phi_n,\eta}x_n)\\ &=\omega_\eta(x_0\Delta^{z_1}_{\phi_1,\eta}x_1\cdots x_{j-1}\Delta^{w_j}_{\psi_j,\eta}y_j\cdots\Delta^{w_m}_{\psi_m,\eta}y_m)\end{aligned} \tag{2.8}$$

for all $x_0,\cdots,x_{j-1}\in M$, $z_j=z'_j+z''_j$, $w_j=z'_j+w''_j$, $\phi_l\in M^+_*$, Re $z_l\geqq 0$ $(l=1,\cdots,j)$, Re $w_j\geqq 0$ *and*

$$\sum_{l=1}^{n}\mathrm{Re}\, z_l\leqq 1,\quad \sum_{l=1}^{j-1}\mathrm{Re}\, z_l+\sum_{l=j}^{m}\mathrm{Re}\, w_l\leqq 1\,.$$

Proof. If Re $z'_j\geqq 0$ and $\sum_{l=1}^{j}\mathrm{Re}\, z_l\leqq 1/2$, then η is also in the domain of B given by (2.4) due to Lemma A. Hence (2.5) and the assumption $A_1\eta=A_2\eta$ imply (2.8). General case follows from this case by analytic continuation.

Notation 2.3.

(1) The set of all formal expressions $A=u\Delta^{1/p}_{\phi,\eta}$ with $\phi\in M^+_*$ and a partial isometry u satisfying $u^*u=s(\phi)$ (the support projection of ϕ) will be denoted by $\mathcal{L}_p(M,\eta)$.

(2) The set of all formal expressions,

$$B=x_0\Delta^{z_1}_{\phi_1,\eta}x_1\cdots\Delta^{z_n}_{\phi_n,\eta}x_n \tag{2.9}$$

with $x_j\in M$ $(j=0,\cdots,n)$, $\phi_j\in M^+_*$ $(j=1,\cdots,n)$, $z=(z_1,\cdots,z_n)\in I^{(n)}_{1-(1/p)}$ will be denoted by $\mathcal{L}^*_p(M,\eta)$ where

$$I^{(n)}_a=\{z\in \boldsymbol{C}^n: \mathrm{Re}\, z_j\geqq 0\ \ (j=1,\cdots,n),\ \ \sum_{j=1}^{n}\mathrm{Re}\, z_j\leqq a\}. \tag{2.10}$$

(3) For $A\in\mathcal{L}_p(M,\eta)$ and $B\in\mathcal{L}^*_p(M,\eta)$,

$$\langle B,A\rangle_{(\eta)}=\omega_\eta(1\Delta^{1/p}_{\phi,\eta}(u^*x_0)\Delta^{z_1}_{\phi_1,\eta}x_1\cdots\Delta^{z_n}_{\phi_n,\eta}x_n), \tag{2.11a}$$

$$\langle A,B\rangle_{(\eta)}=\omega_\eta(x^*_n\Delta^{z_n}_{\phi_n,\eta}\cdots x^*_1\Delta^{\bar z_1}_{\phi_1,\eta}(x^*_0u)\Delta^{1/p_1}_{\phi,\eta}). \tag{2.11b}$$

Since $\mathcal{L}_{p'}(M,\eta)$ with $(p')^{-1}=1-p^{-1}$ is a subset of $\mathcal{L}^*_p(M,\eta)$, this definition applies also for $B\in\mathcal{L}_{p'}(M,\eta)$.

(4) For $1\leqq p\leqq 2$, A and B in $\mathcal{L}^*_p(M,\eta)$ are said to be eqivalent

if $A\eta=B\eta$. (By Lemma A, η is in the domain of A and B.) For $2\leqq p\leqq\infty$, A and B in $\mathcal{L}_p^*(M,\eta)$ are said to be equivalent if $\langle C,A\rangle_{(\eta)}=\langle C,B\rangle_{(\eta)}$ for all C in $\mathcal{L}_p(M,\eta)$.

Lemma 2.4. *If* $2\leqq p<\infty$, *then* $\mathcal{L}_p(M,\eta)\subset L_p(M,\eta)$ *in the sense that* $u\Delta_{\phi,\eta}^{1/p}\eta\in L_p(M,\eta)$ *with*

$$\|u\Delta_{\phi,\eta}^{1/p}\eta\|_p^{(\eta)}=\phi(1)^{1/p}. \tag{2.12}$$

Proof. The bound for $\|\Delta_{\xi,\eta}^{(1/2)-(1/p)}x\Delta_{\phi,\eta}^{1/p}\eta\|^2$ given by Lemma A (iii) implies

$$\|u\Delta_{\phi,\eta}^{1/p}\eta\|_p^{(\eta)}\leqq\phi(1)^{1/p}, \tag{2.13}$$

in view of the definition (1.4). Let ξ_1 be the vector representative of ϕ and $\xi=u\xi_1$. Then the relations (Theorem C.1 (β4))

$$\Delta_{\xi,\eta}^{z}=u\Delta_{\phi,\eta}^{z}u^*, \tag{2.14}$$

$$u^*u\Delta_{\phi,\eta}^{z}=\Delta_{\phi,\eta}^{z}, \tag{2.15}$$

imply

$$\|\Delta_{\xi,\eta}^{(1/2)-(1/p)}u\Delta_{\phi,\eta}^{1/p}\eta\|=\|\Delta_{\phi,\eta}^{1/2}\eta\|=\phi(1)^{1/2}. \tag{2.16}$$

(See (1.1) and (1.2) for the last equality.) Since $\Delta_{\xi,\eta}^{z}=\phi(1)^z\Delta_{\xi',\eta}^{z}$ for $\xi'=\xi/\|\xi\|$ ($\|\xi\|^2=\phi(1)$), we have

$$\|u\Delta_{\phi,\eta}^{1/p}\eta\|_p^{(\eta)}\geqq\|\Delta_{\xi',\eta}^{(1/2)-(1/p)}u\Delta_{\phi,\eta}^{1/p}\eta\|=\phi(1)^{1/p}. \tag{2.17}$$

Combining (2.17) with (2.13), we obtain (2.12).

Lemma 2.5. *If* $2\leqq p\leqq\infty$, $A\in\mathcal{L}_p(M,\eta)$ *and* $B\in\mathcal{L}_p^*(M,\eta)$ *as in Notation* 2.3 *then,*

$$|\langle B,A\rangle_{(\eta)}|\leqq\|A\eta\|_p^{(\eta)}\Big(\prod_{l=0}^{n}\|x_l\|\Big)\Big(\prod_{l=1}^{n}\phi_l(1)^{\mathrm{Re}\,z_l}\Big)\omega_\eta(1)^{\mathrm{Re}\,z_0-(1/p)} \tag{2.18}$$

where $z_0=1-\sum_{l=1}^{n}z_l$.

Proof. Immediate from Lemma A (iii).

Lemma 2.6. *If* $B\in\mathcal{L}_p^*(M,\eta)$, $A_i\in\mathcal{L}_p(M,\eta)$, $\eta\in D(A_i)$ $(i=1,$

$\cdots, n)$, *and* $\sum_{i=1}^{n} A_i\eta = 0$, *then*

(2.19) $$\sum_{i=1}^{n} \langle B, A_i\rangle_{(\eta)} = 0 .$$

Proof. The same as the proof of Lemma 2.2.

Remark 2.7. Later Lemma 4.1. (i) asserts $\mathcal{L}_p(M, \eta) = L_p(M, \eta)$ for $2 \leqq p < \infty$. Lemma 2.5 and Lemma 2.6 then show $\mathcal{L}_p^*(M, \eta) \subset L_p(M, \eta)^*$ for such p (modulo the equivalence relation).

Lemma 2.8. *Let* $2 \leqq p, q, r \leqq \infty$, $p^{-1} + q^{-1} + r^{-1} = 1$, $A_1 \in \mathcal{L}_p(M, \eta)$, $A_2 \in \mathcal{L}_q(M, \eta)$ *and* $A_3 \in \mathcal{L}_r(M, \eta)$. *Then the formal product* A_1A_2 *is in* $\mathcal{L}_r^*(M, \eta)$ *and,*

(2.20) $$|\langle A_1A_2, A_3\rangle_{(\eta)}| \leqq \|A_1\eta\|_p^{(\eta)} \|A_2\eta\|_q^{(\eta)} \|A_3\eta\|_r^{(\eta)} .$$

Proof. The inequality follows from Lemma 2.4 and 2.5.

Lemma 2.9. *Let* $p^{-1} + (p')^{-1} = 1$, $1 < p < \infty$, $A = u\Delta_{\phi,\eta}^{1/p} \in \mathcal{L}_p(M, \eta)$, *and* $B = v\Delta_{\psi,\eta}^{1/p'} \in \mathcal{L}_{p'}(M, \eta) \subset \mathcal{L}_p^*(M, \eta)$. *Then*

(2.21) $$\phi(1)^{1/p} = \max\{|\langle B, A\rangle_{(\eta)}|, \ \psi(1) \leqq 1\}.$$

(*The maximum is attained.*) *If* $A = u\Delta_{\phi,\eta} \in \mathcal{L}_1(M, \eta)$ *and* $C = x \in \mathcal{L}_1^*(M, \eta)$, *then*

(2.22) $$\phi(1) = \max\{|\langle C, A\rangle_{(\eta)}|, \ \|x\| \leqq 1\},$$

(2.23) $$\|x\| = \sup\{|\langle C, A\rangle_{(\eta)}|, \ \phi(1) \leqq 1\}.$$

Proof. Lemma A (iii) implies inequality $\geqq$ in (2.21)-(2.23). The equality in (2.21) is obtained by setting $v = u$ and $\psi = \phi(1)^{-1}\phi$. The equalities in (2.22) and (2.23) follow from $\langle C, A\rangle_{(\eta)} = \phi(u^*x)$.

§ 3. Polar Decomposition in $D(\Delta_\eta^{(1/2)-2\alpha})$

The aim of this section is to show the existence of polar decomposition in $D(\Delta_\eta^{(1/2)-2\alpha})$. We consider the involution operator

$$J_\alpha^{(\eta)} = J\Delta_\eta^{(1/2)-2\alpha} \tag{3.1}$$

where α is real and J is the modular conjugation $J_{\eta,\eta}$. Due to $J\Delta_\eta J = \Delta_\eta^{-1}$ and $J^2 = 1$, $D(\Delta_\eta^{(1/2)-2\alpha})$ is invariant under the action of $J_\alpha^{(\eta)}$ and $(J_\alpha^{(\eta)})^2 \subset 1$. With a fixed element ζ of $D(\Delta_\eta^{(1/2)-2\alpha})$, we associate two operators T_0 and R_0 defined by,

$$T_0 y\eta = \sigma_{2i\alpha}'^{\eta}(y)\zeta \tag{3.2}$$

$$R_0 y\eta = \sigma_{2i\alpha}'^{\eta}(y) J_\alpha^{(\eta)}\zeta \tag{3.3}$$

where $\sigma_t'^{\eta}(y) = \Delta_\eta^{-it} y \Delta_\eta^{it}$ for $y \in M'$ and y is in the set M_0' of all entire analytic elements of M' with respect to the modular automorphisms $\sigma_t'^{\eta}$. Note that the domain $D(T_0) = D(R_0) = M_0'\eta$ is dense in H.

Lemma 3.1. *T_0 and R_0 are closable operators. Their closures T and R satisfy,*

$$T^* \supset R,\ R^* \supset T. \tag{3.4}$$

Proof. It is sufficient to prove that for any y_1, y_2 in M_0',

$$(T_0 y_1\eta, y_2\eta) = (y_1\eta, R_0 y_2\eta). \tag{3.5}$$

By definitions of T_0 and R_0, the two sides of (3.5) are computed as follows;

$$(T_0 y_1\eta, y_2\eta) = (\sigma_{2i\alpha}'^{\eta}(y_1)\zeta, y_2\eta) = (\zeta, \sigma_{2i\alpha}'^{\eta}(y_1)^* y_2\eta) \tag{3.6}$$

$$\begin{aligned}(y_1\eta, R_0 y_2\eta) &= (y_1\eta, \sigma_{2i\alpha}'^{\eta}(y_2) J\Delta_\eta^{(1/2)-2\alpha}\zeta) \\ &= (\sigma_{2i\alpha}'^{\eta}(y_2)^* y_1\eta, J\Delta_\eta^{(1/2)-2\alpha}\zeta) \\ &= (\Delta_\eta^{(1/2)-2\alpha}\zeta, J\sigma_{2i\alpha}'^{\eta}(y_2)^* y_1\eta) \\ &= (\Delta_\eta^{(1/2)-2\alpha}\zeta, \Delta_\eta^{-(1/2)} y_1^* \sigma_{2i\alpha}'^{\eta}(y_2)\eta).\end{aligned} \tag{3.7}$$

The proof is completed by the following formula,

$$\Delta_\eta^{-(1/2)} y_1^* \sigma_{2i\alpha}'^{\eta}(y_2)\eta = \Delta_\eta^{-(1/2)+2\alpha} \sigma_{-2i\alpha}'^{\eta}(y_1^*) y_2\eta \tag{3.8}$$

which is an analytic continuation of the following identity from real t to pure imaginary $2i\alpha$.

$$\Delta_\eta^{-(1/2)} y_1^* \sigma_t'^{\eta}(y_2)\eta = \Delta_\eta^{-(1/2)-it} \sigma_{-t}'^{\eta}(y_1^*) y_2\eta\,. \tag{3.9}$$

Next, we consider the polar decomposition of the closed operator $T=u|T|$.

Lemma 3. 2. *For any* $y\in M_0'$,

$$|T|^2\sigma'^{\eta}_{-4i\alpha}(y)\supset y|T|^2 . \tag{3.10}$$

Proof. For arbitrary elements y_1, y_2 in M_0', we have,

$$\begin{aligned} T_0y_1y_2\eta &= \sigma'^{\eta}_{2i\alpha}(y_1y_2)\zeta=\sigma'^{\eta}_{2i\alpha}(y_1)\sigma'^{\eta}_{2i\alpha}(y_2)\zeta \\ &= \sigma'^{\eta}_{2i\alpha}(y_1)T_0y_2\eta . \end{aligned} \tag{3.11}$$

This implies $T_0y_1\supset\sigma'^{\eta}_{2i\alpha}(y_1)T_0$. By taking the closure, we obtain, for any y in M_0',

$$Ty\supset\sigma'^{\eta}_{2i\alpha}(y)T . \tag{3.12}$$

Taking the adjoint of this relation and replacing y^* by y, we obtain,

$$T^*\sigma'^{\eta}_{-2i\alpha}(y)\supset yT^* . \tag{3.13}$$

Combining these formulas, we obtain the following:

$$T^*T\sigma'^{\eta}_{-4i\alpha}(y)\supset T^*\sigma'^{\eta}_{-2i\alpha}(y)T\supset yT^*T \quad (y\in M_0') .$$

Lemma 3. 3.

(1) *Let* $p=s(T)$ *be the support projection of* T. *Let,*

$$|T|^z=[\exp\{z(\log|T|)p\}]p . \tag{3.14}$$

Then,

$$y|T|^z\subset|T|^z\sigma'^{\eta}_{-2i\alpha z}(y) \tag{3.15}$$

for any $y\in M_0'$ *and any complex* z. *For pure imaginary* z, *equality holds for any* $y\in M'$.

(2) p *and the partial isometry* u *in the polar decomposition of* T *belong to* M.

Proof. Because of (3. 12), $\xi\in\ker T$ implies $y\xi\in D(T)$ and $Ty\xi=0$ for any $y\in M_0'$. So, $(1-p)H$ is M_0'-invariant. The σ-weak density of M_0' in M' then implies $(1-p)\in M$ and hence $p\in M$.

To show the formula (3. 15), we consider the function, (3.16) $$f(z)=(y|T|^{2z}\xi_1,\xi_2)-(\sigma'^{\eta}_{-4i\alpha z}(y)\xi_1,|T|^{2\bar z}\xi_2)$$

of a complex number z, where $y\in M'$ has a compact support with respect to the spectrum of σ'^{η}_t and ξ's have compact supports with respect to the spectrum of $(\log|T|)p$. The following three properties of f implies $f\equiv 0$ due to Carlson's Theorem (Boas [14]).

(α) By the choice of y, ξ_1, and ξ_2, $f(z)$ is exponentially bounded for $\mathrm{Re}\, z\geqq 0$.

(β) The estimate

(3.17) $$\begin{aligned}|f(\pm ir)|&\leqq \||T|^{\pm 2ir}\xi_1\|\,\|y^*\xi_2\|+\|\sigma'^{\eta}_{\pm 4\alpha r}(y)\xi_1\|\,\||T|^{\mp 2ir}\xi_2\|\\&\leqq 2\|y\|\,\|\xi_1\|\,\|\xi_2\|\end{aligned}$$

implies that

(3.18) $$h\left(\pm\frac{\pi}{2}\right)\equiv\overline{\lim_{r\to\infty}}\frac{1}{r}\log|f(\pm ir)|\leqq 0\,.$$

(γ) Due to Lemma 3.2,

(3.19) $$|T|^{2n}\sigma'^{\eta}_{-4in\alpha}(y)\supset y|T|^{2n}$$

and hence $f(n)=0$ for any non negative integer n.

Since the set of ξ's we have used is a core of $|T|^z$ (for any z), we obtain (3.15) from $f=0$.

Lastly, we show $u\in M$. By (3.12) and (3.15), we have $[u,y]|T|=0$ for all $y\in M_0'$ and hence $[u,y]p=0$. Since $up=u$ and $[y,p]=0$, we obtain $[u,y]=0$ and hence $u\in M$.

Lemma 3.4. *Let T_0 be defined by* (3.2) *and $T=\overline{T}_0$ for non-zero real α. There exists a unique normal semifinite weight ϕ such that*

(3.20) $$|T|=\Delta_{\phi,\eta}^{2\alpha}\,.$$

Proof. We consider the following one parameter family of partial isometries;

(3.21) $$u_t=|T|^{(1/2\alpha)it}\Delta_\eta^{-it}\quad(t\in\boldsymbol{R})\,.$$

Then u_t is strongly continuous and belongs to M due to

$$(3.22) \qquad \begin{aligned} yu_t &= y|T|^{(1/2\alpha)it}\Delta_\eta^{-it} \\ &= |T|^{(1/2\alpha)it}\sigma_t'^{\eta}(y)\Delta_\eta^{-it} \\ &= |T|^{(1/2\alpha)it}\Delta_\eta^{-it}y \\ &= u_t y \end{aligned}$$

for any $y \in M'$. Moreover,

$$(3.23) \qquad \begin{aligned} u_{s+t} &= |T|^{(1/2\alpha)i(s+t)}\Delta_\eta^{-i(s+t)} \\ &= |T|^{(1/2\alpha)is}\Delta_\eta^{-is}\Delta_\eta^{is}|T|^{(1/2\alpha)it}\Delta_\eta^{-it}\Delta_\eta^{-is} \\ &= u_s\sigma_s^\eta(u_t), \end{aligned}$$

$$(3.24) \qquad u_t^* u_t = \Delta_\eta^{it} p \Delta_\eta^{-it} = \sigma_t^\eta(p),$$

$$(3.25) \qquad u_t u_t^* = p.$$

According to the characterization of normal semifinite weight by M-valued σ_t^η one cocycle (Theorem B.1 in Appendix B), there exists a unique normal semifinite weight ϕ such that,

$$(3.26) \qquad u_t = (D\phi : D\omega_\eta)_t = \Delta_{\phi,\eta}^{it}\Delta_\eta^{-it}.$$

It follows that $|T|^{(1/2\alpha)it} = \Delta_{\phi,\eta}^{it}$ and (3.20) follows.

Lemma 3.5. *Let* $0 < \alpha \leqq 1/2$. *For any normal semifinite weight* ϕ, $\Delta_{\phi,\eta}^{2\alpha}\eta$ *belongs to* V_η^α, *if* η *is in the domain of* $\Delta_{\phi,\eta}^{2\alpha}$.

Proof. By the property of Radon-Nikodym cocycle,

$$(3.27) \qquad \Delta_{\phi,\eta}^{2\alpha}y \supset \sigma_{2i\alpha}'^{\eta}(y)\Delta_{\phi,\eta}^{2\alpha}$$

for any $y \in M_0'$. If y runs over M_0', then

$$(3.28) \qquad \Delta_\eta^{-\alpha}y^*y\eta = \sigma_{-i\alpha}'^{\eta}(y^*y)\eta$$

is dense in $V_\eta^{(1/2)-\alpha}$. Since V_η^α is the polar of $V_\eta^{(1/2)-\alpha}$, it is enough to show the following:

$$(3.29) \qquad (\Delta_{\phi,\eta}^{2\alpha}\eta, \sigma_{-i\alpha}'^{\eta}(y^*y)\eta) \geqq 0$$

for any $y \in M_0'$. By (3.27),

$$\begin{aligned} (\Delta_{\phi,\eta}^{2\alpha}\eta, \sigma_{-i\alpha}'^{\eta}(y^*y)\eta) &= (\sigma_{i\alpha}'^{\eta}(y)\Delta_{\phi,\eta}^{2\alpha}\eta, \sigma_{-i\alpha}'^{\eta}(y)\eta) \\ &= (\Delta_{\phi,\eta}^{2\alpha}\sigma_{-i\alpha}'^{\eta}(y)\eta, \sigma_{-i\alpha}'^{\eta}(y)\eta) \geqq 0. \end{aligned}$$

Lemma 3.6.

(1) *For any* $\zeta \in D(\Delta_\eta^{(1/2)-2\alpha})$ *with non-zero real* α, *there exists a partial isometry* u *in* M *and a normal semifinite weight* ϕ *such that* $\eta \in D(\Delta_{\phi,\eta}^{2\alpha})$, $\zeta = u\Delta_{\phi,\eta}^{2\alpha}\eta$ *and* $uu^* = s^M(\zeta)$ (*or equivalently* $u^*u = s(\phi)$).

(2) *For any* $\zeta \in D(\Delta_\eta^{1/2})$, *there exists a closed operator* T *affiliated with* M *such that* $\eta \in D(T)$ *and* $\zeta = T\eta$.

Proof. By (3.2), $\zeta = T\eta$. By Lemma 3.3, $T = u|T|$ with a partial isometry u in M satisfying $uu^* = s(T^*)$ (and $u^*u = s(T)$). Since η is separating, $s^M(\zeta) = s(T^*) = uu^*$. If $\alpha \neq 0$, $|T| = \Delta_{\phi,\eta}^{2\alpha}$ for a normal semifinite weight ϕ by Lemma 3.4 and if $\alpha = 0$, $|T|$ is a non-negative selfadjoint operator affiliated with M by (3.15) for $\alpha = 0$.

§ 4. Polar Decomposition in $L_p(M, \eta)$, $2 \leqq p < \infty$

In this section, we shall apply the polar decomposition in $D(\Delta_\eta^\alpha)$, $(0 \leqq \alpha < 1/2)$ to elements in $L_p(M, \eta)$, $2 \leqq p < \infty$. The polar decomposition for the case $1 \leqq p < 2$ will be given in Section 10.

Lemma 4.1.

(1) *Let* $\zeta \in L_p(M, \eta)$, $2 \leqq p < \infty$. *Then there exists a unique* $\phi \in M_*^+$ *and a partial isometry* $u \in M$ *such that*

$$\zeta = u\Delta_{\phi,\eta}^{1/p}\eta. \tag{4.1}$$

In this case, $\|\zeta\|_p^{(\eta)} = \phi(1)^{1/p}$.

(2) *Let* $1 \leqq p < 2$, $\eta \in D(\Delta_{\phi,\eta}^{1/p})$ *for a normal semifinite weight* ϕ *and* $\zeta = u\Delta_{\phi,\eta}^{1/p}\eta$ *where* u *is a partial isometry in* M *such that* $u^*u = s(\phi)$. *Then* ϕ *is bounded and* $\|\zeta\|_p^{(\eta)} = \phi(1)^{1/p}$.

Proof. (1) Taking $\xi = \eta$ in the definition (1.3), any $\zeta \in L_p(M, \eta)$ is in $D(\Delta_\eta^{(1/2)-(1/p)})$ and hence ζ is of the form $\zeta = u\Delta_{\phi,\eta}^{1/p}\eta$ by Lemma 3.6. By definition (1.3), $\zeta \in D(\Delta_{\eta_1,\eta}^{(1/2)-(1/p)})$ for any $\eta_1 \in H$. For any $\eta_1 \in H$ satisfying $s^M(\eta_1) \leqq s(\phi_u)$ where $\phi_u(x) = \phi(u^*xu)$, we prove the following consequence in Lemma 4.2 below:

$$u^*\eta_1 \in D(\Delta_{\phi,\eta_1}^{1/p}), \tag{4.2}$$

$$(4.3)\qquad \|\Delta_{\eta_1,\eta}^{(1/2)-(1/p)}u\Delta_{\phi,\eta}^{(1/p)}\eta\|=\|\Delta_{\phi,\eta_1}^{1/p}u^*\eta_1\|.$$

Therefore for any such η_1 of unit length,

$$(4.4)\qquad \|\Delta_{\phi,\eta_1}^{1/p}u^*\eta_1\|\leqq\|\zeta\|_p^{(\eta)}<\infty .$$

Since ϕ and hence ϕ_u are normal, there exists an increasing net of $\tilde{\phi}_\alpha\in M_*^+$ with $\sup\tilde{\phi}_\alpha=\phi_u$. By Lemma C. 3,

$$(4.5)\qquad \|\Delta_{\tilde{\phi}_\alpha,\eta_1}^{1/p}\eta_1\|\leqq\|\Delta_{\phi_u,\eta_1}^{1/p}\eta_1\|=\|\Delta_{\phi,\eta_1}^{1/p}u^*\eta_1\|.$$

(Last equality is due to Theorem C. 1.) If we take $\eta_1=\xi_\alpha/\|\xi_\alpha\|$ for the vector representative $\xi_\alpha\in\mathcal{P}_\eta^\natural$ of $\tilde{\phi}_\alpha$, then

$$(4.6)\qquad \|\Delta_{\tilde{\phi}_\alpha,\eta_1}^{1/p}\eta_1\|=\tilde{\phi}_\alpha(1)^{1/p}\|\Delta_{\eta_1}^{1/p}\eta_1\|=\tilde{\phi}_\alpha(1)^{1/p} .$$

Combining (4. 6), (4. 5) and (4. 4), we obtain

$$(4.7)\qquad \phi(1)^{1/p}=\phi_u(1)^{1/p}=\sup_\alpha\tilde{\phi}_\alpha(1)^{1/p}\leqq\|\zeta\|_p^{(\eta)}<\infty .$$

This proves the existence of the decomposition (4. 1) with $\phi\in M_*^+$. Then, owing to Lemma C. 2, $\|\Delta_{\eta_1,\eta}^{(1/2)-(1/p)}u\Delta_{\phi,\eta}^{1/p}\eta\|=\|\Delta_{\phi_u,\eta_1}^{1/p}\eta_1\|$. Hence

$$(4.8)\qquad \|\zeta\|_p^{(\eta)}=\sup_{\|\eta_1\|=1}\|\Delta_{\phi_u,\eta_1}^{1/p}\eta_1\|\leqq\phi_u(1)^{1/p}=\phi(1)^{1/p} ,$$

due to $\|\Delta_{\phi_u,\eta_1}^{1/2}\eta_1\|=\phi_u(1)^{1/2}$ and the three line theorem. This shows $\|\zeta\|_p^{(\eta)}=\phi(1)^{1/p}$ due to (4. 7).

To prove the uniqueness of the decomposition, assume $v\Delta_{\psi,\eta}^{1/p}\eta=\zeta$ with a partial isometry $v\in M$ and $\psi\in M_*^+$ satisfying $v^*v=s(\psi)$. Let $T=u\Delta_{\phi,\eta}^{1/p}$. For $y\in M_0'$ we have

$$(4.9)\qquad Ty\eta=\sigma_{i(1/p)}'^{\eta}(y)\zeta=\sigma_{i(1/p)}'^{\eta}(y)v\Delta_{\psi,\eta}^{1/p}\eta=v\Delta_{\psi,\eta}^{1/p}y\eta .$$

By definition, $M_0'\eta$ is a core of T. If $M_0'\eta$ is also a core of $\Delta_{\psi,\eta}^{1/p}$, then we obtain $T=v\Delta_{\psi,\eta}^{1/p}$ and by uniqueness of polar decomposition, we obtain $u=v$ and $\phi=\psi$. Hence the decomposition is unique.

To see that $M_0'\eta$ is the core of $\Delta_{\psi,\eta}^\alpha$ for $0<\alpha\leqq1/2$, it is enough to prove it for $\alpha=1/2$ because of the inequality $\|A^\nu\zeta\|\leqq\|A\zeta\|^\nu\|\zeta\|^{1-\nu}$ for any $A\geqq0$ and $0\leqq\nu\leqq1$. For $x\in M_0$, $x\eta=J\Delta_\eta^{1/2}x^*\eta=y\eta$ where $y=J\sigma_{-(i/2)}^\eta(x^*)J\in M_0'$. Therefore $M_0'\eta\supset M_0\eta$. (Actually equality holds.) By definition, $M\eta$ is a core of $\Delta_{\psi,\eta}^{1/2}$. Since $\|\Delta_{\psi,\eta}^{1/2}x\eta\|=\|x^*\xi(\psi)\|$ and M_0 is $*$-strongly dense in M, $M_0\eta$ is also a core of $\Delta_{\psi,\eta}^{1/2}$. This proves that $M_0'\eta$ is the core of $\Delta_{\psi,\eta}^{1/2}$ and hence that of $\Delta_{\psi,\eta}^\alpha$ for $0<\alpha\leqq1/2$.

