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Time Dependent Representations of the Stationary Wave Operators for "Oscillating" Long-Range Potentials

By

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Introduction

Since the original paper of Dollard [7], the long-range scattering theory for the Schrödinger operators $-\Delta+V(x)$ has been studied by many authors (e.g., Buslaev-Matveev [5], Amrein-Martin-Misra [2], Alsholm-Kato [1], Hörmander [9], Kitada [13], Ikebe-Isozaki [10] and Kako [11]). These works treat the case that the potential $V(x)$ approaches zero without too much oscillation at infinity:

$$(0.1)\qquad \nabla^{\alpha}V(x)=0(r^{-|\alpha|-\delta})\,(|\alpha|=0,1,2,\ldots)\quad\text{for some }\delta>0$$

($\nabla=\nabla_x$ is the gradient in $\mathbb{R}^n$, $r=|x|$ and $\alpha=(\alpha_1,\ldots,\alpha_n)$ are multi-indices with $|\alpha|=\alpha_1+\cdots+\alpha_n$), and prove the existence ([1], [2], [5], [9], [11]) and the completeness ([10], [13]) of the modified wave operators

$$(0.2)\qquad W_D^{\pm}=s\text{-}\lim_{t\to\pm\infty}\exp\{iLt\}\exp\{-iL_0t-iX_{\pm}(p,t)\}\quad\text{in }L^2(\mathbb{R}^n),$$

where $L_0=-\Delta$, $L=-\Delta+V(x)$ on $L^2(\mathbb{R}^n)$, $i=\sqrt{-1}$, $p=-i\nabla_x$ and $X_{\pm}(\xi,t)$, $\xi\in\mathbb{R}^n$, solve the equations

$$(0.3)\qquad \partial_t X_{\pm}(\xi,t)=V(2\xi t+\nabla_{\xi}X_{\pm}(\xi,t))\quad(\partial_t=\partial/\partial t)$$

near $t=\pm\infty$. The selfadjoint operators $X_{\pm}(p,t)$ are called time dependent modifiers for L.

Stationary modifiers $Y_{\pm}(x,\lambda)$, $\lambda\in\mathbb{R}-\{0\}$, solve the equation

$$(0.4)\qquad \mp2\sqrt{\lambda}\partial_r Y_{\pm}(x,\lambda)+|\nabla Y_{\pm}(x,\lambda)|^2+V(x)=0(r^{-1-\delta})$$

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near infinity. As we see in [10] and [13], $Y_{\pm}(x, \lambda)$ can be obtained from $X_{\pm}(\xi, t)$ by a kind of Legendre transformation in classical mechanics and are used to establish the completeness of $W_D^{\pm}$. On the other hand, $Y_{\pm}(x, \lambda)$ are directly used in [11] to obtain another formulation of the modified wave operators. Let $\mathscr{E}_0(\lambda)$, $\lambda \in \mathbb{R}$, be the spectral measure of L_0. Then in [11] is proved the following: For any pre-compact set $e \Subset (0, \infty)$, the limits

$$W_J^{\pm}(e) = s\text{-}\lim_{t \to \pm\infty} \exp\{iLt\} J_{\pm}(e) \exp\{-iL_0 t\} \mathscr{E}_0(e) \text{ in } L^2(\mathbb{R}^n) \tag{0.5}$$

exist, are isometry on $\mathscr{E}_0(e)L^2(\mathbb{R}^n)$ and coincide with $W_D^{\pm}\mathscr{E}_0(e)$, where $J_{\pm}(e)$: $\mathscr{E}_0(e)L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$ are identification operators (cf. Kato [12]) defined by

$$J_{\pm}(e)f = (2\pi)^{-n/2} \int_{G(e)} \exp\{ix \cdot \xi - iY_{\pm}(x, |\xi|^2)\} \hat{f}(\xi) d\xi \tag{0.6}$$

with $\hat{f}(\xi)$ being the Fourier transform of $f(x)$ and $G(e) = \{\xi;\ |\xi|^2 \in e\}$.

In this paper we shall partly extend the above mentioned results to a class of "oscillating" long-range potentials settled in our previous papers [14] and [15] (the definite conditions on $V(x)$ will be given in Section 1). Our main purpose is to show that modified wave operators of the form (0.5) exist and are complete for each $e \Subset (\Lambda_{\bar{\delta}}, \infty)$, where the real number $\Lambda_{\bar{\delta}}$ depends on the asymptotic conditions at infinity of $V(x)$. The results will be summarized in Theorem of Section 4.

Our "oscillating" long-range class includes the following examples:

$$V(x) = c(x) + V_s(x), \tag{E.1}$$

$$V(x) = \frac{c(x)}{\log r} + V_s(x) \qquad (r = |x|), \tag{E.2}$$

$$V(x) = c(x)\sin(\log r) + V_s(x), \tag{E.3}$$

$$V(x) = \frac{\lambda(x)\sin \mu r}{r} + V_s(x), \tag{E.4}$$

where $c(x)$ (real) satisfies the conditions

$$c(x) = 0(1),\ \nabla^{\alpha} c(x) = 0(r^{-1-|\alpha|\delta}) \quad (|\alpha| = 1, 2,\ 1/2 < \delta < 1)$$

near infinity, $\lambda(x)$ (real) satisfies the conditions

$$\nabla^{\alpha}\lambda(x) = 0(r^{-|\alpha|\delta}) \quad (|\alpha| = 0, 1, 2,\ 1/2 < \delta < 1)$$

near infinity, μ is a real number and $V_s(x)$ (real) is short-range, i.e., $V_s(x) = 0(r^{-1-\delta_0})$ $(\delta_0 > 0)$ near infinity. Note that (E.1) generalizes the usual $\frac{1}{2}$ long-range potential which satisfies (0.1) with $\delta>1/2$. Namely, by the terminology "*oscillating*" *long-range potentials* we never exclude ones which are in the frame of *ordinary* long-range potentials.

