Publ. RIMS, Kyoto Univ.
18 (1982), 439-468

Actions on Invariant Spheres around Isolated Fixed Points of Actions of Cyclic Groups

By

Masayoshi KAMATA*

§ 1. Introduction

Fix a prime number p and let Z_p be a cyclic group of order p. We consider a pair (M, ϕ) consisting of a compact simply connected almost complex manifold M without boundary and a smooth Z_p-action ϕ: $Z_p \times M \to M$ preserving the almost complex structure of M. We suppose that M is given an invariant Riemannian metric. If $a\,(\in M)$ is an isolated fixed point, then the induced action of Z_p on the tangent space at a gives a complex Z_p-module V_a which has no trivial irreducible factor. Let ξ: $EZ_p \to BZ_p$ be a universal principal Z_p-bundle and let $\xi(V_a): EZ_p \times_{Z_p} V_a \to BZ_p$ be the V_a-bundle associated with ξ. If a and b are isolated fixed points, we compare the cobordism Euler classes $e(\xi(V_a))$ and $e(\xi(V_b))$ which belong to the complex cobordism group $MU^*(BZ_p)$ of the classifying space BZ_p of Z_p. Let F_U be the universal formal group law over MU^*, and write

$$x +_F y = F_U(x, y).$$

For a positive integer n, $[n]_F(x)$ is inductively defined by

$$[1]_F(x) = x$$

and

$$[n]_F(x) = [n-1]_F(x) +_F x.$$

It is known that the cobordism ring $MU^*(BZ_p)$ is formal power series algebra $MU^*[[x]]$ over MU^* modulo an ideal generated by $[p]_F(x)$

Communicated by N. Shimada, August 3, 1981.
* Department of Mathematics, College of General Education, Kyushu University, Fukuoka 810, Japan.

[15]. Let us write

$$[p]_F(x)=px+a_1^{(p)}x^2+a_2^{(p)}x^3+\cdots,$$

where $a_i^{(p)}\in MU^{-2i}$, and

$$\langle p\rangle_F(x)=p+a_1^{(p)}x+a_2^{(p)}x^2+\cdots.$$

Let S denote the multiplicative set in $MU^*(BZ_p)$ consisting of cobordism Euler classes $e(\xi(V))$, V the non trivial complex Z_p-module, and let λ: $MU^*(BZ_p)\to S^{-1}MU^*(BZ_p)$ be the canonical map [9]. In this paper we show the following

Theorem A. *Assume that* $H^i(BZ_p; \{\pi_i(M)\})\cong 0$ *for* $1\leqq i\leqq 2n-1$ (cf. [4, p. 355]), *and* $\lambda(\alpha)=e(\xi(V_a))/e(\xi(V_b))$. *Then for any Landweber-Novikov operation* S_ω^U, $\omega\neq(0)$ [14], [17], $S_\omega^U(\alpha)$ *belongs to an ideal generated by* x^n *and* $\langle p\rangle_F(x)$ *in* $MU^*(BZ_p)$, *where* $x=e(\xi(L))$ *and L is the canonical one dimensional complex* Z_p*-module with an action of* Z_p *given by multiplication by* $\rho=\exp(2\pi i/p)$ *on* C^1.

The action of Z_p on M induces a natural action on a unit sphere $S(V_a)$ in a tangent space V_a at an isolated fixed point a which is equivalent to the action of Z_p on a sphere around the fixed point. The action ϕ_a: $Z_p\times S(V_a)\to S(V_a)$ determines a weakly complex bordism class $[S(V_a),\phi_a]$ of the bordism group $MU_*(Z_p)$ of fixed point free Z_p actions preserving a weakly complex structure, which is generated as an MU_*-module by the set of Z_p-manifolds $\{[S^{2n+1},\tilde\phi]\}$, where the action $\tilde\phi$ of Z_p on a sphere $S^{2n+1}\subset C^{n+1}$ is defined by $\tilde\phi(g,z)=\rho z$, g a generator of Z_p [6], [11]. Kasparov in [13] showed that the weakly complex bordism class $[S(V_a),\phi_a]$ is computable. By making use the Kasparov theorem and Theorem A, we obtain the following

Theorem B. *Assume that* $H^i(BZ_p; \{\pi_i(M)\})\cong 0$ *for* $1\leqq i\leqq 2n-1$. *If* $V_a=L^{l_1}\oplus\cdots\oplus L^{l_k}$ *and* $V_b=L^{m_1}\oplus\cdots\oplus L^{m_k}$, *then*

$$l_1\cdots l_k[S(V_a),\phi_a]-m_1\cdots m_k[S(V_b),\phi_b]$$
$$=\tilde\mu_1[S^{2k-3},\tilde\phi]+\tilde\mu_2[S^{2k-5},\tilde\phi]+\cdots+\tilde\mu_{k-1}[S^1,\tilde\phi]$$

where $\tilde\mu_1,\tilde\mu_2,\cdots,\tilde\mu_{n-1}$ *belong to an ideal generated by* p, $a_1^{(p)}, a_2^{(p)},\cdots$.

$a_i^{(p)}, \cdots$ *in* MU^*.

In Section 2 we investigate S^1-actions on a product space $S^{2n+1}\times S^{2m+1}$ of spheres and equivariant maps between the S^1-spaces. In Section 3 the Umkehr homomorphism of some map between the orbit spaces $(S^{2n+1}\times S^{2m+1})/S^1$ is computed to give a slightly different proof of the Kasparov theorem [13] in Section 4. In Section 5 we discuss about relations among cobordism characteristic classes [7] of $\xi(V_a)$ and $\xi(V_b)$ and give a proof of Theorem A. Section 6 is devoted to prove Theorem B. In Section 7 we study the isolated fixed point set of Z_3-actions.

Bredon in Section 10 of Chapter VI of [4] compared representations at two fixed points of a smooth action, by using equivariant K-theory.

§ 2. On Orbit Spaces of $S^{2m+1}\times S^{2n+1}$ with Respect to S^1

We define $\phi(l_0, l_1, \cdots, l_n): S^1\times S^{2m+1}\times S^{2n+1}\to S^{2m+1}\times S^{2n+1}$ by

$$\phi(l_0, l_1, \cdots, l_n)(z, (u_0, u_1, \cdots, u_m), (v_0, v_1, \cdots, v_n))$$
$$= ((zu_0, zu_1, \cdots, zu_m), (z^{l_0}v_0, z^{l_1}v_1, \cdots, z^{l_n}v_n)).$$

This is differentiable and the orbit space $(S^{2m+1}\times S^{2n+1})/\phi(l_0, \cdots, l_n)$ is an orientable smooth manifold. Let S^1 act on $S^{2m+1}\times C^1$ by

$$z\cdot((u_0, \cdots, u_m), v) = ((zu_0, \cdots, zu_m), zv).$$

The orbit space induces a complex line bundle over the complex projective space

$$\pi: S^{2m+1}\times_{S^1}C^1\to S^{2m+1}/S^1 = CP^m,\ \pi([u, v]) = [u]$$

which is denoted by η. The total space $S(\eta^{l_0}\oplus\cdots\oplus\eta^{l_n})$ of the sphere bundle associated with $\eta^{l_0}\oplus\cdots\oplus\eta^{l_n}$ is diffeomorphic to $(S^{2m+1}\times S^{2n+1})/\phi(l_0, \cdots, l_n)$. The structure of the integral cohomology group $H^*(S(\eta^{l_0}\oplus\cdots\oplus\eta^{l_n}))$ is determined as follows in [18].

Proposition 2.1. (1) *If* $m\leqq n$, *then* $H^{2j}(S(\eta^{l_0}\oplus\cdots\oplus\eta^{l_n}))\cong H^{2j}(CP^m)$ *and* $H^{2j-1}(S(\eta^{l_0}\oplus\cdots\oplus\eta^{l_n}))\cong H^{2j-2(n+1)}(CP^m)$.

(2) *If* $m>n$, *then*

$$H^{2j}(S(\eta^{l_0}\oplus\cdots\oplus\eta^{l_n}))\cong\begin{cases}0, & j>m\\ Z/(l_0\cdots l_n), & n+1\leqq j\leqq m\\ H^{2j}(CP^m), & j\leqq n\end{cases}$$

$$H^{2j-1}(S(\eta^{l_0}\oplus\cdots\oplus\eta^{l_n}))\cong\begin{cases}0, & 0\leqq j\leqq m\\ H^{2j-2(n+1)}(CP^m), & m+1\leqq j\,.\end{cases}$$

The map $f\colon S^{2m+1}\times S^{2n+1}\to S^{2m+1}\times S^{2n+1}$ defined by

$$f((u_0, \cdots, u_m), (v_0, \cdots, v_n)) = \left((u_0, \cdots, u_m), \frac{1}{r}(v_0^{l_0}, \cdots, v_n^{l_n})\right),$$

$$r=\sqrt{|v_0|^{2l_0}+\cdots+|v_n|^{2l_n}},$$

induces a map of the orbit spaces

$$\tilde{f}\colon (S^{2m+1}\times S^{2n+1})/\phi(1, \cdots, 1)\to(S^{2m+1}\times S^{2n+1})/\phi(l_0, \cdots, l_n).$$

Denote by $[M]$ the fundamental class of a compact orientable manifold M. Then we have

Proposition 2.2. $\tilde{f}_*[(S^{2m+1}\times S^{2n+1})/\phi(1, \cdots, 1)]$
$$=l_0l_1\cdots l_n[(S^{2m+1}\times S^{2n+1})/\phi(l_0, \cdots, l_n)].$$

Proof. $\tilde{f}$ is a fiber preserving map of sphere bundles $S((n+1)\eta)$ and $S(\eta^{l_0}\oplus\cdots\oplus\eta^{l_n})$, as $\eta^{l_0}\oplus\cdots\oplus\eta^{l_n}$ is isomorphic to a bundle of an orbit space of an S^1-action on $S^{2m+1}\times C^{n+1}$ defined by

$$z\cdot(u, (v_0, \cdots, v_n)) = (zu, (z^{l_0}v_0, \cdots, z^{l_n}v_n)).$$

Let f_1 be a fiber preserving map from $(n+1)\eta$ to $\eta^{l_0}\oplus\cdots\oplus\eta^{l_n}$ defined by

$$f_1(u, (v_0, \cdots, v_n)) = (u, (v_0^{l_0}, \cdots, v_n^{l_n}))$$

which induces a map between the Thom complexes

$$\tilde{f}_1\colon T(1, \cdots, 1)\to T(l_0, \cdots, l_n),$$

where $T(l_0, \cdots, l_n)=E(l_0, \cdots, l_n)/\{E(l_0, \cdots, l_n)$ *-the zero section*$\}$, and $E(l_0, \cdots, l_n)$ is the total space of $\eta^{l_0}\oplus\cdots\oplus\eta^{l_n}$. $S(\eta^{l_0}\oplus\cdots\oplus\eta^{l_n})$ and $E(l_0, \cdots, l_n)$ - $\{$*the zero section*$\}$ are of the same homotopy type, and the following diagram is homotopy commutative

$$\begin{array}{ccc} E(1,\cdots,1)-\{\textit{the zero section}\} & \xrightarrow{f_1} & E(l_0,\cdots,l_n)-\{\textit{the zero section}\} \\ \cup\uparrow\simeq & & \cup\uparrow\simeq \\ S((n+1)\eta) & \xrightarrow{\tilde{f}} & S(\eta^{l_0}\oplus\cdots\oplus\eta^{l_n}). \end{array}$$

Let $t(l_0, \cdots, l_n)$ be the Thom class of $\eta^{l_0}\oplus\cdots\oplus\eta^{l_n}$. Then we have $\tilde{f}_1^*(t(l_0, \cdots, l_n)) = l_0 l_1\cdots l_n t(1, \cdots, 1)$. Since the coboundary homomorphism δ: $H^{2m+2n+1}(S(\eta^{l_0}\oplus\cdots\oplus\eta^{l_n}))\to\tilde{H}^{2m+2n+2}(T(l_0, \cdots, l_n))$ is isomorphic, the fundamental class of $(S^{2m+1}\times S^{2n+1})/\phi(l_0, \cdots, l_n)$ is the dual class of $\delta^{-1}\{\pi^*([CP^m]^*)\cup t(l_0, \cdots, l_n)\}$, where π: $E(l_0, \cdots, l_n)\to CP^m$ is the projection and $[CP^m]^*$ is the dual of $[CP^m]$. Then the assertion follows.