For the assertion (2), the boundedness of ϕ follows from Lemma C. 4. To prove the equality, we start from $\xi\in\mathcal{P}_\eta^\natural$ such that $\phi=\omega_\xi$. Let $\eta_1=j(u)u\xi/\|\xi\|$ where $j(u)=JuJ$. By Lemma C. 2,

$$J_{\eta_1,\eta}\Delta_{\eta_1,\eta}^{(1/2)-(1/p)}u\Delta_{\xi,\eta}^{1/p}\eta=u^*\Delta_{j(u)u\xi,\eta_1}^{1/p}\eta_1 \tag{4.10}$$
$$=\|\xi\|^{2/p}u^*\Delta_{\eta_1}^{1/p}\eta_1=\|\xi\|^{2/p}u^*\eta_1,$$

and hence,

$$\begin{aligned}\|\zeta\|_p^{(\eta)}&=\inf\{\|\Delta_{\tilde\eta,\eta}^{(1/2)-(1/p)}\zeta\|:\|\tilde\eta\|=1,\ s^M(\tilde\eta)\geqq s^M(\zeta)\}\\&\leqq\|\Delta_{\eta_1,\eta}^{(1/2)-(1/p)}\zeta\|\\&=\|\xi\|^{2/p}\|u^*\eta_1\|=\phi(1)^{1/p}.\end{aligned} \tag{4.11}$$

The inequality $\|\zeta\|_p^{(\eta)}\geqq\phi(1)^{1/p}$ now follows from

$$\begin{aligned}\phi(1)^{1/p}&=\|j(u)u\xi\|^{2/p}\\&\leqq\|\Delta_{j(u)u\xi,\eta_1}^{1/p}\eta_1\|\end{aligned} \tag{4.12}$$

for any $\eta_1\in H$ such that $s^M(\eta_1)\geqq s^M(u\Delta_{\xi,\eta}^{1/p}\eta)=s^M(uj(u)\xi)$, $\|\eta_1\|=1$ and $\eta_1\in D(\Delta_{j(u)u\xi,\eta_1}^{1/p})$. The inequality in (4.12) follows from $\|\Delta_{j(u)u\xi,\eta_1}^{1/2}\eta_1\|=\|s^M(\eta_1)j(u)u\xi\|=\|j(u)u\xi\|$ and the Hölder inequality $\|A\eta\|^\alpha\leqq\|A^\alpha\eta_1\|$ for $A\geqq0$, $\alpha\geqq1$ and $\|\eta_1\|=1$. Hence we have $\|\zeta\|_p^{(\eta)}=\phi(1)^{1/p}$.

Lemma 4. 2. *Let $p\geqq2$, ϕ be a normal semifinite weight on M, u be a partial isometry in M satisfying $u^*u=s(\phi)$, $\eta_1\in\mathcal{P}_\eta^\natural$ (the natural positive cone) and $s^M(\eta_1)\leqq uu^*$. Assume that*

$$\eta\in D(\Delta_{\eta_1',\eta}^{(1/2)-(1/p)}u\Delta_{\phi,\eta}^{1/p}) \tag{4.13}$$

for all $\eta_1'\in H$ and, if $\|\eta_1'\|=1$,

$$\|\Delta_{\eta_1',\eta}^{(1/2)-(1/p)}u\Delta_{\phi,\eta}^{1/p}\eta\|\leqq A \tag{4.14}$$

for a constant A independent of η_1'. Then (4.2) *and* (4.3) *holds.*

Proof. Let

$$v(t)=\Delta_{\eta_1,\eta}^{-it}\Delta_{\phi_u,\eta}^{it}\in M\ . \tag{4.15}$$

Then $v(t)v(t)^*=s^M(\eta_1)$ and

$$v(t)^*\Delta_{\eta_1,\eta}^zv(t)=\Delta_{\eta_1(t),\eta}^z\quad(z\in\boldsymbol{C}) \tag{4.16}$$

for $\eta_1(t)=v(t)^*j(v(t)^*)\eta_1$ by Theorem C. 1 (β4). Hence the following expression makes sense by the assumtion of Lemma:

$$(4.17)\qquad v(t)^*\Delta_{\eta_1,\eta}^{(1/2)-(1/p)}v(t)u\Delta_{\phi,\eta}^{1/p}\eta$$
$$=v(t)^*\Delta_{\eta_1,\eta}^{(1/2)-(1/p)-it}\Delta_{\phi_u,\eta}^{it}u\Delta_{\phi,\eta}^{1/p}\eta$$
$$=v(t)^*\Delta_{\eta_1,\eta}^{(1/2)-(1/p)-it}u\Delta_{\phi,\eta}^{(1/p)+it}\eta\,.$$

Since $\Delta_{\eta_1(t),\eta}^{z}=\|\eta_1\|^{2z}\Delta_{\eta_1',\eta}^{z}$ for $\eta_1'=\eta_1(t)/\|\eta_1\|$ and $\|\eta_1'\|=1$, we have

$$(4.18)\qquad \|\Delta_{\eta_1,\eta}^{(1/2)-z}u\Delta_{\phi,\eta}^{z}\eta\|\leqq\|\eta_1\|^{1-(2/p)}A$$

for $z=(1/p)+it$ and any $t\in\boldsymbol{R}$. For any $\xi\in H$ with a compact support relative to the spectrum of $\Delta_{\eta_1,\eta}$, we set

$$(4.19)\qquad f_\xi(z)=(u\Delta_{\phi,\eta}^{z}\eta,\ \Delta_{\eta_1,\eta}^{(1/2)-\bar z}\xi)\,.$$

Then $f(z)$ is holomorphic in the strip region $0<\mathrm{Re}\,z<1/p$, continuous on its closure and satisfies

$$(4.20)\qquad |f_\xi(z)|\leqq\{\|\eta_1\|^{1-(2/p)}A\}^{p\,\mathrm{Re}\,z}\|\xi\|\,\|\eta_1\|^{1-p\,\mathrm{Re}\,z}$$

by (4.18), $\|\Delta_{\eta_1,\eta}^{(1/2)-it}u\Delta_{\phi,\eta}^{it}\eta\|=\|w_t^*u^*\eta_1\|\leqq\|\eta_1\|$ for $w_t=\Delta_{\phi,\eta}^{it}\Delta_\eta^{-it}\in M$, and three line theorem. It follows that the mapping $\xi\mapsto f_\xi(z)$ is norm continuous. Hence $u\Delta_{\phi,\eta}^{z}\eta\in D(\Delta_{\eta_1,\eta}^{(1/2)-z})$, $f_\xi(z)=(\Delta_{\eta_1,\eta}^{(1/2)-z}u\Delta_{\phi,\eta}^{z}\eta,\xi)$ for any z satisfying $0\leqq\mathrm{Re}\,z\leqq1/p$ and

$$(4.21)\qquad \|\Delta_{\eta_1,\eta}^{(1/2)-z}u\Delta_{\phi,\eta}^{z}\eta\|\leqq A^{p\mathrm{Re}\,z}\quad\text{for}\quad\|\eta_1\|=1\,.$$

Hence $z\mapsto\Delta_{\eta_1,\eta}^{(1/2)-z}u\Delta_{\phi,\eta}^{z}\eta$ is weakly holomorphic for $0<\mathrm{Re}\,z<1/p$ and weakly continuous on the closure.

For any $\xi\in H$ which has a compact support with respect to the spectrum of Δ_{ϕ,η_1}, we put

$$(4.22)\qquad g_\xi^{(1)}(z)=(J_{\eta_1,\eta}\Delta_{\eta_1,\eta}^{(1/2)-\bar z}u\Delta_{\phi,\eta}^{\bar z}\eta,\ \xi),$$

$$(4.23)\qquad g_\xi^{(2)}(z)=(u^*\eta_1,\ \Delta_{\phi,\eta_1}^{\bar z}\xi)\,.$$

Then $g_\xi^{(1)}(z)$ and $g_\xi^{(2)}(z)$ are holomorphic for $0<\mathrm{Re}\,z<1/p$ and continuous on the closure. Furthermore

$$(4.24)\qquad g_\xi^{(1)}(it)=(J_{\eta_1,\eta}\Delta_{\eta_1,\eta}^{(1/2)+it}u\Delta_{\phi,\eta}^{-it}\eta,\ \xi)$$
$$=(\Delta_{\phi,\eta_1}^{it}u^*\eta_1,\ \xi)$$
$$=g_\xi^{(2)}(it)$$

due to Lemma C. 2. It follows $g_\xi^{(1)}(1/p)=g_\xi^{(2)}(1/p)$. Hence (4. 2) holds and

$$J_{\eta_1,\eta}\Delta_{\eta_1,\eta}^{(1/2)-(1/p)}u\Delta_{\phi,\eta}^{1/p}\eta=\Delta_{\phi,\eta_1}^{1/p}u^*\eta_1\,. \tag{4.25}$$

This implies (4. 3) due to $J_{\eta_1,\eta}^*J_{\eta_1,\eta}=s^M(\eta_1)=s(\Delta_{\eta_1,\eta}^{(1/2)-(1/p)})$.

Lemma 4. 3.

$$L_p^+(M,\eta)=\{\Delta_{\phi,\eta}^{1/p}\eta:\phi\in M_*^+\},\quad 2\leqq p<\infty\,. \tag{4.26}$$

Proof. Let $\zeta=\Delta_{\phi,\eta}^{1/p}\eta$, $\phi\in M_*^+$. By Lemma 3. 5, $\zeta\in V_\eta^{1/(2p)}$. Conversely, let $\zeta\in L_p^+(M,\eta)$ and $\zeta=u\Delta_{\phi,\eta}^{1/p}\eta$ be the polar decomposition. Then for $y\in M_0'$,

$$\begin{aligned}(u\Delta_{\phi,\eta}^{1/p}y\eta,\,y\eta)&=(\sigma'^{\eta}_{i(1/p)}(y)\zeta,\,y\eta)\\&=(\zeta,\,\sigma'^{\eta}_{-i(1/p)}(y^*)y\eta)\geqq 0\end{aligned} \tag{4.27}$$

due to

$$\begin{aligned}\sigma'^{\eta}_{-i(1/p)}(y^*)y\eta&=\Delta_\eta^{-1/(2p)}\sigma'^{\eta}_{i/(2p)}(y)^*\sigma'^{\eta}_{i/(2p)}(y)\eta\\&\in V_\eta^{(1/2)-(1/(2p))}\,.\end{aligned} \tag{4.28}$$

By the proof of Lemma 4. 1, $M_0'\eta$ is a core of $u\Delta_{\phi,\eta}^{1/p}$. Hence $u\Delta_{\phi,\eta}^{1/p}\geqq 0$.

Since ζ is assumed to be in V_η^α, $\alpha=1/(2p)$, we have $J_\alpha^{(\eta)}\zeta=\zeta$ by Theorem 3 of [2] where $J_\alpha^{(\xi)}$ is defined by (3. 1). On the other hand, due to Lemma C. 2 and Theorem C. 1 (β4), we have

$$J_\alpha^{(\eta)}u\Delta_{\phi,\eta}^{1/p}\eta=u^*\Delta_{\phi_u,\eta}^{1/p}\eta\,. \tag{4.29}$$

Hence the uniqueness in Lemma 4. 1 (1) implies

$$u\Delta_{\phi,\eta}^{1/p}=u^*\Delta_{\phi_u,\eta}^{1/p}=\Delta_{\phi,\eta}^{1/p}u^*\,. \tag{4.30}$$

Hence $u\Delta_{\phi,\eta}^{1/p}$ is self-adjoint and positive as was shown above. Therefore $u\Delta_{\phi,\eta}^{1/p}=\Delta_{\phi,\eta}^{1/p}$ by the uniqueness of polar decomposition.

Lemma 4. 4. *For $x\in M$ and $2\leqq p<\infty$,*

$$\|x\|\,\|\zeta\|_p^{(\eta)}\geqq\|x\zeta\|_p^{(\eta)}\,. \tag{4.31}$$

Proof. By Lemma 4. 1 (1), $\zeta=u\Delta_{\phi,\eta}^{1/p}\eta$ with $\|\zeta\|_p^{(\eta)}=\phi(1)^{1/p}$. Hence

(1.25) implies (4.31) in view of the definition (1.3).

§ 5. Special Cases $p=1, \infty$

In this section, we shall give canonical isomorphisms of $L_\infty(M,\eta)$ with M and of $L_1(M,\eta)$ with M_*.

Lemma 5.1. *Let $\zeta\in L_\infty(M,\eta)$. Then there exists a unique $x\in M$ satisfying $\zeta=x\eta$ and $\|\zeta\|_\infty^{(\eta)}=\|x\|$. Under the correspondence $x\in M\mapsto x\eta\in L_\infty(M,\eta)$, $L_\infty(M,\eta)$ is isomorphic to M as a Banach space.*

Proof. By (1.3), $\zeta\in D(\Delta_\eta^{1/2})$. By Lemma 3.6 (2), there exists a closed operator T affiliated with M, satisfying the relation $\zeta=T\eta$. For any unit vector $\eta_1\in H$ and any $y\in M'$,

$$(5.1)\qquad \begin{aligned}(J_{\eta_1,\eta}\Delta_{\eta_1,\eta}^{1/2}\zeta, y\eta) &= (S_{\eta_1,\eta}^* y\eta, \zeta)\\ &= (y^*\eta_1, \zeta)\\ &= (\eta_1, y\zeta)\\ &= (\eta_1, Ty\eta)\end{aligned}$$

where $S_{\eta_1,\eta}$ is given by (1.2), which implies $S_{\eta_1,\eta}^* y\eta=y^*\eta_1$ for $y\in M'$. Since $M'\eta$ is a core of T, we obtain $\eta_1\in D(T^*)$, $J_{\eta_1,\eta}\Delta_{\eta_1,\eta}^{1/2}\zeta=T^*\eta_1$ and

$$(5.2)\qquad \|\zeta\|_\infty^{(\eta)}=\sup_{\|\eta_1\|=1}\|\Delta_{\eta_1,\eta}^{1/2}\zeta\|=\sup_{\|\eta_1\|=1}\|T^*\eta_1\|=\|T^*\|.$$

It follows that T^* and hence T are in M. This proves $\zeta=x\eta$, $x=T\in M$, and $\|x\|=\|\zeta\|_\infty^{(\eta)}$. Since η is separating, $x\in M$ satisfying $\zeta=x\eta$ is unique.

Conversely, if $\zeta=x\eta$ with $x\in M$, then $\zeta\in D(\Delta_{\xi,\eta}^{1/2})$ for any $\xi\in H$ and,

$$(5.3)\qquad \sup_{\|\xi\|=1}\|\Delta_{\xi,\eta}^{1/2}\zeta\|=\sup_{\|\xi\|=1}\|x^*\xi\|=\|x^*\|=\|x\|.$$

Lemma 5.2. $L_\infty^+(M,\eta)=M_+\eta$.

Proof. $M_+\eta\subset V_\eta^0$. Hence $M_+\eta\subset L_\infty^+(M,\eta)$. Let $\zeta\in L_\infty^+(M,\eta)$. By

Lemma 5.1, there exists unique $x\in M$ such that $\zeta=x\eta$ and $(xy\eta, y\eta) = (\zeta, y^*y\eta)\geqq 0$ for any $y\in M_0'$ due to $y^*y\eta\in V_\eta^{1/2}$. Hence $x\geqq 0$ and $L_\infty^+(M,\eta)\subset M_+\eta$.

Lemma 5.3. *Let* $\zeta\in H$ *and* $w_{\zeta,\eta}(x)\equiv(\zeta, x^*\eta)$ *for* $x\in M$. *Then*

$$\|\zeta\|_1^{(\eta)}=\|w_{\zeta,\eta}\|\,. \tag{5.4}$$

Proof. Let $w_{\zeta,\eta}(x)=\phi(xu)$ with a partial isometry u in M and $\phi\in M_*^+$ satisfying $u^*u=s(\phi)$ be the known polar decomposition of $w_{\zeta,\eta}\in M_*$. (Theorem 1.14.4 of [19].) Let $\xi\in\mathcal{P}_\eta^\natural$ be a vector representative for ϕ.

Since $1-uu^*$ is the largest projection $p\in M$ satisfying $w_{\zeta,\eta}(xp)=0$ for all $x\in M$, it is $1-s^M(\zeta)$ and hence $s^M(\zeta)=uu^*$. Therefore

$$\begin{aligned}(u^*\zeta, u^*x^*\eta) &= (\zeta, x^*\eta) = (xu\xi, \xi)\\ &= (J_{\xi,\eta}\Delta_{\xi,\eta}^{1/2}u^*x^*\eta, J_{\xi,\eta}\Delta_{\xi,\eta}^{1/2}\eta)\\ &= (\Delta_{\xi,\eta}^{1/2}\eta, \Delta_{\xi,\eta}^{1/2}u^*x^*\eta)\,.\end{aligned} \tag{5.5}$$

If Ψ is in $(1-u^*u)H$, then it is obviously orthogonal to $u^*\zeta$. It is also in $\ker\Delta_{\xi,\eta}^{1/2}$ because $s^M(\xi)=s(\phi)=u^*u$. Hence

$$(u^*\zeta, u^*ux_1\eta+\Psi) = (\Delta_{\xi,\eta}^{1/2}\eta, \Delta_{\xi,\eta}^{1/2}(u^*ux_1\eta+\Psi)) \tag{5.6}$$

for all $x_1\in M$. Since $u^*uM\eta+(1-u^*u)H\supset M\eta$ is a core for $\Delta_{\xi,\eta}^{1/2}$, (5.6) implies $\Delta_{\xi,\eta}^{1/2}\eta\in D(\Delta_{\xi,\eta}^{1/2})$ and

$$u^*\zeta=\Delta_{\xi,\eta}\eta\,. \tag{5.7}$$

Therefore $\eta\in D(\Delta_{\xi,\eta})$ and

$$\zeta=u\Delta_{\xi,\eta}\eta \tag{5.8}$$

with $u^*u=s^M(\xi)$. By Lemma 4.1 (2), we have

$$\|\zeta\|_1^{(\eta)}=\|\xi\|^2=\|\phi\|=\|w_{\zeta,\eta}\|\,. \tag{5.9}$$

Lemma 5.4. $L_1(M,\eta)$ *and* M_* *are isomorphic as Banach spaces through the unique continuous extension of the mapping*

$$\zeta\in H\mapsto w_{\zeta,\eta}\in M_*\,. \tag{5.10}$$

Proof. By Lemma 5.3, it remains to prove that the set of $w_{\zeta,\eta}$ is norm dense in M_*. Since $M'\eta$ is dense in H, $w_{uy^*y\eta,\eta}=w_{uy\eta,y\eta}$ with $y\in M'$ and a partial isometry $u\in M$ is norm dense in M_*. (Note that $w_{y\eta,y\eta}$ is norm dense in M_*^+.).

Lemma 5.5. *Through the identification of* $L_1(M,\eta)$ *with* M_*,

$$L_1^+(M,\eta)=M_*^+,\quad L_1^+(M,\eta)\cap H=V_\eta^{1/2}. \tag{5.11}$$

Proof. If $\zeta\in V_\eta^{1/2}(=\mathcal{P}_\eta^\flat)$, then $(\zeta, x^*\eta)\geqq 0$ for $x\in M^+$ by the duality of V_η^α and $V_\eta^{(1/2)-\alpha}$. Hence $V_\eta^{1/2}\subset M_*^+$. Since the relation $\langle\zeta, x^*\eta\rangle_{(\eta)}\geqq 0$ for $x\in M^+$ is stable under limit, we have $L_1^+(M,\eta)$ (as the closure of $V_\eta^{1/2}$) in M_*^+. On the other hand, $w_{y^*y\eta,\eta}=w_{y\eta}$ with $y\in M'$ is norm dense in M_*^+. Hence we have $L_1^+(M,\eta)=M_*^+$. By the proof of Lemma 5.3, $w_{\zeta,\eta}\in M_*^+$ implies $\zeta=\Delta_{\xi,\eta}\eta\in V_\eta^{1/2}$ due to Lemma 3.5.

Remark 5.6. Any $\zeta\in H$ has a polar decomposition $\zeta=u|\zeta|$ with $|\zeta|\in V_\eta^{1/2}=\mathcal{P}_\eta^\flat$ satisfying $u^*u=s^M(|\zeta|)$. By applying J, this is the same as the existence of a vector representative in $\mathcal{P}_\eta^\sharp$ for any state.

Remark 5.7. We have $\mathcal{L}_1(M,\eta)=L_1(M,\eta)$ via Lemma 5.4, the identification of $u\Delta_{\phi,\eta}\in\mathcal{L}_1(M,\eta)$ with $u\phi\in M_*$ due to

$$\langle u\Delta_{\phi,\eta}, x^*\rangle_{(\eta)}=(\Delta_{\phi,\eta}^{1/2}\eta, \Delta_{\phi,\eta}^{1/2}u^*x^*\eta)=\phi(xu)=u\phi(x) \tag{5.12}$$

for all $x\in M$ and polar decomposition $\psi=u\phi$ for any $\psi\in M_*$. (See Theorem 1.14.4 of [19].) Then $\|u\Delta_{\phi,\eta}\|_1^{(\eta)}=\phi(1)$. If $\phi(xu)=(\Psi, x^*\eta)$ for some $\Psi\in H$, then the proof of Lemma 5.3 implies $\eta\in D(\Delta_{\phi,\eta})$ and $\Psi=u\Delta_{\phi,\eta}\eta$.

§ 6. Completeness of $L_p(M,\eta)$, $2\leqq p<\infty$

Lemma 6.1.

(1) *For* $\zeta\in L_p(M,\eta)$ *and* $2\leqq p\leqq\infty$,

$$\|\zeta\|_p^{(\eta)}\|\eta\|^{1-(2/p)}\geqq\|\zeta\|. \tag{6.1}$$

(2) *If* $1\leqq p\leqq 2$, *and* $\zeta\in H$, *then*

(6.2) $$\|\zeta\|_p^{(\eta)} \leqq \|\zeta\| \, \|\eta\|^{(2/p)-1}$$

(3) *For* $2 \leqq p \leqq \infty$, $\|\cdot\|_p^{(\eta)}$ *is a norm and* $L_p(M, \eta)$ *is complete.*

Proof. (1) The case $p=\infty$ follows from $\|x\eta\| \leqq \|x\| \, \|\eta\|$ and Lemma 5.1. Let $2 \leqq p < \infty$ and $\zeta = u\Delta_{\xi,\eta}^{1/p}\eta$ be the polar decomposition of $\zeta \in L_p(M, \eta)$ given by Lemma 4.1 (1). Then $\|\zeta\|_p^{(\eta)} = \|\xi\|^{2/p}$ and, due to $\|\Delta_{\xi,\eta}^{1/2}\eta\| = \|\xi\|$, we obtain $\|\zeta\| = \|\Delta_{\xi,\eta}^{1/p}\eta\| \leqq \|\xi\|^{2/p} \|\eta\|^{1-(2/p)}$ by the Hölder inequality.

(2) We compute as follows: let $\zeta = u|\zeta|$, $u \in M^{\text{p.i.}}$, $|\zeta| \in \mathcal{P}^\natural$.

(6.3) $$\begin{aligned} \|\zeta\|_p^{(\eta)} &= \inf\{\|\Delta_{\eta_1,\eta}^{(1/2)-(1/p)}\zeta\| : \|\eta_1\| = 1, s^M(\eta_1) \geqq s^M(\zeta)\} \\ &\leqq \|\Delta_{\zeta/\|\zeta\|,\eta}^{(1/2)-(1/p)}\zeta\| \\ &\leqq \|\zeta\|^{(2/p)-1} \|\Delta_{\zeta,\eta}^{(1/2)-(1/p)}\zeta\|. \end{aligned}$$

Due to the Hölder inequality,

(6.4) $$\|\Delta_{\zeta,\eta}^{-\alpha}\zeta\| \leqq \|\zeta\|^{1-2\alpha} \|\Delta_{\zeta,\eta}^{-1/2}\zeta\|^{2\alpha}$$

where $0 \leqq \alpha = (1/p) - (1/2) \leqq 1/2$ and $\Delta_{\zeta,\eta}^{-1/2}\zeta = s^M(\zeta) u\eta$. Therefore,

(6.5) $$\begin{aligned} \|\zeta\|_p^{(\eta)} &\leqq (1/\|\zeta\|)^{1-(2/p)} \|\zeta\|^{2-(2/p)} \|s^M(\zeta) u\eta\|^{(2/p)-1} \\ &\leqq \|\zeta\| \, \|\eta\|^{(2/p)-1}. \end{aligned}$$

(3) Definition (1.4) and positive definiteness of $\Delta_{\xi,\eta}$ for separating ξ imply that $\|\cdot\|_p^{(\eta)}$ is a norm. We now prove the completeness. Let ζ_n be a Cauchy sequence in $L_p(M, \eta)$ with respect to $\|\cdot\|_p^{(\eta)}$-norm. Then,

(6.6) $$\sup_{\|\eta_1\|=1} \|\Delta_{\eta_1,\eta}^{(1/2)-(1/p)}(\zeta_n - \zeta_m)\| \to 0 \quad \text{as} \quad n, m \to \infty$$

and hence, for each η_1, there exists the limit

(6.7) $$f(\eta_1) = \lim_{n\to\infty} \Delta_{\eta_1,\eta}^{(1/2)-(1/p)}\zeta_n,$$

and satisfies,

(6·8) $$\sup_{\|\eta_1\|=1} \|\Delta_{\eta_1,\eta}^{(1/2)-(1/p)}\zeta_n - f(\eta_1)\| \to 0 \quad \text{as} \quad n \to \infty.$$

On the other hand, (6.1) and (6.6) imply that ζ_n is a Cauchy sequence in H. Let $\zeta = \lim_{n\to\infty} \zeta_n$. It then follows from (6.7) and the closedness of $\Delta_{\eta_1,\eta}^{(1/2)-(1/p)}$ that $\zeta \in D(\Delta_{\eta_1,\eta}^{(1/2)-(1/p)})$ and $f(\eta_1) = \Delta_{\eta_1,\eta}^{(1/2)-(1/p)}\zeta$. Furthermore (6.8) implies, (6.9) $$\sup_{\|\eta_1\|=1} \|\Delta_{\eta_1,\eta}^{(1/2)-(1/p)}\zeta\| = \lim_{n\to\infty} \|\zeta_n\|_p^{(\eta)} < \infty .$$

Hence, $\zeta \in L_p(M, \eta)$ and $\lim_{n\to\infty} \|\zeta - \zeta_n\|_p^{(\eta)} = 0$ by (6.8).

Remark 6.2. Since $\|\Delta_{\eta_1,\eta}^{(1/2)-(1/p)} x\eta\| \leqq \|\Delta_{\eta_1,\eta}^{1/2} x\eta\|^{1-(2/p)} \|x\eta\|^{2/p} = \|x^*\eta_1\|^{1-(2/p)} \|x\eta\|^{2/p} \leqq \|x\| \|\eta\|^{2/p}$ for any $\|\eta_1\| = 1$, we have $M\eta \subset L_p(M, \eta)$, $2 \leqq p \leqq \infty$, and hence $L_p(M, \eta)$, $2 \leqq p \leqq \infty$ is dense in H with respect to the topology of H.

Lemma 6.3. *For* $x \in M$ *and* $1 \leqq p \leqq \infty$,

(6.10) $$\|x\eta\|_p^{(\eta)} \leqq \|x\| \|\eta\|^{2/p} .$$

Proof. Since $\eta = \Delta_{\eta,\eta}^{1/p}\eta$, $\|\eta\|_p^{(\eta)} = \|\eta\|^{2/p}$ by Lemma 4.1. Hence (6.10) follows from Lemma 4.4 for $2 \leqq p \leqq \infty$ and from (6.2) together with $\|x\eta\| \leqq \|x\| \ \|\eta\|$ for $1 \leqq p \leqq 2$.

Lemma 6.4. *Any* $\zeta \in H$ *may be identified with an element of* $L_p(M, \eta)^*$ $(2 \leqq p \leqq \infty)$ *through the inner product* (ζ, ζ') *in* H *for* $\zeta' \in L_p(M, \eta)$ $(\subset H)$ *and, for* $p = \infty$, $\zeta \in L_\infty(M, \eta)_* = L_1(M, \eta)$.

Proof. By Lemma 6.1 (1), $|(\zeta, \zeta')| \leqq \|\zeta\| \|\zeta'\| \leqq c\|\zeta'\|_p^{(\eta)}$ and hence $\zeta \in L_p(M, \eta)^*$. The case $p = \infty$ has already been proved in Lemmas 5.3 and 5.4.

§ 7. A Sesquilinear Form between L_p and $L_{p'}$

In this section, we shall introduce the sesquilinear form between $L_p(M, \eta)$ and $L_{p'}(M, \eta)$ for $p^{-1} + (p')^{-1} = 1$ and imbed $L_{p'}(M, \eta)$, $\mathcal{L}_{p'}(M, \eta)$ and $\mathcal{L}_p^*(M, \eta)$ into the dual space $L_p(M, \eta)^*$ of $L_p(M, \eta)$.

Lemma 7.1.