Now, in (0.5) we take $L_0=-\Delta+\Lambda_{\bar{\delta}}$ on $L^2(\mathbb{R}^n)$. This choice of the free Hamiltonian mainly depends on the fact that we allow the case $\Lambda_{\bar{\delta}}<0$. In fact, for the above examples $\Lambda_{\bar{\delta}}$ is given by (cf. (1.2) and Assumption 2 of Section 1)

$$(0.7)\qquad \Lambda_{\bar{\delta}}=\begin{cases} c_\infty & \text{for (E.1)},\\ 0 & \text{for (E.2)},\\ |c_\infty|\sqrt{1+\varepsilon^{-2}} & \text{for (E.3)},\\ \lambda_\infty|\mu|/\varepsilon+\mu^2/4 & \text{for (E.4)},\end{cases}$$

where $c_\infty=\lim_{r\to\infty} c(x)$, $\lambda_\infty=\limsup_{r\to\infty}\dfrac{\lambda(x)\mu\cos\mu r}{|\mu|}$ and $\varepsilon=4\min\{\delta_0, 2\delta-1, 1/2\}$, and so we have $\Lambda_{\bar{\delta}}<0$ for (E.1) if $c_\infty<0$. Further, in our case, equation (0.4) does not work well, and it is necessary to define $J_\pm(e)$ in a different manner. Let $\rho_\pm(x,\lambda)$, $\lambda>\Lambda_{\bar{\delta}}$, be two solutions, specified in [15], of the equation

$$(0.8)\qquad \partial_r^2\rho+\frac{n-1}{r}\partial_r\rho-(\partial_r\rho)^2+V(x)-\lambda=0(r^{-1-\bar{\delta}})$$

near infinity. Then our identification operators $J_\pm(e)$, $e\Subset(\Lambda_{\bar{\delta}},\infty)$, are defined by

$$(0.9)\qquad J_\pm(e)f=\frac{\pm 1}{2i\sqrt{\pi}}\int_e \exp\{-\rho_\pm(x,\lambda)\}[\mathscr{F}_0 f](\lambda,\tilde{x})d\lambda,$$

where $\tilde{x}=x/|x|$ and $\mathscr{F}_0$: $L^2(\mathbb{R}^n)\to L^2((\Lambda_\delta,\infty)\times S^{n-1})$ (S^{n-1} being the unit sphere in $\mathbb{R}^n$) is a spectral representation of $L_0=-\Delta+\Lambda_{\bar{\delta}}$:

$$(0.10)\qquad [\mathscr{F}_0 f](\lambda,\tilde{x})=\frac{1}{\sqrt{2}}(\lambda-\Lambda_\delta)^{(n-2)/4}\\ \times(2\pi)^{-n/2}\int_{\mathbb{R}^n}\exp\{-i\sqrt{\lambda-\Lambda_\delta}\tilde{x}\cdot y-i\pi(n-3)/4\}f(y)dy.$$

We shall show that the operators $\exp\{iLt\}J_\pm(e)\exp\{-iL_0t\}\mathscr{E}_0(e)$ are bounded in $\mathscr{E}_0(e)L^2(\mathbb{R}^n)$ and strongly converge as $t\to\pm\infty$ to the stationary wave operators $U_\pm(e)$. Here the existence and completeness of $U_\pm(e)$ are already established in [15]. Our argument essentially bases on [14] and [15], whose results are summarized in Section 1. To show the boundedness of $J_\pm(e)$ we shall follow a method of Calderón-Vaillancourt [6] on the L^2-boundedness of pseudo-differential operators (Sections 2 and 3). On the other hand, for the proof of the convergence the stationary phase method will play an important role (Sections 4 and 5).

As we see in (0.7) for (E.3) and (E.4), $(\Lambda_{\mathfrak{z}}, \infty)$ does not in general cover the essential spectrum of $L=-\Delta+V(x)$. In this sense it remains some ambiguousness in our theory.

Here we note that potentials of the special form

$$V(x)=V(r)=\frac{\lambda \sin \mu r^{\alpha}}{r^{\beta}}+V_s(r) \quad (\lambda, \mu \text{ are real constants}) \tag{0.11}$$

including the case $\alpha=\beta=1$ (cf. (E.4)) have been studied by Dollard-Friedman [8], Ben-Artzi-Devinatz [4] and others. They reduce the problem to the study of ordinary differential operators on the half line $\boldsymbol{R}_+=(0, \infty)$, and prove the absolute continuity of the positive spectrum $(0, \infty)$ of $-\Delta+V(r)$ except for one possible eigenvalue $\mu^2/4$, and the existence and completeness of the Møller wave operators. In this paper, we do not assume that $V(x)$ is spherically symmetric. However, our results for the concrete potential (0.11) with $\alpha=\beta=1$ (von Neumann-Wigner's adiabatic oscillator) is weaker than theirs. Also we have not shown whether or not our wave operators are equivalent to the ordinary Møller ones.

In case $V(x)=V_s(x)$, we can see that our modified wave operators coincide with the Møller wave operators modulo some simple unitary operators. Similar results can also be expected to the potential $V(x)$ which is improper integrable in $r=|x|\in\boldsymbol{R}_+$. It remains as an open problem so far.