Suppose that M^m and N^n are orientable manifolds. A continuous map h: $M^m\to N^n$ determines the Umkehr homomorphism

$$h_!: H^k(M^m) \overset{D}{\cong} H_{m-k}(M^m) \xrightarrow{h_*} H_{m-k}(N^n) \overset{D^{-1}}{\cong} H^{n-m+k}(N^n)$$

where D is the Poincare duality.

Proposition 2.3. *Assume that g is an embedding of $(S^{2m+1}\times S^{2n+1})/\phi(1, \cdots, 1)$ into S^N for a large N. Then the Umkehr homomorphism of*

$F=\tilde{f}\times g$: $(S^{2m+1}\times S^{2n+1})/\phi(1, \cdots, 1)\to(S^{2m+1}\times S^{2n+1})/\phi(l_0, \cdots, l_n)\times S^N$,
$\tilde{f}\times g(x) = (\tilde{f}(x), g(x))$, *satisfies*

$$F_!(\tilde{f}^*(y)) = l_0\cdots l_n y\times[S^N]^*$$

where $[S^N]^*$ *is the dual of* $[S^N]$.

Proof. The Umkehr homomorphism satisfies $F_!(F^*(a)\cup b) = a\cup F_!(b)$ [8]. We calculate using Proposition 2.2,

$$\begin{aligned} &F_!(\tilde{f}^*(y)) \\ &= (y\times 1)\cup F_!(1) \\ &= (y\times 1)\cup D^{-1}(\tilde{f}\times g)_*[(S^{2m+1}\times S^{2n+1})/\phi(1, \cdots, 1)] \\ &= (y\times 1)\cup D^{-1}((l_0\cdots l_n)[(S^{2m+1}\times S^{2n+1})/\phi(l_0, \cdots, l_n)]\times 1) \\ &= (y\times 1)\cup l_0\cdots l_n(1\times[S^N]^*). \end{aligned}$$

Q.E.D.

If $m \leqq n$, then we get a short exact sequence

$$0 \to MU^*(CP^m) \xrightarrow{\pi^*} MU^*(S(\eta^{l_0} \oplus \cdots \oplus \eta^{l_n}))$$
$$\xrightarrow{\delta} \widetilde{MU}^*(T(l_0, \cdots, l_n)) \to 0$$

and $\delta\colon MU^{2n+1}(S(\eta^{l_0} \oplus \cdots \oplus \eta^{l_n})) \to \widetilde{MU}^{2n+2}(T(l_0, \cdots, l_n))$ is isomorphic. In this case we may determine the ring structure of $MU^*((S^{2m+1} \times S^{2n+1})/\phi(l_0, \cdots, l_n))$ (cf. [18]).

Proposition 2.4. *If $m \leqq n$, then $MU^*((S^{2m+1} \times S^{2n+1})/\phi(l_0, \cdots, l_n))$ is $MU^*[x, y]/(x^{m+1}, y^2)$ where x is the first cobordism Chern class $c_U^1(\pi^! \eta)$ and y is an element of $MU^{2n+1}(S(\eta^{l_0} \oplus \cdots \oplus \eta^{l_n}))$ such that δy is the Thom class of $\eta^{l_0} \oplus \cdots \oplus \eta^{l_n}$.*

Proof. $MU^*(S(\eta^{l_0} \oplus \cdots \oplus \eta^{l_n}))$ is isomorphic to the direct sum of $MU^*(CP^m)$ and $\widetilde{MU}^*(T(l_0, \cdots, l_n))$. We have

$$(-1)^{\deg a} \delta(\pi^* a \cup b) = \pi^* a \cup \delta b$$

(cf. Chapter 13 of [20]), and $MU^*(S(\eta^{l_0} \oplus \cdots \oplus \eta^{l_n}))$ is a free MU^*-module generated by $\{(\pi^* x)^i,\ i = 1, 2, \cdots, m\}$ and $\{(\pi^* x)^i \cup y,\ i = 1, 2, \cdots, m\}$. It follows from Proposition 2.1 that $MU^{2(2n+1)}(S(\eta^{l_0} \oplus \cdots \oplus \eta^{l_n}))$ is zero. Q.E.D.

§ 3. On the Umkehr Homomorphism of $\tilde{f}$ with the MU^*-Orientation

For any set $\omega = (i_1, \cdots, i_r)$ of positive integers, let $\sum t_1^{i_1} \cdots t_r^{i_r}$ be the symmetric polynomial of variable t_j, $1 \leqq j \leqq n$ to be the smallest symmetric polynomial containing the monomial $t_1^{i_1} \cdots t_r^{i_r}$, which is expressible uniquely as a polynomial with integral coefficients in the elementary symmetric polynomials $\mathfrak{S}_1, \mathfrak{S}_2, \cdots, \mathfrak{S}_n$ of the t's and write

$$P_\omega(\mathfrak{S}_1, \mathfrak{S}_2, \cdots, \mathfrak{S}_n) = \sum t_1^{i_1} \cdots t_r^{i_r}\,.$$

For an n-dimensional complex vector bundle ζ over X, we define

$$c_\omega^H(\zeta) = P_\omega(c_H^1(\zeta), c_H^2(\zeta), \cdots, c_H^n(\zeta))$$

and $c_{(0,\cdots,0)}^H(\zeta) = 1$, where $c_H^i(\zeta)$ are the ordinary cohomology Chern classes.

Suppose that $x\in MU^k(X)$ is represented by

$$g: S^{2N-k}X^+\to MU(N).$$

We define

$$S_\omega^H(x)=\sigma^{k-2N}g^*\Phi c_\omega^H(\gamma_N),$$

where Φ: $H^*(BU(N))\to\widetilde{H}^*(MU(N))$ is the Thom isomorphism, σ^{k-2N} denotes $(k-2N)$-fold iterated suspension isomorphism and γ_N is the N-dimensional universal complex vector bundle. The ring $H_*(MU)$ is isomorphic to $Z[t_1, t_2, \cdots]$. Let

$$\omega=(\underbrace{1,\cdots,1}_{i_1},\underbrace{2,\cdots,2}_{i_2},\cdots,\underbrace{k,\cdots,k}_{i_k})$$

and we define

$$|\omega|=i_1+2i_2+\cdots+ki_k$$

and

$$t^\omega=t_1^{i_1}t_2^{i_2}\cdots t_k^{i_k}.$$

There exists a multiplicative natural transformation

$$\beta_H:\ MU^*(X)\to(H\wedge MU)^*(X)=H^*(X)[[t_1, t_2, \cdots]]$$

defined by

$$\beta_H(x)=\sum_\omega s_\omega^H(x)\ t^\omega$$

which is called Boardman map (cf. [1]). β_H: $MU^*(S^0)\to H_*(MU)$ is the Hurewicz homomorphism which is injective [16]. Given $x\in MU^*(X)$ with $x=[g: S^{2N-k}X^+\to MU(N)]$, the Thom homomorphism μ: $MU^k(X)\to H^k(X)$ is defined by $\mu(x)=\sigma^{k-2N}g^*\Phi(1)=S_{(0,\cdots,0)}^H(x)$.

Proposition 3.1. *Suppose that a finite CW-complex X has no torsion in its integral cohomology, then the Boardman map β_H is injective.*

Proof. Since the cohomology of X has no torsion, the Thom homomorphism is surjective. Suppose that $y_1^{(n)}, y_2^{(n)}, \cdots, y_{i_n}^{(n)}$ are the basis of $H^n(X)$, then we can take $u_j^{(n)}$ with $\mu(u_j^{(n)})=y_j^{(n)}$. The correspondence

$\sum y_j^{(n)} \otimes b_j^{(n)} \to \sum b_j^{(n)} u_j^{(n)}$ yields an isomorphism $H^*(X) \otimes MU^* \cong MU^*(X)$ (cf. [5]). We see

$$\beta_H(\sum b_j^{(n)} u_j^{(n)}) = \sum \beta_H(b_j^{(n)}) \{y_j^{(n)} + \sum_{|\omega|>0} S_\omega^H(u_j^{(n)}) t^\omega\}.$$

Let $\beta_H(\sum b_j^{(n)} u_j^{(n)}) = 0$, and we can derive inductively that $\beta_H(b_j^{(n)}) = 0$ and $b_j^{(n)} = 0$. Q.E.D.

For an n-dimensional complex vector bundle ζ over X, consider a formal power series of t's:

$$c_t^H(\zeta) = \sum_\omega c_\omega^H(\zeta) t^\omega .$$

This satisfies the naturality and $c_t^H(\zeta_1 \oplus \zeta_2) = c_t^H(\zeta_1) c_t^H(\zeta_2)$. Suppose that X and M are weakly almost complex manifolds. An embedding h: $M \to X$ with the normal vector bundle ν equipped with the complex structure induces the Umkehr homomorphisms:

$$h_!: MU^*(M) \to MU^*(X),$$

and

$$h_!^H: H^*(M)[[t_1, t_2, \cdots]] \to H^*(X)[[t_1, t_2, \cdots]].$$

Now we recall the following (cf. [19])

Theorem 3.2. $\beta_H(h_!(1)) = h_!^H(c_t^H(\nu))$.

Proof. A composition of a collapsing map c of the Thom construction and a classifying map g_ν for ν

$$\tilde{g}_\nu: X \xrightarrow{c} T(\nu) \xrightarrow{g_\nu} MU(k)$$

represents $h_!(1) \in MU^*(X)$. By making use of the following commutative diagram:

$$\begin{array}{ccccc} H_*(X) & \overset{D}{\underset{\cong}{\longleftarrow}} & H^*(X) & \overset{c^*}{\longleftarrow} & \tilde{H}^*(T(\nu)) \\ \uparrow h_* & & & & \cong \uparrow \Phi \\ H_*(M) & & \overset{D}{\underset{\cong}{\longleftarrow}} & & H^*(M) \end{array}$$

we calculate

$$\beta_H(h_!(1)) = \sum_\omega S^H_\omega[g_\nu c]t^\omega$$
$$= \sum_\omega c^*\Phi_H c^H_\omega(\nu)t^\omega$$
$$= h^H_!(\sum_\omega c^H_\omega(\nu)t^\omega).$$

Q.E.D.