(1) *For* $1 \leqq p \leqq 2$ *and for any* $\zeta \in H$, $\|\zeta\|_p^{(\eta)} < \infty$

(2) *If* $p^{-1} + (p')^{-1} = 1$, $\zeta \in L_p(M, \eta) \cap H$ *and* $\zeta' \in L_p(M, \eta) \cap H$, *then*

(7.1) $$|(\zeta, \zeta')| \leqq \|\zeta\|_p^{(\eta)} \|\zeta'\|_{p'}^{(\eta)} .$$

Proof. (1) There exists a partial isometry u in M and $|\zeta|\in\mathcal{P}_\eta^\natural$, such that $\zeta=u|\zeta|$ and $u^*u=s^M(|\zeta|)$ by eq. (7.3) of [2]. Then $|\zeta|=J|\zeta|=\Delta_{|\zeta|,\eta}^{1/2}\eta$ and $\Delta_{\zeta,\eta}^z=u\Delta_{|\zeta|,\eta}^z u^*$. Hence $\Delta_{\zeta,\eta}^{-(1/2)}\zeta=u\eta$, which implies $\zeta\in D(\Delta_{\zeta,\eta}^\alpha)$ for any $-(1/2)\leqq\alpha\leqq 0$ and in particular for $\alpha=(1/2)-(1/p)$ if $1\leqq p\leqq 2$.
(2) We may assume $1\leqq p\leqq 2$. If $\zeta\in H$ and $\varepsilon>0$, there exists η_1 with $\|\eta_1\|=1$ such that $\|\Delta_{\eta_1,\eta}^{(1/2)-(1/p)}\zeta\|\leqq\|\zeta\|_p^{(\eta)}+\varepsilon$. Since $(2^{-1}-p^{-1})+(2^{-1}-(p')^{-1})=0$, we obtain for any $\zeta'\in L_{p'}(M,\eta)$,

$$|(\zeta,\zeta')|=|(\Delta_{\eta_1,\eta}^{(1/2)-(1/p)}\zeta,\ \Delta_{\eta_1,\eta}^{(1/2)-(1/p')}\zeta')| \leqq(\|\zeta\|_p^{(\eta)}+\varepsilon)\|\zeta'\|_{p'}^{(\eta)}. \tag{7.2}$$

Since ε is arbitrary, $|(\zeta,\zeta')|\leqq\|\zeta\|_p^{(\eta)}\|\zeta'\|_{p'}^{(\eta)}$.

Lemma 7.2. *For $1\leqq p\leqq 2$ and $\zeta=u\Delta_{\xi,\eta}^{1/p}\eta\ (\eta\in D(\Delta_{\xi,\eta}^{1/p}))$ with a partial isometry $u\in M$ satisfying $u^*u=s^M(\xi)$,*

$$\|\zeta\|_p^{(\eta)}=\sup\{|(\zeta,\zeta')|:\zeta'\in L_{p'}(M,\eta),\ \|\zeta'\|_p^{(\eta)}\leqq 1,\ p^{-1}+(p')^{-1}=1\}. \tag{7.3}$$

Proof. By Lemma 4.1 (2), $\|\zeta\|_p^{(\eta)}=\|\xi\|^{2/p}$. The equality is attained in (7.1) by the homogeneity of relative modular operator if we set $\zeta'=u\Delta_{\tilde{\xi},\eta}^{1/p'}\eta$ where $\tilde{\xi}=\xi/\|\xi\|$ and $p^{-1}+(p')^{-1}=1$. Hence (7.3) holds.

Lemma 7.3. *If $2\leqq p\leqq\infty$ and $p^{-1}+(p')^{-1}=1$, then any element in $\mathcal{L}_{p'}^*(M,\eta)$ can be viewed as an element of $L_p(M,\eta)$ in the sense that $\eta\in D(A)$ and $A\eta\in L_p(M,\eta)$ for any*

$$A=x_0\Delta_{\phi_1,\eta}^{z_1}x_1\cdots\Delta_{\phi_n,\eta}^{z_n}x_n\in\mathcal{L}_{p'}^*(M,\eta)\quad(z\in I_{1/p}^{(n)}), \tag{7.4}$$

(see Notation 2.3 (2) for definition of $\mathcal{L}_{p'}^*(M,\eta)$ and $I_{1/p}^{(n)}$), and

$$\|A\eta\|_p^{(\eta)}\leqq\Big(\prod_{l=0}^n\|x_l\|\Big)\Big(\prod_{l=1}^n\phi_l(1)^{\mathrm{Re}\,z_l}\Big)\omega_\eta(1)^{(1/p)-\sum_{l=1}^n\mathrm{Re}\,z_l}. \tag{7.5}$$

If $1\leqq p\leqq 2$, $p^{-1}+(p')^{-1}=1$ and A is given by (7.4), then A can be viewed as an element of $L_{p'}(M,\eta)^*$ through the inner product $\langle A,B\rangle_{(\eta)}$ for $B\in\mathcal{L}_{p'}(M,\eta)=L_{p'}(M,\eta)$ and (7.5) holds.

Proof. First let $2\leqq p\leqq\infty$. By Lemma A, η is in the domain of A and $A\eta$ is in the domain of $\Delta_{\xi,\eta}^{(1/2)-(1/p)}$ for any $\xi\in H$. Furthermore the estimate (1.25) of Lemma A (iii) implies that $A\eta$ is in $L_p(M,\eta)$

and (7.5) holds. Next consider the case $1 \leqq p \leqq 2$. By Remark 2.7 and Lemma 4.1 (1), we may view $\mathcal{L}^*_{p'}(M, \eta)$ as an element of $L_{p'}(M, \eta)^*$ and (7.5) follows from (1.25).

Lemma 7.4. *For* $2 \leqq p < \infty$, $(p')^{-1} = 1 - p^{-1}$ *and* $A = u\Delta_{\xi,\eta}^{1/p'} \in \mathcal{L}_{p'}(M, \eta)$ *with* $u^*u = s^M(\xi)$,

$$\|\xi\|^{2/p'} = \max\{|\langle B, A\rangle_{(\eta)}|: B \in \mathcal{L}_p(M, \eta),\ \|B\eta\|_p^{(\eta)} = 1\}. \tag{7.6}$$

(The maximum is attained.)

Proof. By Lemmas 2.9 and 4.1 (1).

Remark 7.5. This Lemma shows that the norm $\|A\|_{p'}^{(\eta')}$ of A as elements in the dual space $L_p(M, \eta)^*$ is $\|\xi\|^{2/p'}$ for $2 \leqq p < \infty$. In view of Lemmas 2.9 and 4.4 (1), there exists $A \in \mathcal{L}^*_p(M, \eta)$ for any given $B \in \mathcal{L}_p(M, \eta)$ $(2 \leqq p \leqq \infty)$ such that $\langle B, A\rangle_{(\eta)} = \|B\|_p^{(\eta)} \|A\|_{p'}^{(\eta')}$.

Notation 7.6. Let $\mathcal{L}^*_{p,0}(M, \eta)$ be the set of all formal expression (2.9) satisfying $\sum_{j=1}^{n} \operatorname{Re} z_j = 1 - (1/p)$ in addition to all conditions stated below (2.9). The adjoint $B^* \in \mathcal{L}^*_{p,0}(M, \eta)$ for B in $\mathcal{L}^*_{p,0}(M, \eta)$ given by (2.9) is defined as

$$B^* = x_n^* \Delta_{\phi_n,\eta}^{\bar{z}_n} \cdots x_1^* \Delta_{\phi_1,\eta}^{\bar{z}_1} x_0^*\ . \tag{7.7}$$

The product $BC \in \mathcal{L}^*_{r,0}(M, \eta)$ of $B \in \mathcal{L}^*_{p,0}(M, \eta)$ and $C \in \mathcal{L}^*_{q,0}(M, \eta)$ is defined if $r^{-1} = p^{-1} + q^{-1} - 1$ and $1 \leqq r, p, q \leqq \infty$ as the expression obtained by writing expressions for B and C together in that order and combine the last x in B and the top x in C according to the product operation in M.

Lemma 7.7.

(1) *Any element* $B \in \mathcal{L}^*_p(M, \eta)$ *is equivalent to an element in* $\mathcal{L}^*_{p,0}(M, \eta)$.

(2) *If* $B_i \in \mathcal{L}^*_{p,0}(M, \eta)$ $i = 1, \cdots, n$ *and* $\sum_{i=1}^{n} B_i = 0$ *either as elements in* $L_{p'}(M, \eta)$ *for* $1 \leqq p \leqq 2$ *and* $(p')^{-1} = 1 - p^{-1}$ *(Lemma 7.3) or as elements in* $L_p(M, \eta)^*$ *for* $2 \leqq p \leqq \infty$ *(Remark 2.7), then* $\sum_{i=1}^{n} B_i^* = 0$ *in*

the same sense and $\sum_{i=1}^{n} B_i C = \sum_{i=1}^{n} CB_i = 0$ *in* $L_{r'}(M,\eta)$ *for* $1 \leqq r \leqq 2$ *and* $(r')^{-1} = 1 - r^{-1}$ *or in* $L_r(M,\eta)^*$ *for* $2 \leqq r \leqq \infty$ *where* $C \in \mathcal{L}^*_{q,0}(M,\eta)$, $r^{-1} = p^{-1} + q^{-1} - 1$ *and* $1 \leqq r, p, q \leqq \infty$.

Proof. (1) Let B be given by (2.9) with $z \in I^{(n)}_{1-(1/p)}$, $w = 1 - (1/p) - \sum_{j=1}^{n} z_j$, and $B' = B\Delta^w_\eta$. Then B is equivalent to B' (due to $\Delta^w_\eta \eta = \eta$) in $\mathcal{L}^*_p(M,\eta)$ and $B' \in \mathcal{L}^*_{p,0}(M,\eta)$.

(2) First consider the case $1 \leqq p \leqq 2$. Then $\sum B_i \eta = 0$. For $x \in M_0$, we have

$$(7.8) \qquad (x\eta, \sum B_i^* \eta) = (j(\sigma^\eta_{-i/2}(x^*))\eta, \sum B_i^* \eta)$$
$$= (j(\sigma^\eta_{(i/2)-(i/p)}(x^*)) \sum B_i \eta, \eta) = 0\,.$$

Hence we have $\sum B_i^* \eta = 0$. If $1 \leqq r \leqq 2$ in addition, we have $\sum CB_i \eta = 0$ from $\sum B_i \eta = 0$. Combining with the preceding result, we obtain $\sum C^* B_i^* = 0$ and hence $\sum B_i C = \sum (C^* B_i^*)^* = 0$. If $2 \leqq r \leqq \infty$ and $A \in \mathcal{L}_r(M,\eta) = L_r(M,\eta)$, then $\sum \langle C^*A, B_i \rangle_{(\eta)} = 0$ by Lemma 2.6. Since $\langle C^*A, B_i \rangle_{(\eta)} = \langle A, CB_i \rangle_{(\eta)}$, we have $\sum CB_i = 0$ in $L_r(M,\eta)^*$. Combining with the next result, this implies $\sum B_i C = 0$ as before.

We now consider the case $2 \leqq p \leqq \infty$. Then $\sum \langle B_i, A_1 \rangle_{(\eta)} = 0$ for any $A_1 \in \mathcal{L}_p(M,\eta) = L_p(M,\eta)$. Since $x_1 \eta \in L_\infty(M,\eta) \subset L_p(M,\eta)$ for all $x_1 \in M$, we take A_1 such that $A_1 \eta = x_1 \eta$. Then Lemma 2.2 and (1.27) imply

$$(7.9) \qquad \overline{\langle B_i, A_1 \rangle}_{(\eta)} = \omega_\eta(B_i^* A_1) = \omega_\eta(B_i^* x_1)$$
$$= \omega_\eta(x_1 \Delta_\eta^{1/p} B_i^*) = \omega_\eta(x^* B_i^*)$$

where $x \in M_0$ and x_1 is taken to be $\sigma^\eta_{-i/p}(x^*)$.

Hence

$$(7.10) \qquad \langle A, \sum B_i^* \rangle_{(\eta)} = 0$$

for all $A \in \mathcal{L}_p(M,\eta)$ such that $A\eta = x\eta$ for $x \in M_0$.

If x_α tends to x $*$-strongly in M with $\|x_\alpha\| \leqq \|x\|$, then $\omega_\eta(x_\alpha^* B_i^*)$ tends to $\omega_\eta(x^* B_i^*)$ by Lemma $A(vi)$. Since any $x \in M$ can be approximated by such $x_\alpha \in M_0$, (7.10) holds for A such that $A\eta = x\eta$ for $x \in M$. In particular we may take $x = u\Delta^{it}_{\xi,\eta}\Delta^{-it}_\eta \in M$ where u is a partial isometry in M. Since $x\eta = A(t)\eta$ for $A(t) = u\Delta^{it}_{\xi,\eta}$, we have (7.10) for $A = A(t)$.

By an analytic continuation and continuity, we obtain (7.10) for $A=A(-i/p)$. Hence $\sum B_i^*=0$ in $L_p(M,\eta)^*$.

Since $C^*A\eta\in L_p(M,\eta)$ for $A\in\mathcal{L}_r(M,\eta)$, we obtain

$$(7.11)\qquad \langle A, \sum CB_i^*\rangle_{(\eta)}=\langle C^*A, \sum B_i^*\rangle_{(\eta)}=0$$

and hence $\sum CB_i^*=0$ in $L_r(M,\eta)^*$. From $\sum C^*B_i^*=0$, we then obtain $\sum B_iC=\sum(C^*B_i^*)^*=0$.

Lemma 7.8. *For $2\leqq p\leqq\infty$, $x_i\in M$, $\xi_i\in H$ and $A_i=x_i\Delta_{\xi_i,\eta}^{1/p}$, $i=1,\cdots,n$, we have*

$$(7.12)\qquad (\|\sum_i^n A_i\eta\|_p^{(\eta)})^2=\|\sum_{i,j}^n A_i^*A_j\|_{p/2}^{(\eta,*)}$$

where $A_i^*A_j=\Delta_{\xi_i,\eta}^{1/p}x_i^*x_j\Delta_{\xi_j,\eta}^{1/p}\in\mathcal{L}_q^*(M,\eta)$ with $q^{-1}=1-2p^{-1}$ are considered as elements of $L_q(M,\eta)^*$ if $2\leqq p\leqq 4$ and as elements of $L_{p/2}(M,\eta)$ if $4\leqq p\leqq\infty$. The norm $\|\cdot\|_{p/2}^{(\eta,*)}$ denotes the norm in the dual space $L_q(M,\eta)^*$ if $2\leqq p\leqq 4$ and $\|\cdot\|_{p/2}^{(\eta)}$ if $4\leqq p\leqq\infty$. (The two coincides for $p=4$.)

Proof. If $p=\infty$, the statement is a property of C^*-norm. Let $2\leqq p<\infty$. By Lemma 4.1 (1), there exists a partial isometry u in M and $\xi\in H$ such that

$$(7.13)\qquad \sum_i^n A_i\eta=A\eta,\quad A=u\Delta_{\xi,\eta}^{1/p},$$

$u^*u=s^M(\xi)$ and $\|\sum_i^n A_i\eta\|_p^{(\eta)}=\|\xi\|^{2/p}$. By Lemma 7.7, we have $\sum_i A_i^*A_j-A^*A_j=0$ for all j and hence $\sum_{i,j}A_i^*A_j-A^*A=0$ in $\mathcal{L}_q^*(M,\eta)$. If $4\leqq p$, this implies $\sum_{i,j}^n A_i^*A_j\eta=A^*A\eta=\Delta_{\xi,\eta}^{2/p}\eta$ and (7.12) holds due to $\|\Delta_{\xi,\eta}^{2/p}\eta\|_{p/2}^{(\eta)}=\|\xi\|^{4/p}$ (Lemma 4.1 (1)).

Now let $2\leqq p<4$. For $x\in M_0$, we have $x\eta\in L_q(M,\eta)$, $x\eta=y\eta$ for $y=j(\sigma_{-i/2}^{(\eta)}(x^*))\in M_0'$ (elements of M' entire analytic for $\sigma_t'^{\eta}(y)=\Delta_\eta^{-it}y\Delta_\eta^{it}$) and

$$(7.14)\qquad \langle x, \sum_{i,j}^n A_i^*A_j\rangle_{(\eta)}=\omega_\eta(\sum A_j^*A_ix)$$

$$=\sum_{i,j}^n(A_ix\eta, A_j\eta)=\sum_{i,j}^n(A_iy\eta, A_j\eta)$$

$$= \sum_{i,j} (\sigma'^{\eta}_{i/p}(y) A_i\eta, A_j\eta) = (\sigma'^{\eta}_{i/p}(y) A\eta, A\eta)$$

$$= (Ay\eta, A\eta) = \langle x, A^*A\rangle_{(\eta)}$$

$$= \langle x, \Delta^{2/p}_{\xi,\eta}\rangle_{(\eta)} .$$

By the same proof as the preceding Lemma, (7.14) for $x \in M_0$ implies

$$\langle B, \sum_{i,j}^{n} A_i^* A_j\rangle_{(\eta)} = \langle B, \Delta^{2/p}_{\xi,\eta}\rangle_{(\eta)} \tag{7.15}$$

for all $B \in L_q(M, \eta)$. Hence $\sum_{i,j}^{n} A_i^* A_j = \Delta^{2/p}_{\xi,\eta}$ in $L_q(M, \eta)^*$. For $2 < p \leqq 4$, we have $2 \leqq q < \infty$ and hence we obtain by Lemma 7.4

$$\| \sum_{i,j}^{n} A_i^* A_j \|^{(\eta^*)}_{p/2} = \| \Delta^{2/p}_{\xi,\eta} \|^{(\eta^*)}_{p/2} = \| \xi \|^{4/p} = (\| \sum A_i\eta \|^{(\eta)}_{p})^2 . \tag{7.16}$$

If $p=2$, then (7.14) for $x \in M_0$ implies the same for $x \in M$ by the approximation argument of the preceding Lemma. Together with

$$\langle x\eta, \Delta_{\xi,\eta}\rangle_{(\eta)} = (\Delta^{1/2}_{\xi,\eta} x\eta, \Delta^{1/2}_{\xi,\eta}\eta) = (\xi, x^*\xi), \tag{7.17}$$

(7.14) for $x \in M$ implies

$$\| \sum_{i,j}^{n} A_i^* A_j \|_1^{(\eta,*)} = \| \xi \|^2 = (\| \sum A_i\eta \|_2^{(\eta)})^2 .$$

§ 8. Clarkson's Inequality

In this section, we shall show Clarkson's inequality for $L_p(M, \eta)$, $2 \leqq p < \infty$. It implies the reflexivity of the Banach space $L_p(M, \eta)$, $1 < p < \infty$.

Lemma 8.1. *For* $2 \leqq p < \infty$ *and* $\zeta_1, \zeta_2 \in L_p(M, \eta)$, *the following inequality holds,*

$$(\| \zeta_1 + \zeta_2 \|^{(\eta)}_p)^p + (\| \zeta_1 - \zeta_2 \|^{(\eta)}_p)^p \leqq 2^{p-1} \{ (\| \zeta_1 \|^{(\eta)}_p)^p + (\| \zeta_2 \|^{(\eta)}_p)^p \}. \tag{8.1}$$

Proof. The following inequality is the key point of the proof:

$$|\langle \zeta_1 + \zeta_2, \zeta_1' \rangle_{(\eta)} + \langle \zeta_1 - \zeta_2, \zeta_2' \rangle_{(\eta)}| \tag{8.2}$$

$$\leqq 2^{1-(1/p)} \{ (\| \zeta_1 \|^{(\eta)}_p)^p + (\| \zeta_2 \|^{(\eta)}_p)^p \}^{1/p} \{ (\| \zeta_1' \|^{(\eta^*)}_{p'})^{p'} + (\| \zeta_2' \|^{(\eta^*)}_{p'})^{p'} \}^{1/p'} ,$$

where $p^{-1} + (p')^{-1} = 1$ and ζ_1' and ζ_2' in $\mathcal{L}_{p'}(M, \eta)$. Before proving (8.2), we derive (8.1) from (8.2). By Notation 2.3, Remark 7.5, and Lemma 4.1 (1), there exist ζ_1' and ζ_2' in $\mathcal{L}_{p'}(M,\eta)$ such that

$$(8.3)\qquad \langle\zeta_1+\zeta_2,\zeta_1'\rangle_{(\eta)}=\|\zeta_1+\zeta_2\|_p^{(\eta)}\ \|\zeta_1'\|_{p'}^{(\eta^*)},$$

$$(8.4)\qquad \langle\zeta_1-\zeta_2,\zeta_2'\rangle_{(\eta)}=\|\zeta_1-\zeta_2\|_p^{(\eta)}\ \|\zeta_2'\|_{p'}^{(\eta^*)}.$$

We have still freedom of choosing $s=\|\zeta_1'\|_{p'}^{(\eta^*)}$ and $t=\|\zeta_2'\|_{p'}^{(\eta^*)}$ and hence we choose them such that $|s|+|t|\neq 0$ and

$$(8.5)\qquad \|\zeta_1+\zeta_2\|_p^{(\eta)}s+\|\zeta_1-\zeta_2\|_p^{(\eta)}t$$
$$=\{(\|\zeta_1+\zeta_2\|_p^{(\eta)})^p+(\|\zeta_1-\zeta_2\|_p^{(\eta)})^p\}^{1/p}(s^{p'}+t^{p'})^{1/p'}.$$

Substituting (8.3) and (8.4) into the left hand side of (8.2) and using (8.5), we obtain

$$(8.6)\qquad \{(\|\zeta_1+\zeta_2\|_p^{(\eta)})^p+(\|\zeta_1-\zeta_2\|_p^{(\eta)})^p\}^{1/p}$$
$$\leqq 2^{1-(1/p)}\{(\|\zeta_1\|_p^{(\eta)})^p+(\|\zeta_2\|_p^{(\eta)})^p\}^{1/p}.$$

We now prove the inequality (8.2). For $\zeta_i=u_i\Delta_{\xi_i,\eta}^{1/p}\eta$ and $\zeta_i'=u_i'\Delta_{\xi_i',\eta}^{1/p}$ $(i=1,2)$ (cf. Lemma 3.6), we consider the following function,

$$(8.7)\qquad F(z)=\omega_\eta(\Delta_{\xi_1',\eta}^{1-z}u_1'^*u_1\Delta_{\xi_1,\eta}^z)+\omega_\eta(\Delta_{\xi_1',\eta}^{1-z}u_1'^*u_2\Delta_{\xi_2,\eta}^z)$$
$$+\omega_\eta(\Delta_{\xi_2',\eta}^{1-z}u_2'^*u_1\Delta_{\xi_1,\eta}^z)-\omega_\eta(\Delta_{\xi_2',\eta}^{1-z}u_2'^*u_2\Delta_{\xi_2,\eta}^z).$$

By Lemma A, $F(z)$ is a continuous and bounded function of z for $0\leqq\operatorname{Re}z\leqq 1/2$, holomorphic in the interior of this strip region. We have

$$(8.8)\qquad |F(1/p)|=|\langle\zeta_1+\zeta_2,\zeta_1'\rangle_{(\eta)}+\langle\zeta_1-\zeta_2,\zeta_2'\rangle_{(\eta)}|$$

where we have used Lemma 2.6 for $\langle\zeta_1\pm\zeta_2,\zeta_k'\rangle_{(\eta)}=\langle\zeta_1,\zeta_k'\rangle_{(\eta)}\pm\langle\zeta_2,\zeta_k'\rangle_{(\eta)}$,

$$(8.9)\qquad |F(it)|=|\omega_\eta(\Delta_{\xi_1',\eta}^{1-it}u_1'^*u_1\Delta_{\xi_1,\eta}^{it})+\omega_\eta(\Delta_{\xi_1',\eta}^{1-it}u_1'^*u_2\Delta_{\xi_2,\eta}^{it})$$
$$+\omega_\eta(\Delta_{\xi_2',\eta}^{1-it}u_2'^*u_1\Delta_{\xi_1,\eta}^{it})-\omega_\eta(\Delta_{\xi_2',\eta}^{1-it}u_2'^*u_2\Delta_{\xi_2,\eta}^{it})|$$
$$\leqq 2(\|\xi_1'\|^2+\|\xi_2'\|^2)$$
$$=2\{(\|\zeta_1'\|_{p'}^{(\eta^*)})^{p'}+(\|\zeta_2'\|_{p'}^{(\eta^*)})^{p'}\}$$

where we have used (1.25), and

$$(8.10)\qquad |F((1/2)+it)|$$
$$=|(u_1\Delta_{\xi_1,\eta}^{(1/2)+it}\eta+u_2\Delta_{\xi_2,\eta}^{(1/2)+it}\eta,\ u_1'\Delta_{\xi_1',\eta}^{(1/2)+it}\eta)$$
$$+(u_1\Delta_{\xi_1,\eta}^{(1/2)+it}\eta-u_2\Delta_{\xi_2,\eta}^{(1/2)+it}\eta,\ u_2'\Delta_{\xi_1',\eta}^{(1/2)+it}\eta)|$$

$$\begin{aligned}
&\leqq \| u_1 \Delta_{\xi_1,\eta}^{(1/2)+it}\eta + u_2 \Delta_{\xi_2,\eta}^{(1/2)+it}\eta \| \, \| \Delta_{\xi_1',\eta}^{1/2} \| \\
&\quad + \| u_1 \Delta_{\xi_1,\eta}^{(1/2)+it}\eta - u_2 \Delta_{\xi_2,\eta}^{(1/2)+it}\eta \| \, \| \Delta_{\xi_2',\eta}^{1/2}\eta \| \\
&\leqq \{ \| u_1 \Delta_{\xi_1,\eta}^{(1/2)+it}\eta + u_2 \Delta_{\xi_2,\eta}^{(1/2)+it}\eta \|^2 \\
&\quad + \| u_1 \Delta_{\xi_1,\eta}^{(1/2)+it}\eta - u_2 \Delta_{\xi_2,\eta}^{(1/2)+it}\eta \|^2 \}^{1/2} \\
&\quad \times \{ \| \Delta_{\xi_1',\eta}^{1/2}\eta \|^2 + \| \Delta_{\xi_2',\eta}^{1/2}\eta \|^2 \}^{1/2} \\
&= \{ 2 (\| u_1 \Delta_{\xi_1,\eta}^{(1/2)+it}\eta \|^2 + \| u_2 \Delta_{\xi_2,\eta}^{(1/2)+it}\eta \|^2) \}^{1/2} \\
&\quad \times \{ \| \xi_1' \|^2 + \| \xi_2' \|^2 \}^{1/2} \\
&\leqq \sqrt{2} \{ \| \xi_1 \|^2 + \| \xi_2 \|^2 \}^{1/2} \{ \| \xi_1' \|^2 + \| \xi_2' \|^2 \}^{1/2} \\
&= \sqrt{2} \{ (\| \zeta_1 \|_p^{(\eta)})^p + (\| \zeta_2 \|_p^{(\eta)})^p \}^{1/2} \\
&\quad \times \{ (\| \xi_1' \|_{p'}^{(\eta^*)})^{p'} + (\| \zeta_2' \|_{p'}^{(\eta^*)})^{p'} \}^{1/2} .
\end{aligned}$$

By the three line theorem, the inequality (8. 2) follows.

A Banach space X with norm $\|\cdot\|$ is said to be uniformly convex if for each ε with $0<\varepsilon\leqq 2$ there exists a $\delta(\varepsilon)>0$ such that $x, y\in X$, $\|x\|\leqq 1$, $\|y\|\leqq 1$ and $\|x-y\|\geqq\varepsilon$ imply $\|(x+y)/2\|\leqq 1-\delta(\varepsilon)$.

Proposition 8. 2. *$L_p(M,\eta)$ is uniformly convex for $2\leqq p<\infty$.*

Proof. From Clarkson's inequality

$$(\|\zeta_1+\zeta_2\|_p^{(\eta)})^p + (\|\zeta_1-\zeta_2\|_p^{(\eta)})^p \leqq 2^p \tag{8. 11}$$

for $\zeta_1, \zeta_2 \in L_p(M,\eta)$, $2\leqq p<\infty$ satisfying $\|\zeta_j\|_p^{(\eta)}\leqq 1$, $j=1, 2$. If $\|\zeta_1-\zeta_2\|_p^{(\eta)} \geqq\varepsilon$, we obtain

$$\begin{aligned}
\|(\zeta_1+\zeta_2)/2\|_p^{(\eta)} &\leqq \{1-(\|\zeta_1-\zeta_2\|_p^{(\eta)}/2)^p\}^{1/p} \\
&\leqq \{1-(\varepsilon/2)^p\}^{1/p},
\end{aligned}$$

which shows the uniform convexity of $L_p(M,\eta)$, $2\leqq p<\infty$.

Corollary 8. 3. *$L_p(M,\eta)$ $(2\leqq p<\infty)$ is a reflexive Banach space.*

Proof. By Proposition 8. 2 and Milman's theorem (§ 26, 6. (4) of [14]).

§ 9. Uniform Strong Differentiability of the Norm for $1<p<\infty$

Let X be a Banach space. Its norm $\|\cdot\|$ is said to be uniformly strongly differentiable if for any $x\in X$ satisfying $x\neq 0$ and $\|x\|\leqq 1$ there exists a continuous real linear functional u_x on X and a monotone increasing function $\delta_x(\rho)$ $(\rho>0)$ such that $\lim_{\rho\to 0}\delta_x(\rho)=0$ and

$$(9.1)\qquad (\|x+y\|-\|x\|-\langle u_x, y\rangle)\leqq\|y\|\delta_x(\|y\|)$$

for all y. In this section, we show the uniform strong differentiability of the norm $\|\cdot\|_p^{(\eta)}$, $1<p<\infty$. The uniform strong differentiability of the norm is equivalent to the uniform convexity of dual norm in the dual space (§ 26, 10. (12) of [14]) and implies the reflexivity of the space. Therefore we have only to consider the case $2<p<\infty$.