§ 1. Assumptions and Preliminaries

Let Ω be an infinite domain in $\boldsymbol{R}^n$ with smooth compact boundary $\partial\Omega$ lying inside some sphere $S(R_0)=\{x;\ |x|=R_0\}$. We consider in Ω the Schrödinger operator $-\Delta+V(x)$, where Δ is the Laplacian and $V(x)$ is a potential function. We assume

Assumption 1. $V(x)=V_1(x)+V_s(x)$, where $V_1(x)$ is a real-valued function satisfying the "Stummel condition" for some $\mu>0$:

$$\begin{cases} \sup_{x\in\Omega}\int_{|x-y|<1}|V_1(y)|^2|x-y|^{-n+4-\mu}dy<\infty & (\text{if } \ n\geq 4), \\ \sup_{x\in\Omega}\int_{|x-y|<1}|V_1(y)|^2dy<\infty & (\text{if } \ n\leq 3), \end{cases}$$

and $V_s(x)$ is a real-valued bounded measurable function in Ω. Moreover, the unique continuation property holds for both $-\Delta+V(x)$ and $-\Delta+V_1(x)$.

Assumption 2. $V_1(x)$ is an "oscillating" long-range potential; that is, there exist some constants $C_1>0$, $R_1\geq R_0$, $a\geq 0$ and $1/2<\delta_j<1$ ($j=1, 2$) such that for any $x\in B(R_1)=\{x;\ |x|>R_1\}$,

(i) $|V_1(x)|\leq C_1$,

(ii) $|\partial_r V_1(x)|\leq C_1 r^{-1}$,

(iii) $|\partial_r^2 V_1(x)+aV_1(x)|\leq C_1 r^{-1-\delta_1}$,

(iv) $|(\nabla-\tilde{x}\partial_r)V_1(x)|\leq C_1 r^{-1-\delta_2}$,

(v) $|(\nabla-\tilde{x}\partial_r)\partial_r V_1(x)|\leq C_1 r^{-1-\delta_1}$,

(vi) $|(\nabla-\tilde{x}\partial_r)\cdot(\nabla-\tilde{x}\partial_r)V_1(x)|\leq C_1 r^{-1-2\delta_2}$.

On the other hand, $V_s(x)$ is a short-range potential; that is, there exist some constants $C_2>0$ and $0<\delta_0<1$ such that for any $x\in B(R_1)$,

(vii) $|V_s(x)|\leq C_2 r^{-1-\delta_0}$.

In the following we put $\delta=\min\{\delta_0, \delta_1, \delta_2\}$ and $\tilde{\delta}=\min\{\delta, 2\delta_2-1\}$. Note that the condition $\delta_j<1$ ($j=0, 1, 2$) does not restrict the generality.

We put

$$E(\gamma)=\limsup_{r\to\infty}\frac{1}{\gamma}\{r\partial_r V_1(x)+\gamma V_1(x)\}\qquad\text{for}\quad \gamma>0, \tag{1.1}$$

and define Λ_σ, $\sigma>0$, as follows:

$$\Lambda_\sigma=E(\min\{4\sigma, 2\})+a/4, \tag{1.2}$$

where $a\geq 0$ is the constant given in (iii) of Assumption 2. Then as is discussed in [14; §8], we have the

Lemma 1.1. *Λ_σ is non-increasing and continuous in $\sigma>0$, and*

$$\Lambda_{1/2}=\min_{\sigma>0}\Lambda_\sigma\geq\limsup_{r\to\infty}V_1(x)+a/4, \tag{1.3}$$

where

$$|V_1(x)|\leq C_3 r^{-1}\quad \textit{in}\quad B(R_1)\quad \textit{if}\quad a>0. \tag{1.4}$$

We put

$$\eta(\lambda)=4\lambda/(4\lambda-a)\quad\text{for}\quad \lambda>\Lambda_{1/2}. \tag{1.5}$$

Note that $\eta(\lambda)\equiv 1$ if $a=0$. Then by means of (1.3) and (1.4) we can easily prove the following

Lemma 1.2. *Let ε be any constant satisfying $0<\varepsilon<1$. Then there exist some constants $C_4>0$ and $R_1'\geq R_1$ depending only on ε such that for any $(x, \lambda)\in B(R_1')\times[\Lambda_{1/2}+\varepsilon, \infty)$,*

$$\varepsilon/2\leq\lambda-\eta(\lambda)V_1(x)\leq|\lambda|+C_4, \tag{1.6}$$

$$1/2\leq\partial_\lambda\{\lambda-\eta(\lambda)V_1(x)\}\leq 2 \qquad (\partial_\lambda=\partial/\partial\lambda), \tag{1.7}$$

$$|\partial_\lambda^l\{\lambda-\eta(\lambda)V_1(x)\}|\leq C_4r^{-1} \qquad (l=2\sim 6). \tag{1.8}$$

For some $R_2\geq R_1'$ and any $(x, \lambda)\in B(R_2)\times[\Lambda_{1/2}+\varepsilon, \infty)$ we put

$$\begin{aligned}\rho_\pm(x, \lambda)\\ =\mp i\int_{R_2}^r\sqrt{\lambda-\eta(\lambda)V_1(s\tilde{x})}ds+\frac{n-1}{2}\log r+\frac{1}{4}\log\{\lambda-\eta(\lambda)V_1(x)\}.\end{aligned} \tag{1.9}$$

Then by a straight calculation (cf. Lemma 1.1 of [15]) we have the

Lemma 1.3. *There exists a constant $C_5>0$ depending on ε such that for any $(x, \lambda)\in B(R_2)\times[\Lambda_{1/2}+\varepsilon, \infty)$,*

$$|\partial_r^2\rho_\pm+\frac{n-1}{r}\partial_r\rho_\pm-(\partial_r\rho_\pm)^2+V_1(x)-\lambda|\leq C_5r^{-1-\delta_1}, \tag{1.10}$$

$$|(\nabla-\tilde{x}\partial_r)\rho_\pm|\leq C_5r^{-\delta_2}, \tag{1.11}$$

$$|(\nabla-\tilde{x}\partial_r)\partial_r\rho_\pm|\leq C_5r^{-1-\min\{\delta_1,\delta_2\}}, \tag{1.12}$$

$$|(\nabla-\tilde{x}\partial_r)\cdot(\nabla-\tilde{x}\partial_r)\rho_\pm|\leq C_5r^{-2\delta_2}. \tag{1.13}$$