$MU^*(BU(1))$ is isomorphic to $MU^*[[x_{MU}]]$, $x_{MU}=c^1_U(\gamma_1)$. The first cobordism Chern class $c^1_U(\gamma^k_1)$ of the k-fold tensor product of γ_1 is described as

$$c^1_U(\gamma^k_1) = [k]_F(x_{MU})$$
$$= kx_{MU} + a^{(k)}_1 x^2_{MU} + a^{(k)}_2 x^3_{MU} + \cdots .$$

Let g: $X \to BU(1)$ be a classifying map for a complex line bundle ζ over X. We see

$$\langle k\rangle_F(c^1_U(\zeta)) = g^*\{k + a^{(k)}_1 x_{MU} + a^{(k)}_2 x^2_{MU} + \cdots\}.$$

The map $\tilde{f}$: $(S^{2m+1}\times S^{2n+1})/\phi(1, \cdots, 1) \to (S^{2m+1}\times S^{2n+1})/\phi(l_0, \cdots, l_n)$ defined by

$$\tilde{f}([(u_0, \cdots, u_m), (v_0, \cdots, v_n)]$$
$$= \left[(u_0, \cdots, u_m), \frac{1}{r}(v_0^{l_0}, \cdots, v_n^{l_n})\right],$$

$$r = \sqrt{|v_0|^{2l_0} + \cdots + |v_n|^{2l_n}},$$

and an embedding h: $(S^{2m+1}\times S^{2n+1})/\phi(1, \cdots, 1) \to S^{2N}$ for a large N determine a bordism class $[(S^{2m+1}\times S^{2n+1})/\phi(1, \cdots, 1), \tilde{f}\times h]$ of $MU_*((S^{2m+1}\times S^{2n+1})/\phi(l_0, \cdots, l_n)\times S^{2N})$. The projection π: $(S^{2m+1}\times S^{2n+1})/\phi(l_0, \cdots, l_n) \to CP^m$ is defined by $\pi[u, v] = [u]$. Then we have

Theorem 3.3. *Suppose that* $m \leqq n$. *Then it follows that*

$$[(S^{2m+1}\times S^{2n+1})/\phi(1, \cdots, 1), \tilde{f}\times h]$$
$$= D_{MU}\pi^*(\langle l_0\rangle_F(c^1_U(\eta))\langle l_1\rangle_F(c^1_U(\eta))\cdots\langle l_n\rangle_F(c^1_U(\eta))) \times [P\subset S^{2N}]$$

where $P = \{a\ point\}$ *and* D_{MU} *is the Atiyah-Poincare isomorphism* [3].

Proof. If $m\leqq n$, then $H^*((S^{2m+1}\times S^{2n+1})/\phi(l_0, \cdots, l_n))$ has no torsion from Propositions 2.1 and 3.1 implies that

$$\beta_H\colon MU^*((S^{2m+1}\times S^{2n+1})/\phi(l_0, \cdots, l_n)\times S^{2N})$$
$$\rightarrow H^*((S^{2m+1}\times S^{2n+1})/\phi(l_0, \cdots, l_n)\times S^{2N})[t_1, t_2, \cdots]$$

is injective. The tangent bundle of $(S^{2m+1}\times S^{2n+1})/\phi(l_0, \cdots, l_n)$ is stably isomorphic to $\pi^!(\tau(CP^m)\oplus\eta^{l_0}\oplus\cdots\oplus\eta^{l_n})$ where η is the Hopf bundle over CP^m and $\tau(M)$ denotes the tangent bundle of M [18]. The normal vector bundle ν for $\tilde{f}\times h$ satisfies that $\nu\oplus\tau((S^{2m+1}\times S^{2n+1})/\phi(1, \cdots, 1))$ is isomorphic to $\tilde{f}^!\tau((S^{2m+1}\times S^{2n+1})/\phi(l_0, \cdots, l_n))\oplus 2N\varepsilon$, where ε is a trivial real line bundle. It follows directly from the definition that

$$c_t^H(\eta)=1+xt_1+x^2t_2+\cdots+x^mt_m, \quad x=c_H^1(\eta)$$

and

$$c_t^H(\nu)=\pi^*\left\{\frac{c_t^H(\eta^{l_0})\cdots c_t^H(\eta^{l_n})}{\{c_t^H(\eta)\}^{n+1}}\right\},$$

since the following diagram is commutative

$$\begin{array}{ccc}(S^{2m+1}\times S^{2n+1})/\phi(1, \cdots, 1) & \xrightarrow{\tilde{f}} & (S^{2m+1}\times S^{2n+1})/\phi(l_0, \cdots, l_n) \\ & {\scriptstyle\pi}\searrow \quad \swarrow{\scriptstyle\pi} & \\ & CP^m & \end{array}.$$

By using Theorem 3.2 and Proposition 2.3 we have

$$\beta_H((\tilde{f}\times h)_!(1))=(\tilde{f}\times h)_!^H c_t^H(\nu)$$
$$=\pi^*\left\{\frac{l_0\cdots l_n c_t^H(\eta^{l_0})\cdots c_t^H(\eta^{l_n})}{\{c_t^H(\eta)\}^{n+1}}\right\}\times[S^{2N}]^*.$$

On the other hand, we see that

$$\beta_H(c_U^1(\eta^k))=c_H^1(\eta^k)\,c_t^H(\eta^k)=kc_H^1(\eta)\,c_t^H(\eta^k)$$

and

$$\beta_H(c_U^1(\eta^k))=\beta_H(\langle k\rangle_F(c_U^1(\eta))\cdot c_U^1(\eta))=\beta_H(\langle k\rangle_F(c_U^1(\eta)))\beta_H(c_U^1(\eta)).$$

Therefore we have

$$\beta_H(\langle k\rangle_F(c_H^1(\eta)))=\frac{kc_t^H(\eta^k)}{c_t^H(\eta)}.$$

Noting that β_H maps $D_{MU}^{-1}([P\subset S^{2N}])$ to $[S^{2N}]^*$, we obtain

$$\beta_H(\pi^*\{\langle l_0\rangle_F(c_U^1(\eta))\cdots\langle l_n\rangle_F(c_U^1(\eta))\}\times D_{MU}^{-1}([P\subset S^{2N}])$$
$$=\beta_H((\tilde{f}\times h)_!(1)).$$

This completes the proof.

§ 4. Another Proof of the Kasparov Theorem

Let $l_0, l_1, \cdots, l_n$ be integers prime to p. An action of Z_p on $S^{2m+1}\times S^{2n+1}$ is defined by

$$\phi_p(l_0, \cdots, l_n)(g, ((u_0, \cdots, u_m), (v_0, \cdots, v_n)))$$
$$=((\rho u_0, \cdots, \rho u_m), (\rho^{l_0}v_0, \cdots, \rho^{l_n}v_n)),$$

where $\rho=\exp(2\pi i/p)$ and g is a generator of Z_p. The map f: $S^{2m+1}\times S^{2n+1}\to S^{2m+1}\times S^{2n+1}$ with

$$f((u_0, \cdots, u_m), (v_0, \cdots, v_n))=\left((u_0, \cdots, u_m), \frac{1}{r}(v_0^{l_0}, \cdots, v_n^{l_n})\right),$$

$$r=\sqrt{|v_0|^{2l_0}+\cdots+|v_n|^{2l_n}},$$

induces a map of orbit spaces:

$$\tilde{f}_p\colon (S^{2m+1}\times S^{2n+1})/\phi_p(1, \cdots, 1)\to(S^{2m+1}\times S^{2n+1})/\phi_p(l_0, \cdots, l_n).$$

Let $\pi\colon (S^{2m+1}\times S^{2n+1})/\phi_p(l_0, \cdots, l_n)\to(S^{2m+1}\times S^{2n+1})/\phi(l_0, \cdots, l_n)$ be the natural projection. We take up a differentiable embedding

$$h\colon (S^{2m+1}\times S^{2n+1})/\phi(1, \cdots, 1)\to S^{2N}$$

for a sufficiently large N.

Proposition 4.1. *In the following commutative diagram*

$$\begin{array}{ccc}(S^{2m+1}\times S^{2n+1})/\phi_p(1, \cdots, 1) & \xrightarrow{\pi} & (S^{2m+1}\times S^{2n+1})/\phi(1, \cdots, 1)\\ \downarrow \tilde{f}_p\times h\pi & & \downarrow \tilde{f}\times h\\ (S^{2m+1}\times S^{2n+1})/\phi_p(l_0, \cdots, l_n)\times S^{2N} & \xrightarrow{\pi\times id} & (S^{2m+1}\times S^{2n+1})/\phi(l_0, \cdots, l_n)\times S^{2N}\end{array}$$

(1) $\tilde{f}_p\times h\pi$ *and* $\tilde{f}\times h$ *are embeddings*

(2) $\pi\times id$ *is transverse regular to* $(\tilde{f}\times h)((S^{2m+1}\times S^{2n+1})/\phi(1,$

$\cdots, 1))$

(3) $(\pi\times id)^{-1}(\tilde{f}\times h)\,((S^{2m+1}\times S^{2n+1})/\phi(1,\cdots,1))$
$= (\tilde{f}_p\times h\pi)\,((S^{2m+1}\times S^{2n+1})/\phi_p(1,\cdots,1))$.

Proof. A tangent vector at a point of $(S^{2m+1}\times S^{2n+1})/\phi_p(1,\cdots,1)$ is described as $\vec{v}+\vec{w}$ with $\vec{v}\in$ {*the tangent space along the base space of the smooth fiber bundle* π: $(S^{2m+1}\times S^{2n+1})/\phi_p(1,\cdots,1)\to(S^{2m+1}\times S^{2n+1})/\phi(1,\cdots,1)$} and $\vec{w}\in$ {*the tangent space along the fiber*}. Let $d(\tilde{f}_p\times h\pi)(\vec{v}+\vec{w})=0$, then $d(\tilde{f}\times h)(\vec{v})=0$. Since $\tilde{f}\times h$ is an embedding, $\vec{v}=0$. On the other hand, $d\tilde{f}_p$ is injective on each tangent space along the fiber, and $\vec{w}=0$. This implies that $\tilde{f}_p\times h\pi$ is embedding, because $\tilde{f}_p\times \mathrm{h}\pi$ is injective. The differentiable fibration $\pi\times id$ is transverse regular to any submanifold of $(S^{2m+1}\times S^{2n+1})/\phi(l_0,\cdots,l_n)\times S^{2N}$. Q.E.D.

Considering the geometric interpretation of the cobordism group [19], we can see that Proposition 4.1 implies

Proposition 4.2. *The induced homomorphism* $(\pi\times id)^*$: $MU^*((S^{2m+1}\times S^{2n+1})/\phi(l_0,\cdots,l_n)\times S^{2N})\to MU^*((S^{2m+1}\times S^{2n+1})/\phi_p(l_0,\cdots,l_n)\times S^{2N})$ *sends* $D_{MU}^{-1}[(S^{2m+1}\times S^{2n+1})/\phi(1,\cdots,1),\tilde{f}\times h]$ *to* $D_{MU}^{-1}[(S^{2m+1}\times S^{2n+1})/\phi_p(1,\cdots,1),\tilde{f}_p\times h\pi]$.