Lemma 9.1. *The norm* $\|\cdot\|_p^{(\eta)}$ $(2<p<\infty)$ *is uniformly strongly differentiable.*

Proof. Let $n<p\leqq 2n$, $n=2, 3, \cdots$. We prove by induction on n.

As a preliminary remark, $L_{q'}(M, \eta)$ for $2\leqq q'<\infty$ is uniformly convex by Proposition 8.2 and hence the norm of its dual $L_{q'}(M, \eta)^*$ is uniformly strongly differentiable.

Let $\zeta_1, \zeta_2\in L_p(M, \eta)$ and $\zeta_j=u_j\Delta_{\xi_j,\eta}^{1/p}\eta$ be the polar decomposition given by Lemma 4.1 (1). Then each term in

$$(9.2)\qquad \zeta=|u_1\Delta_{\xi_1,\eta}^{1/p}+u_2\Delta_{\xi_2,\eta}^{1/p}|^2$$
$$=\Delta_{\xi_1,\eta}^{2/p}+\Delta_{\xi_2,\eta}^{1/p}u_2^*u_1\Delta_{\xi_1,\eta}^{1/p}+\Delta_{\xi_1,\eta}^{1/p}u_1^*u_2\Delta_{\xi_2,\eta}^{1/p}+\Delta_{\xi_2,\eta}^{2/p}$$

is in $L_q(M, \eta)$ with $q=p/2$ if $n>2$ due to Lemma 7.3 and in $L_{q'}(M, \eta)^*$ $((q')^{-1}+q^{-1}=1)$ if $n=2$ due to Remark 2.7. Since $2\leqq q'<\infty$ for $n=2$, the norm of $L_{q'}(M, \eta)^*$ is uniformly strongly differentiable. For other n, the norm of $L_q(M, \eta)$ is uniformly strongly differentiable by inductive assumption. In either case, we have

$$(9.3)\qquad (\|\zeta_1+\zeta_2\|_p^{(\eta)})^2=\|\zeta\|_q^{(\eta,*)},\quad (\|\zeta_1\|_p^{(\eta)})^2=\|\Delta_{\xi_1,\eta}^{1/q}\|_q^{(\eta,*)}$$

by Lemma 7.8 and the uniform strong differentiability implies

$$(9.4)\qquad \left| \|\zeta\|_q^{(\eta,*)} - \|\Delta_{\xi_1,\eta}^{1/q}\|_q^{(\eta,*)} - u(\zeta') \right| \leqq \|\zeta'\|_q^{(\eta,*)}\delta(\|\zeta'\|_q^{(\eta,*)})$$

where u is a continuous real linear functional on $L_q(M,\eta)$ (or on $L_{q'}(M,\eta)^*$ if $2\leqq p\leqq 4$),

$$(9.5)\qquad \zeta' = \Delta_{\xi_2,\eta}^{1/p}u_2^*u_1\Delta_{\xi_1,\eta}^{1/p} + \Delta_{\xi_1,\eta}^{1/p}u_1^*u_2\Delta_{\xi_2,\eta}^{1/p} + \Delta_{\xi_2,\eta}^{1/q} \in \mathrm{Lin}(\mathcal{L}_{q'}^*(M,\eta))$$

where the linear hull $\mathrm{Lin}(\mathcal{L}_{q'}^*(M,\eta))$ is in $L_q(M,\eta)$ if $4\leqq p\leqq\infty$ or in $L_{q'}(M,\eta)^*$ if $2<p\leqq 4$, and $\delta(\rho)$ is a monotone increasing function of $\rho>0$ vanishing as $\rho\to 0$. Both u and δ may depend on ζ_1 through ξ_1 but they are independent of ζ_2. We have

$$(9.6)\qquad \|\Delta_{\xi_2,\eta}^{1/q}\|_q^{(\eta,*)} = \|\xi_2\|^{4/p} = (\|\zeta_2\|_p^{(\eta)})^2,$$

$$(9.7)\qquad \|\zeta'\|_q^{(\eta,*)} \leqq 2\|\xi_2\|^{2/p}\|\xi_1\|^{2/p} + \|\xi_2\|^{4/p} = (2\|\zeta_1\|_p^{(\eta)} + \|\zeta_2\|_p^{(\eta)})\|\zeta_2\|_p^{(\eta)},$$

where equalities are due to Lemma 4.1 (1) and Lemma 7.4, the inequality for $2<p\leqq 4$ is due to Lemma 2.8 and Lemma 7.4 and the inequality for $4\leqq p$ is due to Lemma 7.3. Hence

$$(9.8)\qquad \left| (\|\zeta_1+\zeta_2\|_p^{(\eta)})^2 - (\|\zeta_1\|_p^{(\eta)})^2 - v(\zeta_2) \right| \leqq \|\zeta_2\|_p^{(\eta)}\delta_1(\|\zeta_2\|_p^{(\eta)}),$$

$$(9.9)\qquad \delta_1(\rho) \equiv (2\|\zeta_1\|_p^{(\eta)} + \rho)\,\delta(2\|\zeta_1\|_p^{(\eta)}\rho + \rho^2) + \rho\|u\|,$$

$$(9.10)\qquad v(\zeta_2) \equiv u(\Delta_{\xi_2,\eta}^{1/p}u_2^*u_1\Delta_{\xi_1,\eta}^{1/p} + \Delta_{\xi_1,\eta}^{1/p}u_1^*u_2\Delta_{\xi_2,\eta}^{1/p}).$$

Then v is a real linear functional of ζ_2 (for fixed ζ_1) by Lemma 7.7 and is continuous by Lemma 7.3 for $4\leqq p\leqq\infty$ and by Lemma 2.8 (in view of Lemma 2.6) for $2<p\leqq 4$. From this we obtain

$$(9.11)\qquad \left| \|\zeta_1+\zeta_2\|_p^{(\eta)} - \|\zeta_1\|_p^{(\eta)} \right| \leqq \|\zeta_2\|_p^{(\eta)}\sigma(\|\zeta_2\|_p^{(\eta)}),$$

$$(9.12)\qquad \sigma(\rho) = (\|\zeta_1\|_p^{(\eta)})^{-1}(\|v\| + \delta_1(\rho)).$$

Therefore, we obtain

$$(9.13)\qquad \begin{aligned} &\left| \|\zeta_1+\zeta_2\|_p^{(\eta)} - \|\zeta_1\|_p^{(\eta)} - (2\|\zeta_1\|_p^{(\eta)})^{-1}v(\zeta_2) \right| \\ &\leqq \left| (2\|\zeta_1\|_p^{(\eta)})^{-1} - (\|\zeta_1+\zeta_2\|_p^{(\eta)} + \|\zeta_1\|_p^{(\eta)})^{-1} \right| |v(\zeta_2)| \\ &\quad + (\|\zeta_1+\zeta_2\|_p^{(\eta)} + \|\zeta_1\|_p^{(\eta)})^{-1}\|\zeta_2\|_p^{(\eta)}\delta_1(\|\zeta_2\|_p^{(\eta)}) \\ &\leqq \|\zeta_2\|_p^{(\eta)}\delta_2(\|\zeta_2\|_p^{(\eta)}), \end{aligned}$$

$$(9.14)\qquad \delta_2(\rho) = (\|\zeta_1\|_p^{(\eta)})^{-1}\delta_1(\rho) + \|v\|\sigma(\rho)\rho(2\{\|\zeta_1\|_p^{(\eta)}\}^2)^{-1}.$$

Since $\delta_2(\rho)\to 0$ as $\rho\to 0$ and $\delta_2(\rho)$ is monotone increasing, we have the uniform strong differentiability.

Corollary 9.2. *$L_p(M,\eta)^*$ is uniformly convex for $2<p<\infty$.*

Proof. By the equivalence of the uniform strong differentiability derived in Lemma 9.1 and the uniform convexity of the dual space, as quoted before Lemma 9.1.

§ 10. Polar Decomposition in $L_p(M,\eta)$, $1\leqq p<2$

Lemma 10.1. *Let $2\leqq p<\infty$ and $p^{-1}+(p')^{-1}=1$.*

(1) *For $\zeta_1,\zeta_2\in\mathcal{L}_{p'}(M,\eta)$, $\zeta_1=\zeta_2$ in $L_p(M,\eta)^*$ if and only if $\zeta_1=\zeta_2$ (i.e. $u_1=u_2$, $\phi_1=\phi_2$ for $\zeta_j=u_j\Delta_{\phi_j,\eta}^{1/p'}\mathcal{L}$, $j=1,2$). (Uniqueness of polar decomposition)*

(2) *$\mathcal{L}_{p'}(M,\eta)=L_p(M,\eta)^*$. (Existence of polar decomposition.)*

Proof. (1) Let $\zeta_j=u_j\Delta_{\phi_j,\eta}^{1/p'}\in\mathcal{L}_{p'}(M,\eta)$ and $\zeta'_j=u_j\Delta_{\phi_j,\eta}^{1/p}\in L_p(M,\eta)$ $(j=1,2)$. Then $(\zeta_j,\zeta'_j)=\phi_j(1)=\|\zeta_j\|_{p'}^{(\eta^*)}\|\zeta'_j\|_p^{(\eta)}$ due to Lemmas 4.1 (1) and 7.4. Since $L_p(M,\eta)$ is uniformly convex (Proposition 8.2), ζ'_j satisfying such a relation and with a given p-norm is uniquely determined by ζ_j. If $\zeta_1=\zeta_2$, then $\|\zeta'_1\|_p^{(\eta)}=\phi_1(1)^{1/p}=(\|\zeta_1\|_{p'}^{(\eta^*)})^{p'/p}=(\|\zeta_2\|_{p'}^{(\eta^*)})^{p'/p}=\|\zeta'_2\|_p^{(\eta)}$ and hence $\zeta'_1=\zeta'_2$. The uniqueness of the polar decomposition in $L_p(M,\eta)$ (Lemma 4.1 (1)) then implies $u_1=u_2$, $\phi_1=\phi_2$.

(2) We already know that $\mathcal{L}_{p'}(M,\eta)$ can be imbedded in $L_p(M,\eta)^*$ (Remark 2.7 and Lemma 7.4). Let $\zeta\in L_p(M,\eta)^*$. Since $u=0$, $\phi=0$ gives $0=u\Delta_{\phi,u}^{1/p'}\in\mathcal{L}_{p'}(M,\eta)$, *we assume* $\zeta\neq 0$. Then there exists a nonzero $\zeta'\in L_p(M,\eta)$ such that $(\zeta,\zeta')=\|\zeta\|_p^{(\eta^*)}\|\zeta'\|_p^{(\eta)}$. Let $\zeta'=u\Delta_{\phi,\eta}^{1/p}\eta$ be the polar decomposition (Lemma 4.1 (1)). Then $\zeta''=u\Delta_{\phi,\eta}^{1/p'}\in\mathcal{L}_{p'}(M,\eta)$ satisfies $\langle\zeta'',\zeta'\rangle_{(\eta)}=\|\zeta''\|_p^{(\eta^*)}\|\zeta'\|_p^{(\eta)}$. By the uniform convexity of $L_p(M,\eta)^*$ given by Corollary 9.2, such ζ'' is unique up to multiplication by a positive number r, i.e. $\zeta=r\zeta''$. Let $\psi=r^{p'}\phi$. Then $\zeta=u\Delta_{\psi,\eta}^{1/p'}$ as is easily seen from the formula $\Delta_{s\phi,\eta}=s\Delta_{\phi,\eta}$. Therefore any $\zeta\in L_p(M,\eta)^*$ is in $\mathcal{L}_{p'}(M,\eta)$.

Remark 10.2. Lemma holds also for $p'=1$ if we replace $L_p(M,\eta)^*$ by M_* due to the known polar decomposition of $\phi\in M_*$ (Theorem 1.14.4 of [19]) and the correspondence given by Remark 5.7.

Lemma 10.3. *Let $2\leqq p\leqq\infty$ and $(p')^{-1}=1-p^{-1}$. If $\Psi\in H$ and $\zeta\in\mathcal{L}_{p'}(M,\eta)$ coincide as elements of $L_p(M,\eta)^*$ (i.e. $(\Psi,\zeta'\eta)_H=\langle\zeta,\zeta'\rangle_{(\eta)}$ for all $\zeta'\in\mathcal{L}_p(M,\eta)$), then η is in the domain of ζ and $\Psi=\zeta\eta$.*

Proof. Let $\zeta=u\Delta_{\phi,\eta}^{1/p'}$, $u^*u=s(\phi)$ and take the special elements $\zeta'=x\eta\in L_p(M,\eta)$ with $x\in M$. We have

$$(\Psi,x\eta)=\langle\zeta,\zeta'\rangle_{(\eta)}=(\Delta_{\phi,\eta}^{1/2}\eta,\Delta_{\phi,\eta}^{(1/p')-(1/2)}u^*x\eta).\tag{10.1}$$

Since the set of $x\in M$ with $u^*x=0$ is $(1-uu^*)M$ and η is cyclic, $(1-uu^*)\Psi=0$. Hence $(\Psi,x\eta)=(u^*\Psi,u^*x\eta)$ and (10.1) implies

$$(\Delta_{\phi,\eta}^{1/2}\eta,\Delta_{\phi,\eta}^{(1/p')-(1/2)}\Phi)=(u^*\Psi,\Phi)\tag{10.2}$$

whenever $\Phi=u^*x\eta+\Phi'$ with $\Phi'\in(1-u^*u)H$ because $\Delta_{\phi,\eta}\Phi'=0$ due to $s(\Delta_{\phi,\eta})=s(\phi)$. Since $u^*M\eta=u^*uM\eta$ is dense in u^*uH and $u^*uM\eta+(1-u^*u)H$ (which contains $M\eta$) is a core of $\Delta_{\phi,\eta}^{(1/p')-(1/2)}$, we have $\Delta_{\phi,\eta}^{1/2}\eta\in D(\Delta_{\phi,\eta}^{(1/p')-(1/2)})$ and,

$$\Delta_{\phi,\eta}^{1/p'}\eta=\Delta_{\phi,\eta}^{(1/p')-(1/2)}\Delta_{\phi,\eta}^{1/2}\eta=u^*\Psi\,.\tag{10.3}$$

Therefore $\Psi=uu^*\Psi=u\Delta_{\phi,\eta}^{1/p'}\ \eta$.

Lemma 10.4. *Let $2\leqq p\leqq\infty$ and $p^{-1}+(p')^{-1}=1$. Under the identification of $\mathcal{L}_p(M,\eta)$ with $L_p(M,\eta)$ and $\mathcal{L}_{p'}(M,\eta)$ with $L_p(M,\eta)^*$, $\langle\zeta',\zeta\rangle_{(\eta)}$ defined by (1.29) for $\zeta\in\mathcal{L}_p(M,\eta)$, $\zeta'\in\mathcal{L}_{p'}(M,\eta)$ is a continuous sesquilinear form on $L_p(M,\eta)\otimes L_p(M,\eta)^*$ coinciding with the inner product in H if ζ' is in H, and hermitian in the sense $\overline{\langle\zeta,\zeta'\rangle_{(\eta)}}=\langle\zeta',\zeta\rangle_{(\eta)}$.*

Proof. Hermiticity follows from the definition (1.22) of ω_η. Then conjugate linearity of $\langle\zeta',\zeta\rangle_{(\eta)}$ in ζ follows from Lemma 2.6 while the linearity in ζ' follows from the identification of $\mathcal{L}_{p'}(M,\eta)$ with $L_p(M,\eta)^*$ through this form. The continuity $|\langle\zeta',\zeta\rangle_{(\eta)}|\leqq\|\zeta'\|_{p'}^{(\eta^*)}\|\zeta\|_p^{(\eta)}$ precedes the identification of $\mathcal{L}_{p'}(M,\eta)$ with $L_p(M,\eta)^*$ (Lemma 7.4). By

Corollary 2.1, $\langle\zeta',\zeta\rangle_{(\eta)}$ coincides with the inner product in H if ζ' is in H (Lemma 10.3).

Lemma 10.5. *For* $1<p\leqq 2$ *and* $(p')^{-1}=1-p^{-1}$, $\Psi\in H$ *can be identified with an element of* $L_{p'}(M,\eta)^*$ *through the inner product in* H. *Then* $\Psi_1=\Psi_2$ *in* $L_{p'}(M,\eta)^*$ *only if* $\Psi_1=\Psi_2$ *in* H,

$$\|\Psi\|_p^{(\eta)}=\|\Psi\|_p^{(\eta')}, \tag{10.4}$$

H *is dense in* $L_{p'}(M,\eta)^*$ *and* $L_p(M,\eta)=L_{p'}(M,\eta)^*$, *where the equality* (10.4) *holds for all* $\Psi\in L_p(M,\eta)$. *For* $A=u\Delta_{\phi,\eta}^{1/p}\in\mathcal{L}_p(M,\eta)$, $\|A\|=\phi(1)^{1/p}$.

Proof. By Lemma 7.1 (2), $\Psi\in H$ is in $L_{p'}(M,\eta)^*$ and $\|\Psi\|_p^{(\eta)}\geqq\|\Psi\|_p^{(\eta')}$. By Lemmas 10.1 and 10.3, there exists $\zeta\in\mathcal{L}_{p'}(M,\eta)$ such that $\Psi=\zeta\eta$. Lemmas 7.4 and 4.1 (2) imply (10.4). It now follows that $\|\zeta\|_p^{(\eta)}$ is a seminorm on H. Since $L_{p'}(M,\eta)$ is dense in H (Remark 6.2), it must be a norm.

Since H separates $L_{p'}(M,\eta)$ $(\subset H)$, H is weakly dense subspace of $L_{p'}(M,\eta)^*$ ($L_{p'}(M,\eta)$ is the dual of $L_{p'}(M,\eta)^*$). By the Hahn-Banach separation theorem the norm closure of H must coincide with its weak closure. Therefore the completion of H relative to $\|\cdot\|_p^{(\eta)}$ can be identified with $L_{p'}(M,\eta)^*$.

By (10.4) and Lemma 7.4, we have $\|A\|_p^{(\eta)}=\phi(1)^{1/p}$.

Lemma 10.6. *Let* $1<p\leqq 2$. *The subset* $\mathcal{L}_p^+(M,\eta)$ *of* $\mathcal{L}_p(M,\eta)$ *consisting of all* $\Delta_{\phi,\eta}^{1/p}$, $\phi\in M_*^+$, *coincides with* $L_p^+(M,\eta)$ *through the identification of* $\mathcal{L}_p(M,\eta)$ *and* $L_p^*(M,\eta)$.

Proof. By Lemmas 10.3 and 3.5, $\mathcal{L}_p^+(M,\eta)\cap H$ is contained in $V_\eta^{1/(2p)}$. The set of vector states $\omega_{y\eta}$ with $y\in M_0$ is norm dense in M_+^* because $M_0\eta$ is dense in H. If $\|\phi_n-\phi\|\to 0$ in M_*, $(p')^{-1}=1-p^{-1}$ and $\zeta\in\mathcal{L}_{p'}(M,\eta)$,

$$|\langle\Delta_{\phi_n,\eta}^{1/p},\zeta\rangle_{(\eta)}-\langle\Delta_{\phi,\eta}^{1/p},\zeta\rangle_{(\eta)}| \tag{10.5}$$

tends to 0 due to Lemma A (vi). Since $\|\Delta_{\phi_n,\eta}^{1/p}\|_p^{(\eta)}=\phi_n(1)^{1/p}$ is uniformly bounded, the weak closure of $V_\eta^{1/(2p)}$ in $L_p(M,\eta)$ contains $\mathcal{L}_p^+(M,\eta)$. Again the norm closure of the convex set $V_\eta^{1/(2p)}$ coincides with its weak closure. It remains to prove the converse. For this purpose we use two properties of $\Psi\in V_\eta^{1/(2p)}$.

By Theorem 3 (2) of [2], Ψ is in the domain of $J_p(\eta,\eta)$ defined by (11.3) and invariant under $J_p(\eta,\eta)$. Since $J_p(\eta,\eta)$ has the unique continuous extension $J_p(\eta,\eta)$ to $L_p(M,\eta)$ as a conjugate linear isometry, as will be shown in Lemma 11.2, the invariance property will be preserved in the closure of $V_\eta^{1/(2p)}$. As will be shown in the same Lemma, $u\Delta_{\phi,\eta}^{1/p}\in\mathcal{L}_p(M,\eta)$ will be mapped by this isomorphism to $u^*\Delta_{\phi_u,\eta}^{1/p}$ and the invariance implies $u=u^*$ and $\phi(u^*xu)=\phi(x)$ for all $x\in M$ (i.e. u commutes with $\Delta_{\phi,\eta}$).

For $y\in M_0'$ and $(p')^{-1}=1-p^{-1}$,

$$\sigma'^{\eta}_{-i/p}(y^*)y\eta=\Delta_\eta^{-1/(2p)}\{\sigma'^{\eta}_{i/(2p)}(y)\}^*\sigma'^{\eta}_{i/(2p)}(y)\eta \in V_\alpha^{1/(2p')} \tag{10.6}$$

by the definition $V_\eta^\alpha=(\Delta_\eta^\alpha M_+\eta)^-=(\Delta_\eta^{\alpha-(1/2)}M'_+\eta)^-$ for $0\leqq\alpha\leqq 1/2$. (Note that $y\eta=\Delta_\eta^{1/2}j(y^*)\eta$ for $y\in M'$.) By Theorem 3 (5) of [2], $\Psi\in V_\eta^{1/(2p)}$ satisfies

$$(\Psi,\Phi)\geqq 0\,,\quad \Phi=\sigma'^{\eta}_{-i/p}(y^*)y\eta\,. \tag{10.7}$$

Since $\Phi=x\eta$ with $x=j(\sigma'^{\eta}_{i/2}(\sigma'^{\eta}_{-i/p}(y^*)y))^*\in M$, we have $\Phi\in L_\infty(M,\eta)\subset L_{p'}(M,\eta)$. By Lemma 10.5, we obtain

$$\langle\Psi,x\rangle_{(\eta)}\geqq 0$$

for the above x and any Ψ in the closure of $V_\eta^{1/(2p)}$ in $L_p(M,\eta)$. By definition (1.22) and (2.11), we have for such $\Psi=u\Delta_{\phi,\eta}^{1/p}\in L_p(M,\eta)$

$$\begin{aligned}\langle\Psi,x\rangle_{(\eta)}&=(\Delta_{\phi,\eta}^{1/(2p)}\eta,\ \Delta_{\phi,\eta}^{1/(2p)}u^*x\eta)\\&=(\Delta_{\phi,\eta}^{1/(2p)}\eta,\ \Delta_{\phi,\eta}^{1/(2p)}u^*\sigma'^{\eta}_{-i/p}(y^*)y\eta)\\&=(\Delta_{\phi,\eta}^{1/(2p)}y\eta,\ \Delta_{\phi,\eta}^{1/(2p)}u^*y\eta)\\&=(u\Delta_{\phi,\eta}^{1/(2p)}y\eta,\ \Delta_{\phi,\eta}^{1/(2p)}y\eta),\end{aligned}$$

where we have used the first result above that u commutes with $\Delta_{\phi,\eta}$. Since $M_0'\eta=M_0\eta$ is a core of $\Delta_{\phi,\eta}^{1/2}$ (by the definition of $\Delta_{\phi,\eta}$ and by $\|\Delta_{\phi,\eta}^{1/2}x\eta\|=\|x^*\eta\|$), and since $s(\Delta_{\phi,\eta})=s(\phi)=u^*u$, $\Delta_{\phi,\eta}^{1/(2p)}M_0'\eta$ is dense in u^*uH.

Therefore $u \geqq 0$ and hence $u = s(\phi)$. Therefore any Ψ in the closure of $V_\eta^{1/(2p)}$ in $L_p(M, \eta)$ is of the form $\Delta_{\phi,\eta}^{1/p}\eta$.

Remark 10.7. In the first part of the above proof, $\|\Delta_{\phi_n,\eta}^{1/p}\|_p^{(\eta)} = \phi_n(1)^{1/p} \to \phi(1)^{1/p} = \|\Delta_{\phi,\eta}^{1/p}\|_p^{(\eta)}$. Therefore $\Delta_{\phi_n,\eta}^{1/2}$ actually converges to $\Delta_{\phi,\eta}^{1/p}$ in L_p-norm due to uniform convexity.

§ 11. Change of the Reference Vector η

In this section, we discuss the change of reference vector η and the associated isomorphism $\tau_p(\eta_2, \eta_1)$ from $L_p(M, \eta_1)$ to $L_p(M, \eta_2)$. Let η_1 and η_2 be two cyclic and separating vectors.

Lemma 11.1. *Let $2 \leqq p \leqq \infty$. The mapping $J_p(\eta_2, \eta_1)$ defined by*

$$J_p(\eta_2, \eta_1)\zeta = J_{\eta_2,\eta_1}\Delta_{\eta_2,\eta_1}^{(1/2)-(1/p)}\zeta \tag{11.1}$$

for $\zeta \in L_p(M, \eta_1)$ is a conjugate linear isometric map of Banach spaces $L_p(M, \eta_1)$ and $L_p(M, \eta_2)$. If $2 \leqq p < \infty$, it maps $u\Delta_{\phi,\eta_1}^{1/p} \in \mathcal{L}_p(M, \eta_1)$ (with $u^*u = s(\phi)$) to $u^*\Delta_{\phi_u,\eta_2}^{1/p}$ $(= \{u\Delta_{\phi,\eta_2}^{1/p}\}^*) \in \mathcal{L}_p(M, \eta_2)$, where $\phi_u(x) = \phi(u^*xu)$. If $p = \infty$, it maps $x\eta_1 \in L_\infty(M, \eta_1)$ to $x^*\eta_2 \in L_\infty(M, \eta_2)$.

Proof. Let $2 \leqq p < \infty$ and $\zeta = u\Delta_{\phi,\eta_1}^{1/p}\eta_1$ be the polar decomposition given by Lemma 4.1 (1). By Lemma C.2,

$$J_{\eta_2,\eta_1}\Delta_{\eta_2,\eta_1}^{(1/2)-(1/p)}(u\Delta_{\phi,\eta_1}^{1/p}\eta_1) = \Delta_{\phi,\eta_2}^{1/p}u^*\eta_2 = u^*\Delta_{\phi_u,\eta_2}^{1/p}\eta_2\,. \tag{11.2}$$

It follows that

$$\|\zeta\|_p^{(\eta_1)} = \phi(1)^{1/p} = \phi_u(1)^{1/p} = \|J_p(\eta_2, \eta_1)\zeta\|_p^{(\eta_2)}\,.$$

The case $p = \infty$ follows from $\overline{S_{\eta_2,\eta_1}} = J_{\eta_2,\eta_1}\Delta_{\eta_2,\eta_1}^{1/2}$.

Lemma 11.2. *Let $1 \leqq p \leqq 2$. The mapping $J_p(\eta_2, \eta_1)$ defined on $\zeta \in D(\Delta_{\eta_2,\eta_1}^{(1/2)-(1/p)})$ by*

$$J_p(\eta_2, \eta_1)\zeta = J_{\eta_2,\eta_1}\Delta_{\eta_2,\eta_1}^{(1/2)-(1/p)}\zeta \tag{11.3}$$

has a unique extension (again denoted by $J_p(\eta_2, \eta_1)$) to a conjugate linear isometric map of $L_p(M, \eta_1)$ onto $L_p(M, \eta_2)$. It maps $u\Delta_{\phi,\eta_1}^{1/p} \in$

$\mathcal{L}_p(M,\eta_1)$ to $u^*\Delta_{\phi_u,\eta_2}^{1/p}\in\mathcal{L}_p(M,\eta_2)$. Moreover $J_p(\eta_2,\eta_1)$ and $J_{p'}(\eta_1,\eta_2)$ are adjoint of each other relative to the form $\langle\zeta,\zeta'\rangle_{(\eta)}$ for $\zeta\in L_p(M,\eta)$ and $\zeta'\in L_{p'}(M,\eta)$, where $p^{-1}+(p')^{-1}=1$.