For any real number μ and $G\subset\Omega$, let $L_\mu^2(G)$ denote the space of all functions $f(x)$ such that

$$\|f\|_{\mu,G}^2=\int_G(1+r)^{2\mu}|f(x)|^2dx<\infty. \tag{1.14}$$

If $\mu=0$ or $G=\Omega$, the subscript μ or G will be omitted. Let α, β be a pair of positive constants satisfying

$$0<\alpha\leq\beta\leq 1 \quad \text{and} \quad \alpha+\beta\leq 2\delta. \tag{1.15}$$

For $\lambda>\Lambda_{\beta/2}(\geq\Lambda_\delta)$ and $f\in L_{(1+\beta)/2}^2(\Omega)$ let us consider the exterior boundary-value problem

$$\begin{cases}(-\Delta+V(x)-\lambda)u(x)=f(x) & \text{in} \quad \Omega\\ Bu=\begin{cases}u\\ \text{or}\\ \nu(x)\cdot\nabla u+d(x)u\end{cases}=0 & \text{on} \quad \partial\Omega,\end{cases} \tag{1.16}$$

where $\nu(x)=(\nu_1(x),\dots, \nu_n(x))$ is the outer unit normal to the boundary $\partial\Omega$ and

$d(x)$ is a real-valued smooth function on $\partial\Omega$. The outgoing $(+)$ or incoming $(-)$ solution of (1.16) will be distinguished by the radiation condition

$$(1.17)_{\pm} \qquad u \in L^2_{-(1+\alpha)/2}(\Omega) \quad \text{and} \quad \partial_r u + (\partial_r \rho_{\pm}(x, \lambda))u \in L^2_{(-1+\beta)/2}(B(R_2)).$$

Now let L be the selfadjoint operator in $L^2(\Omega)$ defined by

$$(1.18) \qquad \begin{cases} \mathscr{D}(L) = \{u \in L^2(\Omega);\ \Delta u \in L^2(\Omega) \quad \text{and} \quad Bu = 0 \quad \text{on} \quad \partial\Omega\} \\ Lu = -\Delta u + V(x)u \quad \text{for} \quad u \in \mathscr{D}(L), \end{cases}$$

and let $R(\zeta)$ ($\zeta \in \boldsymbol{C} - \boldsymbol{R}$) and $\mathscr{E}(\lambda)$ ($\lambda \in \boldsymbol{R}$) be its resolvent and spectral measure, respectively. Then the main results of [14] and [15] can be summarized in the following propositions. To show Proposition 1.1 we require (1.10) and (1.12) (see Theorems 1—5 of [14]). To show Proposition 1.2 we require (1.10) and (1.11) (see Theorem 2.1 of [15]). (1.13) is used to show Proposition 1.3 (see Theorems 3.1, 4.1 and 6.1 of [15]).

Proposition 1.1. (a) *Let α, β be any pair satisfying* (1.15), *and let ε and N be any constants satisfying $0 < \varepsilon < 1 < N < \infty$. Then there exists a constant $C_6 > 0$ such that for any $f \in L^2_{(1+\beta)/2}(\Omega)$ (which is dense in $L^2(\Omega)$), $\lambda \in [\Lambda_{\beta/2} + \varepsilon, \Lambda_{\beta/2} + N]$ and $\tau \in (0, 1)$,*

$$(1.19) \qquad \|R(\lambda \pm i\tau)f\|_{-(1+\alpha)/2} \leq C_6 \|f\|_{(1+\beta)/2}.$$

Moreover, $R(\lambda \pm i\tau)f$ converges in $L^2_{-(1+\alpha)/2}(\Omega)$ to the unique outgoing [or incoming] solution $u_{\pm} = R_{\pm}(\lambda)f$ of (1.16) *as $\tau \downarrow 0$.*

(b) *The above convergence is uniform in $\lambda \in [\Lambda_{\beta/2} + \varepsilon, \Lambda_{\beta/2} + N]$. Thus, $R_{\pm}(\lambda)f$ is continuous in $L^2_{-(1+\alpha)/2}(\Omega)$ with respect to $(\lambda, f) \in (\Lambda_{\beta/2}, \infty) \times L^2_{(1+\beta)/2}(\Omega)$.*

(c) *Let $R^*_{\pm}(\lambda)$: $L^2_{(1+\alpha)/2}(\Omega) \to L^2_{-(1+\beta)/2}(\Omega)$ be the adjoint of $R_{\pm}(\lambda)$. Then we have for any $f \in L^2_{(1+\beta)/2}(\Omega)$ ($\subset L^2_{(1+\alpha)/2}(\Omega)$) and $\lambda \in (\Lambda_{\beta/2}, \infty)$,*

$$(1.20) \qquad R^*_{\pm}(\lambda)f = R_{\mp}(\lambda)f.$$

(d) *For any pre-compact set $e \Subset (\Lambda_{\beta/2}, \infty)$ and $f, g \in L^2_{(1+\beta)/2}(\Omega)$ we have*

$$(1.21) \qquad (\mathscr{E}(e)f, g) = (2\pi i)^{-1} \int_e (R_+(\lambda)f - R_-(\lambda)f, g)\, d\lambda,$$

where (,) denotes the inner product in $L^2(\Omega)$, or more generally, the duality between $L^2_{-(1+\alpha)/2}(\Omega)$ and $L^2_{(1+\alpha)/2}(\Omega)$. Thus, the part of L in $\mathscr{E}((\Lambda_\delta, \infty))L^2(\Omega)$ is absolutely continuous with respect to the Lebesgue measure on $\lambda \in (\Lambda_\delta, \infty)$.