Let $\psi_p(l_0,\cdots,l_n): Z_p\times S^{2n+1}\to S^{2n+1}$ be an action of Z_p on S^{2n+1} defined by

$$\psi_p(l_0,\cdots,l_n)\,(g,(v_0,\cdots,v_n))=(\rho^{l_0}v_0,\cdots,\rho^{l_n}v_n).$$

We have a complex line bundle $\hat{\xi}(L): S^{2n+1}\times_{Z_p}C^1\to S^{2n+1}/\psi_p(l_0,\cdots,l_n)$ by taking the orbit space of an action of Z_p on $S^{2n+1}\times C^1$

$$g\cdot((u_0,\cdots,u_n),z)=((\rho^{l_0}u_0,\cdots,\rho^{l_n}u_n),\rho z)$$

where g is a generator of Z_p. Denote by

$$\hat{\xi}(L): S^{2n+1}\times_{Z_p}C^1\to S^{2n+1}/Z_p$$

a line bundle over a standard lens space which is the orbit space of an action of Z_p on $S^{2n+1}\times C^1$ defined by $g\cdot((u_0,\cdots,u_n),z)=((\rho u_0,\cdots,\rho u_n),\rho z)$. The bordism class of $\tilde{f}_p\times\tilde{h}$: $(S^{2m+1}\times S^{2n+1})/\phi_p(1,\cdots,1)\to(S^{2m+1}\times$

$S^{2n+1})/\phi_p(l_0, \cdots, l_n) \times S^{2N}$ with the embedding $\tilde{h}$ for a large N is described as follows.

Proposition 4.3. *Suppose that* $m \leqq n$. *Then*

$$[(S^{2m+1}\times S^{2n+1})/\phi_p(1, \cdots, 1), \tilde{f}_p \times \tilde{h}]$$
$$= D_{MU}\{\pi^*\{\langle l_0\rangle_F(c_U^1(\tilde{\xi}(L))\cdots\langle l_n\rangle_F(c_U^1(\tilde{\xi}(L))\}\} \times [P\subset S^{2N}],$$

in $MU_*((S^{2m+1}\times S^{2n+1})/\phi_p(l_0, \cdots, l_n))$, *where* $P=\{a\ point\}$ *and* π: $(S^{2m+1} \times S^{2n+1})/\phi_p(l_0, \cdots, l_n) \to S^{2m+1}/\psi_p(1, \cdots, 1)$ *is the natural projection.*

Proof. Theorem 3.3 and Proposition 4.2 imply that

$$[(S^{2m+1}\times S^{2n+1})/\phi_p(1, \cdots, 1), \tilde{f}_p \times h\pi]$$
$$= D_{MU}\{\pi^*\{\langle l_0\rangle_F(c_U^1(\tilde{\xi}(L)))\cdots\langle l_n\rangle_F(c_U^1(\tilde{\xi}(L)))\}\} \times [P\subset S^{2N}].$$

But $h\pi$ is homotopic to $\tilde{h}$, and the bordism class is homotopy invariant, and hence the proposition follows.

The map f: $S^{2n+1}\to S^{2n+1}$ with $f(v_0, \cdots, v_n) = \dfrac{1}{r}(v_0^{l_0}, \cdots, v_n^{l_n})$, r the norm of $(v_0^{l_0}, \cdots, v_n^{l_n})$, induces a map of orbit spaces

$$\hat{f}_p\colon S^{2n+1}/\psi_p(1, \cdots, 1) \to S^{2n+1}/\psi_p(l_0, \cdots, l_n).$$

Theorem 4.4. *In* $MU_*(S^{2n+1}/\psi_p(l_0, \cdots, l_n))$, $[S^{2n+1}/\psi_p(1, \cdots, 1), \hat{f}_p] = D_{MU}\{\langle l_0\rangle_F(c_U^1(\hat{\xi}(L)))\cdots\langle l_n\rangle_F(c_U^1(\hat{\xi}(L)))\}$.

Proof. Define π_2: $(S^{2n+1}\times S^{2n+1})/\phi_p(l_0, \cdots, l_n) \to S^{2n+1}/\psi_p(l_0, \cdots, l_n)$ by $\pi_2[u, v]=[v]$ and take a differentiable embedding h: $S^{2n+1}/\psi_p(l_0, \cdots, l_n) \to S^{2N}$ for a sufficiently large N. In the commutative diagram

$$\begin{array}{ccc}
(S^{2n+1}\times S^{2n+1})/\phi_p(1, \cdots, 1) & \xrightarrow{\pi_2} & S^{2n+1}/\psi_p(1, \cdots, 1) \\
\Big\downarrow \tilde{f}_p \times h\pi_2 & & \Big\downarrow \hat{f}_p \times h \\
(S^{2n+1}\times S^{2n+1})/\phi_p(l_0, \cdots, l_n)\times S^{2N} & \xrightarrow{\pi_2\times id} & S^{2n+1}/\psi_p(l_0, \cdots, l_n)\times S^{2N}
\end{array}$$

$\tilde{f}_p \times h\pi_2$ is an embedding and $\pi_2 \times id$ is transverse regular to $(\hat{f}_p \times h)(S^{2n+1}/\psi_p(1, \cdots, 1))$. Thus it follows that

$$(\pi_2\times id)^*D_{MU}^{-1}[S^{2n+1}/\psi_p(1,\cdots,1),\hat{f}_p\times h]$$
$$=D_{MU}^{-1}[(S^{2n+1}\times S^{2n+1})/\phi_p(1,\cdots,1),\tilde{f}_p\times h\pi_2].$$

We now note that the induced bundle $\pi^!\hat{\tilde{\xi}}(L)$ by the projection π: $(S^{2n+1}\times S^{2n+1})/\phi_p(l_0,\cdots,l_n)\to S^{2n+1}/\psi_p(1,\cdots,1)$ is isomorphic to the induced bundle $\pi_2^!\hat{\xi}(L)$ by the natural projection π_2: $(S^{2n+1}\times S^{2n+1})/\phi_p(l_0,\cdots,l_n)\to S^{2n+1}/\psi_p(l_0,\cdots,l_n)$. Proposition 4.3 implies that

$$(\pi_2\times id)^*D_{MU}^{-1}([S^{2n+1}/\psi_p(1,\cdots,1),\hat{f}_p\times h])$$
$$=\pi_2^*\{\langle l_0\rangle_F(c_U^1(\hat{\xi}(L)))\cdots\langle l_n\rangle_F(c_U^1(\hat{\xi}(L)))\}\times D_{MU}^{-1}[P\subset S^{2N}].$$

Since $(\pi_2\times id)^*$ is injective, it follows that

$$[S^{2n+1}/\psi_p(1,\cdots,1),\hat{f}_p\times h]$$
$$=D_{MU}\{\langle l_0\rangle_F(c_U^1(\hat{\xi}(L)))\cdots\langle l_n\rangle_F(c_U^1(\hat{\xi}(L)))\}\times[P\subset S^{2N}].$$

Applying the homomorphism $MU_*(S^{2n+1}/\psi_p(l_0,\cdots,l_n)\times S^{2N})\to MU_*(S^{2n+1}/\psi_p(l_0,\cdots,l_n))$ induced by the projection, we obtain the assertion.

Theorem 4.5. *Let* $\hat{g}_p$: $S^{2n+1}/\psi_p(l_0,\cdots,l_n)\to S^{2n+1}/\psi_p(1,\cdots,1)$ *be the map of orbit spaces defined by*

$$\hat{g}_p[v_0,\cdots,v_n]=\left[\frac{1}{r}(v_0^{l_0'},\cdots,v_n^{l_n'})\right]$$

where $l_jl_j'\equiv 1$ *modulo* p *and* r *is the norm of* $(v_0^{l_0'},\cdots,v_n^{l_n'})$. *Then*

$$D_{MU}^{-1}[S^{2n+1}/\psi_p(l_0,\cdots,l_n),\hat{g}_p]$$
$$\equiv\langle l_0'\rangle_F([l_0]_F(x))\cdots\langle l_n'\rangle_F([l_n]_F(x))\quad modulo\quad(\langle p\rangle_F(x))$$

where $\langle p\rangle_F(x)\in MU^*(S^{2n+1}/\psi_p(1,\cdots,1))$ *and* $x=c_U^1(\hat{\tilde{\xi}}(L))$.

Proof. Consider the natural injection j: $S^{2n+1}/\psi_p(1,\cdots,1)\to S^{2n+3}/\psi_p(1,\cdots,1)$. We can see that $j\hat{g}_p\hat{f}_p\simeq j$ and $\hat{g}_p^!(\hat{\tilde{\xi}}(L))\cong\hat{\xi}(L)$. We note that the Atiyah-Poincare isomorphism D_{MU}: $MU^*(X)\to MU_*(X)$, X a weakly almost complex manifold, is given by

$$D_{MU}(z)=z\cap[X,\ identity].$$

We put $U=[S^{2n+1}/\psi_p(1,\cdots,1),\ identity]\in MU_{2n+1}(S^{2n+1}/\psi_p(1,\cdots,1))$ and $\tilde{U}=[S^{2n+1}/\psi_p(l_0,\cdots,l_n),\ identity]\in MU_{2n+1}(S^{2n+1}/\psi_p(l_0,\cdots,l_n))$. Let us compute with Theorem 4.4

$$\begin{aligned} j_*(U) &= j_*\hat{g}_{p*}\hat{f}_{p*}(U) \\ &= j_*\hat{g}_{p*}\{[S^{2n+1}/\psi_p(1,\cdots,1),\hat{f}_p]\} \\ &= j_*\hat{g}_{p*}\{\hat{g}_p^*\{\langle l_0\rangle_F(x)\cdots\langle l_n\rangle_F(x)\}\cap\tilde{U}\} \\ &= j_*\{\langle l_0\rangle_F(x)\cdots\langle l_n\rangle_F(x)\cap\hat{g}_{p*}(\tilde{U})\} \\ &= j_*\{\{\langle l_0\rangle_F(x)\cdots\langle l_n\rangle_F(x)\cup D_{MU}^{-1}(\hat{g}_{p*}(\tilde{U}))\}\cap U\}. \end{aligned}$$

Hence $\langle l_0\rangle_F(x)\cdots\langle l_n\rangle_F(x)\cup D_{MU}^{-1}(\hat{g}_{p*}(\tilde{U}))-1$ belongs to $D_{MU}^{-1}(j_*^{-1}(0))$. We recall the following commutative diagram:

$$\begin{array}{ccccc} MU_*(S^{2n+3}/\psi_p(1,\cdots,1)) & \underset{\cong}{\overset{D_{MU}}{\longleftarrow}} & MU^*(S^{2n+3}/\psi_p(1,\cdots,1)) & \overset{c^*}{\longleftarrow} & \widetilde{MU}^*(T(\xi(L)) \\ \uparrow j_* & & & & \Phi_U \uparrow \\ MU_*(S^{2n+1}/\psi_p(1,\cdots,1)) & & \underset{\cong}{\overset{D_{MU}}{\longleftarrow}} & & MU^*(S^{2n+1}/\psi_p(1,\cdots,1)) \end{array}$$

where Φ_U is the Thom isomorphism and c is the canonical collapsing map. Since $\Phi_U^{-1}c^{*-1}(0)$ is generated by $\langle p\rangle_F(x)$ (cf. [12]), $\langle l_0\rangle_F(x)\cdots\langle l_n\rangle_F(x)\cup D_{MU}^{-1}(\hat{g}_{p*}(\tilde{U}))-1$ belongs to the ideal generated by $\langle p\rangle_F(x)$ in $MU^*(S^{2n+1}/\psi_p(1,\cdots,1))$. On the other hand, since $\{\tilde{\xi}(L)^{l_j}\}^{l'_j}\cong\tilde{\xi}(L)$, we get

$$\{\langle l'_j\rangle_F([l_j]_F(x))\}\,[l_j]_F(x)=x$$

and it follows from Lemma 5 of [9] that $\{\langle l'_j\rangle_F([l_j]_F(x))\}\langle l_j\rangle_F(x)-1$ belongs to an ideal generated by $\langle p\rangle_F(x)$. Then we have

$$\begin{aligned} & D_{MU}^{-1}\hat{g}_{p*}(\tilde{U}) \\ &\quad \equiv\{\langle l_0\rangle_F(x)\cdots\langle l_n\rangle_F(x)\}\{\langle l'_0\rangle_F([l_0]_F(x))\cdots \\ &\qquad \langle l'_n\rangle_F([l_n]_F(x))\cup D_{MU}^{-1}(\hat{g}_{p*}(\tilde{U}))\} \quad \textit{modulo}\ (\langle p\rangle_F(x)), \end{aligned}$$

and

$$D_{MU}^{-1}(\hat{g}_{p*}(\tilde{U}))\equiv\langle l'_0\rangle_F([l_0]_F(x))\cdots\langle l'_0\rangle_F([l_n]_F(x))\ \text{modulo}\ (\langle p\rangle_F(x)).$$

Q.E.D.