Proof. Let $\zeta'\in L_{p'}(M,\eta_2)$ and $\zeta\in D(\Delta_{\eta_2,\eta_1}^{(1/2)-(1/p)})$. By the relation $\Delta_{\eta_2,\eta_1}^{(1/2)-(1/p)}J_{\eta_1,\eta_2}=J_{\eta_1,\eta_2}\Delta_{\eta_1,\eta_2}^{(1/2)-(1/p')}$, we obtain

$$\begin{aligned}(11.4)\qquad \langle J_p(\eta_2,\eta_1)\zeta,\zeta'\rangle_{(\eta_2)}&=(J_{\eta_2,\eta_1}\Delta_{\eta_2,\eta_1}^{(1/2)-(1/p)}\zeta,\zeta')\\&=(J_{\eta_1,\eta_2}\zeta',\Delta_{\eta_2,\eta_1}^{(1/2)-(1/p)}\zeta)\\&=(J_{p'}(\eta_1,\eta_2)\zeta',\zeta).\end{aligned}$$

By Lemma 11.1 and the formula (1.6) proved in Lemma 10.5, we have

$$(11.5)\qquad \|J_p(\eta_2,\eta_1)\zeta\|_p^{(\eta_2)}=\|\zeta\|_p^{(\eta_1)}$$

for $\zeta\in D(\Delta_{\eta_2,\eta_1}^{(1/2)-(1/p)})$, which is a dense subset of H, hence dense in H relative to $\|\cdot\|_p^{(\eta_1)}$ due to Lemma 6.1 (2) and therefore dense in $L_p(M,\eta_1)$ due to the density of H proved in Lemma 10.4. Since $J_p(\eta_2,\eta_1)$ is conjugate linear on $D(\Delta_{\eta_2,\eta_1}^{(1/2)-(1/p)})$, this proves the first assertion of Lemma. At the same time, (11.4) implies

$$(11.6)\qquad \langle J_p(\eta_2,\eta_1)\zeta,\zeta'\rangle_{(\eta_2)}=\langle J_{p'}(\eta_1,\eta_2)\zeta',\zeta\rangle_{(\eta_1)}$$

for all $\zeta\in L_p(M,\eta_1)$ and $\zeta'\in L_{p'}(M,\eta_2)$ and hence the last assertion of Lemma.

Let $1<p\leqq 2$, $A=u\Delta_{\phi,\eta_1}^{1/p}\in\mathcal{L}_p(M,\eta_1)$ and $B=v\Delta_{\psi,\eta_2}^{1/p'}\in L_{p'}(M,\eta_2)$. By (11.6) and Lemma 11.1, we obtain

$$\begin{aligned}(11.7)\qquad \langle J_p(\eta_2,\eta_1)A,B\rangle_{(\eta_2)}&=\langle v^*\Delta_{\psi_v,\eta_1}^{1/p'},A\rangle_{(\eta_1)}\\&=\omega_{\eta_1}(\Delta_{\phi,\eta_1}^{1/p}u^*v^*\Delta_{\psi_v,\eta_1}^{1/p'})=\omega_{\eta_2}(\Delta_{\phi,\eta_2}^{1/p}u^*v^*\Delta_{\phi_v,\eta_2}^{1/p'})\\&=\omega_{\eta_2}(u^*\Delta_{\phi_u,\eta_2}^{1/p}1\Delta_{\psi,\eta_2}^{1/p'}v^*)=\omega_{\eta_2}(\Delta_{\psi,\eta_2}^{1/p'}v^*u^*\Delta_{\phi_u,\eta_2}^{1/p})\\&=\langle u^*\Delta_{\phi_u,\eta_2}^{1/p},B\rangle_{(\eta_2)}\end{aligned}$$

where the third equality is due to Lemma A(vii), the fourth equality utilizes $u^*\Delta_{\phi_u,\eta_2}^{\alpha}=\Delta_{\phi,\eta_2}^{\alpha}u^*$ and $v^*\Delta_{\psi_v,\eta_1}^{\alpha'}=\Delta_{\psi,\eta_1}^{\alpha'}v^*$ in the definition (1.22) and (1.26) of ω_η and the fifth equality is due to (1.27). This proves $J_p(\eta_2,\eta_1)A=u^*\Delta_{\phi_u,\eta_2}^{1/p}$ for $1<p\leqq 2$.

Let $p=1$, $A=u\Delta_{\phi,\eta_1}\in\mathcal{L}_p(M,\eta_1)$ and $B=x\eta_2\in L_\infty(M,\eta_2)$ with $x\in M$.

The same computation as (11.7) shows

$$\langle J_p(\eta_2, \eta_1) A, B\rangle_{(\eta_2)} = \langle u^* \Delta^{1/p}_{\phi_u, \eta_2}, B\rangle_{(\eta_2)} \tag{11.8}$$

and hence $J_p(\eta_2, \eta_1) A = u^* \Delta^{1/p}_{\phi_u, \eta_2}$.

Remark 11.3. If $\eta_1 = \eta_2$, this mapping $J_p(\eta, \eta)$ corresponds to the complex conjugation in the commutative case and to the adjoint * in the case of L_p spaces defined by trace. By the explicit description of the map $J_p(\eta_2, \eta_1)$ given in Lemmas 11.1 and 11.2, we see that $J_p(\eta_1, \eta_2) = J_p(\eta_2, \eta_1)^{-1}$ and $J_p(\eta, \eta)^2 = 1$.

Lemma 11.4. *If* $\zeta \in D(\Delta_\eta^{(1/2)-(1/p)})$ $(1 \leqq p \leqq 2)$ *or* $\zeta \in L_p(M, \eta)$ $(2 \leqq p \leqq \infty)$, *then*

$$J_p(\eta, \eta)\zeta = J^{(\eta)}_{1/(2p)}\zeta \tag{11.9}$$

where $J^{(\eta)}_\alpha$ *is defined by* (3.1).

Proof. For $p \neq \infty$, $\zeta = u\Delta^{1/p}_{\phi,\eta}\eta$ $(\eta \in D(\Delta^{1/p}_{\phi,\eta}))$ by Lemmas 10.1 (2) and 10.3 for $1 \leqq p \leqq 2$ and by Lemma 4.1 (1) for $2 \leqq p < \infty$. Then Lemma C.2 implies (11.9) due to an explicit description for $J_p(\eta, \eta)\zeta$ given by Lemmas 11.1 and 11.2. For $p = \infty$, $\zeta = x\eta$ with $x \in M$ and $J^{(\eta)}_0\zeta = J\Delta^{1/2}_\eta x\eta$ $x^*\eta = J_\infty(\eta, \eta)\zeta$ (see Lemma 11.1).

We define $\tau_p(\eta_2, \eta_1)$ by (1.9).

Lemma 11.5.

(1) $\tau_p(\eta_2, \eta_1)$ *is an isomorphism of* $L_p(M, \eta_1)$ *onto* $L_p(M, \eta_2)$ *and is independent of* η.

(2) $\tau_p(\eta_3, \eta_2)$ $\tau_p(\eta_2, \eta_1) = \tau_p(\eta_3, \eta_1)$, *where* η_j $(j = 1, \cdots, 3)$ *are any cyclic and separating vectors.*

(3) *Let* $1 \leqq p < \infty$. $\zeta = u\Delta^{1/p}_{\phi,\eta_1} \in \mathcal{L}_p(M, \eta_1)$. *Then* $\tau_p(\eta_2, \eta_1)\zeta = u\Delta^{1/p}_{\phi,\eta_2} \in \mathcal{L}_p(M, \eta_2)$.

(4) *Let* $\zeta = x\eta_1 \in L_\infty(M, \eta_1)$. *Then* $\tau_p(\eta_2, \eta_1)\zeta = x\eta_2$.

Proof. By Lemmas 11.1 and 11.2, $J_p(\eta_2, \eta_1)$ maps $u\Delta^{1/p}_{\phi,\eta_1} \in \mathcal{L}_p(M, \eta_1)$ onto $u^*\Delta^{1/p}_{\phi,\eta_2} \in \mathcal{L}_p(M, \eta_2)$ for $1 \leqq p < \infty$ and $x\eta_1 \in L_\infty(M, \eta_1)$ onto $x^*\eta_2$

$\in L_\infty(M, \eta_2)$ for $p=\infty$. Hence $\tau_p(\eta_2, \eta_1)$ maps $u\Delta_{\phi,\eta_1}^{1/p} \in \mathcal{L}_p(M, \eta_1)$ onto $u\Delta_{\phi,\eta_2}^{1/p} \in \mathcal{L}_p(M, \eta_2)$ for $1 \leqq p < \infty$ and $x\eta_1 \in L_\infty(M, \eta_1)$ onto $x\eta_2 \in L_\infty(M, \eta_2)$. Hence the assertions follow.

§ 12. Product and Hölder Inequality

Let us recall Notation 7.6 for $\mathcal{L}_{p,0}^*(M, \eta)$ $(1 \leqq p \leqq \infty)$, adjoint and product. By Lemma 7.3, we may identify elements of $\mathcal{L}_{p,0}^*(M, \eta)$ (modulo induced equivalence) with elements of $L_{p'}(M, \eta)$ (directly for $2 \leqq p' \leqq \infty$, through duality $L_{p'}(M, \eta) = L_p(M, \eta)^*$ for $1 < p' \leqq 2$ and through $L_{p'}(M, \eta) \subset L_p(M, \eta)^*$ together with *-strong continuity on bounded sets in Lemma A (vi) for $p'=1$.

Lemma 12.1. *Let* $1 \leqq p, q, r \leqq \infty$, $p^{-1} + (p')^{-1} = q^{-1} + (q')^{-1} = r^{-1} + (r')^{-1} = 1$, $p^{-1} + q^{-1} = r^{-1}$.

(1) *If* A_1 *and* A_2 *in* $\mathcal{L}_{p',0}^*(M, \eta)$ *are equal as elements of* $L_p(M, \eta)$, *then* $A_1^* = A_2^*$ *in* $L_p(M, \eta)$, $A_1 B = A_2 B$ *and* $BA_1 = BA_2$ *in* $L_r(M, \eta)$ *where* $B \in \mathcal{L}_{q',0}^*(M, \eta)$.

(2) A^* *is conjugate linear in* A *and* AB *is bilinear in* (A, B).

(3) *The product is associative and* $(AB)^* = B^* A^*$.

(4) $\|AB\|_r^{(\eta)} \leqq \|A\|_p^{(\eta)} \|B\|_q^{(\eta)}$.

Proof. Viewing $\beta \in \boldsymbol{C}$ as an element of $\mathcal{L}_{1,0}^*(M, \eta)$, it is easy to check $(\beta A)^* = \bar{\beta} A^*$, $(\beta A) C = A(\beta C) = \beta AC$ and the equivalence of $A_1 = A_2$ with $A_1 + (-1) A_2 = 0$ in $L_p(M, \eta)$ form the definition and linear dependence of ω_η on x's. (Lemma A (v).) Therefore Lemma 7.7 (2) implies (1) as well as (2). (3) follows directly from the definition. To prove (4), we may restrict $A \in \mathcal{L}_p(M, \eta)$ and $B \in \mathcal{L}_q(M, \eta)$ due to (1) because $\mathcal{L}_s(M, \eta)$ is a subset of $\mathcal{L}_{s'}^*(M, \eta)$ on one hand and $\mathcal{L}_s(M, \eta) = L_s(M, \eta)$ on the other where $s = p$ *or* q. Then (4) follows from

$$|\langle AB, C\rangle_{(\eta)}| = |\omega_\eta(C^* AB)| \leqq \|A\|_p^{(\eta)} \|B\|_q^{(\eta)} \|C\|_{r'}^{(\eta)} \tag{12.1}$$

for any $C \in \mathcal{L}_{r'}(M, \eta) = L_{r'}(M, \eta)$ due to Lemma A (iii), $\|u\Delta_{\phi,\eta}^{1/p}\|_p^{(\eta)} = \phi(1)^{1/p}$ (proven in Lemmas 4.1 (i) for $2 \leqq p < \infty$, in Lemmas 7.4 and 10.4 for $1 < p \leqq 2$ and Remark 5.7 for $p=1$) and $\|x\eta\|_\infty^{(\eta)} = \|x\|$ (Lemma 5.1).

Remark 12.2. $\mathcal{L}_p(M, \eta)$ is in $\mathcal{L}^*_{p',0}(M, \eta)$ and $\mathcal{L}_p(M, \eta)$ exhausts $L_p(M, \eta)$ for $1 \leqq p < \infty$ while $\mathcal{L}^*_{1,0}(M, \eta)$ exhausts $L_\infty(M, \eta)$ under the above identification. Hence the adjoint is defined as a conjugate linear involution in $L_p(M, \eta)$ and product is defined as a bilinear map from $L_p(M, \eta) \otimes L_q(M, \eta)$ into $L_r(M, \eta)$. In particular the adjoint coincides with the map $J_p(\eta, \eta)$ as is seen explicitly on $\mathcal{L}_p(M, \eta)$ for $1 \leqq p < \infty$ and on $\mathcal{L}^*_{1,0}(M, \eta)$ for $p = \infty$ due to Lemmas 11.1 and 11.2.

Lemma 12.3. *The multiplication of $x \in M = \mathcal{L}^*_{1,0}(M, \eta)$ with $B \in \mathcal{L}^*_{p'}(M, \eta)$ makes $L_p(M, \eta)$ an M-module ($p^{-1} + (p')^{-1} = 1$). If there exists $\Psi \in H$ coinciding with B as an element of $L_p(M, \eta)$ $(= L_{p'}(M, \eta)^*)$, then xB coincides with $x\Psi$ as a multiplication of $x \in M$ on a vector Ψ in H.*

Proof. The special case of Lemma 12.1 shows that $L_p(M, \eta)$ is an M-module. For $2 \leqq p \leqq \infty$, $\Psi = B\eta$ and xB coincides with $x\Psi = xB\eta$ by definition. Let $1 \leqq p < 2$ and $A \in \mathcal{L}_p(M, \eta)$ coincide with Ψ as an element of $L_p(M, \eta)$. (Lemma 10.1 (1).) By Lemma 10.3, $\eta \in D(A)$ and $\Psi = A\eta$. Then $\eta \in D(xA)$ and hence xA coincides with $(xA)\eta = x\Psi$ (product in H) by (2.5). Since $xB = xA$ in $L_p(M, \eta)$ by Lemma 12.1, xB coincides with $x\Psi$ in $L_p(M, \eta)$.

Remark 12.4. Even if there exists $\Psi \in H$ coinciding with $B \in \mathcal{L}^*_{p'}(M, \eta)$, η is not necessarily in the domain of B in contrast to Lemma 10.3.

§ 13. Linear Polar Decomposition

Lemma 13.1. *Let $\zeta \in \mathcal{L}_p(M, \eta)$ such that $J_p(\eta, \eta)\zeta = \zeta$. Then there exists $\zeta_+ \geqq 0$ and $\zeta_- \geqq 0$ such that*

$$\zeta = \zeta_+ - \zeta_- . \tag{13.1}$$

This decomposition is unique under the condition,

$$s^M(\zeta_+) \perp s^M(\zeta_-) \tag{13.2}$$

where $s^M(\zeta)$ is the smallest projection $P \in M$ satisfying $P\zeta = \zeta$ in the

M-module $L_p(M, \eta)$.

Proof. As is noticed in Remark 12.2 $J_p(\eta, \eta)$ maps $\zeta \in L_p(M, \eta)$ to ζ^*, i.e. $\zeta = u\Delta_{\phi,\eta}^{1/p}$, $u^* = u$, $s(\phi) = u^2$ and $\phi_u = \phi$ (equivalently $u\Delta_{\phi,\eta}^{\alpha} = \Delta_{\phi,\eta}^{\alpha} u$) for $p \neq \infty$ and $\zeta = x\eta$, $x^* = x$ for $p = \infty$. For $p = \infty$, the unique decomposition $x = x_+ - x_-$, $x_\pm \in M_+$ implies the existence of decomposition as well as the uniqueness because η is separating for M.

In the case $p \neq \infty$, let $E_\pm$ be the spectral projection of u for ± 1 and $\phi_\pm \equiv \phi \circ E_\pm$. Then $E_+ + E_- = s(\phi)$, $s(\Delta_{\phi_\pm,\eta}) = s(\phi_\pm) = E_\pm$ and $\Delta_{\phi,\eta} = \Delta_{\phi_+,\eta} + \Delta_{\phi_-,\eta}$. Hence

$$\begin{aligned} \Delta_{\phi,\eta}^{1/p} \pm u\Delta_{\phi,\eta}^{1/p} &= (1 \pm u)\Delta_{\phi,\eta}^{1/p} \\ &= 2E_\pm \Delta_{\phi,\eta}^{1/p} \\ &= 2\Delta_{\phi_\pm,\eta}^{1/p}. \end{aligned} \tag{13.3}$$

Therefore

$$u\Delta_{\phi,\eta}^{1/p} = \Delta_{\phi_+,\eta}^{1/p} - \Delta_{\phi_-,\eta}^{1/p}, \tag{13.4}$$

which proves the existence of the decomposition.

To prove the uniqueness of the decomposition for $p \neq \infty$, we assume $\zeta = \zeta'_+ - \zeta'_-$, $\zeta'_\pm \in L_p^+(M, \eta)$ be another such decomposition satisfying $s^M(\zeta'_+) \perp s^M(\zeta'_-)$. By Lemma 4.3 ($2 \leqq p < \infty$), Lemma 10.6 ($1 < p \leqq 2$), Lemma 5.4 and Remark 5.7 ($p = 1$), $\zeta'_\pm = \Delta_{\phi'_\pm,\eta}^{1/p}$ for some $\phi'_\pm \in M_*^+$ satisfying $s^M(\zeta'_\pm) = s(\phi'_\pm)$. If we define a partial isometry u' such that u' is 1 on $s^M(\zeta'_+)$, -1 on $s^M(\zeta'_-)$ and 0 on their orthogonal complement, then

$$u'\zeta' = \zeta'_+ - \zeta'_- = \zeta, \quad s^M(\zeta') = (u')^2 \tag{13.5}$$

where $\zeta' = \zeta'_+ + \zeta'_- = \Delta_{\phi'_+ + \phi'_-,\eta}^{1/p}$. By the uniqueness of the polar decomposition, $u' = u$ and $\zeta' = \Delta_{\phi,\eta}^{1/p}$. This means $E_\pm \Delta_{\phi,\eta}^{1/p} = \Delta_{\phi'_\pm,\eta}^{1/p}$. This shows $\zeta'_\pm = \Delta_{\phi_\pm,\eta}^{1/p}$ and the uniqueness of the decomposition follows.

Corollary 13.2. *Any* $\zeta \in L_p(M, \eta)$ $(1 \leqq p \leqq \infty)$ *has a unique decomposition* $\zeta = (\zeta_{r+} - \zeta_{r-}) + i(\zeta_{i+} - \zeta_{i-})$ *such that* $\zeta_{\tau\sigma} \in L_p^+(M, \eta)$ *and* $s^M(\zeta_{\tau+}) \perp s^M(\zeta_{\tau-})$ *where* $\tau = r, i$ *and* $\sigma = +, -$.

Proof. Relative to the conjugate linear involutive isometry $J_p(\eta, \eta)$,

$\zeta \in L_p(M, \eta)$ has a unique decomposition

$$\zeta = (\mathrm{Re}\,\zeta) + i(\mathrm{Im}\,\zeta), \tag{13.6}$$

$$\mathrm{Re}\,\zeta = (\zeta + J_p(\eta, \eta)\zeta)/2, \ \mathrm{Im}\,\zeta = (\zeta - J_p(\eta, \eta)\zeta)/(2i), \tag{13.7}$$

where $\mathrm{Re}\,\zeta$ and $\mathrm{Im}\,\zeta$ are uniquely determined by their $J_p(\eta, \eta)$-invariance and (13.6). By applying Lemma 13.1 to $\mathrm{Re}\,\zeta$ and $\mathrm{Im}\,\zeta$, we obtain Corollary.

Lemma 13.3. *Any $\zeta \in D(\Delta_\eta^{(1/2)-2\alpha})$ $(0 \leqq \alpha \leqq 1/2)$ has a decomposition*

$$\zeta = \zeta_{r+} - \zeta_{r-} + i(\zeta_{i+} - \zeta_{i-}) \tag{13.8}$$

such that $\zeta_{\tau\sigma} \in V_\eta^\alpha$ $(\tau = r, i, \sigma = \pm)$. This decomposition is unique if we impose the following condition.

$$s^M(\zeta_{\tau+}) \perp s^M(\zeta_{\tau-}) \quad \text{if} \quad \alpha \geqq 1/4, \tag{13.9}$$

$$s^{M'}(\zeta_{\tau+}) \perp s^{M'}(\zeta_{\tau-}) \quad \text{if} \quad \alpha \leqq 1/4. \tag{13.10}$$

Proof. First consider the case $1/2 \geqq \alpha \geqq 1/4$ and let $\zeta \in D(\Delta_\eta^{(1/2)-2\alpha})$. By Lemmas 6.4 and 10.1 (2), we may apply the proof of decomposition in Lemma 13.1 $\zeta \in \mathcal{L}_p(M, \eta) = L_p(M, \eta)$ for $p = (2\alpha)^{-1}$. Since $J_p(\eta, \eta)$ coincides with $J_{1/(2p)}^{(\eta)} = J_\alpha^{(\eta)}$ on $D(\Delta_\eta^{(1/2)-2\alpha})$ by Lemma 11.4, and since the range of $J_\alpha^{(\eta)}$ is again in $D(\Delta_\eta^{(1/2)-2\alpha})$ $(J_\alpha^{(\eta)} = \Delta_\eta^{2\alpha-(1/2)} J_\eta)$, both $\mathrm{Re}\,\zeta$ and $\mathrm{Im}\,\zeta$ are in $D(\Delta_\eta^{(1/2)-2\alpha})$. If $\zeta \in D(\Delta_\eta^{(1/2)-2\alpha})$ is $J_\alpha^{(\eta)}$-invariant, then $\zeta = u\Delta_{\phi,\eta}^{1/p}\eta$ with $\eta \in D(\Delta_{\phi,\eta}^{1/p})$ by Lemmas 10.1 (2) and 10.3 and $u^* = u$, $\phi_u = \phi$ as in the proof of Lemma 13.1. In the same proof, $\Delta_{\phi_\pm,\eta}^{1/p} = \Delta_{\phi,\eta}^{1/p} s(\phi_\pm) \supset s(\phi_\pm) \Delta_{\phi,\eta}^{1/p}$ and hence $\eta \in D(\Delta_{\phi_\pm,\eta}^{1/p})$. Therefore $\zeta_{\tau\pm}$ in Lemma 13.1 belongs to V_η^α. The uniqueness of the decomposition is a special case of Lemma 13.1.

Next consider the case $1/4 \geqq \alpha \geqq 0$. If $\zeta \in D(\Delta_\eta^{(1/2)-2\alpha})$, then $J\zeta \in D(\Delta_\eta^{2\alpha-(1/2)})$ due to $J\Delta_\eta = \Delta_\eta^{-1} J$. We can apply the above proof for $\alpha' = (1/2) - \alpha$ $(2\alpha - (1/2) = (1/2) - 2\alpha')$ and obtain a decomposition

$$J\zeta = \zeta'_{r+} - \zeta'_{r-} + i(\zeta'_{i+} - \zeta'_{i-}) \tag{13.11}$$

with $s^M(\zeta'_{\tau+}) \perp s^M(\zeta'_{\tau-})$. Therefore we obtain the decomposition (13.8) satisfying (13.10) with $\zeta_{\tau\sigma} = J\,\zeta'_{\tau\sigma}$ due to $J\,V_\eta^{\alpha'} = V_\eta^\alpha$ (Theorem 3 (4) in [2]) and $s^{M'}(J\,\xi) = j\,(s^M(\xi))$. Conversely, the decomposition (13.8) satisfying (13.10) implies the decomposition (13.11) satisfying $s^M(\zeta'_{\tau+}) \perp s^M(\zeta'_{\tau-})$ with $\zeta'_{\tau\sigma} = J\zeta_{\tau\sigma}$ and hence the uniqueness of decomposition for the present case follows from the same for the first case.

§ 14. Proof of Theorems

Theorem 1. (1) For $2 \leqq p \leqq \infty$, $L_p(M, \eta)$ is a Banach space by Lemma 6.1 (3) and is an M-module by Lemma 4.4. For $1 \leqq p < 2$, $\|\cdot\|_p^{(\eta)}$ is a norm by Lemma 10.5 and $L_p(M, \eta)$ is a Banach space by definition. In either case $L_p(M, \eta)$ is an M-module by Lemma 12.3.

(2) and (3): $\langle\zeta, \zeta'\rangle_{(\eta)}$ is a continuous sesquilinear form on $L_p(M, \eta) \times L_{p'}(M, \eta)$ for $p^{-1} + p'^{-1} = 1$ satisfying (1.6) by Lemmas 10.4 and 10.5 for $1 < p < \infty$ and by (5.12), Lemma 5.1 and Lemma 5.4 for $p = 1$ or ∞. It coincides with (ζ, ζ') in H whenever ζ and ζ' are in H by Lemma 10.4 for $1 < p < \infty$ and by Remark 5.7 for $p = 1$ or ∞. Since $H \cap L_p(M, \eta)$ is either whole $L_p(M, \eta)$ or a dense subset (Lemma 10.5), $\langle\zeta, \zeta'\rangle_{(\eta)}$ can be obtained as the unique continuous extension of (ζ, ζ'). By Corollary 8.3, $L_p(M, \eta)$ is reflexive for $2 \leqq p < \infty$ and by Lemma 10.5 $L_{p'}(M, \eta) = L_p(M, \eta)^*$ for $2 \leqq p < \infty$ and $(p')^{-1} + p^{-1} = 1$. By Lemmas 5.1 and 5.4, $L_1(M, \eta)^* = (M_*)^* = M = L_\infty(M, \eta)$.

(4) By Lemma 8.1.

Theorem 2. By Lemmas 11.1, 11.2 and 11.5 where (1.9) is used as a definition.

Theorem 3. (1) (3) and (4): By Lemmas 4.1 (1), 4.3 and 3.5 for $2 \leqq p < \infty$, Lemmas 10.1, 10.5 and 10.6 for $1 < p \leqq 2$, by Lemma 5.1, Lemma 5.2 and polar decomposition of $x \in M$ for $p = \infty$ and by Lemma 5.5 and Remark 5.7 for $p = 1$. Note that (1.18) is given by (1.22).

(2) By Lemmas 11.1 and 11.2.

Theorem 4. By Lemmas 5.1 and 5.4.

Lemma A. Proved in Appendix A.

Theorem 5. By Lemma 12.1. (Note that (1.30) is obtained by repeated application of Lemma 12.1 (4) and the equality $\|x\|=\|x\|_\infty^{(\eta)}$ for $x\in\mathcal{L}_{1,0}^*(M,\eta)$.

Theorem 6. By Corollary 13.2.

Theorem 7. (1) The first half is by Lemmas 3.5 and 3.6, except for $|T|\eta\in V_\eta^0$ for T affiliated with M, which follows from the duality of $V_\eta^{1/2}$ and V_η^0 along with $(y^*y\eta, |T|\eta)=(y\eta, |T|y\eta)\geqq 0$ for $y\in M'$. For the second half, we use $J\zeta\in V_\eta^{\alpha'}$ if $\alpha'+\alpha=1/2$ and $\zeta\in V_\eta^\alpha$ (Theorem 3 (4) of [2]) and apply the first half to $J\zeta$ to obtain $J\zeta=u|J\zeta|_{\alpha'}$ and hence $\zeta=u'|\zeta|'_\alpha$ with $u'=j(u)\in M'$, $|\zeta|'_\alpha=J|J\zeta|_{\alpha'}\in V_\eta^\alpha$ and $u'u'^*=j(uu^*)=j(s^M(J\zeta))=s^{M'}(\zeta)$.

(2) The first half follows from polar decomposition $\zeta=u|\zeta|_\alpha$ in $L_{1/(2\alpha)}(M,\eta)$, which implies $|\zeta|_\alpha=u^*\zeta$ due to $u^*u=s^M(|\zeta|_\alpha)$, and therefore $|\zeta|_\alpha\in V_\eta^\alpha$ due to $\zeta\in H$, Lemma 12.3 (hence $|\zeta|_\alpha\in H$) and Theorem 1 (5). The second half is obtained by applying the first half to $J\zeta$ to obtain $J\zeta=u|J\zeta|_\alpha$ $(\alpha=(1/2)-\alpha')$ and hence $\zeta=u'|\zeta|_{\alpha'}$ as in (1).

(3) Any $\zeta\in V_\eta^\alpha\subset L_p^+(M,\eta)$ $(p=(2\alpha)^{-1})$ is of the form $\Delta_{\phi,\eta}^{1/p}\eta$ for a $\phi\in M_*^+$ by Lemmas 10.6 and 10.3. If $\zeta\in V_\eta^\alpha$ is of the form $\zeta=\Delta_{\phi,\eta}^{1/p}\eta$, then $\Delta_{\phi,\eta}^{1/p}\in\mathcal{L}_p(M,\eta)$ coincides with ζ (in $L_{p'}(M,\eta)^*$) by Lemma 2.2, hence the uniqueness of ϕ for a given ζ.

(4) By Lemma 13.3.

Corollary. Any $\phi\in M_*^+$ has a vector representative $\zeta\in H$, to which we apply the second half of Theorem 7 (2) to obtain $\zeta=u'|\zeta|_\alpha$ with $|\zeta|_\alpha\in V_\eta^\alpha$ for any $0\leqq\alpha\leqq 1/4$. Since $u'^*u'|\zeta|_\alpha=|\zeta|_\alpha$, the vector states by $|\zeta|_\alpha$ and by ζ is the same and are ϕ. Conversely, any two representative ζ_1 and ζ_2 of the same ϕ are related by $\zeta_1=u'\zeta_2$ where u' is a partial isometry in M' satisfying $u'u'^*=s^{M'}(\zeta_1)$, $u'^*u'=s^{M'}(\zeta_2)$. Hence by the uniqueness of polar decomposition in Theorem 7 (2), we obtain the uniqueness if ζ_1 and ζ_2 are in V_η^α.