Remark 1.1. Let $R_{1,\pm}(\lambda)$ be the operator $R_{\pm}(\lambda)$ with $V_s(x)\equiv 0$. In this case we can choose $\beta=1$ in (1.15), where α should be chosen as

$$0<\alpha\leq 2\min\{\delta_1,\delta_2\}-1. \tag{1.22}$$

Proposition 1.2. (a) *For any* α, β *satisfying* (1.15), *let* $f\in L^2_{(1+\beta)/2}(\Omega)$ *and* $\lambda\geq\Lambda_{\beta/2}+\varepsilon$. *Then there exists a sequence* $r_l=r_l(\alpha,\beta,f,\lambda)$ *diverging to* ∞ *such that*

$$\lim_{l\to\infty}\int_{S(r_l)}\{r^{-\alpha}|R_{1,\pm}(\lambda)f|^2+r^{\beta}|(\nabla+\tilde{x}\partial_r\rho_{\pm})R_{1,\pm}(\lambda)f|^2\}dS=0. \tag{1.23}$$

(b) *For any* $\lambda\in[\Lambda_{1/2}+\varepsilon,\infty)$ *and* $f\in L^2_1(\Omega)$, *let* $r_l=r_l(\alpha,1,f,\lambda)$, *where* α *satisfies* (1.22). *Then*

$$\mathscr{F}_{1,\pm}(\lambda, r_l)f\equiv\frac{1}{\sqrt{\pi}}\exp\{\rho_{\pm}(r_l\cdot,\lambda)\}[R_{1,\pm}(\lambda)f](r_l\cdot) \tag{1.24}$$

strongly converges in $L^2(S^{n-1})$ *as* $l\to\infty$. *Let* $\mathscr{F}_{1,\pm}(\lambda)$: $L^2_1(\Omega)\to L^2(S^{n-1})$ *be defined by*

$$\mathscr{F}_{1,\pm}(\lambda)f=\operatorname*{s-lim}_{l\to\infty}\mathscr{F}_{1,\pm}(\lambda,r_l)f \quad \text{in} \quad L^2(S^{n-1}). \tag{1.25}$$

Then we have

$$\|\mathscr{F}_{1,\pm}(\lambda)f\|^2_{L^2(S^{n-1})}=(2\pi i)^{-1}(R_{1,+}(\lambda)f-R_{1,-}(\lambda)f,f). \tag{1.26}$$

Moreover, $\mathscr{F}_{1,\pm}(\lambda)$ *is independent of the choice of* r_l.

(c) *Let* $\tilde{\alpha}, \tilde{\beta}$ *satisfy*

$$0<\tilde{\alpha}\leq\tilde{\beta}\leq 1 \quad \text{and} \quad \tilde{\alpha}+\tilde{\beta}\leq 2\tilde{\delta}=\min\{2\delta, 4\delta_2-2\} \tag{1.27}$$

((1.27) *is a stronger condition than* (1.15)). *Then for any* $\lambda\in[\Lambda_{\tilde{\beta}/2}+\varepsilon,\infty)$ *the operator* $\mathscr{F}_{1,\pm}(\lambda)$ *can be extended to a bounded operator from* $L^2_{(1+\tilde{\beta})/2}(\Omega)$ *to* $L^2(S^{n-1})$ *by continuity. Denoting the extended operator by* $\mathscr{F}_{1,\pm}(\lambda)$ *again, we have for any* $f\in L^2_{(1+\tilde{\beta})/2}(\Omega)$, $\phi\in L^2(S^{n-1})$ *and* $\lambda\in[\Lambda_{\tilde{\beta}/2}+\varepsilon,\infty)$,

$$(\mathscr{F}_{1,\pm}(\lambda)f,\phi)_{L^2(S^{n-1})}=\lim_{l\to\infty}(\mathscr{F}_{1,\pm}(\lambda,r_l)f,\phi)_{L^2(S^{n-1})}, \tag{1.28}$$

where $r_l=r_l(\tilde{\alpha},\tilde{\beta},f,\lambda)$.

Remark 1.2. In [15] we neglect the fact that R_1' in Lemma 1.2 depends on ε, and then $\mathscr{F}_{1,\pm}(\lambda)$ depends on ε and $R_2\geq R_1'$ by (1.9) and (1.25). So the above and the following propositions are corrections of [15]. Let ε' and R_1'' be another pair and let $\mathscr{F}'_{1,\pm}(\lambda)$ be the operator $\mathscr{F}_{1,\pm}(\lambda)$ corresponding to ε' and some $R_2'\geq R_1''$. Then for $\lambda\in[\Lambda_{\tilde{\beta}/2}+\varepsilon,\infty)\cap[\Lambda_{\tilde{\beta}/2}+\varepsilon',\infty)$, $\mathscr{F}'_{1,\pm}(\lambda)$ coincides with

$\mathscr{F}_{1,\pm}(\lambda)$ modulo a unitary operator on $L^2(S^{n-1})$:

$$(1.29)\quad [\mathscr{F}'_{1,\pm}(\lambda)f](\tilde{x})=\exp\left\{\pm i\int_{R_2}^{R'_2}\sqrt{\lambda-\eta(\lambda)V_1(s\tilde{x})}ds\right\}[\mathscr{F}_{1,\pm}(\lambda)f](\tilde{x})$$

for $f\in L^2_{(1+\tilde{\beta})/2}(\Omega)$.

Remark 1.3. Let $\mathscr{F}_0(\lambda)$ be the operator $\mathscr{F}_{1,+}(\lambda)$ corresponding to the selfadjoint operator $L_0=-\Delta+\Lambda_{\tilde{\delta}}$ on $L^2(\boldsymbol{R}^n)$. In this case, $\eta(\lambda)V_1(x)\equiv\Lambda_{\tilde{\delta}}$ being constant, we can choose $R_2=0$. Then for any $\lambda>\Lambda_{\tilde{\delta}}$, $\mathscr{F}_0(\lambda)f$ is represented by the right side of (0.10) (see Remark 6.2 of [15]).