Let us consider the composite

$$j_*: MU_*(S^{2n+1}/\psi_p(1,\cdots,1)) \xrightarrow{i_*} MU_*(BZ_p) \overset{\vartheta}{\cong} MU_*(Z_p)$$

where i_* is the MU_*-homomorphism induced from the natural injection and ϑ is the natural isomorphism given in [5]. Now we shall prove the Kasparov theorem.

Theorem 4.6. *Assume that $l_j l'_j \equiv 1$ modulo p. Then*

$$[S^{2n+1}, \psi_p(l_0,\cdots,l_n)]$$
$$= j_* D_{MU}\{\langle l'_0\rangle_F([l_0]_F(x))\cdots\langle l'_n\rangle_F([l_n]_F(x))\},$$

where $x = c^1_U(\tilde{\xi}(L)) \in MU^2(S^{2n+1}/\psi_p(1,\cdots,1))$.

Proof. From Theorem 4.5 there exists $h(x) \in MU^2(S^{2n+1}/\psi_p(1,\cdots,1))$ such that

$$D^{-1}_{MU}[S^{2n+1}/\psi_p(l_0,\cdots,l_n), \hat{g}_p]$$
$$= \langle l'_0\rangle_F([l_0]_F(x))\cdots\langle l'_n\rangle_F([l_n]_F(x)) + \langle p\rangle_F(x)h(x)$$

and

$$[S^{2n+1}/\psi_p(l_0,\cdots,l_n), \hat{g}_p]$$
$$= \{\langle l'_0\rangle_F([l_0]_F(x))\cdots\langle l'_n\rangle_F([l_n]_F(x)) + \langle p\rangle_F(x)h(x)\} \cap U$$

where $U = [S^{2n+1}/\psi_p(1,\cdots,1), \mathit{identity}]$. Let $\bar{x}$ be the first cobordism Chern class of the canonical line bundle $\tilde{\xi}(L)$ over $S^{2n+3}/\psi_p(1,\cdots,1)$ and let

$$\bar{U} = [S^{2n+3}/\psi_p(1,\cdots,1), \mathit{identity}]$$

which belongs to $MU_{2n+3}(S^{2n+3}/\psi_p(1,\cdots,1))$. Then we have

$$\bar{x} \cap \bar{U} = i_* U \quad \text{(cf. [11])}.$$

Noting that $[p]_F(\bar{x}) = 0$, we calculate

$$i_*[S^{2n+1}/\psi_p(l_0,\cdots,l_n), \hat{g}_p]$$
$$= i_*\{i^*\{\langle l'_0\rangle_F([l_0]_F(\bar{x}))\cdots\langle l'_n\rangle_F([l_n]_F(\bar{x})) + \langle p\rangle_F(\bar{x})h(\bar{x})\} \cap U\}$$
$$= \{\langle l'_0\rangle_F([l_0]_F(\bar{x}))\cdots\langle l'_n\rangle_F([l_n]_F(\bar{x})) + \langle p\rangle_F(\bar{x})h(\bar{x})\} \cap i_*(U)$$
$$= \langle l'_0\rangle_F([l_0]_F(\bar{x}))\cdots\langle l'_n\rangle_F([l_n]_F(\bar{x})) \cap i_*(U)$$

$$=i_*\{D_{MU}(\langle l_0'\rangle_F([l_0]_F(x))\cdots\langle l_n'\rangle_F([l_n]_F(x))\}.$$

Q.E.D.

§ 5. Characteristic Classes of $\xi(V_a)$

The product space $I\times X$ of a Z_p-space X and an interval $I=[0,1]$ has a Z_p-action with $g\cdot(t,x)=(t,g\cdot x)$, and we have Z_p-spaces

$S(X)$: the usual suspension of X

$C^+(X)=X\times[1/2,1]/X\times\{1\}$

$C^-(X)=X\times[0,1/2]/X\times\{0\}$.

Denote by p_0 and p_1 vertices obtained by the identification of $X\times 0$ and $X\times 1$ in these spaces. A map ε_i: $EZ_p\times_{Z_p}\{p\}\to EZ_p\times_{Z_p}S(X)$ is defined to be $\varepsilon_i(x,p)=(x,p_i)$, and a map π: $EZ_p\times_{Z_p}X\to EZ_p\times_{Z_p}\{p\}=BZ_p$ is defined to be $\pi(y,x)=(y,p)$. We can derive the following propositions after the fashion of Proposition 10.1 and Theorem 10.2 of [4].

Proposition 5.1. *Suppose that X is a compact Z_p-space. Then there exists an exact sequence:*

$$MU^*(EZ_p\times_{Z_p}S(X))\xrightarrow{\varepsilon_1^*-\varepsilon_0^*}MU^*(BZ_p)\xrightarrow{\pi^*}MU^*(EZ_p\times_{Z_p}X).$$

Proof. $\widetilde{MU}^*((EZ_p)^+\wedge_{Z_p}-)$ is an equivariant cohomology theory described in [10]. Consider the Mayer-Vietoris exact sequence for a triple $(\{S(X)\}^+;\ \{C^+(X)\}^+,\ \{C^-(X)\}^+)$

$$\longrightarrow MU^*(EZ_p\times_{Z_p}S(X))\xrightarrow{j^*}MU^*(EZ_p\times_{Z_p}C^+(X))$$
$$\oplus MU^*(EZ_p\times_{Z_p}C^-(X))$$
$$\xrightarrow{k^*}MU^*(EZ_p\times_{Z_p}X)\longrightarrow$$

where $j^*(x)=(j_1^*(x),j_0^*(x))$ and $k^*(x_1,x_0)=i_1^*(x_1)-i_0^*(x_0)$, and j_s and i_s are natural inclusions. The isomorphisms $MU^*(EZ_p\times_{Z_p}C^+(X))\cong MU^*(BZ_p)$ and $MU^*(EZ_p\times_{Z_p}C^-(X))\cong MU^*(BZ_p)$ yield the proposition.

Let Ψ: $\mathrm{Vect}_C(-)\to MU^*(-)$ be a natural transformation assigning a complex vector bundle over X to an element of $MU^*(X)$ which satisfies

$$\Psi(f^!\zeta)=f^*\Psi(\zeta).$$

Consider complex vector bundles

$$\xi(V_a)\,;\ EZ_p\times_{Z_p}V_a\to BZ_p$$

where V_a is the complex Z_p-module obtained by the tangent space at an isolated fixed point a of an almost complex Z_p-manifold M. Then we have

Proposition 5.2. *Suppose that a and b are isolated fixed points of a simply connected almost complex Z_p-manifold. If $H^i(BZ_p;\{\pi_i(M)\})\cong 0$ for $1\leqq i\leqq 2n-1$, then $\Psi(\xi(V_a))-\Psi(\xi(V_b))$ belongs to an ideal generated by x^n in $MU^*(BZ_p)\cong MU^*[[x]]/([p]_F(x))$, where $x=c_U^1(\xi(L))$, L the canonical one dimensional complex Z_p-module.*

Proof. The $(2n-1)$-skeleton of EZ_p can be taken to be S^{2n-1} with the action given by the complex n-dimensional Z_p-module nL. We take an invariant subspace $EZ_p\times\{0,1\}$ is a Z_p-space $EZ_p\times I$ with $g\cdot(e,t)=(g\cdot e,t)$. Consider the constant maps

$$h_0\colon EZ_p\to\{b\}\quad\text{and}\quad h_1\colon EZ_p\to\{a\}$$

which induce maps

$$\tilde h_0\colon S^{2n-1}\subset EZ_p\to\{b\}\ \text{ and }\ \tilde h_1\colon S^{2n-1}\subset EZ_p\to\{a\}.$$

We can construct an equivariant homotopy $h\colon S^{2n-1}\times I\to M$ between $\tilde h_0$ and $\tilde h_1$, by using the condition for the cohomology $H^i(BZ_p;\{\pi_i(M)\})$, and an equivariant map $\tilde h\colon S(S^{2n-1})\to M$ (cf. [4, p. 355]). Since

$$\xi(V_a)=\varepsilon_1^!(id\times_{Z_p}\tilde h)^!\tilde\tau\quad\text{and}\quad \xi(V_b)=\varepsilon_0^!(id\times_{Z_p}\tilde h)^!\tilde\tau,$$

where $\tilde\tau$ denotes a vector bundle $EZ_p\times_{Z_p}E(\tau(M))\to EZ_p\times_{Z_p}M$, it follows from Proposition 5.1 that $\pi^*(\Psi(\xi(V_a))-\Psi(\xi(V_b)))=0$. By using the Gysin exact sequence

$$\to MU^*(BZ_p)\xrightarrow{\cdot x^n}MU^{*+2n}(BZ_p)\xrightarrow{\pi^*}MU^{*+2n}(EZ_p\times_{Z_p}S^{2n-1})\to$$

we complete the proof.