§ 15. Discussion

The L_p-space $L_p(M)$ of Haagerup ([18]) is defined as: the set of all τ-measurable operator $\tilde{T}$ affiliated with N satisfying $\theta_t(\tilde{T}) = e^{-t/p}\tilde{T}$, where $N = M \times_\sigma \boldsymbol{R}$ is the crossed product of M with the modular action σ_t of $\boldsymbol{R}$ induced by ω_η, θ_t is the dual action, and τ is the canonical trace on N. The spatial L_p-spaces of Hilsum $L_p(M, \omega_\eta(J \cdot J))$ (see [12]) consists of operators $T = u\Delta_{\phi,\eta}^{p/1}$ with the norm $\|T\| = \phi(1)^{1/p}$. Our $\mathcal{L}_p(M, \eta)$ is seen to be the same as $L_p(M, \omega_\eta(J \cdot J))$.

We note that Hilsum's theory uses the L_p-spaces of Haagerup through an isomorphism and Haagerup's construction of L_p-spaces goes through the crossed product of M with the modular action. In contrast, our construction is directly on the Hilbert space H (without using trace anywhere) and reveals a close relation between the positive part of L_p-spaces and the positive cones V_η^α associated with the von Neumann algebra M. Another advantage of our method is that the linear structure of L_p-spaces is clear from its construction in contrast to the discussion of Hilsum where it is discovered by finding an isomorphism with the L_p-spaces of Haagerup. Our discussion of positive cones is closely related to recent results of Kosaki [16], [17].

Appendix A. *N* point Analytic Function

In this section we give the proof of Lemma A in Section 1.

Lemma A. 1. *Let $\phi_j \in M_*^+$ and $x_j \in M$ $(j = 0, \cdots, n)$. Let $\xi = \xi(\phi_0)$ be the representative vector of ϕ_0 (in $\mathcal{P}_\eta^\natural$). Then*

$$\zeta(z) = x_0 \Delta_{\phi_1,\xi}^{z_1} \cdots x_{n-1} \Delta_{\phi_n,\xi}^{z_n} x_n \xi \tag{A. 1}$$

is defined for $z = (z_1, \cdots, z_n) \in I_{1/2}^{(n)}$ (in the sense that ξ is in the domain of the product of operators in front), holomorphic in the interior of $I_{1/2}^{(n)}$ and strongly continuous on $I_{1/2}^{(n)}$ with the bound

$$\|\zeta(z)\| \leqq \left(\prod_{j=0}^{n} \|x_j\| \|\phi_j\|^{\mathrm{Re}\, z_j}\right) \tag{A. 2}$$

where $\|\phi_j\| = \phi_j(1)$, $\|\phi_0\| = \|\xi\|^2$, $z_0 \equiv (1/2) - \sum_{j=1}^{n} z_j$ and $I_a^{(n)}$ is defined by (1.21) where 1 is to be replaced by $a \geqq 0$.

Proof. The tube domain $I_a^{(n)}$ has the following distinguished boundaries corresponding to extremal points of its base:

(A. 3) $\partial_0 I_a^{(n)} = \{z\colon \operatorname{Re} z_j = 0,\ j=1, \cdots, n\}$,

(A. 4) $\partial_k I_a^{(n)} = \{z\colon \operatorname{Re} z_j = 0\ (j \neq k),\ \operatorname{Re} z_k = a\},\ k=1, \cdots, n$.

The expression (A. 1) is well-defined and (A. 2) holds on $\partial_0 I_a^{(n)}$ obviously and on $\partial_k I_a^{(n)}$ $(k=1, \cdots, n)$ due to the following formulas:

(A. 5) $\zeta(z) = x_0 \Delta_{\phi_1,\xi}^{it_1} \cdots x_{k-1} \zeta_k$,

(A. 6) $\zeta_k = \Delta_{\phi_k,\xi}^{1/2} y_k \xi = J s^M(\xi) y_k^* \xi(\phi_k)$,

(A. 7) $y_k = u_k \sigma_{t_k}^{\eta}(x_k u_{k+1} \cdots u_{n-1} \sigma_{t_{n-1}}^{\eta}(x_{n-1} u_n \sigma_{t_n}^{\eta}(x_n)) \cdots) w_k$,

(A. 8a) $\Delta_{\phi_j,\xi}^{it_j} \Delta_{\eta,\xi}^{-it_j} = u_j s^{M'}(\xi),\quad u_j = (D\phi_j\colon D\eta)_{t_j} \in M$,

(A. 8b) $\Delta_{\eta,\xi}^{it} x \Delta_{\eta,\xi}^{-it} = \sigma_t^{\eta}(x) s^{M'}(\xi)\quad (x \in M)$,

(A. 8c) $w_k = (D\eta : D\xi)_t,\quad t = t_k + \cdots + t_n$,

(A. 9)
$$\begin{aligned}&(\Delta_{\phi_k,\xi}^{it_k} \Delta_{\eta,\xi}^{-it_k}) \Delta_{\eta,\xi}^{it_k}(x_k \cdots (x_{n-1}(\Delta_{\phi_n,\xi}^{it_n} \Delta_{\eta,\xi}^{-it_n})\\ &\qquad \Delta_{\eta,\xi}^{it_n} x_n \Delta_{\eta,\xi}^{-it_n}) \cdots) \Delta_{\eta,\xi}^{-it_k}(\Delta_{\eta,\xi}^{it} \Delta_{\xi,\xi}^{-it})\\ &= y_k s^{M'}(\xi).\end{aligned}$$

Here η is any faithful normal semifinite weight, σ_t^{η} is its modular automorphism, the formula (A. 6) is due to (C. 1, 3, 4 and 12), the formula (A. 8a) due to (C. 5), the formula (A. 8b) due to Theorem C1 (β1), $\xi(\phi_k)$ is the unique vector representative of ϕ_k in $\mathcal{P}_\eta^\natural$ and the rest is a straightforward computation.

Therefore, if the expression (A. 1) is defined for $z \in I_{1/2}^{(n)}$, holomorphic in the interior of $I_{1/2}^{(n)}$ and weakly continuous on $I_{1/2}^{(n)}$, then (A. 2) follows by the generalized three line theorem for several complex variables (Theorem 2. 1 in [3]) applied to

(A. 10) $\|\zeta(z)\| = \sup\{|(\zeta, \zeta_1)|\colon \|\zeta_1\| \leqq 1\}$.

To show that ξ is in the domain of the operator in (A. 1) as well as holomorphy and weak continuity, we use mathematical induction on n. The case $n=1$ is known due to $x\xi \in D(\Delta_{\phi,\xi}^{1/2})$. Assume the assertion for n. Let $z = (w, z_1, \cdots, z_n)$ be in $I_{1/2}^{(n+1)}$, $\phi \in M_*^+$ and $\zeta_1 \in D(\Delta_{\phi,\xi}^{1/2})$. We consider the function

(A. 11) $G(z) = (x_0 \Delta_{\phi_1,\xi}^{z_1} \cdots \Delta_{\phi_n,\xi}^{z_n} x_n \xi,\ \Delta_{\phi,\xi}^{\bar{w}} \zeta_1)$, which is holomorphic in the interior of $I_{1/2}^{(n+1)}$ and continuous on $I_{1/2}^{(n+1)}$ (by holomorphy and strong continuity of $\Delta_{\phi,\xi}^{w}\zeta_1$ as well as by inductive assumption) with the bound

$$\text{(A. 12)} \qquad |G(z)| \leqq \|\zeta_1\| \|x_0\| \|\phi\|^{\mathrm{Re}\, w} \|\phi_0\|^{\mathrm{Re}\, z_0 - \mathrm{Re}\, w} \prod_{j=1}^{n} (\|x_j\| \|\phi_j\|^{\mathrm{Re}\, z_j})$$

due to the generalized three line theorem and estimates at distinguished boundaries similar to (A. 5) and (A. 6). Hence $G(z)$ is a continuous conjugate linear functional of ζ_1 and there exists $\tilde{\zeta} \in H$ such that $G(z) = (\tilde{\zeta}, \zeta_1)$. Hence $\zeta(z)$ is in the domain of $\Delta_{\phi,\eta}^{w}$ (hence of $x\Delta_{\phi,\eta}^{w}$ for $x \in M$) if $(w, z_1, \cdots, z_n)$ is in $I_{1/2}^{(n+1)}$. Due to the uniform bound on $\tilde{\zeta} = \Delta_{\phi,\eta}^{w}\zeta(z)$ given by (A. 12), this also shows holomorphy of $\tilde{\zeta}$ as a function of $(w, z_1, \cdots, z_n)$ in the interior of $I_{1/2}^{(n+1)}$ as well as weak continuity on $I_{1/2}^{(n+1)}$.

The strong continuity can be proved again by induction on n. Step from n to $n+1$ is as follows. We use the formula

$$\text{(A. 13)} \qquad \Delta_{\phi,\xi}^{(1/2)-\Sigma z_j}\zeta = J_{\phi,\xi}^{*} x_n^{*} \Delta_{\phi_n,\xi(\phi)}^{\bar{z}_n} \cdots x_1^{*} \Delta_{\phi_1,\xi(\phi)}^{\bar{z}_1} x^{*} \xi(\phi),$$

$$\text{(A. 13a)} \qquad \zeta = x \Delta_{\phi_1,\xi}^{z_1} x_1 \cdots \Delta_{\phi_n,\xi}^{z_n} x_n \xi,$$

which is obtained for pure imaginary z's from the formula (A. 6) (with an appropriate change of notation such as $\phi_k \to \phi$, $k+1 \to 1$, $t \to 0$, $w_k \to 1$, $t_k \to t = -\sum t_j$, and $u_k \to u = (D\phi : D\eta)_t$) by using the first formula of (A. 8a) and the formula (A. 8b) (both depending on ξ only through $s^{M'}(\xi)$) with a change $\xi \to \xi(\phi)$ for replacing $\sigma_{t_j}^{\eta}(\cdot) u_j^{*} s^{M'}(\xi(\phi))$ in y^{*} by $\Delta_{\eta,\xi(\phi)}^{it_j} \Delta_{\phi_j,\xi(\phi)}^{-it_j}$ where $s^{M'}(\xi(\phi))$ is to be supplied from $\xi(\phi)$ by $s^{M'}(\xi(\phi))\xi(\phi) = \xi(\phi)$ and commutativity of $s^{M'}(\xi(\phi))$ with $x_l \in M$ and $\Delta_{\phi_l,\xi(\phi)}^{-it_l}$ and for a similar replacement of $\sigma_t^{\eta}(\cdot) u^{*}$, and hence holds for $z \in I_{1/2}^{(n)}$ by analytic continuation and weak continuity (with a help of edge of wedge theorem as applied to the difference of two sides compared with analytic function 0). For $0 \leqq \mathrm{Re}\, z_0 \leqq (1/2) - \mathrm{Re} \sum z_j \equiv w_0$, we have

$$\text{(A. 14)} \qquad \Delta_{\phi,\xi}^{z_0}\zeta = \{\Delta_{\phi,\xi}^{z_0}(1 + \Delta_{\phi,\xi}^{w_0})^{-1}\} (1 + \Delta_{\phi,\xi}^{w_0})\zeta$$

with ζ given by (A. 13a). The first factor on the right hand side is strongly continuous with norm $\leqq 1$ and the rest is strongly continuous by inductive assumption and (A. 13). Therefore we have strong continuity for n.

Lemma $\tilde{\mathrm{A}}$. For the sake of convenience in the proof, we replace η in Lemma A by $\xi=\xi(\phi_0)$ $(\phi_0\in M_*^+)$ everywhere and call it Lemma $\tilde{\mathrm{A}}$. (Namely we drop the assumption that ω_η is faithful. Since we shall use another faithful normal semifinite weight η as an auxiliary tool in the proof, we introduce $\phi_0\in M_*^+$ instead. In our application in the present paper, we need only the faithful case.) In the following proof, η in the statement of Lemma A is understood to be replaced by ξ whenever equations or statements in Lemma A are quoted. In addition $x_j=x'_j s^M(\xi)x''_j$ in (1.27).

Proof of Lemma $\tilde{\mathrm{A}}$. Let the right hand side of (1.22) be F_j. By the holomorphy, strong continuity and boundedness of (A.1) proved above, we see that F_j is holomorphic in the interior of the domain I_j defined by (1.23) and (1.24), continuous on I_j and bounded as in (1.25). Within I_j, F_j depends on z'_j and z''_j only through their sum z_j and $F_j=F_{j+1}$ on $I_j\cap I_{j+1}$, both of which are seen by transposing operators from one member of the inner product to the other. Since $z\in I_j$ for all j if $0\leqq\sum \operatorname{Re} z_j\leqq 1/2$, we have single function $F(z)$ satisfying (1.22), (i), (ii) and (iii).

We use notation (1.26) and let the right hand side of (1.27) be G_j where $x_j=x'_j s^M(\xi)x''_j$ in addition to replacement of η by ξ. If $z_k=it_k$ $(t_k\in\boldsymbol{R})$ for $k\neq j$ and $z_j=1+it_j$ $(t_j\in\boldsymbol{R})$, we have

$$\text{(A.15)}\qquad \begin{aligned} F(z) &= (\Delta_{\phi_j,\xi}^{1/2} y_1\xi,\ \Delta_{\phi_j,\xi}^{1/2} y_2\xi) = (s^M(\xi)y_2^*\xi(\phi_j),\ y_1^*\xi(\phi_j)) \\ &= (s^M(\xi)y_3 s^M(\xi)y_2^*\xi(\phi_j),\ x'^*_j\xi(\phi_j)) \\ &= (\Delta_{\phi_j,\xi}^{1/2} x'_j\xi,\ \Delta_{\phi_j,\xi}^{1/2}(y_3y_2^*)^*\xi) \\ &= G_j(z), \end{aligned}$$

where $y_1=x'_j s^M(\xi)y_3$,

$$\text{(A.16)}\qquad x''_j\Delta_{\phi_{j+1},\xi}^{it_{j+1}}x_{j+1}\cdots\Delta_{\phi_n,\xi}^{it_n}x_n\Delta_{\xi,\xi}^{-i(t_{j+1}+\cdots+t_n)}=s^{M'}(\xi)y_3,$$

$$\text{(A.16a)}\qquad y_3=x''_j u_{j+1}\sigma_{t_{j+1}}^\eta(x_{j+1}\cdots u_{n-1}\sigma_{t_{n-1}}^\eta(x_{n-1}u_n\sigma_{t_n}^\eta(x_n))\cdots)w_j\in M,$$

$$\text{(A.17)}\qquad \Delta_{\phi_j,\xi}^{-it_j}x_{j-1}^*\cdots\Delta_{\phi_1,\xi}^{-it_1}x_0^*\Delta_{\xi,\xi}^{i(t_1+\cdots+t_j)}=s^{M'}(\xi)y_2$$

$$\text{(A.17a)}\qquad y_2=v_j^*\sigma_{-t_j}^\eta(x_{j-1}^*\cdots v_2^*\sigma_{-t_2}^\eta(x_1^*v_1^*\sigma_{-t_1}^\eta(x_0^*))\cdots)w'^*_j\in M,$$

$$\text{(A.18)}\qquad u_k=(D\phi_k:D\eta)_{t_k}\in M,\ v_k=(D\eta:D\phi_k)_{-t_k}\in M,$$

(A. 19) $w_j=(D\eta:D\xi)_{t_{j+1}+\cdots+t_n},\ w'_j=(D\xi:D\eta)_{-t_1-\cdots-t_j},$

(A. 20) $s^{M'}(\xi)y_3y_2^*=x''_j\Delta_{\phi_{j+1},\xi}^{it_{j+1}}x_{j+1}\cdots x_n\Delta_{\xi,\xi}^{-i\Sigma t_k}x_0\cdots x_{j-1}\Delta_{\phi_j,\xi}^{it_j}$

and η is any faithful normal semifinite weight for the purpose of computation. By using continuity of F and G_j and edge of wedge theorem (for $F-G_j$ on one side and 0 on the other), we have $F(z)=G_j(z)$ as an analytic function and hence $F(z)=G_j(z)$ for $z\in I_1^{(n)}$ by continuity. This proves (iv).

In passing we note the following: Using the third member of (A. 15), we have

(A. 21) $F(z)=(y_1y_2^*\xi_j,\xi_j),\ \xi_j\equiv\xi(\phi_j),$

(A. 22)
$$\begin{aligned} y_1y_2^*s^{M'}(\xi_j) &= x_ju_{j+1}\sigma_{t_{j+1}}^{\eta}(x_{j+1}\cdots u_n\sigma_{t_n}^{\eta}(x_n)\cdots)w_jw'_j\cdots \\ &\quad\cdots v_js^{M'}(\xi_j) \\ &= x_j\Delta_{\phi_{j+1},\xi_j}^{it_{j+1}}x_{j+1}\cdots x_n\Delta_{\xi,\xi_j}^{-i(t_1+\cdots+t_n)} \\ &\quad\times x_0\Delta_{\phi_1,\xi_j}^{it_1}\cdots x_{j-1}\Delta_{\xi_j,\xi_j}^{it_j}. \end{aligned}$$

Therefore, denoting $z_0=1-\sum_{k=1}^{n}z_k$, we have

(A. 23) $F(z)=\omega_{\xi_j}(x_j\Delta_{\phi_{j+1},\xi_j}^{z_{j+1}}\cdots x_n\Delta_{\phi_0,\xi_j}^{z_0}x_0\cdots\Delta_{\phi_{j-1},\xi_j}^{z_{j-1}}x_{j-1})$

for $z_k=it_k$ $(k\neq j)$ and $z_j=1+it_j$. Since

(A. 24) $z^{(j)}\equiv(z_{j+1},\cdots,z_n,z_0,z_1,\cdots,z_{j-1})\in I_1^{(n)}$

if and only if $z\in I_1^{(n)}$, (A. 23) holds for $z\in I_1^{(n)}$ again by edge of wedge theorem.

If $\sum z_j=1$, then $z_0=0$ in (A. 23) and hence information on ϕ_0 vanishes from the right hand side of (A. 23). Therefore $F(z)$ is independent of ϕ_0 if $\sum z_j=1$, which shows (vii).

(v) is immediate from definition.

To prove (vi), let us write $F(z;\nu)$ instead of $F(z)$ where ν indicates x's and ϕ's together. Suppose that $\nu_\alpha\to\nu$ in some sense and for any K,

(A. 25) $\sup\{|F(z;\nu)-F(z;\nu_\alpha)|:\ z\in\partial I_1^{(n)},\ \sum|z_j|^2<K\}\to0.$

Then we have

(A. 26) $\sup\{|e^{z^2}(F(z;\nu)-F(z;\nu_\alpha))|:\ z\in\partial I_1^{(n)}\}\to0$

where $z^2=\sum z_j^2$. By the maximum principle for analytic functions, we have

(A. 27) $$\sup\{|e^{z^2}(F(z;\nu)-F(z;\nu_\alpha))|:\ z\in I_1^{(n)}\}\to 0,$$

namely

(A. 28) $$\sup\{|F(z;\nu)-F(z;\nu_\alpha)|:\ z\in I_1^{(n)},\ \sum|z_j|^2<K\}\to 0.$$

We use the original definition (such as (1. 26)) on $\partial_0 I_1^{(n)}$ and the right hand side of (A. 23) on $\partial_j I_1^{(n)}$ where all z_j are pure imaginary. If x's are restricted to a bounded set, all operators in sight are therefore uniformly bounded.

For boundary values, we have the following type of estimates when $\|y_k\|\leqq\overline{K}$ and $\|y_{k\alpha}\|\leqq\overline{K}$:

(A. 29) $$\|y_k\Delta_k^{it_k}\cdots y_1\Delta_1^{it_1}y_0\xi-y_{k\alpha}\Delta_{k\alpha}^{it_k}\cdots y_{1\alpha}\Delta_{1\alpha}^{it_1}y_{0\alpha}\xi\|$$
$$\leqq\sum_{j=0}^{k}\overline{K}^{k-j}\|(y_j-y_{j\alpha})\zeta_j\|+\sum_{j=1}^{k}\overline{K}^{k-j+1}\|(\Delta_j^{it_j}-\Delta_{j\alpha}^{it_j})\zeta_j'\|$$

(A. 30) $$\zeta_j=\Delta_j^{it_j}y_{j-1}\cdots y_1\Delta_1^{it_1}y_0\xi,\quad \zeta_j'=y_{j-1}\zeta_{j-1}.$$

We note that $\Delta_{\tilde{\phi}_\alpha}^{it}\to\Delta_{\tilde{\phi}}^{it}$ uniformly in $t\in[-T,T]$ if $\|\tilde{\phi}_\alpha-\tilde{\phi}\|\to 0$ (see proof of Theorem C. 1 for $\tilde{\phi}$) by proof of Theorem 10 of [2], and hence the same holds for $\Delta_{\phi_{j\alpha},\xi_\alpha}^{it}\to\Delta_{\phi_j,\xi}^{it}$. To deal with uniformity in t's appearing in ζ and ζ', we use a finite number of t_{1l} such that any $t_1\in[-T,T]$ has some t_{1l} such that $\|\Delta_1^{it_1}y_0\xi-\Delta_1^{it_{1l}}y_0\xi\|<\varepsilon$. After replacing t_1 by t_{1l}, we proceed with approximation of $t_2\in[-T,T]$ by a finite number of points. We can then approximate (A. 30) for $t_k\in[-T,T]$ by a finite number of vectors (up to ε) and hence the convergence of (A. 29) is uniform over $(t_1,\cdots,t_k)$ provided that t's are bounded. This proves (vi).

Appendix B. Partially Isometric Radon-Nikodym Cocycles

Let ϕ_0 and ϕ be normal semifinite weights on M, ϕ_0 be faithful, the relative modular operator Δ_{ϕ,ϕ_0} be defined by

(B. 1) $$\Delta_{\phi,\phi_0}=S_{\phi,\phi_0}^*\overline{S_{\phi,\phi_0}},$$

(B. 2) $$S_{\phi,\phi_0}\eta_{\phi_0}(x)=\eta_\phi(x^*)\quad(x\in N_{\phi_0}\cap N_\phi^*)$$

where N_ψ is the set of all $x\in M$ with $\psi(x^*x)<\infty$ and $\eta_\psi(x)$ is the vector in the GNS construction associated with ψ satisfying $(\eta_\psi(x_1), \eta_\psi(x_2)) = \psi(x_2^* x_1)$, and

(B. 3) $$u_t = (D\phi : D\phi_0)_t = \Delta_{\phi,\phi_0}^{it} \Delta_{\phi_0}^{-it}$$

where $\Delta_{\phi,\phi_0}^{it}$ is defined as the sum of 0 on $(1-s(\phi))H$ and the usual power of Δ_{ϕ,ϕ_0} on $s(\phi)H$.

Theorem B. 1. *u_t defined by* (B. 3) *is a continuous one-parameter family of partial isometries in M satisfying the cocycle condition*

(B. 4) $$u_t \sigma_t^{\phi_0}(u_s) = u_{t+s}$$

and the support properties

(B. 5) $$u_t u_t^* = P, \quad u_t^* u_t = \sigma_t^{\phi_0}(P)$$

for a projection P in M $(P = s(\phi))$. Conversely, for any continuous one-parameter family of partial isometries u_t in M satisfying (B. 4) *and* (B. 5), *there exists a unique normal semifinite weight ϕ on M such that* (B. 3) *holds. (Then $P = s(\phi)$.)*

Proof. (B. 4) and (B. 5) follow from the definition (B. 3). The fact that u_t belongs to M follows from the Tomita-Takesaki theory for 2×2 matrices over M restricted by the projection $\begin{pmatrix} 1 & 0 \\ 0 & s(\phi) \end{pmatrix}$. (Theorem C. 1 ($\gamma$).)

To prove the converse, let ϕ_1 be a faithful normal semifinite weight on $(1-P)M(1-P)$ and $\phi_2(x) = \phi_1((1-P)x(1-P))$. Let $v_t = (D\phi_2 : D\phi_0)_t$ and $w_t = u_t + v_t$. Then w_t is a unitary σ^{ϕ_0}-cocycle and hence there exists a faithful normal semifinite weight ψ on M such that $w_t = (D\psi : D\phi_0)_t$. (Theorem 1. 2. 4 of [9].)

From (B. 5) for u_t (by assumption) and for v_t with P replaced by $(1-P)$ (by the first half of Theorem), we have

(B. 6) $$\sigma_t^\psi(P) = w_t \sigma_t^{\phi_0}(P) w_t^* = u_t \sigma_t^{\phi_0}(P) u_t^* = P,$$

namely P commutes with ψ and hence with $\Delta_{\psi,\phi_0}^{it}$. Furthermore $(1-P)w_t = v_t$ and $Pw_t = u_t$. From the first equation explicitly written in terms of Δ's, we obtain

(B. 7) $$(1-P)\Delta_{\psi,\phi_0}^{it} = \Delta_{\psi,\phi_0}^{it}(1-P) = \Delta_{\phi_2,\phi_0}^{it}.$$

Since $\Delta_{\phi_2,\phi_0}^{it}\Delta_{\psi,\phi_0}^{-it}$ is independent of ϕ_0, we may replace ϕ_0 by ψ and we obtain

(B. 8) $$\Delta_{\psi}^{z}(1-P)=\Delta_{\phi_2,\psi}^{z}$$

for $z=it$ and hence for all z. By taking $z=1/2$, we see that $x\in N_{\psi}\cap N_{\psi}^{*}$ (which is equivalent to $\eta_{\psi}(x)\in D(\Delta_{\psi}^{1/2})$) implies $\eta_{\psi}((1-P)x)\in D(\Delta_{\psi}^{1/2})$ (because P commutes with ψ), hence $x^{*}\in N_{\phi_2}$ and

(B. 9) $$\psi((1-P)xx^{*}(1-P))=\phi_2(xx^{*})$$

(due to $\|\Delta_{\phi_2,\psi}^{1/2}\eta_{\psi}(x)\|^2=\|\eta_{\phi_2}(x^{*})\|^2$). For any $x_0\in M_{+}$, there exists an increasing net $x_{\alpha}\in M_{+}$ such that $\psi(x_{\alpha})<\infty$ (i.e. $x_{\alpha}^{1/2}\in N_{\psi}\cap N_{\psi}^{*}$) and $x_0=\sup x_{\alpha}$ due to semifiniteness of ψ. If $x_0\in(M_{1-P})_{+}$ in particular, then $x_{\alpha}\in(M_{1-P})_{+}$ and hence, by (B. 9),

(B. 10) $$\phi_2(x_0)=\sup\phi_2(x_{\alpha})=\sup\psi(x_{\alpha})=\psi(x_0).$$

Since the support of ϕ_2 is $1-P$, we have for any $x\in M_{+}$

(B. 11) $$\phi_2(x)=\phi_2((1-P)x(1-P))=\psi((1-P)x(1-P)).$$

Since P commutes with ψ,

(B. 12) $$\phi(x)\equiv\psi(x)-\phi_2(x)=\psi(PxP),\quad x\in M_{+}$$

is a normal semifinite weight on M and Theorem C. 1 implies

(B. 13) $$\Delta_{\psi,\phi_0}^{it}=\Delta_{\phi,\phi_0}^{it}+\Delta_{\phi_2,\phi_0}^{it},$$

with $s(\Delta_{\phi,\phi_0})=P$. Hence $u_t=(D\phi:D\phi_0)_t$.

The uniqueness of ϕ for given u_t follows, for example, from the uniqueness of faithful $\psi=\phi+\phi_2$ for a given $(D\psi:D\phi_0)$. (Theorem 1. 2. 4 in [9].)

Appendix C. Relative Modular Operators

We shall use standard results on Tomita-Takesaki Theory [23]. Let ϕ be a normal semifinite weight, $s(\phi)$ be its support projection, N_{ϕ} be the set of all $x\in M$ satisfying $\phi(x^{*}x)<\infty$, N_{ϕ}^{*} be the set of x^{*} with $x\in N_{\phi}$, M_{ϕ} be the linear hull of $N_{\phi}^{*}N_{\phi}$ (to which ϕ is extended as a finite-valued linear functional), σ_t^{ϕ} be modular automorphisms of $s(\phi)Ms(\phi)$ determined by ϕ, N_{ϕ}^{0} be the set of all $x\in s(\phi)N_{\phi}s(\phi)$ such that x is σ_t^ϕ-entire analytic, $\sigma_z^\phi(x)\in N_\phi$ for all z and $\eta_\phi(\sigma_z^\phi(x))=\Delta_{\eta_\phi}^{iz}\eta_\phi(x)$ (which is dense in $s(\phi)Ms(\phi)$ due to σ^ϕ-invariance of $s(\phi)N_\phi s(\phi)$), $\eta_\phi(x)$ for $x\in N_\phi$ be a GNS-representation vector satisfying $(\eta_\phi(x_1),\eta_\phi(x_2))=\phi(x_2^*x_1)$ and $x_2\eta_\phi(x_1)=\eta_\phi(x_2x_1)$ and $\mathcal{P}_\phi^\natural$ be the closure of the vectors $\Delta_{\phi,\phi}^{1/4}\eta_\phi(x)$ with $x\in N_\phi\cap M_+$ ($\Delta_{\phi,\phi}$ is defined by (C. 2) below), which is a proper convex cone. (Any $\eta(x)$, $x\in N_\phi\cap M_+$, is in the domain of $\Delta_{\phi,\phi}^{1/2}$ and hence of $\Delta_{\phi,\phi}^{1/4}$.) In the following all $\eta_\phi(x)$ is in one Hilbert space H on which M has a standard representation (although all discussions can be carried through even if $\eta_\phi(x)$ for different ϕ are in different Hilbert spaces). For each ϕ, there are many choices of the map $x\in M\mapsto\eta_\phi(x)\in H$ and we shall deal with all possibilities for η_ϕ. Hence we denote the set of all η_ϕ by $\mathcal{H}$ and we introduce a notation $\phi=\omega_\eta$ for any given $\eta=\eta_\phi$. We also write N_η for N_ϕ and σ^η for σ^ϕ if $\eta=\eta_\phi$. If $\phi\in M_*$, then $\eta_\phi(x)=x\eta$ for a vector $\eta=\eta_\phi(1)$ and the vector state ω_η is ϕ. The closure of $\eta(N_\phi)$ is M-invariant and the corresponding projection operator ($\in M'$) is denoted by $s^{M'}(\eta)$, while $s(\omega_\eta)$ ($\in M$) is denoted by $s^M(\eta)$. If $\omega_\eta\in M_*^+$, then they are M'- and M-support of the vector $\eta=\eta(1)$.