Proposition 1.3. *Let $\tilde{\alpha}$, $\tilde{\beta}$ satisfy* (1.27) *and $\Lambda_{\tilde{\delta}}\leq\Lambda_{\tilde{\beta}/2}<\Lambda_{\tilde{\delta}}+\varepsilon$.*

(a) *For $\lambda\in[\Lambda_{\tilde{\beta}/2}+\varepsilon, \infty)$ let*

$$(1.30)\qquad \mathscr{F}_\pm(\lambda)=\mathscr{F}_{1,\pm}(\lambda)\{1-V_sR_\pm(\lambda)\}\,.$$

Then it defines a bounded operator from $L^2_{(1+\tilde{\beta})/2}(\Omega)$ to $L^2(S^{n-1})$. Moreover, it depends continuously on λ.

(b) *Let $\mathscr{F}_\pm$: $L^2_{(1+2\tilde{\delta})/2}(\Omega)\to L^2([\Lambda_{\tilde{\delta}}+2\varepsilon, \infty)\times S^{n-1})$ be defined by*

$$(1.31)\qquad [\mathscr{F}_\pm f](\lambda, \tilde{x})=[\mathscr{F}_\pm(\lambda)f](\tilde{x}),\ (\lambda, \tilde{x})\in[\Lambda_{\tilde{\delta}}+2\varepsilon, \infty)\times S^{n-1}.$$

Then $\mathscr{F}_\pm$ can be extended to a partial isometric operator from $L^2(\Omega)$ onto $L^2([\Lambda_{\tilde{\delta}}+2\varepsilon, \infty)\times S^{n-1})$ with initial set $\mathscr{E}([\Lambda_{\tilde{\delta}}+2\varepsilon, \infty))L^2(\Omega)$. The extended operator will be denoted by $\mathscr{F}_\pm$ again.

(c) (*Spectral representations*) *For any bounded Borel function $b(t)$ on $\boldsymbol{R}$ and any $f\in L^2(\Omega)$, we have*

$$(1.32)\qquad \mathscr{E}([\Lambda_{\tilde{\delta}}+2\varepsilon, \infty))b(L)f=\mathscr{F}^*_\pm b(\lambda)\mathscr{F}_\pm f$$
$$=s\text{-}\lim_{N\to\infty}\int_{\Lambda_{\tilde{\delta}}+2\varepsilon}^{N}\mathscr{F}^*_\pm(\lambda)b(\lambda)[\mathscr{F}_\pm f](\lambda, \cdot)d\lambda\qquad in\quad L^2(\Omega),$$

where $\mathscr{F}^*_\pm$: $L^2([\Lambda_{\tilde{\delta}}+2\varepsilon, \infty)\times S^{n-1})\to L^2(\Omega)$ is the adjoint of $\mathscr{F}_\pm$ and $\mathscr{F}^*_\pm(\lambda)$: $L^2(S^{n-1})\to L^2_{-(1+\tilde{\beta})/2}(\Omega)$ is the adjoint of $\mathscr{F}_\pm(\lambda)$.

(d) (*Stationary wave operators*) *We put for any pre-compact set $e\Subset[\Lambda_{\tilde{\delta}}+2\varepsilon, \infty)$,*

$$(1.33)\qquad U_\pm(e)=\mathscr{F}^*_\pm\mathscr{F}_0\mathscr{E}_0(e)\,,$$

where $\mathscr{E}_0(\lambda)$, $\lambda\in\boldsymbol{R}$, is the spectral measure of L_0. Then each $U_\pm(e)$ is a unitary operator from $\mathscr{E}_0(e)L^2(\boldsymbol{R}^n)$ onto $\mathscr{E}(e)L^2(\Omega)$, which intertwines the operators $\mathscr{E}_0(e)L_0$ and $\mathscr{E}(e)L$. Namely, we have for any bounded Borel function $b(t)$ on $\boldsymbol{R}$,

$$(1.34)\qquad \mathscr{E}(e)b(L)=U_{\pm}(e)\mathscr{E}_0(e)b(L_0)U_{\pm}^*(e),$$

$$(1.35)\qquad \mathscr{E}_0(e)b(L_0)=U_{\pm}^*(e)\mathscr{E}(e)b(L)U_{\pm}(e),$$

where $U_{\pm}^*(e)$: $\mathscr{E}(e)L^2(\Omega)\to\mathscr{E}_0(e)L^2(\mathbb{R}^n)$ *is the adjoint of* $U_{\pm}(e)$.

§2. Expressions of $\mathscr{F}_{\pm}^*(\lambda)$ and the Identification Operators $J_{\pm}(e)$