We consider the symmetric polynomial $P_\omega(\mathfrak{S}_1, \cdots, \mathfrak{S}_n)$ discussed in Section 3, and put $c^U_\omega(\gamma_n) = P_\omega(c^1_U(\gamma_n), \cdots, c^n_U(\gamma_n))$, where $c^i_U(\gamma_n)$ is the i-th cobordism Chern class [7]. The Landweber-Novikov operation

$$S^U_\omega\colon MU^*(X) \to MU^{*+2|\omega|}(X)$$

is defined as follows: for $x=[f]$, $f\colon S^{2n-k}X^+ \to MU(n)$,

$$S^U_\omega(x) = \sigma^{k-2n} f^* \Phi_U(c^U_\omega(\gamma_n)) \quad \text{(cf. [14], [17])}.$$

The Boardman map $\beta_U\colon MU^*(X) \to (MU \wedge MU)^*(X) \cong MU^*(X)[[t_1, t_2, \cdots]]$ is defined by

$$\beta_U(x) = \textstyle\sum_\omega S^U_\omega(x)\, t^\omega \quad \text{(cf. [2], [19])},$$

which is natural and multiplicative. Let $J(G)$ be the set of isomorphism classes of non trivial irreducible complex Z_p-modules, and let $\mathcal{V} = \{V^{k_1}_{j_1} \oplus \cdots \oplus V^{k_l}_{j_l} \mid V_{j_s} \in J(G)$ and k's are non negative integers$\}$. We consider the multiplicative system S consisting of cobordism Euler classes $\{e(EZ_p \times_{Z_p} V) \mid V \in \mathcal{V}\}$ in $MU^*(BZ_p)$. For a Z_p-space X, $MU^*(EZ_p \times_{Z_p} X)$ is a $MU^*(BZ_p)$-module by a map $EZ_p \times_{Z_p} X \to BZ_p \times (EZ_p \times_{Z_p} X)$ sending $[e, x]$ to $([e], [e, x])$. The localized module $S^{-1}MU^*(EZ_p \times_{Z_p} X)$ of the $MU^*(BZ_p)$-module $MU^*(EZ_p \times_{Z_p} X)$ consists of all fractions $\{x/e;\ x \in MU^*(EZ_p \times_{Z_p} X),\ e \in S\}$. For a complex vector bundle ζ over X, we put

$$c^U_t(\zeta) = 1 + \sum_\omega c^U_\omega(\zeta)\, t^\omega$$

which is an invertible element of $MU^*[[t_1, t_2, \cdots]]$. We define $\tilde{\beta}_U\colon S^{-1}MU^*(EZ_p \times_{Z_p} X) \to S^{-1}MU^*(EZ_p \times_{Z_p} X)[[t_1, t_2, \cdots]]$ by

$$\tilde{\beta}_U(y/e(\xi(V))) = \left(\beta_U(y) \cdot \frac{1}{c^U_t(\xi(V))}\right) \Big/ e(\xi(V))$$

which is multiplicative and natural. Moreover, we define

$$\tilde{S}^U_\omega\colon S^{-1}MU^*(EZ_p \times_{Z_p} X) \to S^{-1}MU^*(EZ_p \times_{Z_p} X)$$

by $\tilde{\beta}_U(x/e) = \sum_\omega \tilde{S}^U_\omega(x/e)\, t^\omega$.

Proposition 5.3. *The operation $\tilde{S}^U_\omega$ on $S^{-1}MU^*(EZ_p \times_{Z_p} -)$ have the following properties*:

(1) *$\tilde{S}^U_\omega$ is natural.*

(2) *$\tilde{S}^U_\omega((x_1/e_1) \cdot (x_2/e_2)) = \sum_{\omega=(\omega'\omega'')} \tilde{S}^U_{\omega'}(x_1/e_1)\, \tilde{S}^U_{\omega''}(x_2/e_2)$, where for*

$\omega' = (j'_1, \cdots, j'_s)$ *and* $\omega'' = (j''_1, \cdots, j''_t)$, $(\omega'\omega'')$ *denotes* $(j'_1, \cdots, j'_s, j''_1, \cdots, j''_t)$.

(3) $\tilde{S}^U_\omega(x/1) = S^U_\omega(x)/1$, *where* S^U_ω *is the ordinary Landweber-Novikov operation, i.e.* $\lambda S^U_\omega = \tilde{S}^U_\omega \lambda$, *where* $\lambda: MU^*(EZ_p \times_{Z_p} -) \to S^{-1}MU^*(EZ_p \times_{Z_p} -)$ *is the canonical map.*

(4) For $\omega = (\underbrace{1, \cdots, 1}_{i_1}, \underbrace{2, \cdots, 2}_{i_2}, \cdots, \underbrace{k, \cdots, k}_{i_k})$,

$$\tilde{S}^U_\omega(1/e(\xi(L))) = (-1)^{i_1+\cdots+i_k}\left\{\frac{(i_1+\cdots+i_k)!}{i_1!i_2!\cdots i_k!}e(\xi(L))^{|\omega|-1}\right\}\Big/1.$$

Proof. By making use of the multiplicativity and the naturality of β_U, we derive (1) and (2). For a zero dimensional complex Z_p-module 0, we have $e(\xi(0)) = 1$ and $c^U_t(\xi(0)) = 1$, and

$$\tilde{\beta}_U(x/1) = \beta_U(x) \cdot \frac{1}{c^U_t(\xi(0))}\Big/ e(\xi(0))$$
$$= \beta_U(x)/1$$

which implies (3). To prove (4), we calculate

$$\begin{aligned}
&\tilde{\beta}_U(1/e(\xi(L))) \\
&= \left\{\beta_U(1) \cdot \frac{1}{c^U_t(\xi(L))}\right\}\Big/ e(\xi(L)) \\
&= \left\{\frac{1}{1 + e(\xi(L))t_1 + e(\xi(L))^2 t_2 + \cdots}\right\}\Big/ e(\xi(L)) \\
&= \{\textstyle\sum(-1)^i (e(\xi(L))t_1 + e(\xi(L))^2 t_2 + \cdots)^i\}/e(\xi(L)).
\end{aligned}$$

This completes the proof.

We see easily the following

Proposition 5.4. $S^U_\omega(e(\xi(V))) = e(\xi(V))\ c^U_\omega(\xi(V))$.

Taking two complex Z_p-modules V_a and V_b obtained from tangent spaces at isolated fixed points a and b of an almost complex Z_p-manifold, a fraction $e(\xi(V_a))/e(\xi(V_b))$ is an integral element from the following proposition.

Proposition 5.5. *Suppose that L is a canonical complex one dimensional Z_p-module. Take k_i and l_j such that $(k_i, p)=1$ and $(l_j, p)=1$. Then for $n \geqq m$, $e(\xi(L^{k_1}\oplus\cdots\oplus L^{k_n}))/e(\xi(L^{l_1}\oplus\cdots\oplus L^{l_m}))$ belongs to the image of λ: $MU^*(BZ_p)\to S^{-1}MU^*(BZ_p)$ which sends x to $x/1$.*

Proof. For $x=c_U^1(\xi(L))$,

$$e(\xi(L^k))=[k]_F(x)=kx+a_1^{(k)}x^2+a_2^{(k)}x^3+\cdots$$

and

$$e(\xi(L^k))/x=\langle k\rangle_F(x)/1.$$

Assume that $(l, p)=1$, then there is an integer l' such that $l'l\equiv 1$ modulo p and

$$x=\langle l'\rangle_F([l]_F(x))\cdot[l]_F(x).$$

Therefore we have

$$\begin{aligned}&\frac{e(\xi(L^{k_1}\oplus\cdots\oplus L^{k_n}))}{e(\xi(L^{l_1}\oplus\cdots\oplus L^{l_m}))}\\ &\qquad=\langle l_1'\rangle_F([l_1]_F(x))\cdots\langle l_m'\rangle_F([l_m]_F(x))\langle k_1\rangle_F(x)\cdots\\ &\qquad\qquad\langle k_m\rangle_F(x)\,[k_{m+1}](x)\cdots[k_n]_F(x)/1\,.\end{aligned}$$

where $l_j'l_j\equiv 1$ module p. Q.E.D.

Proof of Theorem A. For brevity, we put $e_a=e(\xi(V_a))$ and $e_b=e(\xi(V_b))$. We show by induction with respect to the length of the partition ω that

$$\tilde{S}_\omega^U\left(\frac{e_a}{e_b}\right)=\frac{e_a}{e_b}\cdot\frac{h_\omega(x)\cdot x^n}{1}$$

where $h_\omega(x)\in MU^*(BZ_p)$. By using (2) of Proposition 5.3 we obtain

$$\tilde{S}_{(i)}^U\left(\frac{e_a}{1}\right)=\tilde{S}_{(i)}^U\left(\frac{e_a}{e_b}\right)\cdot\frac{e_b}{1}+\frac{e_a}{e_b}\cdot\tilde{S}_{(i)}^U\left(\frac{e_b}{1}\right).$$

Hence it follows from (3) of Propositions 5.3 and 5.4 that

$$\tilde{S}^U_{(i)}\left(\frac{e_a}{e_b}\right)=\frac{e_a}{e_b}\cdot\frac{c^U_{(i)}(\xi(V_a))-c^U_{(i)}(\xi(V_b))}{1}.$$

Proposition 5.2 implies that there is an element $h_{(i)}(x)\in MU^*(BZ_p)$ such that $c^U_{(i)}(\xi(V_a))-c^U_{(i)}(\xi(V_b))=h_{(i)}(x)x^n$, and

$$\tilde{S}^U_{(i)}\left(\frac{e_a}{e_b}\right)=\frac{e_a}{e_b}\cdot\frac{h_{(i)}(x)x^n}{1}.$$

Suppose the result is proved for ω' whose length is less than the length of ω. By using (2) of Proposition 5.3 with the inductive hypothesis we calculate

$$\tilde{S}^U_\omega\left(\frac{e_a}{1}\right)=\tilde{S}^U_\omega\left(\frac{e_a}{e_b}\cdot\frac{e_b}{1}\right)$$

$$=\tilde{S}^U_\omega\left(\frac{e_a}{e_b}\right)\cdot\frac{e_b}{1}+\tilde{S}^U_\omega\left(\frac{e_b}{1}\right)\cdot\frac{e_a}{e_b}+\sum_{\omega=(\omega'\omega'')}\frac{e_a}{e_b}\cdot\frac{h_{\omega'}(x)x^nS^U_{\omega''}(e_b)}{1}$$

where $h_{\omega'}(x)\in MU^*(BZ_p)$. Moreover it follows from Propositions 5.4 and 5.2 that there exists an element $\tilde{h}_\omega(x)\in MU^*(BZ_p)$ such that

$$\tilde{S}^U_\omega\left(\frac{e_a}{e_b}\right)=\frac{e_a}{e_b}\tilde{h}_\omega(x)x^n/1-\sum_{\omega=(\omega'\omega'')}\frac{e_a}{e_b}\{h_{\omega'}(x)x^nc^U_{\omega''}(\xi(V_b))\}/1,$$

and there is an element $h_\omega(x)\in MU^*(BZ_p)$ such that

$$\tilde{S}^U_\omega\left(\frac{e_a}{e_b}\right)=\frac{e_a}{e_b}h_\omega(x)x^n/1.$$

It is pointed out by [9] that the canonical map λ: $MU^*(BZ_p)\to S^{-1}MU^*(BZ_p)$ with $\lambda(x)=x/1$ has the kernel which is an ideal generated by $\langle p\rangle_F(x)$. We then complete the proof.

§ 6. On the Bordism Classes of Actions on Invariant Spheres around the Isolated Fixed Points

The Thom homomorphism μ: $MU^*(-)\to H^*(-)$ is the multiplicative natural transformation with the following properties.

Proposition 6.1. *Let ζ be a complex vector bundle over X. Then*

(1) $\mu c_\omega^U(\zeta) = c_\omega^H(\zeta)$

(2) $\mu\Phi_U(x) = \Phi(\mu(x))$, *where* Φ_U: $MU^*(X) \to \widetilde{MU}^*(T(\zeta))$ *and* Φ: $H^*(X) \to \tilde{H}^*(T(\zeta))$ *are the Thom homomorphisms.*

Recall the following property of the Umkehr homomorphism [8].

Proposition 6.2. $g_!(g^*(x) \cup y) = x \cup g_!(y)$.

We observe S_ω^H: $MU^*(X) \to H^*(X)$ for a weakly complex manifold X.