For η_1 and η_2 in $\mathcal{H}$, we define

$$S_{\eta_1,\eta_2}(\eta_2(x)+(1-s^{M'}(\eta_2))\zeta)=s^M(\eta_2)\eta_1(x^*) \tag{C. 1}$$

for all $x\in N_{\eta_2}\cap N_{\eta_1}^*$ and $\zeta\in H$. If $\eta_2(x)+(1-s^{M'}(\eta_2))\zeta=0$, then each term (having mutually orthogonal M'-support) vanishes and hence $xs^M(\eta_2)=0$, which implies the vanishing of the right hand side. Therefore S_{η_1,η_2} is a well-defined, conjugate linear operator. We shall see below that it is closable and has a dense domain. By polar decomposition of the closure $\bar{S}_{\eta_1,\eta_2}$, we obtain the relative modular operator

$$\Delta_{\eta_1,\eta_2}=S_{\eta_1,\eta_2}^*\bar{S}_{\eta_1,\eta_2} \tag{C. 2}$$

and the associated partially isometric conjugate linear operator J_{η_1,η_2}:

$$\bar{S}_{\eta_1,\eta_2}=J_{\eta_1,\eta_2}\Delta_{\eta_1,\eta_2}^{1/2}, \tag{C. 3}$$

and $J_{\eta_1,\eta_2}^*J_{\eta_1,\eta_2}=s^M(\eta_1)s^{M'}(\eta_2)$, $J_{\eta_1,\eta_2}J_{\eta_1,\eta_2}^*=s^M(\eta_2)s^{M'}(\eta_1)$.

In the following, A^z for a positive selfadjoint operator A denotes the sum of 0 on $(1-s(A))H$ and usual power $A^z=\exp(z\log A)$ on $s(A)H$.

Theorem C. 1. (α) *S_{η_1,η_2} is a densely defined closable operator with its support $s(\bar{S}_{\eta_1,\eta_2})=s^M(\eta_1)s^{M'}(\eta_2)$ and the closure of its range $s(S^*_{\eta_1,\eta_2})=s^M(\eta_2)s^{M'}(\eta_1)$.*

(β) *Δ_{η_1,η_2} is a positive selfadjoint operator with the support*

$$s(\Delta_{\eta_1,\eta_2})=s^M(\eta_1)s^{M'}(\eta_2), \tag{C. 4}$$

depends on η_1 only through the weight ω_{η_1} and have the following properties.

($\beta 1$) $\Delta^{it}_{\eta_1,\eta_2}x\Delta^{-it}_{\eta_1,\eta_2}=\sigma^{\eta_1}_t(x)s^{M'}(\eta_2)$ *for* $x\in s^M(\eta_1)Ms^M(\eta_1)$

($\beta 2$) *There exists a continuous one-parameter family of elements $(D\phi_1:D\phi_2)_t$ of $s(\phi_1)Ms(\phi_2)$ depending only on $\phi_j=\omega_{\eta_j}$ $(j=1,2)$ and satisfying*

$$\Delta^{it}_{\eta_1,\eta}\Delta^{-it}_{\eta_2,\eta}=(D\phi_1:D\phi_2)_t s^{M'}(\eta) \tag{C. 5}$$

for all η.

($\beta 3$) *If $x\in N_{\eta_2}\cap N^*_{\eta_1}$ and $0\leqq\alpha\leqq 1/2$, then*

$$\|\Delta^{1/2}_{\eta_1,\eta_2}\eta_2(x)\|=\|s^M(\eta_2)\eta_1(x^*)\|, \tag{C. 6}$$

$$\|\Delta^{\alpha}_{\eta_1,\eta_2}\eta_2(x)\|\leqq\|s^M(\eta_1)\eta_2(x)\|^{1-2\alpha}\|s^M(\eta_2)\eta_1(x^*)\|^{2\alpha}. \tag{C. 7}$$

($\beta 4$) *If u is a partial isometry in M such that u^*u commutes with ω_{η_1} (in particular, if $u^*u\geqq s^M(\eta_1)$), then*

$$u\Delta^{it}_{\eta_1,\eta_2}u^*=\Delta^{it}_{u\circ\eta_1,\eta_2} \tag{C. 8}$$

where we define $(u\circ\eta_1)(x)=\eta_1(xu)$ and hence $\omega_{u\circ\eta_1}(x)=\omega_{\eta_1}(u^*xu)$.

($\beta 5$) $J_{\eta_1,\eta_2}\Delta_{\eta_1,\eta_2}J_{\eta_2,\eta_1}=\Delta^{-1}_{\eta_2,\eta_1}$.

(γ) *If $s(\phi_1)$ and $s(\phi_2)$ commute, $(D\phi_1:D\phi_2)_t$ is a partial isometry with initial and final projections $\sigma^{\phi_2}_t(s(\phi_1)s(\phi_2))$ and $\sigma^{\phi_1}_t(s(\phi_1)s(\phi_2))$, having the following properties.*

($\gamma 1$) $(D\phi_1:D\phi_2)^*_t=(D\phi_2:D\phi_1)_t$.

($\gamma 2$) *If $s(\phi_2)\geqq s(\phi_1)$, then*

$$(D\phi_1:D\phi_2)_s\sigma^{\phi_2}_s((D\phi_1:D\phi_2)_t)=(D\phi_1:D\phi_2)_{s+t}. \tag{C. 9}$$

($\gamma 3$) *If $[s(\phi_1),s(\phi_2)]=0$ and $x\in M_{s(\phi_1)s(\phi_2)}$, then*

$$(D\phi_1:D\phi_2)_t\sigma^{\phi_2}_t(x)(D\phi_1:D\phi_2)^*_t=\sigma^{\phi_1}_t(x). \tag{C. 10}$$

(γ4) *If either* $s(\phi_2)\geqq s(\phi_1)$ *or* $s(\phi_2)\geqq s(\phi_3)$, *then*

$$(D\phi_1 : D\phi_2)_t (D\phi_2 : D\phi_3)_t = (D\phi_1 : D\phi_3)_t . \tag{C.11}$$

(δ) *If* ω_{η_0} *is faithful, then* $\mathcal{P}^{\natural}_{\eta_0}$ *is a selfdual convex cone having the following properties.*

(δ1) *Any normal semifinite weight* ϕ *has a unique* $\eta=\eta^0_\phi$ *such that* $\phi=\omega_\eta$ *and* $\mathcal{P}^{\natural}_{\eta}$ *is contained in (if* ϕ *is faithful, identical with)* $\mathcal{P}^{\natural}_{\eta_0}$.

(δ2) *Any other* $\eta=\eta_\phi$ *satisfying* $\phi=\omega_\eta$ *is related to* η^0_ϕ *by* $\eta_\phi(x) = u'\eta^0_\phi(x)$ *with a unique partially isometric operator* u' *in* M' *having initial and final projections* $u'^*u' = s^{M'}(\eta^0_\phi)$ *and* $u'u'^* = s^{M'}(\eta_\phi)$.

(δ3) *Any* $\phi\in M_*^+$ *has the unique representative vector* $\xi(\phi) = \eta^0_\phi(1)$ *in* $\mathcal{P}^{\natural}_{\eta_0}$.

(δ4) *For* $\xi,\zeta\in\mathcal{P}^{\natural}_{\eta_0}$, $\|\xi-\zeta\|^2\leqq\|\omega_\xi-\omega_\zeta\|$.

(ε) *Let* $J=J_{\eta_0,\eta_0}$ *for a fixed* η_0 *for which* ω_{η_0} *is faithful. Then* J *is a conjugate unitary involution,* $j(x)=JxJ\in M'$ *for any* $x\in M$, $j(y) = JyJ\in M$ *for any* $y\in M'$ *and* J *has the following properties.*

(ε1) *Let* $\eta_j=u'_j\eta^0_{\phi_j}$ *for* $\phi_j=\omega_{\eta_j}$ $(j=1,2)$ *where* $\eta^0_{\phi_j}$ *is given by* (δ1). *Then*

$$J_{\eta_1,\eta_2} = u'_1 J u'^*_2 , \tag{C.12}$$

$$s(J_{\eta_1,\eta_2}) = s^M(\eta_1)\, s^{M'}(\eta_2),\quad s(J^*_{\eta_1,\eta_2}) = s^M(\eta_2)\, s^{M'}(\eta_1), \tag{C.13}$$

$$J^*_{\eta_1,\eta_2} = J_{\eta_2,\eta_1} . \tag{C.14}$$

(ε2) *If* u' *is a partial isometry in* M' *and* $u'^*u'\geqq s^{M'}(\eta_2)$, *then*

$$u'\Delta^{it}_{\eta_1,\eta_2}u'^* = \Delta^{it}_{\eta_1,u'\eta_2} \tag{C.15}$$

(ε3) *For* $x\in N_{\eta_0}$, $j(x)\eta_0(x)\in\mathcal{P}^{\natural}_{\eta_0}$,

(ε4) *The set of* $j(x)\eta_0(x)$, $x\in N^0_{\eta_0}$, *is dense in* $\mathcal{P}^{\natural}_{\eta_0}$.

(ε5) *Any* $\xi\in\mathcal{P}^{\natural}_{\eta_0}$ *satisfies* $J\xi=\xi$.

(ε6) *For any* $\xi\in\mathcal{P}^{\natural}_{\eta_0}$ *and* $x\in M$, $xj(x)\xi\in\mathcal{P}^{\natural}_{\eta_0}$.

Remark. In the situation of (δ1), $\mathcal{P}^{\natural}_{\eta} = s(\phi)j(s(\phi))\,\mathcal{P}^{\natural}_{\eta_0}$.

Proof. (α) Let M_n be the $n\times n$ full matrix algebra with matrix units u_{ij} and M'_n be its commutant with matrix units v_{ij} acting on a

Hilbert space of n^2 dimension with an orthonormal basis e_{ij} satisfying $u_{kl}e_{ij}=\delta_{li}e_{kj}$ and $v_{kl}e_{ij}=\delta_{lj}e_{ik}$. Let $M_{(n)}\equiv M\otimes M_n$ and consider $\widetilde{M}=(M_{(n)})_E$ with $E=\sum s^M(\eta_i)\otimes u_{ii}$, a faithful normal semifinite weight $\widetilde{\phi}$ on $\widetilde{M}$ given in terms of weights ϕ_i on M by

$$\widetilde{\phi}\left(\sum x_{ij}\otimes u_{ij}\right)=\sum \phi_i(x_{ii}) \tag{C. 16}$$

and its GNS representation given in terms of η_j satisfying $\phi_j=\omega_{\eta_j}$ by

$$\widetilde{\eta}\left(\sum x_{ij}\otimes u_{ij}\right)=\sum \eta_j(x_{ij})\otimes e_{ij} \tag{C. 17}$$

on $\widetilde{H}\equiv\sum s^M(\eta_i)s^{M'}(\eta_j)H\otimes e_{ij}$, where $x_{ij}\in s^M(\eta_i)N_{\phi_j}s^M(\eta_j)$. (Note that $\eta_j(s^M(\eta_i)N_{\phi_j}s^M(\eta_j))=s^M(\eta_i)\eta_j(N_{\phi_j})$ is dense in $s^M(\eta_i)s^{M'}(\eta_j)H$.)

By the Tomita-Takesaki Theory, we have

$$S_{\widetilde{\eta}}\widetilde{\eta}(x)=\widetilde{\eta}(x^*)\quad (x\in N_{\widetilde{\phi}}\cap N_{\widetilde{\phi}}^*), \tag{C. 18}$$

$$\Delta_{\widetilde{\eta}}=S_{\widetilde{\eta}}^*\bar{S}_{\widetilde{\eta}},\quad \bar{S}_{\widetilde{\eta}}=J_{\widetilde{\eta}}\Delta_{\widetilde{\eta}}^{1/2}. \tag{C. 19}$$

Since $1\otimes u_{ii}$ commutes with $\widetilde{\phi}$, it commutes with $\Delta_{\widetilde{\eta}}$,

$$(1\otimes u_{ii})\widetilde{J}(1\otimes u_{jj})\widetilde{\eta}(x)=\widetilde{\eta}((1\otimes u_{ii})x(1\otimes u_{jj}))\equiv\eta_j(x_{ij})\otimes e_{ij} \tag{C. 20}$$

for $x\in N_{\widetilde{\phi}}\cap N_{\widetilde{\phi}}^*$ (which implies $\widetilde{\eta}(\widetilde{x}_{ij})\in D(S_{\widetilde{\eta}})$ and hence $\widetilde{x}_{ij}\in N_{\widetilde{\phi}}\cap N_{\widetilde{\phi}}^*$ for $\widetilde{x}_{ij}=(1\otimes u_{ii})x(1\otimes u_{jj})=x_{ij}\otimes u_{ij}$), and vectors (C. 20) are dense in $(1\otimes u_{ii})\widetilde{J}(1\otimes u_{jj})\widetilde{H}$. From (C. 20), $\widetilde{J}(1\otimes u_{jj})=1\otimes v_{jj}$ and $\eta_j(x_{ij})$ with $x_{ij}\otimes u_{ij}\in N_{\widetilde{\phi}}\cap N_{\widetilde{\phi}}^*$ are dense in $s^M(\eta_i)s^{M'}(\eta_j)H$. Since $\widetilde{\eta}(x_{ij}\otimes u_{ij})=\eta_j(x_{ij}\otimes e_{ij}$ and $\widetilde{\eta}((x_{ij}\otimes u_{ij})^*)=\eta_i(x_{ij}^*)\otimes e_{ji}$, we have $x_{ij}\in N_{\phi_j}\cap N_{\phi_i}^*$. Therefore $\eta_2(x)$ with $x\in N_{\eta_2}\cap N_{\eta_1}^*\cap s(\phi_1)Ms(\phi_2)$ are dense in $s^M(\eta_1)s^{M'}(\eta_2)H$.

Since $N_{\phi_2}=N_{\phi_2}s(\phi_2)+M(1-s(\phi_2))$ and N_{ϕ_2} is a left ideal in M, $(1-s(\phi_1))N_{\phi_2}s(\phi_2)$ is in N_{ϕ_2}. It is also in $N_{\phi_1}^*$ because $N_{\phi_1}\supset M(1-s(\phi_1))$. Thus $(1-s(\phi_1))N_{\phi_2}s(\phi_2)$ is in $N_{\phi_2}\cap N_{\phi_1}^*$ and $\eta_2((1-s(\phi_1))N_{\phi_2}s(\phi_2))=(1-s(\phi_1))\eta_2(N_{\phi_2})$ is dense in $(1-s(\phi_1))s^{M'}(\eta_2)H$. Combining with the above, we see that $\eta_2(N_{\phi_2}\cap N_{\phi_1}^*)$ is dense in $s^{M'}(\eta_2)H$ and S_{η_1,η_2} is densely defined.

By definition, S_{η_1,η_2} is 0 on $(1-s(\phi_1))\eta_2(N_{\phi_2})$ and on $(1-s^{M'}(\eta_2))H$. It is closable on $s(\phi_1)\eta_2(N_{\phi_2})$ and its closure has zero kernel on $s(\phi_1)s^{M'}(\eta_2)H$ because of the same known property for $S_{\widetilde{\eta}}$. Therefore S_{η_1,η_2} is closable and the support of its closure is $s^M(\eta_1)s^{M'}(\eta_2)$.

By interchanging ϕ_1 and ϕ_2, $\eta_1(x^*)$ with $x^*\in N_{\phi_1}\cap N_{\phi_2}^*$ is dense in

$s^{M'}(\eta_1)H$. Therefore $s(S^*_{\eta_1,\eta_2})=s^{M'}(\eta_1)s^M(\eta_2)$.

(β) By definition and the above result for $s(\bar{S}_{\eta_1,\eta_2})$, Δ_{η_1,η_2} is a positive selfadjoint operator with its support given by (C. 4). If η_1' and η_1 give the same weight ϕ_1, then define u' as the sum of 0 on $(1-s^{M'}(\eta_1))H$ and the closure of $u'\eta_1(x)=\eta_1'(x)$, $x\in N_{\phi_1}$, on $s^{M'}(\eta_1)H$. Then u' is partially isometric and commutes with $x_1\in M$ due to

$$u'x_1\eta_1(x)=u'\eta_1(x_1x)=\eta_1'(x_1x)=x_1\eta_1'(x)=x_1u'\eta_1(x). \tag{C. 21}$$

Therefore $u'\in M'$. We obtain from (C. 1)

$$S_{\eta_1',\eta_2}=u'S_{\eta_1,\eta_2}. \tag{C. 22}$$

Since

$$\begin{aligned} s(J^*_{\eta_1,\eta_2}) &= s(S^*_{\eta_1,\eta_2})=s^M(\eta_2)s^{M'}(\eta_1) \\ &\leqq s^{M'}(\eta_1)=u'^*u', \end{aligned} \tag{C. 23}$$

$u'J_{\eta_1,\eta_2}$ is partially isometric and we obtain

$$\Delta_{\eta_1',\eta_2}=\Delta_{\eta_1,\eta_2} \tag{C. 24}$$

as well as

$$J_{\eta_1',\eta_2}=u'J_{\eta_1,\eta_2}. \tag{C. 25}$$

(β1) By comparing definition of $\Delta_{\tilde\eta}$ and Δ_{η_i,η_j}, we have

$$\Delta_{\tilde\eta}=\sum\Delta_{\eta_i,\eta_j}\otimes u_{ii}v_{jj}. \tag{C. 26}$$

For $x_{11}\in s^M(\eta_1)Ms^M(\eta_1)$, we have

$$\Delta_{\tilde\eta}^{it}(x_{11}\otimes u_{11})\Delta_{\tilde\eta}^{-it}=\sigma_t^{\tilde\eta}(x_{11}\otimes u_{11}). \tag{C. 27}$$

Since $1\otimes u_{11}$ commutes with $\tilde\phi$ and the restriction of $\tilde\phi$ to $\tilde M_{1\otimes u_{11}}=M_{s(\phi_1)}\otimes u_{11}\sim M_{s(\phi_1)}$ is ϕ_1, the characterization of modular automorphisms by KMS condition implies

$$\sigma_t^{\tilde\eta}(x_{11}\otimes u_{11})=\sigma_t^{\phi_1}(x_{11})\otimes u_{11}. \tag{C. 28}$$

By restricting (C. 27) to $(1\otimes u_{11}v_{22})\tilde H$, we obtain ($\beta$1) on $s^M(\eta_1)s^{M'}(\eta_2)H$ and hence on H (due to the support property of two sides of the equation).

(β2) Since $1\otimes u_{kk}$ ($k=i$ or j) is $\sigma^{\tilde\phi}$-invariant, $u_{ii}u_{ij}=u_{ij}u_{jj}=u_{ij}$ implies

(C. 29) $$\sigma_t^{\bar{\eta}}(s(\phi_i)s(\phi_j)\otimes u_{ij}) = U_t\otimes u_{ij}$$

for some $U_t\in s(\phi_i)Ms(\phi_j)$. By (C. 26),

(C. 30) $$U_t\otimes u_{ij}=\sum_k \Delta^{it}_{\eta_i,\eta_k}\Delta^{-it}_{\eta_j,\eta_k}\otimes u_{ij}v_{kk}\,.$$

By multiplying $1\otimes v_{kk}$, we obtain (C. 5) with $U_t=(D\phi_i: D\phi_j)_t$ and $\eta=\eta_k$ on $s^M(\eta_j)s^{M'}(\eta_k)H$ and hence on H. Since $\Delta_{\eta_i,\eta}$ depends on η_i only through ϕ_i, U_t depends only on ϕ_i and ϕ_j.

(β3) The first equation follows from

(C. 31) $$\|s^M(\eta_2)\eta_1(x^*)\| = \|S_{\phi_1,\phi_2}\eta_2(x)\| = \|\Delta^{1/2}_{\phi_1,\phi_2}\eta_2(x)\|.$$

Then (C. 7) is due to Hölder inequality. (Note that $\Delta^0_{\eta_1,\eta_2}=s^M(\eta_1)s^{M'}(\eta_2)$ and $s^{M'}(\eta_2)\eta_2(x)=\eta_2(x)$.)

(β4) We have

(C. 32) $$\begin{aligned} &S_{u\circ\eta_1,\eta_2}(\eta_2(x)+(1-s^{M'}(\eta_2))\zeta)\\ &=s^M(\eta_2)\eta_1(x^*u)=S_{\eta_1,\eta_2}(\eta_2(u^*x)+(1-s^{M'}(\eta_2))u^*\zeta)\\ &=S_{\eta_1,\eta_2}u^*(\eta_2(x)+(1-s^{M'}(\eta_2))\zeta)\,. \end{aligned}$$

Therefore

(C. 33) $$S_{u\circ\eta_1,\eta_2}=S_{\eta_1,\eta_2}u^*\,.$$

Hence we have

(C. 34) $$\Delta_{u\circ\eta_1,\eta_2}=u\Delta_{\eta_1,\eta_2}u^*\,.$$

Since $s(\Delta_{\eta_1,\eta_2})$ commute with the initial projection u^*u of u, we have (C. 8).

(β5) From definition $\bar{S}_{\bar{\eta}}=J_{\bar{\eta}}\Delta^{1/2}_{\bar{\eta}}$, we have

(C. 35) $$J_{\bar{\eta}}(\sum\zeta_{ij}\otimes e_{ij})=\sum(J_{\eta_i,\eta_j}\zeta_{ij})\otimes e_{ji}\,.$$

Hence (β5) follows from $J_{\bar{\eta}}\Delta_{\bar{\eta}}J_{\bar{\eta}}=\Delta^{-1}_{\bar{\eta}}$.

(γ) If $s(\phi_1)$ and $s(\phi_2)$ commute, then $s(\phi_1)s(\phi_2)\otimes u_{12}$ is partially isometric and hence (C. 29) shows that $(D\phi_1: D\phi_2)_t$ is a partially isometry. Since $s(\phi_1)s(\phi_2)\otimes u_{ii}$ ($i=2$ and 1) are initial and final projections for $s(\phi_1)s(\phi_2)\otimes u_{12}$, $\sigma_t^{\bar{\eta}}(s(\phi_1)s(\phi_2)\otimes u_{ii})=\sigma_t^{\phi_i}(s(\phi_1)s(\phi_2))\otimes u_{ii}$ ($i=2$ and 1) are those for $U_t\otimes u_{12}$. Hence (γ) holds.

(γ1)$\sim$(γ4) follows from (C. 5).

(ε) This is a standard result of the Tomita-Takesaki theory.

(ε1) (C. 13) follows from (α). (C. 35) and $J_{\bar\eta}^2=1$ implies $J_{\eta_1,\eta_2}J_{\eta_2,\eta_1}=s^M(\eta_2)s^{M'}(\eta_1)$ on $s^M(\eta_2)s^{M'}(\eta_1)H$ and hence on H. Multiplying J_{η_1,η_2}^*, we obtain (C. 14).

Due to $J_{\bar\eta}(s(\phi_1)s(\phi_3)\otimes u_{13})J_{\bar\eta}\in\widetilde{M}'$, we have

(C. 36) $$w=J_{\eta_1,\eta_2}s(\phi_1)s(\phi_3)J_{\eta_2,\eta_3}=J_{\eta_1,\eta_2}J_{\eta_2,\eta_3}\in s^{M'}(\eta_1)M's^{M'}(\eta_3)$$

on $s^M(\eta_2)H$ and w is independent of η_2. Hence

(C. 37) $$J_{\eta_1,\eta_2}s^M(\eta_3)=wJ_{\eta_3,\eta_2}$$

on H. Taking $\eta_2=\eta_3=\eta_0$, we obtain

(C. 38) $$J_{\eta_1,\eta_0}=w_1J$$

with a partial isometry w_1 in $s^{M'}(\phi_1)M'$. By taking adjoint, we have $J_{\eta_0,\eta_1}=Jw_1^*$. Taking $\eta_3=\eta_0$ in (C. 37), we then obtain

(C. 39) $$J_{\eta_1,\eta_2}=w_1J_{\eta_0,\eta_2}=w_1Jw_2^*\,.$$

We have $w_1w_1^*=J_{\eta_1,\eta_0}J_{\eta_1,\eta_0}^*=s^{M'}(\eta_1)$. Hence $\eta_{10}\equiv w_1^*\eta_1$ satisfies $\omega_{\eta_{10}}=\omega_{\eta_1}$, $\eta_1=w_1\eta_{10}$ and $J_{\eta_{10},\eta_0}=w_1^*J_{\eta_1,\eta_0}=w_1^*w_1J=s^{M'}(\eta_{10})J$. After proof of ($\delta$1), we prove that $J_{\eta_{10}',\eta_0}=s^{M'}(\eta_1)J$ for $\eta_{10}'=\eta_{\phi_1}^0$ given by (δ1). Let u' be given by (δ2) satisfying $\eta_{10}=u'\eta_{10}'$. By definition, we have $S_{\eta_{10},\eta_0}=u'S_{\eta_{10}',\eta_0}$ and hence $J_{\eta_{10},\eta_0}=u'J_{\eta_{10}',\eta_0}$. This implies $u'=s^{M'}(\eta_{10})$ and hence $\eta_{10}'=u'^*\eta_{10}=\eta_{10}$. Therefore $w_1=u_1'$. Similarly $w_2=u_2'$. Thus we obtain (C. 12).

(ε2) By replacing u' by $u's^{M'}(\eta_2)$, we may assume that $u'^*u'=s^{M'}(\eta_2)$. Then $s^{M'}(u'\eta_2)=u'u'^*$ and $s^M(u'\eta_2)=s^M(\eta_2)$ (due to $\omega_{u'\eta_2}=\omega_{\eta_2}$). Therefore

(C. 40) $$\begin{aligned}&S_{\eta_1,u'\eta_2}(u'\eta_2(x)+(1-s^{M'}(u'\eta_2))\zeta)\\&=s^M(u'\eta_2)\eta_1(x^*)=s^M(\eta_2)\eta_1(x^*)\\&=S_{\eta_1,\eta_2}(\eta_2(x)+(1-s^{M'}(\eta_2))u'^*\zeta)\\&=S_{\eta_1,\eta_2}u'^*(u'\eta_2(x)+(1-s^{M'}(u'\eta_2))\zeta).\end{aligned}$$

This implies

(C. 41) $$S_{\eta_1,u'\eta_2}=S_{\eta_1,\eta_2}u'^*$$

and hence (C. 15).

(ε3) If $x_n\in N_{\eta_0}^0\cap(N_{\eta_0}^0)^*=N_{\eta_0}^0$ (due to $x_n^*\in N_{\eta_0}$ and $\eta_0(x_n^*)=$

$J\eta_0(\sigma^{\eta_0}_{-i/2}(x_n))$ for $x_n \in N^0_{\eta_0}$), then

(C. 42) $$J\eta_0(x_n) = \Delta^{1/2}_{\eta_0}\eta_0(x_n^*) = \eta_0(\sigma^{\eta}_{-i/2}(x^*)).$$

Hence

(C. 43) $$\begin{aligned}\Delta^{1/4}_{\eta_0}\eta_0(x_n^* x_n) &= \eta_0(\sigma^{\eta_0}_{-i/2}((\sigma^{\eta_0}_{-i/4}(x_n^* x_n))^*)) \\ &= J\eta_0(\sigma^{\eta_0}_{-i/4}(x_n^*)\sigma^{\eta_0}_{-i/4}(x_n)) \\ &= j(\sigma^{\eta_0}_{-i/4}(x_n^*))J\eta_0(\sigma^{\eta_0}_{-i/4}(x_n)) \\ &= j(\sigma^{\eta_0}_{-i/4}(x_n^*))\eta_0(\sigma^{\eta_0}_{-i/4}(x_n^*)) \\ &= j(x_{n1})\eta_0(x_{n1})\end{aligned}$$

with $x_{n1} = \sigma^{\eta_0}_{-i/4}(x_n^*)$. Let $x \in N_{\eta_0}$ and

(C. 44) $$x_n = (n/\pi)^{1/2}\int \sigma^{\eta_0}_t(x^*)\exp(-n(t+(i/4))^2)\,dt.$$

Then x_n has the property mentioned above and

(C. 45) $$x_{n1} = \varepsilon^{\eta_0}_n(x) \equiv (n/\pi)^{1/2}\int \sigma^{\eta_0}_t(x)\exp(-nt^2)\,dt,$$

(C. 46) $$\eta_0(x_{n1}) = (n/\pi)^{1/2}\int \Delta^{it}_{\eta_0}\eta_0(x)\exp(-nt^2)\,dt,$$

which converges strongly to x and $\eta_0(x)$ respectively as $n \to \infty$. Therefore $j(x)\eta_0(x)$ for $x \in N_{\eta_0}$ is in $\mathcal{P}^{\natural}_{\eta_0}$ as the limit of $\eta_0(\sigma^{\eta_0}_{-i/4}(x_n^* x_n)) \in \mathcal{P}^{\natural}_{\eta_0}$.