Let $\rho_{\pm}$ be as given in (1.9) with some $R_2\geq R_1'$ and $e=(\lambda_1, \lambda_2)$ be a bounded interval in $[\Lambda_{\delta}+2\varepsilon, \infty)$. For any $\phi(\lambda, \tilde{x})\in C_0^{\infty}(e\times S^{n-1})$ we put

$$(2.1)\qquad v_{\phi,\pm}(x, \lambda)=\begin{cases}\dfrac{1}{\sqrt{\pi}}\exp\{-\rho_{\pm}(x, \lambda)\}\phi(\lambda, \tilde{x})\psi(r), & |x|=r>R_2+1\\ 0, & |x|=r<R_2+1,\end{cases}$$

$$(2.2)\qquad g_{\phi,\pm}(x, \lambda)=\{-\Delta+V(x)-\lambda\}v_{\phi,\pm}(x, \lambda),$$

where $\psi(r)$ is a smooth function of $r>0$ such that $0\leq\psi(r)\leq 1$, $\psi(r)=0$ for $r<R_2+1$ and $=1$ for $r>R_2+2$. Note that

$$(2.3)\qquad g_{\phi,\pm}=\frac{1}{\sqrt{\pi}}\exp\{-\rho_{\pm}\}[\{\partial_r^2\rho_{\pm}+\frac{n-1}{r}\partial_r\rho_{\pm}-(\partial_r\rho_{\pm})^2+V-\lambda\}\phi\psi$$
$$+\{(\nabla-\tilde{x}\partial_r)\cdot(\nabla-\tilde{x}\partial_r)\rho_{\pm}-((\nabla-\tilde{x}\partial_r)\rho_{\pm})^2\}\phi\psi-\{\psi''+\frac{n-1}{r}\psi'$$
$$-2(\partial_r\rho_{\pm})\psi'\}\phi-\{(\nabla-\tilde{x}\partial_r)\cdot(\nabla-\tilde{x}\partial_r)\phi-2(\nabla-\tilde{x}\partial_r)\rho_{\pm}\cdot\nabla\phi\}\psi].$$

Here $(\nabla-\tilde{x}\partial_r)\cdot(\nabla-\tilde{x}\partial_r)\phi=0(r^{-2})$ and $\nabla\phi=0(r^{-1})$ near infinity. Then as is easily seen from (1.9), (vii) and Lemma 1.3, we have

Lemma 2.1. *There exists a constant* $C_7>0$ *such that for any* $(x, \lambda)\in B(R_2+1)\times e$,

$$(2.4)\qquad |g_{\phi,\pm}(x, \lambda)|\leq C_7 r^{-(n-1)/2}r^{-(1+\tilde{\delta})},$$

$$(2.5)\qquad |v_{\phi,\pm}(x, \lambda)|\leq C_7 r^{-(n-1)/2}.$$

Moreover, we have

$$(2.6)\qquad \{\partial_r+\partial_r\rho_{\pm}(x, \lambda)\}v_{\phi,\pm}(x, \lambda)=0\quad \textit{in}\quad (x, \lambda)\in B(R_2+2)\times e,$$

$$(2.7)\qquad Bv_{\phi,\pm}(x, \lambda)=0\quad \textit{on}\quad (x, \lambda)\in\partial\Omega\times e.$$

Let $\tilde{\alpha}$, $\tilde{\beta}$ be as given in Proposition 1.3. Then (2.4) implies that $g_{\phi,\pm}\in L^2_{(1+\tilde{\beta})/2}(\Omega)$, and it follows from (2.5)—(2.7) that $v_{\phi,\pm}$ determines an outgoing [incoming] solution of (1.16) and $(1.17)_{\pm}$ with $f=g_{\phi,\pm}$, $\alpha=\tilde{\alpha}$ and $\beta=\tilde{\beta}$. Namely, we have

$$v_{\phi,\pm}(\cdot,\lambda)=[R_{\pm}(\lambda)g_{\phi,\pm}(\cdot,\lambda)](\cdot),\ \lambda\in e. \tag{2.8}$$

Proposition 2.1. *For any* $\phi(\lambda,\tilde{x})\in C_0^\infty(e\times S^{n-1})$, $\mathscr{F}_\pm^*(\lambda)\phi(\lambda,\cdot)\in L^2_{-(1+\tilde{\beta})/2}(\Omega)$ *is expressed as follows*:

$$[\mathscr{F}_\pm^*(\lambda)\phi(\lambda,\cdot)](x)=\frac{\pm 1}{2i}\{v_{\phi,\pm}(x,\lambda)-[R_\mp(\lambda)g_{\phi,\pm}(\cdot,\lambda)](x)\}. \tag{2.9}$$

Proof. For $f\in L^2_{(1+\tilde{\beta})/2}(\Omega)$ and $\lambda\in e$ let

$$u_\pm(\cdot,\lambda)=R_\pm(\lambda)f=R_{1,\pm}(\lambda)\{1-V_sR_\pm(\lambda)\}f. \tag{2.10}$$

Noting $\{1-V_sR_\pm(\lambda)\}f\in L^2_{(1+\tilde{\beta})/2}(\Omega)$, we choose a sequence $r_l=r_l(\tilde{\alpha},\tilde{\beta},\{1-V_sR_\pm(\lambda)\}f,\lambda)$ diverging to ∞ as in Proposition 1.2 (a). Let $\Omega(r_l)=\{x\in\Omega;\ |x|<r_l\}$. Then by the Green formula, (2.6) and (2.7) we have

$$\begin{aligned}\frac{\pm 1}{2i}&\int_{\Omega(r_l)}\{u_\pm\overline{g_{\phi,\pm}}-f\overline{v_{\phi,\pm}}\}dx \\ &=\frac{\pm 1}{2i}\int_{S(r_l)}\{\partial_r u_\pm\overline{v_{\phi,\pm}}-u_\pm\partial_r\overline{v_{\phi,\pm}}\}dS \\ &=\frac{\pm 1}{2i}\int_{S(r_l)}(\partial_r+\partial_r\rho_\pm)u_\pm\overline{v_{\phi,\pm}}dS\mp\int_{S(r_l)}(\mathrm{Im}\,\partial_r\rho_\pm)u_\pm\overline{v_{\phi,\pm}}dS.\end{aligned} \tag{2.11}$$