Proposition 6.3. *Take an element* $x = [M \xrightarrow{g} X] \in MU_*(X)$, *where* X *is a weakly complex manifold and* g *is a differentiable map. Then,*

$$S_\omega^H D_{MU}^{-1}(x) = \sum_{\omega=(\omega'\omega'')} c_{\omega'}^H(\tilde{\tau}(X))\, g_!(c_{\omega''}^H(\nu))$$

where ν *is the normal bundle of* M *in a Euclidean space with the complex structure and* $\tilde{\tau}(X)$ *is the Whitney sum of* $\tau(X)$ *and some trivial bundle which is a complex bundle.*

Proof. Let $\tilde{g}$: $M \to X \times R^l$ be an embedding with the normal bundle $\tilde{\nu}$ equipped with a complex structure and $\tilde{g} \simeq g$. $D_{MU}^{-1}(x)$ is represented by the composition

$$S^l \wedge X^+ \xrightarrow{c} T(\tilde{\nu}) \xrightarrow{\hat{g}} MU(k)$$

which c is the collapsing map and $\hat{g}$ is the map induced by the classifying map for ν. The Whitney sum $\tilde{\nu} \oplus \tau(M)$ is stably equivalent to $g^! \tau(X)$ and

$$c_t^H(\tilde{\nu}) \cdot c_t^H(\tilde{\tau}(M)) = g^* c_t^H(\tilde{\tau}(X)).$$

Hence we have that $c_t^H(\tilde{\nu}) = g^* c_t^H(\tilde{\tau}(X)) \cdot c_t^H(\nu)$. We calculate with Propositions 6.1 and 6.2

$$\begin{aligned} S_\omega^H D_{MU}^{-1}(x) &= \mu S_\omega^U D_{MU}^{-1}(x) = \sigma^{-l} c^* \{\Phi(c_\omega^H(\tilde{\nu}))\} \\ &= g_!(c_\omega^H(\tilde{\nu})) \\ &= g_!\Big(\sum_{\omega=(\omega'\omega'')} g^*(c_{\omega'}^H(\tilde{\tau}(X))\, c_{\omega''}^H(\nu))\Big) \end{aligned}$$

$$=\sum_{\omega=(\omega'\omega'')} c^H_{\omega'}(\tilde{\tau}(X))\, g_!(c^H_{\omega''}(\nu)).$$

Q.E.D.

MU^k is isomorphic to MU_{-k} and a bordism class $[M]$ of a weakly almost complex manifold can be regarded to be in MU^*. Directly Proposition 6.3 implies

Corollary 6.4. *$\mu S^U_\omega[M]=\langle c^H_\omega(\nu), [M]\rangle$, where ν is the normal vector bundle of M in a Euclidean space which is equipped with the complex structure, where $c^H_{(i_1,\dots,i_l)}$ is the Chern class for $\sum t_1^{i_1}\cdots t_l^{i_l}$.*

We consider the ideal $\mathcal{I}_p$ in MU^* which is generated by p, $a_1^{(p)}$, $a_2^{(p)}, \cdots, a_k^{(p)}, \cdots$ which are coefficients of

$$[p]_F(x)=px+a_1^{(p)}x^2+a_2^{(p)}x^3+\cdots.$$

We recall the following property of $\mathcal{I}_p$.

Proposition 6.5 (cf. [9]). *$[M]$ belongs to $\mathcal{I}_p$ if and only if $c^H_\omega[M]=\langle c^H_\omega(\tau(M)), [M]\rangle\equiv 0$ modulo p, for any ω, where p is prime.*

Proof. Let $y=c^1_U(\eta)$ be the cobordism first Chern class of the Hopf bundle η over CP^∞. It is known (cf. [14], [17]) that

$$S^U_\omega([p]_F(y))=\begin{cases}\{[p]_F(y)\}^{i+1} & \text{if } \omega=(i)\\ 0 & \text{otherwise.}\end{cases}$$

We see $S^H_\omega([p]_F(y))\equiv 0$ modulo p, and

$$S^H_\omega(py+a_1^{(p)}y^2+a_2^{(p)}y^3+\cdots)\equiv 0 \quad \text{modulo } p.$$

Then we can deduce that $S^H_\omega(a_i^{(p)})\equiv 0$ modulo p. Therefore we have that the Chern numbers of $[N]$ are zero modulo p if $[N]$ belongs to $\mathcal{I}_p$. The Hopf bundle $\tilde{\eta}$ over CP^n satisfies that

$D_{MU}(c^1_U(\tilde{\eta}^q))=q[CP^{n-1}\subset CP^n]+a_1^{(q)}[CP^{n-2}\subset CP^n]+\cdots+a_{n-1}^{(q)}[P\subset CP^n]$, in $MU_*(CP^n)$. Let $D_{MU}(c^1_U(\tilde{\eta}^q))=[V^{n-1}_{(q)}\subset CP^n]$, then

$$(*)\quad [V^{n-1}_{(q)}]=q[CP^{n-1}]+a_1^{(q)}[CP^{n-2}]+\cdots+a_{n-1}^{(q)}.$$

We note that $V^{n-1}_{(q)}$ is a U-submanifold dual to $c^1_H(\tilde{\eta}^q)$ (cf. [7, p. 81]), and the fundamental classes of $V_{(q)}^{n-1}$ and CP^n satisfy that $i_*[V_{(q)}^{n-1}]=c_H^1(\tilde{\eta}^q)\cap[CP^n]$, i: $V_{(q)}^{n-1}\subset CP^n$. Noting that the normal bundle ν of $V_{(q)}^{n-1}$ in CP^n is isomorphic to $i^!\tilde{\eta}^q$, we have that $c_{(n-1)}^H(\tau(V_{(q)}^{n-1}))=i^*\{(n+1)-q^{n-1}\}\tilde{y}^{n-1}$, where $\tilde{y}=c_H^1(\tilde{\eta})$. Therefore it follows that the Chern number $c_{(n-1)}^H[V_{(q)}^{n-1}]=q(n+1)-q^n$. Using (*) and $c_{(n-1)}^H[CP^{n-1}]=n$, we have $c_{(n-1)}^H[a_{n-1}^{(q)}]=q-q^n$. For prime q, we take

$$[W_{q^k-1}]=a_{q^k-1}^{(q)}+q^b[CP^u],\ b=q^k-k \ \text{ and } \ u=q^k-1$$

whose Chern number $c_{(q^k-1)}^H[W_{q^k-1}]$ equals to q. Take a $2i$-dimensional weakly almost complex manifold W_i, $i\neq q^k-1$ for any prime q, such that $c_{(i)}^H[W_i]=1$. According to [16], $MU^*=Z[[W_1],\ [w_2],\cdots]$. Assume that $c_\omega^H[M]\equiv 0$ modulo p for any ω and

$$[M]=\sum a_{i_1\cdots i_n}[W_1]^{i_1}\cdots[W_n]^{i_n}.$$

Noting that

$$S^H_{(\underbrace{1,\cdots,1}_{i_1},\underbrace{2,\cdots,2}_{i_2},\cdots,\underbrace{n,\cdots,n}_{i_n})}[W_1]^{i_1}[W_2]^{i_2}\cdots[W_n]^{i_n}$$
$$=(c_{(1)}^H[W_1])^{i_1}(c_{(2)}^H[W_2])^{i_2}\cdots(c_{(n)}^H[W_n])^{i_n},$$

we inductively deduce that if $i_s=0$ for $s=p^k-1$, then $a_{i_1i_2\cdots i_n}\equiv 0$ modulo p, and $[M]\in\mathcal{I}_p$. Q.E.D.

We now go back to consider the cobordism Euler class of complex vector bundle $\xi(V_a)$: $EZ_p\times_{Z_p}V_a\to BZ_p$, V_a the complex Z_p-module given by the tangent space at the isolated fixed points of a Z_p-manifold.

Proposition 6.6. *Suppose that V_a and V_b are complex Z_p-modules given by tangent spaces at isolated fixed points a and b of a simply connected almost complex Z_p-manifold M, and $\lambda(\alpha)=e(\xi(V_a))/e(\xi(V_b))$, where λ: $MU^*(BZ_p)\to S^{-1}MU^*(BZ_p)$ is the canonical homomorphism. If $H^i(BZ_p;\{\pi_i(M)\})\cong 0$ for $1\leqq i\leqq 2n-1$, then*

$$\alpha=\lambda_0+\lambda_1x+\lambda_2x^2+\cdots$$

where $\lambda_1,\lambda_2,\cdots,\lambda_{n-1}$ belong to $\mathcal{I}_p$.

Proof. Suppose that $|\omega|=2i$, $1\leqq i\leqq n-1$. Then $S_\omega^U\lambda_k\in MU^{2i-2k}$.

Note that μ: $MU^k(P)\to H^k(P)$, $P=\{\text{a point}\}$, is the zero homomorphism for $k>0$, and $S^U_\omega(\lambda_0)=0$ if $\omega\neq(0)$. Suppose that λ_j, $j=1,2,\cdots,i-1$, belong to $\mathcal{J}_p$. Then

$$\mu S^U_\omega(\alpha)=\mu S^U_\omega(\lambda_i)\cdot x^i_H=c^H_\omega[\lambda_i]x^i_H,$$

where $x_H=c^1_H(\xi(L))$. Since $S^U_\omega(\alpha)$ belongs to an ideal generated by x^n and $\langle p\rangle_F(c^1_U(\xi(L)))$ from Theorem A, $c^H_\omega[\lambda_i]x^i_H=0$ in $H^*(BZ_p)$. Proposition 6.5 implies that $\lambda_i\in\mathcal{J}_p$. Q.E.D.

Proof of Theorem B. Let $\tilde{\xi}(V)$ be a complex vector bundle $S^{2k-1}\times_{Z_p}V\to S^{2k-1}/Z_p$, where V is a complex Z_p-module and S^{2k-1} has the Z_p-action $\psi_p(1,\cdots,1)$. Let i: $S^{2k-1}/\psi_p(1,\cdots,1)\to BZ_p$ be the natural injection. Put $x=c^1_U(\xi(L))$ and $\bar{x}=c^1_U(\tilde{\xi}(L))$. Then, $i^!\xi(L)\cong\tilde{\xi}(L)$. We see that in $S^{-1}MU^*(BZ_p)$,

$$l_1\cdots l_k\frac{x^k}{e(\xi(V_a))}-m_1\cdots m_k\frac{x^k}{e(\xi(V_b))}$$

$$=l_1\cdots l_k\frac{x^k}{e(\xi(V_a))}-m_1\cdots m_k\frac{x^k}{e(\xi(V_a))}\cdot\frac{e(\xi(V_a))}{e(\xi(V_b))}.$$

On the other hand it follows from Proposition 6.6 that

$$m_1\cdots m_k\langle l_1\rangle_F(x)\langle m_1'\rangle_F([m_1]_F(x))\cdots\langle l_k\rangle_F(x)\langle m_k'\rangle_F([m_k]_F(x))$$
$$\equiv l_1\cdots l_k+h(x)x^n \quad \text{modulo } \mathcal{J}_p$$

where $m_im_i'\equiv1$ modulo p. Therefore we get

$$l_1\cdots l_k\langle l_1'\rangle_F([l_1]_F(x))\cdots\langle l_k'\rangle_F([l_k]_F(x))$$
$$-m_1\cdots m_k\langle m_1'\rangle_F([m_1]_F(x))\cdots\langle m_k'\rangle_F([m_k]_F(x))$$
$$\equiv\tilde{h}(x)x^n \text{ modulo } \mathcal{J}_p,\ l_il_i'\equiv1 \text{ modulo } p, \text{ where } \tilde{h}(x)\in MU^*(BZ_p).$$

Applying i^* to the above, we have

$$l_1\cdots l_k\langle l_1'\rangle_F([l_1]_F(\bar{x}))\cdots\langle l_k'\rangle_F([l_k]_F(\bar{x}))$$
$$-m_1\cdots m_k\langle m_1'\rangle_F([m_1]_F(\bar{x}))\cdots\langle m_k'\rangle_F([m_k]_F(\bar{x}))$$
$$\equiv\tilde{h}(\bar{x})\bar{x}^n \quad \text{modulo } \mathcal{J}_p \text{ (cf. [12])}.$$

Since $j_*D_{MU}\bar{x}^n=[S^{2(k-n)-1},\phi]$ (cf. [11]), Theorems 4.5 and 4.6 imply the theorem.