(ε4) For $x \in N_{\eta_0} \cap M_+$,

(C. 47) $$\|\Delta^{1/4}_{\eta_0}\eta_0(x)\|^2 = (\eta_0(x), \Delta^{1/2}_{\eta_0}\eta_0(x)) = (\eta_0(x), J\eta_0(x)).$$

Hence $\eta_0(x_\alpha) \to \eta_0(x)$ implies $\Delta^{1/4}_{\eta_0}\eta_0(x_\alpha) \to \Delta^{1/4}_{\eta_0}\eta_0(x)$ for $x_\alpha \in N_{\eta_0} \cap M_+$.

Let $y \in N_{\eta_0}$. Then $\varepsilon^{\eta_0}_n(y) \in N^0_{\eta_0}$, $\eta_0(\varepsilon^{\eta_0}_n(y)) \to \eta_0(y)$, $\varepsilon^{\eta_0}_n(y)^* \to y^*$ and hence $\Delta^{1/4}_{\eta_0}\eta_0(\varepsilon^{\eta_0}_n(y)^*\varepsilon^{\eta_0}_n(y)) \to \Delta^{1/4}_{\eta_0}\eta_0(y^*y)$. In view of (C. 43), $\Delta^{1/4}_{\eta_0}(y^*y)$ is in the closure of the set of $j(x)\eta_0(x)$, $x \in N^0_{\eta_0}$.

Let $e_\alpha = e_\alpha^*$ be a uniformly bounded net in N_{η_0} tending to 1. By approximating $\Delta^{it}_{\eta_0}\xi$ (for a fixed vector ξ) in norm over a compact set of t by a finite number of $t = t_j$, we find that the net $\sigma^{\eta_0}_z(\varepsilon^{\eta_0}_n(e_\alpha))$ for any fixed z and n tends to 1 strongly. For $x \in N_{\eta_0} \cap M_+$, we set $y = x^{1/2}\varepsilon^{\eta_0}_n(e_\alpha)$. By the formula

(C. 48) $$\begin{aligned}\eta_0(xe) &= J\Delta^{1/2}_{\eta_0}\eta_0(e^*x^*) = j(\sigma^{\eta_0}_{-i/2}(e^*))J\Delta^{1/2}_{\eta_0}\eta_0(x^*) \\ &= j(\sigma^{\eta_0}_{-i/2}(e^*))\eta_0(x)\end{aligned}$$

for $e^*=e=\varepsilon_n(e_\alpha)$, we see that $\eta_0(y^*y)=ej(\sigma^{\eta_0}_{-i/2}(e^*))\eta_0(x)\to\eta_0(x)$. By (C. 47), $\Delta^{1/4}_{\eta_0}\eta_0(x)$ for $x\in N^0_{\eta_0}\cap M_+$ is in the closure of the set of $\Delta^{1/4}_{\eta_0}\eta_0(y^*y)$, $y\in N_{\eta_0}$ and hence of $j(x)\eta_0(x)$, $x\in N^0_{\eta_0}$.

(ε5) If $x\in N_{\eta_0}\cap M_+$, then $x\in N_{\eta_0}\cap N^*_{\eta_0}$ and

$$J\Delta^{1/2}_{\eta_0}\eta_0(x)=\eta_0(x). \tag{C. 49}$$

Since $\Delta^{1/4}_{\eta_0}J=J\Delta^{-1/4}_{\eta_0}$, we obtain J-invariance of $\Delta^{1/4}_{\eta_0}\eta_0(x)$.

(ε6) If $y\in N_{\eta_0}$, then (ε3) implies

$$xj(x)\,(j(y)\eta_0(y))=j(xy)\eta_0(xy)\in\mathcal{P}^\natural_{\eta_0} \tag{C. 50}$$

for any $x\in M$. By (ε4), the set of $j(y)\eta_0(y)$, $y\in N^0_{\eta_0}$ is already dense in $\mathcal{P}^\natural_{\eta_0}$. Hence $xj(x)\xi\in\mathcal{P}^\natural_{\eta_0}$ if $\xi\in\mathcal{P}^\natural_{\eta_0}$.

(δ) The rest of Theorem is proved in [22]. For sake of selfcontained exposition, we include here somewhat different proof. $\mathcal{P}^\natural_{\eta_0}$ is a convex cone by definition. For x, $y\in N^0_{\eta_0}$,

$$\begin{aligned} j(x)\eta_0(y) &= Jx\eta_0(\sigma^{\eta_0}_{-i/2}(y^*)) \\ &=\eta_0(\sigma^{\eta_0}_{-i/2}(\{x\sigma^{\eta_0}_{-i/2}(y^*)\}^*)) \\ &=yJ\eta_0(x) \end{aligned} \tag{C. 51}$$

by (C. 42). (The formula holds for any x, $y\in N_{\eta_0}$ through an approximation by $\varepsilon^{\eta_0}_n(x)$ and $\varepsilon^{\eta_0}_n(y)$.) Hence for $x_1, x_2\in N^0_{\eta_0}$,

$$\begin{aligned} (j(x_1)\eta_0(x_1),\ j(x_2)\eta_0(x_2)) &= (\eta_0(x_1),\ j(x_1^*x_2)\eta_0(x_2)) \\ &= (\eta_0(x_1),\ x_2J\eta_0(x_1^*x_2)) \\ &= (\eta_0(x_2^*x_1),\ \Delta^{1/2}_{\eta_0}\eta_0(x_2^*x_1))\geqq 0\,. \end{aligned} \tag{C. 52}$$

By (ε4), $(\xi_1,\xi_2)\geqq 0$ for any $\xi_1,\xi_2\in\mathcal{P}^\natural_{\eta_0}$, i.e.

$$(\mathcal{P}^\natural_{\eta_0})^*\supset\mathcal{P}^\natural_{\eta_0}\,. \tag{C. 53}$$

To prove the converse inclusion, let ζ satisfy $(\zeta,\zeta')\geqq 0$ for all $\zeta'\in\mathcal{P}^\natural_{\eta_0}$. Let $\xi\in\mathcal{P}^\natural_{\eta_0}$. By ($\varepsilon$5), $s^{M'}(\xi)=j(s^M(\xi))$. Let $e=s^M(\xi)j(s^M(\xi))$ and consider M_e on eH. Then ξ is a cyclic and separating vector for M_e on eH, $\Delta_{\xi,\xi}$ is the usual modular operator for ξ on eH (being 0 on $(1-e)H$) and hence the closure of $\Delta^{1/4}_\xi\eta_\xi(x)=\Delta^{1/4}_\xi x\xi=\Delta^{1/4}_\xi exe\xi$, $x\in M$ is $V^{1/4}_\xi=\mathcal{P}^\natural_\xi$ defined in [2]. Hence it is the closure of the set of $xj(x)\xi$, $x\in M_e$, which is a subset of $\mathcal{P}^\natural_{\eta_0}$ by (ε6) and hence contained in $e\mathcal{P}^\natural_{\eta_0}$, which in turn is contained in $\mathcal{P}_{\eta_0}^{\natural}$ by (ε6).

For $\zeta'\in\mathcal{P}_{\xi}^{\natural}\subset\mathcal{P}_{\eta_0}^{\natural}$, we have $(e\zeta,\zeta')=(\zeta,\zeta')\geqq 0$. By the selfduality of $\mathcal{P}_{\xi}^{\natural}$ in eH (Theorem 4 of [2]), we have $e\zeta\in\mathcal{P}_{\xi}^{\natural}\subset\mathcal{P}_{\eta_0}^{\natural}$. The rest is to find $\xi_\alpha\in\mathcal{P}_{\eta_0}^{\natural}$ such that $e_\alpha=s^M(\xi_\alpha)j(s^M(\xi_\alpha))$ (as a net) tends to 1.

By (ε3), $\xi=j(x)\eta_0(x)\in\mathcal{P}_{\eta_0}^{\natural}$. We shall show that $s^M(\xi)=s(x^*)$ for $x\in N_{\eta_0}^0$. Since $N_{\eta_0}^0$ is a dense subset in M, $\{s(x^*): x\in N_{\eta_0}\}$ with usual partial ordering of projections is a net tending to 1. Let $e\in M$ and $ej(x)\eta_0(x)=0$. Then $j(x_1)\eta_0(x_1)=0$ for $x_1=x^*ex\in N_{\eta_0}\cap M_+$. For $y\in N_{\eta_0}$, we have

$$0=(\eta_0(y),j(x_1)\eta_0(x_1))=(j(x_1^*)\eta_0(y),\eta_0(x_1))$$
$$=(yJ\eta_0(x_1^*),\eta_0(x_1))=(y\Delta_{\eta_0}^{1/2}\eta_0(x_1),\eta_0(x_1)). \tag{C. 54}$$

Taking a limit of a net $y=y_\alpha$ tending to 1, we may replace y by 1. Since Δ_{η_0} is positive definite, we have $\eta_0(x_1)=0$. Since η_0 is faithful, this implies $(ex)^*(ex)=x_1=0$. Hence $ex=0$, which shows $s^M(\xi)=s(x^*)$.

(δ1) First we prove the statement for faithful ϕ. There exists some η_1 with $\omega_{\eta_1}=\phi$. By the proof of (ε1), there exists a partial isometry $u'\in M'$ such that $J_{\eta_1,\eta_0}=u'J$. Since $1=s^M(\eta_0)=j(u'^*u')$, u' must be isometric. Let $\eta=(u')^*\eta_1$. Then $\omega_\eta=\phi$ and $J_{\eta,\eta_0}=J$ due to

$$S_{\eta,\eta_0}\eta_0(x)=(u')^*\eta_1(x^*)=(u')^*J_{\eta_1,\eta_0}\Delta_{\eta_1,\eta_0}^{1/2}\eta_0(x)$$
$$=J\Delta_{\eta_1,\eta_0}^{1/2}\eta_0(x). \tag{C. 55}$$

We can now use the formula (C. 51) and (ε4) for both η_0 and η with common j and J. For $x\in N_{\eta_0}$ and $y\in N_\eta$, we obtain

$$(j(x)\eta_0(x),j(y)\eta(y))=(j(y^*x)\eta_0(x),\eta(y))$$
$$=(xJ\eta_0(y^*x),\eta(y))=(J\eta_0(y^*x),\eta(x^*y))$$
$$=(J\eta(x^*y),\eta_0(y^*x))=(\Delta_{\eta_1,\eta_0}^{1/2}\eta_0(y^*x),\eta_0(y^*x))\geqq 0. \tag{C. 56}$$

Hence $\mathcal{P}_{\eta}^{\natural}\subset(\mathcal{P}_{\eta_0}^{\natural})^*=\mathcal{P}_{\eta_0}^{\natural}$ and $\mathcal{P}_{\eta_0}^{\natural}\subset(\mathcal{P}_{\eta}^{\natural})^*=\mathcal{P}_{\eta}^{\natural}$, i.e. $\mathcal{P}_{\eta}^{\natural}=\mathcal{P}_{\eta_0}^{\natural}$.

Now consider a general normal semifinite weight ϕ_1. Let ϕ_2 be a normal semifinite weight with support $s(\phi_2)=1-s(\phi_1)$. Then $\phi=\phi_1+\phi_2$ is faithful and $s(\phi_1)$ commutes with ϕ. Let η be as above and $\eta_1(x)=\eta(xs(\phi_1))$ for $x\in N_{\phi_1}$. Then $\omega_{\eta_1}=\phi_1$.

By $s(\Delta_{\eta_1})=s^M(\eta_1)s^{M'}(\eta_1)=s(\phi_1)j(s(\phi_1))\equiv e_1$, $\mathcal{P}_{\eta_1}^{\natural}$ is in e_1H and is generated by $\Delta_{\eta_1}^{1/4}\eta_1(x)=\Delta_{\eta_1}^{1/4}\eta_1(s(\phi_1)xs(\phi_1))$. The characterization of modular automorphisms by KMS condition shows that $\sigma_t^{\eta}(y)=\sigma_t^{\eta_1}(y)$ for $y\in s(\phi_1)Ms(\phi_1)$ (where $s(\phi_1)$ is σ^{η}-invariant) and hence $\Delta_{\eta}^{it}\eta(y)=\Delta_{\eta_1}^{it}\eta_1(y)$. Therefore $\mathcal{P}_{\eta_1}^{\natural}$ is generated by

$$\Delta_{\eta_1}^{1/4}\eta_1(x)=\Delta_{\eta}^{1/4}\eta(s(\phi_1)xs(\phi_1))\in\mathcal{P}_{\eta}^{\natural} \tag{C.57}$$

with $x\in N_{\eta_1}\cap M_+$, which shows $\mathcal{P}_{\eta_1}^{\natural}\subset\mathcal{P}_{\eta}^{\natural}$ for this η_1.

To prove the uniqueness, let η_1' be such that $\omega_{\eta_1'}=\phi_1$ and $\mathcal{P}_{\eta_1'}^{\natural}\subset\mathcal{P}_{\eta}^{\natural}$. Then there exists a partial isometry $u'\in M'$ such that $\eta_1'(x)=u'\eta_1(x)$ for $x\in N_{\phi_1}$ and $u'^*u'=s^{M'}(\eta_1)$. For any $x\in N_{\phi_1}^0\cap M_+$, $\xi=\eta_1(\sigma_{-i/4}^{\phi_1}(x))$ and $\xi'=u'\xi$. Then $\xi\in\mathcal{P}_{\eta_1}^{\natural}\subset\mathcal{P}_{\eta}^{\natural}$ and $\xi'\in\mathcal{P}_{\eta_1'}^{\natural}\subset\mathcal{P}_{\eta}^{\natural}$. We also have $\omega_{\xi}=\omega_{\xi'}$ and hence $s^M(\xi)=s^M(\xi')$, as well as $s^{M'}(\xi)=j(s^M(\xi))=s^{M'}(\xi')$. If we restrict our attention to M_e on eH with $e=s^M(\xi)s^{M'}(\xi)$, then $e\mathcal{P}_{\eta}^{\natural}$ is $\mathcal{P}_{\xi}^{\natural}$ for M_e and any normal state on M_e has a unique representative in $\mathcal{P}_{\xi}^{\natural}$. In particular $\xi=\xi'$. For $x\in N_{\phi_1}^0$, let $\sigma_{i/4}^{\phi_1}(x)=x_1-x_2+i(x_3-x_4)$ with $x_i\geqq 0$. Linear combination of the above result yields $\eta_1(y)=\eta_1'(y)$ for $y=\sigma_{-i/4}^{\phi_1}(\varepsilon_n^{\eta}(\sigma_{i/4}^{\phi_1}(x)))=\varepsilon_n^{\eta}(x)$ and hence for $y=x$ by taking $n\to\infty$. By substituting $x=\varepsilon_n^{\eta}(x_1)$ with $x_1\in N_{\phi_1}$ and taking $n\to\infty$, we obtain $\eta_1(x_1)=\eta_1'(x_1)$ for all $x_1\in N_{\phi_1}$, which shows the uniqueness.

(ε1, continued) We prove that $J_{\eta_1,\eta_0}=s^{M'}(\eta_1)J$ $(=Js^M(\phi_1))$ for η_1 given above. We have $J\Delta_{\eta,\eta_0}^{1/2}\eta_0(x)=\eta(x^*)$ for $x\in N_{\eta_0}\cap N_{\eta}^*$. Hence $J\Delta_{\eta,\eta_0}^{1/2}s(\phi_1)\eta_0(x)=\eta(x^*s(\phi_1))=\eta_1(x^*)$. Since $s(\phi_1)$ is σ_t^{η}-invariant, we obtain $j(s(\phi_1))J\Delta_{\eta,\eta_0}^{1/2}\eta_0(x)=\eta_1(x^*)$. Therefore $J_{\eta_1,\eta_0}=j(s(\phi_1))J$ (and $\Delta_{\eta_1,\eta_0}=s(\phi_1)\Delta_{\eta_1,\eta_0}$). Due to σ_t^{η}-invariance of $s(\phi_1)$, we have $\eta_1(x)=\eta(xs(\phi_1))=j(s(\phi_1))\eta(x)$. Since $\eta(N_{\eta})$ is dense in H, $s^{M'}(\eta_1)=j(s(\phi_1))$.

(δ2) has been shown in the proof of (β).

(δ3) is a special case of (δ1).

(δ4) Let $\phi=\omega_{\xi}+\omega_{\zeta}$. Then $\phi\in M_*$ has a unique vector representative $\xi(\phi)$ in $\mathcal{P}_{\eta_0}^{\natural}$ satisfying $J_{\xi(\phi),\xi(\phi)}=j(s(\phi))Jj(s(\phi))=eJ$ with $e=s(\phi)j(s(\phi))$ by (ε1). (Note that $s^{M'}(\xi(\phi))=j(s^M(\xi(\phi)))$.) If we restrict our attention to M_e on eH, then $\mathcal{P}_{\xi(\phi)}^{\natural}=e\mathcal{P}_{\eta_0}^{\natural}$ is $V_{\xi(\phi)}^{1/4}$ in [2] and the unique vector representative ξ and ζ (both in $e\mathcal{P}_{\eta_0}^{\natural}$ because $s^M(\xi)\leqq s(\phi)$ due to $\omega_{\xi}\leqq\phi$, $j(s^M(\xi))=s^{M'}(\xi)$ due to $\xi\in\mathcal{P}_{\eta_0}^{\natural}$, hence $e\xi=\xi$ and similarly $e\zeta=\zeta$) which satisfies

$$\|\xi-\zeta\|^2\leqq\|\omega_{\xi}^e-\omega_{\zeta}^e\| \tag{C.58}$$

where ω^e indicates a vector state on M_e. Since $e\xi=\xi$, we have $\omega_\xi(x) = \omega_\xi^e(exe)$ and the same for ζ. Therefore

$$\|\omega_\xi^e - \omega_\zeta^e\| = \|\omega_\xi - \omega_\zeta\| . \tag{C. 59}$$

Lemma C. 2. (1) *Let ϕ be a normal semifinite weight on M, $\xi\in H$, η be a cyclic and separating vector in H and u be a partial isometry in M satisfying $u^*u=s(\phi)$. Then*

$$J_{\xi,\eta}\Delta_{\xi,\eta}^{(1/2)+it}u\Delta_{\phi,\eta}^{-it}\eta = u^*\Delta_{\phi_u,\xi}^{it}\xi . \tag{C. 60}$$

(2) If $\xi\in D(\Delta_{\phi_u,\xi}^{\lambda})$ and $\eta\in D(\Delta_{\phi,\eta}^{\lambda})$ for $0\leqq\lambda\leqq 1/2$, then $u\Delta_{\phi,\eta}^{\lambda}\eta \in D(\Delta_{\xi,\eta}^{(1/2)-\lambda})$ and

$$J_{\xi,\eta}\Delta_{\xi,\eta}^{(1/2)-\lambda}u\Delta_{\phi,\eta}^{\lambda}\eta = u^*\Delta_{\phi_u,\xi}^{\lambda}\xi , \tag{C. 61}$$

where $\phi_u(x) = \phi(u^*xu)$.

Proof. (1) We have

$$\begin{aligned} J_{\xi,\eta}\Delta_{\xi,\eta}^{(1/2)+it}u\Delta_{\phi,\eta}^{-it}\eta &= S_{\xi,\eta}(\Delta_{\xi,\eta}^{it}\Delta_{\phi_u,\eta}^{-it})u\eta \\ &= u^*(\Delta_{\xi,\eta}^{it}\Delta_{\phi_u,\eta}^{-it})^*\xi = u^*(\Delta_{\phi_u,\eta}^{it}\Delta_{\xi,\eta}^{-it})s^{M'}(\xi)\xi \\ &= u^*\Delta_{\phi_u,\xi}^{it}\Delta_{\xi,\xi}^{-it}\xi = u^*\Delta_{\phi_u,\xi}^{it}\xi , \end{aligned} \tag{C. 62}$$

where we have used (C. 8) in the first equality,

$$w_t \equiv \Delta_{\xi,\eta}^{it}\Delta_{\phi_u,\eta}^{-it} = (D\omega_\xi : D\phi_u)_t \in M \tag{C. 63}$$

(due to (C. 5)) in the second equality, and the formula (C. 5) again in the fourth equality.

(2) For $z=it$ $(t\in\boldsymbol{R})$ and $\zeta\in D(\Delta_{\xi,\eta}^{(1/2)})$,

$$(\zeta, J_{\xi,\eta}^*u^*\Delta_{\phi_u,\xi}^{z}\xi) = (\Delta_{\xi,\eta}^{(1/2)-z}\zeta, u\Delta_{\phi,\eta}^{\bar z}\eta) \tag{C. 64}$$

holds due to (C. 60) $\equiv\zeta_1$ and the following computation.

$$\begin{aligned} (\zeta, J_{\xi,\eta}^*\zeta_1) &= (\zeta_1, J_{\xi,\eta}\zeta) = (J_{\xi,\eta}\zeta_2, J_{\xi,\eta}\zeta) \\ &= (s^M(\xi)\zeta, \zeta_2) = (\zeta, \zeta_2) \end{aligned} \tag{C. 65}$$

for $\zeta_2 = \Delta_{\xi,\eta}^{(1/2)+it}u\Delta_{\phi,\eta}^{-it}\eta$ satisfying $s^M(\xi)\zeta_2=\zeta_2$. Both sides of (C. 64) is holomorphic in $\{z\in\boldsymbol{C}: 0<\mathrm{Re}\, z<\lambda\}$ and continuous in the closure. Therefore (C. 64) holds for all $z\in\boldsymbol{C}$, $0\leqq\mathrm{Re}\, z\leqq\lambda$. Hence $u\Delta_{\phi,\eta}^{\lambda}\eta\in D(\Delta_{\xi,\eta}^{(1/2)-\lambda})$ and (C. 61) holds.

Lemma C. 3. *Let η, ϕ and ψ be normal semifinite weights satisfying $\phi\leqq\psi$. Then $D(\Delta_{\psi,\eta}^{\lambda})\subset D(\Delta_{\phi,\eta}^{\lambda})$ and*

$$\|\Delta_{\phi,\eta}^{\lambda}\zeta\|\leqq\|\Delta_{\psi,\eta}^{\lambda}\zeta\| \tag{C. 66}$$

for all $\zeta\in D(\Delta_{\psi,\eta}^{\lambda})$, where $0\leqq\lambda\leqq 1/2$.

Proof. For $\lambda=1/2$, $x\in N_{\eta}\cap N_{\psi}^{*}$ and $\zeta'\in(1-s^{M'}(\eta))H$, we have $x\in N_{\eta}\cap N_{\phi}^{*}$ (due to $N_{\phi}\supset N_{\psi}$) and

$$\begin{aligned}\|\Delta_{\psi,\eta}^{1/2}(\eta(x)+\zeta')\|^{2}&=\psi(|s^{M}(\eta)x^{*}|^{2})\\ &\geqq\phi(|s^{M}(\eta)x^{*}|^{2})=\|\Delta_{\phi,\eta}^{1/2}(\eta(x)+\zeta')\|^{2}.\end{aligned} \tag{C. 67}$$

Since the set of vectors $\eta(x)+\zeta'$ is a core for $\Delta_{\psi,\eta}^{1/2}$, we obtain (C. 66) for $\lambda=1/2$ and for all ζ in $D(\Delta_{\phi,\eta}^{1/2})$. By (D. 2) in Appendix D, we obtain (C. 66).

Lemma C. 4. *Let η be a cyclic and separating vector and ϕ be a normal semifinite weight.*

(1) *$\eta\in D(\Delta_{\phi,\eta}^{1/2})$ if and only if $\phi(1)<\infty$.*

(2) *$\eta\in D(\Delta_{\phi,\eta})$ if there exists some $\lambda>0$* satisfying *$\phi(x)\leqq\lambda\omega_{\eta}(x)$ for all positive x in M.*

Proof. (1) There exists a net $\phi_{\alpha}\in M_{*}^{+}$ such that $\phi=\sup\phi_{\alpha}$. By Lemma C. 3,

$$\phi_{\alpha}(1)=\|\Delta_{\phi_{\alpha},\eta}^{1/2}\eta\|^{2}\leqq\|\Delta_{\phi,\eta}^{1/2}\eta\|^{2} \tag{C. 68}$$

and hence

$$\phi(1)=\sup\phi_{\alpha}(1)\leqq\|\Delta_{\phi,\eta}^{1/2}\eta\|^{2}<\infty. \tag{C. 69}$$

Conversely, if $\phi(1)<\infty$, then $1\in N_{\phi}^{*}\cap N_{\eta}=M$ and $\eta(1)=\eta\in D(\Delta_{\phi,\eta}^{1/2})$.

(2) If $\phi(y)\leqq\lambda\omega_{\eta}(y)$ for $y=x^{*}x$ and $x\in M$, then

$$\begin{aligned}|(\Delta_{\phi,\eta}^{1/2}x\eta,\Delta_{\phi,\eta}^{1/2}\eta)|&=|(\xi(\phi),x^{*}\xi(\phi))|=|(x\xi(\phi),\xi(\phi))|\\ &\leqq\phi(x^{*}x)^{1/2}\phi(1)^{1/2}\leqq\lambda\|\eta\|\|x\eta\|.\end{aligned}$$

Since $M\eta$ is a core of $\Delta_{\phi,\eta}^{1/2}$, we have $\Delta_{\phi,\eta}^{1/2}\eta\in D(\Delta_{\phi,\eta}^{1/2})$, i.e. $\eta\in D(\Delta_{\phi,\eta})$.

Lemma C. 5. (1) *For ξ, $\eta\in H$ and $0\leqq\alpha\leqq 1/2$,*

$$\|\Delta_{\xi,\eta}^{\alpha}\eta\| \leqq \|s^M(\eta)\xi\|^{2\alpha}\|s^M(\xi)\eta\|^{1-2\alpha} \leqq \|\xi\|^{2\alpha}\|\eta\|^{1-2\alpha},$$

(2) *If* $\eta \in D(\Delta_{\xi,\eta}^{\alpha})$ *and* $\alpha > 1/2$, *then*

$$\|s^M(\eta)\xi\|^{2\alpha}\|s^M(\xi)\eta\|^{1-2\alpha} \leqq \|\Delta_{\xi,\eta}^{\alpha}\eta\|.$$

Proof. Since $\|\Delta_{\xi,\eta}^{1/2}\eta\| = \|s^M(\eta)\xi\|$ and $\|\Delta_{\xi,\eta}^{0}\eta\| = \|s^M(\xi)\eta\|$, we obtain (1) and (2) by the Hölder inequality $a_{\beta k} \leqq a_k^{\beta} a_0^{1-\beta}$ $(0 \leqq \beta \leqq 1)$ for $a_k = \|\Delta_{\xi,\eta}^{k}\eta\|^2 = \int \lambda^k d\mu(\lambda)$.

Appendix D

Lemma D. *Let* f *be an operator monotone function on* $[0, \infty)$ *and* A, B *be closed operators such that* $D(A) \subset D(B)$ *and* $\|B\xi\| \leqq \|A\xi\|$ *for any* $\xi \in D(A)$. *Then* $D(f(A^*A)^{1/2}) \subset D(f(B^*B)^{1/2})$ *and*

$$\|f(B^*B)^{1/2}\xi\| \leqq \|f(A^*A)^{1/2}\xi\|, \tag{D.1}$$

for any $\xi \in D(f(A^*A)^{1/2})$. *In particular,*

$$\|(B^*B)^{\lambda/2}\xi\| \leqq \|(A^*A)^{\lambda/2}\xi\|, \tag{D.2}$$

for $\xi \in D((A^*A)^{\lambda/2})$, $0 \leqq \lambda \leqq 1$.

Proof. We may replace A and B by $|A|$ and $|B|$ in the whole discussion. Hence we may assume that A and B are positive selfadjoint without loss of generality. Let E and F be spectral projections of A and B, respectively such that AE and BF are bounded. By the assumption,

$$\|EBFE\xi\| \leqq \|FBE\xi\| \leqq \|BE\xi\| \leqq \|AE\xi\|. \tag{D.3}$$

Hence $0 \leqq (EBFE)^2 \leqq (AE)^2$, which implies

$$f((EBFE)^2) \leqq f((AE)^2). \tag{D.4}$$

By taking the limit $E \to 1$, we see that the uniformly bounded sequence $(EBFE)^2$ converges to $(BF)^2$ and hence (for example, as is clear from a uniform approximation of f, which is continuous due to Theorem 2.2 in [21], over the interval $[0, \|BF\|^2]$ by a polynomial)

$$\|f(B^2)^{1/2}F\xi\| = \|f((BF)^2)^{1/2}\xi\| = \lim_{E \to 1} \|f((EBFE)^2)^{1/2}\xi\| \tag{D.5}$$

$$\leq \lim_{E\to 1} \|f((AE)^2)^{1/2}\xi\| = \lim_{E\to 1} \|f(A^2)^{1/2}E\xi\|$$

$$= \|f(A^2)^{1/2}\xi\|$$

for any $\xi \in D(f(A^2)^{1/2})$. By taking the limit $F\to 1$, we see that $\xi \in D(f(B^2)^{1/2})$ and (D. 1) holds. The function x^{λ} is operator monotone on $[0, \infty)$ for $0 \leqq \lambda \leqq 1$, which proves (D. 2).

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