§ 7. The Isolated Fixed Points of Z_3-Actions

In this section we will consider an complex structure preserving smooth Z_3-action (M^{2k}, ϕ) on a simply connected closed almost complex manifold M^{2k}. Let a and b be isolated fixed points. We describe the induced actions of Z_3 on the tangent spaces at a and b as complex Z_3-modules

$$V_a = sL^2 \oplus (k-s) L$$

and

$$V_b = (s+t) L^2 \oplus (k-s-t) L .$$

Recall that

$$\langle 2 \rangle_F (x) = a_0^{(2)} + a_1^{(2)} x + a_2^{(2)} x^2 + \cdots, \quad a_i^{(2)} \in MU^{-2i}$$

and

$$c_{(n)}^H (a_n^{(2)}) = 2 - 2^{n+1} .$$

In this situation we shall first indicate a lemma which is derived as proof of Theorem B.

Lemma 7.1. *Suppose that* $H^i(BZ_3; \{\pi_i(M^{2k})\}) \cong 0$ *for* $1 \leqq i \leqq 2n-1$. *Then for* $1 \leqq j \leqq n-1$

$$\textstyle\sum_{i_1 + \cdots + i_t = j} a_{i_1}^{(2)} \cdots a_{i_t}^{(2)} \quad \textit{belong to } \mathcal{J}_3 .$$

Proof. In $S^{-1}MU^*(BZ_3)$, $MU^*(BZ_3) \cong MU^*[[x]]/[3]_F(x)$, we have

$$\frac{e(V_a)}{e(V_b)} = \mu_0 + \mu_1 x + \cdots + \mu_k x^k + \cdots, \ \mu_1, \cdots, \mu_{n-1} \in \mathcal{J}_3$$

from Proposition 6.6 and

$$\frac{2^s x^k}{e(V_a)} - \frac{2^{s+t} x^k}{e(V_b)} = \tilde{\mu}_1 x + \tilde{\mu}_2 x^2 + \cdots + \tilde{\mu}_k x^k + \cdots ,$$

$$\tilde{\mu}_1, \cdots, \tilde{\mu}_{n-1} \in \mathcal{J}_3 .$$

Noting the fact that the kernel of the canonical map λ: $MU^*(BZ_3) \rightarrow$

$S^{-1}MU^*(BZ_3)$ is the ideal generated by $\langle 3\rangle_F(x)$, we obtain

$$2^s x^k e(V_b) - 2^{s+t} x^k e(V_a)$$
$$= e(V_a)\, e(V_b)\, \{\tilde{\mu}_1 x + \tilde{\mu}_2 x^2 + \cdots + \tilde{\mu}_k x^k + \cdots\}$$

and

$$2^s(\{\langle 2\rangle_F(x)\}^t - 2^t)$$
$$= \hat{\mu}_1 x + \hat{\mu}_2 x^2 + \cdots + \hat{\mu}_k x^k + \cdots,\ \hat{\mu}_1, \cdots, \hat{\mu}_{n-1} \in \mathcal{I}_3 . \qquad \text{Q.E.D.}$$

Then we obtain the following

Lemma 7.2. *Suppose that* $H^i(BZ_3; \{\pi_i(M^{2k})\}) \cong 0$ *for* $1 \leqq i \leqq 2n - 1$. *Then, for* $1 \leqq m \leqq n-1$ *the binomial coefficients* $\binom{t}{m}$ *are divisible by* 3.

Proof. We take a partition

$$\omega = (\underbrace{k, \cdots, k}_{j_k}, \cdots, \underbrace{2, \cdots, 2}_{j_2}, \underbrace{1, \cdots, 1}_{j_1}, \underbrace{0, \cdots, 0}_{j_0})$$

of k, where

$$|\omega| = 1\cdot j_1 + 2\cdot j_2 + \cdots + k\cdot j_k = k$$

and

$$j_0 + j_1 + \cdots + j_k = t .$$

We define now

$$\|\omega\| = j_1 + \cdots + j_k ,$$
$$a_\omega^{(2)} = \{a_k^{(2)}\}^{j_k} \cdots \{a_1^{(2)}\}^{j_1} \{a_0^{(2)}\}^{j_0}$$

and

$$\lambda_\omega = \frac{t!}{j_k! \cdots j_2!\, j_1!\, j_0!} .$$

Then we have the following

$$\sum_{i_1+\cdots+i_t=j} a_{i_1}^{(2)} \cdots a_{i_t}^{(2)} = \sum_{|\omega|=j} \lambda_\omega a_\omega^{(2)} .$$

We take up the case $k=1$. Since from Lemma 7.1 $2^{t-1} t \cdot a_1^{(2)} = \sum_{i_1+\cdots+i_t=1}$

$a_{i_1}^{(2)}\cdots a_{i_t}^{(2)}$ belongs to $\mathcal{J}_3$, and $c_{(1)}^H(a_1^{(2)}) = -2$, t is divisible by 3. Assume that $m<n$ and $\binom{t}{j}$, $j=1, \cdots, m-1$, are divisible by 3. From Lemma 7.1 $\sum_{\|\omega\|=m}\lambda_\omega a_\omega^{(2)}$ belongs to $\mathcal{J}_3$, and for $\|\omega\|\leqq m-1$

$$\lambda_\omega = \frac{\|\omega\|!}{j_k!\cdots j_2!\,j_1!}\cdot\binom{t}{\|\omega\|}\equiv 0 \quad \text{modulo } 3\,.$$

By induction we complete the proof.

We shall give some information on isolated fixed points of Z_3-actions.

Theorem 7.3. *Let a and b be isolated fixed points of a complex structure preserving smooth action of Z_3 on the simply connected closed almost complex manifold M^{2k}. Suppose that*

$$k=\lambda_u 3^u+\lambda_{u-1}3^{u-1}+\cdots+\lambda_1 3+\lambda_0,\ \ 0\leqq\lambda_j\leqq 2 \ \ \textit{and}\ \ \lambda_u\neq 0$$

and

$$H^i(BZ_3;\{\pi_i(M^{2k})\})\cong 0 \quad \textit{for} \quad 1\leqq i\leqq 2\cdot 3^u+1\,.$$

Then V_a is equivalent to V_b.

Proof. Let $V_a=sL^2\oplus(k-s)L$ and $V_b=(s+t)L^2\oplus(k-s-t)L$. Suppose that $t=\lambda'_u 3^u+\lambda'_{u-1}3^{u-1}+\cdots+\lambda'_1 3+\lambda'_0\leqq k$. It follows from Lemma 7.2 that

$$\lambda'_l\equiv\binom{t}{3^l}\equiv 0 \quad \text{modulo } 3\,.$$

Hence $\lambda'_l=0$ and $t=0$. Q.E.D.

Corollary 7.4. *Suppose that Z_3 acts on a simply connected almost complex closed 2k-dimensional manifold M as a complex structure preserving deffeomorphism with isolated fixed points only. Let $k=\lambda_u 3^u+\cdots+\lambda_1 3+\lambda_0$, $0\leqq\lambda_j\leqq 2$, and $\lambda_u\neq 0$. If $H^i(BZ_3;\{\pi_i(M)\})\cong 0$ for $1\leqq i\leqq 2\cdot 3^u+1$, then the number of fixed points is divisible by $3^{[(k-1)/2]+1}$.*

Proof. Let n be the number of the fixed points. Theorem 7.3

implies that

$$n[S(V_a), \phi_a] = 0 \quad \text{in} \quad MU_*(Z_3)$$

where $V_a = sL^2 + (k-s)L$. The Kasparov theorem (Theorem 4.6) implies that

$$n(l+3m)[S^{2k-1}, \tilde{\phi}] + \mu_1[S^{2k-3}, \tilde{\phi}] + \cdots + \mu_{k-1}[S^1, \tilde{\phi}] = 0$$

where $l \not\equiv 0$ modulo 3 and $\mu_i \in \Gamma(3)$, $\Gamma(3)[[CP^2]] = MU_*$ (cf. [6], [11]). From the result of [6] and [11] we can derive the assersion.

References

[1] Adams, J. F., *Stable homotopy and generalised homology*, Chicago, Univ. of Chicago Math. Lecture Notes, 1974.

[2] Araki, S., *Typical formal group in cobordism and K-theory*, Lecture in Math. Kyoto Univ., Kinokuniya, 1973.

[3] Atiyah, M. F., Bordism and cobordism, *Proc. Camb. Phil. Soc.*, **57** (1961), 200–208.

[4] Bredon, G. E., *Introduction to compact transformation groups*, Academic Press, 1972.

[5] Conner, P. E., and Floyd, E. E., *Differentiable periodic maps*, Springer, Berlin-Heiderberg-New York, 1964.

[6] ———, Periodic maps which preserve a complex structure, *Bull. Amer. Math. Soc.*, **70** (1964), 574–579.

[7] ———, *The relation of cobordism to K-theories, Lecture Notes in Math.*, **28**, Springer, Berlin-Heiderberg-New York, 1966.

[8] Dyer, E., *Cohomology theories*, Benjamin, 1969.

[9] Tom Dieck, T., Actions of finite p-groups without stationary points, *Topology*, **9** (1970), 359–366.

[10] ———, Lokalisierung äquivarianter Kohomologie-Theorien, *Math. Z.*, **121** (1971), 253–262.

[11] Kamata, M., The structure of the bordism group $U_*(BZ_p)$, *Osaka J. Math.*, **7** (1970), 409–416.

[12] ———, Notes on the cobordism group $U^*(L^n(m))$, *Osaka J. Math.*, **9** (1972), 287–292.

[13] Kasparov, G. G., Invariants of classical lens manifolds in cobordism theory, *Izv. Akad. Nauk SSSR. Ser. Math.*, **33** (1969), 753–747, *Math., USSR. Izv.*, **3** (1969), 695–706.

[14] Landweber, P. S., Cobordism operations and Hopf algebras, *Trans. Amer. Math. Soc.*, **129** (1967), 94–110.

[15] ———, Coherence, flatness and cobordism of classifying spaces, *Proc. Adv. Study Inst. Alg. Top., Aarhus* (1970), 256–269.

[16] Milnor, J. W., On the cobordism ring Ω^* and a complex analogue, Part I, *Amer. J. Math.*, **82** (1960), 505–521.

[17] Novikov, S. P., The method of algebraic topology from the viewpoint of cobordism theories, *Izv. Akad. Nauk SSSR*, **31** (1967), *Math. USSR Izv.*, **1** (1967), 827–913.

[18] Ozeki, H. and Uchida, F., Principal circle actions on a product of spheres, *Osaka J, Math.* **9** (1972), 379–390.

[19] Quillen, D., Elementary proofs of some results of cobordism theory using Steenrod operations, *Advances on Math.*, **7** (1971), 29–56.

[20] Switzer, R. M., *Algebraic topology-homotopy and homology*, Springer-Verlag, Berlin-Heiderberg-New York, 1975.