Here by (1.9) and (2.1),

$$\begin{aligned}\mp\int_{S(r_l)}(\mathrm{Im}\,\partial_r\rho_\pm)u_\pm\overline{v_{\phi,\pm}}dS&=\int_{S(r_l)}\sqrt{\lambda-\eta(\lambda)V_1(x)}u_\pm\overline{v_{\phi,\pm}}dS \\ &=\int_{S^{n-1}}\frac{1}{\sqrt{\pi}}\exp\{\rho_\pm(r_l\tilde{x},\lambda)\}u_\pm(r_l\tilde{x},\lambda)\overline{\phi(\lambda,\tilde{x})}dS_{\tilde{x}}.\end{aligned}$$

So, letting $l\to\infty$ in (2.11), we have

$$\begin{aligned}\frac{\pm 1}{2i}&\int_\Omega\{u_\pm\overline{g_{\phi,\pm}}-f\overline{v_{\phi,\pm}}\}dx \\ &=\lim_{l\to\infty}\left(\frac{1}{\sqrt{\pi}}\exp\{\rho_\pm(r_l\cdot,\lambda)\}u_\pm(r_l\cdot,\lambda),\ \phi(\lambda,\cdot)\right)_{L^2(S^{n-1})}.\end{aligned} \tag{2.12}$$

By means of (2.10) and Proposition 1.1 (c), the left side of (2.12) equals $\left(f,\ \frac{\pm 1}{2i}\{v_{\phi,\pm}-R_\mp(\lambda)g_{\phi,\pm}\}\right)$. On the other hand, by means of Proposition 1.2 (c) and (1.30), the right side equals $(f,\ \mathscr{F}_\pm^*(\lambda)\phi(\lambda,\cdot))$. Thus, we obtain (2.9).

q. e. d.

Now we define the operator $K_\pm(e)$ as follows:

$$\begin{aligned}[K_\pm(e)\phi(\cdot,\cdot)](x)&=\frac{\pm 1}{2i}\int_e v_{\phi,\pm}(x,\lambda)d\lambda \\ &=\begin{cases}\dfrac{\pm 1}{2\sqrt{\pi}i}\displaystyle\int_e\exp\{-\rho_\pm(x,\lambda)\}\psi(r)\phi(\lambda,\tilde{x})d\lambda, & |x|>R_2+1\\ 0, & |x|\le R_2+1\end{cases}\end{aligned} \tag{2.13}$$

for $\phi(\lambda, \tilde{x}) \in C_0^\infty(e \times S^{n-1})$. In the next section we shall show that $K_\pm(e)$ can be extended to a bounded operator from $L^2(e \times S^{n-1})$ to $L^2(\Omega)$. The extended operator will be denoted by $K_\pm(e)$ again. Then our identification operators $J_\pm(e)$ are:

$$J_\pm(e)f = K_\pm(e)\mathscr{F}_0 f \qquad \text{for} \quad f \in \mathscr{E}_0(e)L^2(\boldsymbol{R}^n). \tag{2.14}$$

§3. L^2-Boundedness of $K_\pm(e)$

We begin with a lemma which is a slight modification of Calderón-Vaillancourt [6].

Lemma 3.1. *Let I be a bounded interval of $\boldsymbol{R}=(-\infty, \infty)$ and let $A(r)$ $(r \in I)$ be a weakly measurable and uniformly bounded family of operators in a separable Hilbert space $\mathfrak{H}$. If the inequalities*

$$\|A(r)A^*(r')\| \le h^2(r, r') \quad \text{and} \quad \|A^*(r)A(r')\| \le h^2(r, r')$$

hold for $r, r' \in I$ with a non-negative function $h(r, r')$ which is the kernel of a bounded integral operator H_I in $L^2(I)$ ($A^*(r)$ being the adjoint of $A(r)$), then the operator $\int_I A(r)dr$ defined by

$$\left(\int_I A(r)dr\right)f = \int_I A(r)f dr \quad \text{for} \quad f \in \mathfrak{H}$$

is a bounded operator in $\mathfrak{H}$ with norm

$$\left\|\int_I A(r)dr\right\| \le \|H_I\|.$$

Proof. By assumption we can admit $\|A(r)\| \le M$ for any $r \in I$. From the two inequalities

$$\|A(r_1)A^*(r_2)A(r_3)\cdots A^*(r_{2m})\| \le \|A(r_1)A^*(r_2)\|\cdots\|A(r_{2m-1})A^*(r_{2m})\|$$

and

$$\|A(r_1)A^*(r_2)A(r_3)\cdots A^*(r_{2m})\| \le \|A(r_1)\|\,\|A^*(r_2)A(r_3)\|\cdots$$
$$\cdots\|A^*(r_{2m-2})A(r_{2m-1})\|\,\|A^*(r_{2m})\|,$$

we have for $r_i \in I$ $(i=1, 2, \cdots, 2m)$,

$$\|A(r_1)A^*(r_2)A(r_3)\cdots A^*(r_{2m})\| \le Mh(r_1, r_2)h(r_2, r_3)\cdots h(r_{2m-1}, r_{2m}). \tag{3.1}$$

Since $\int_I A(r)dr\left(\int_I A(r)dr\right)^*$ is a bounded selfadjoint operator in $\mathfrak{H}$ and

$\left(\int_I A(r)dr\right)^* = \int_I A^*(r)dr$, we have from (3.1)

$$(3.2)\qquad \left\|\int_I A(r)dr\right\|^2 = \left\|\left[\int_I A(r)dr\left(\int_I A(r)dr\right)^*\right]^m\right\|^{1/m}$$
$$\leq \left(\int\cdots\int_{I^{2m}} \|A(r_1)A^*(r_2)A(r_3)\cdots A^*(r_{2m})\|dr_1\cdots dr_{2m}\right)^{1/m}$$
$$\leq \left(M\iint_{I^2} dr_1 dr_{2m}\int\cdots\int_{I^{2m-2}} h(r_1, r_2)\cdots h(r_{2m-1}, r_{2m})dr_2\cdots dr_{2m-1}\right)^{1/m}$$
$$\leq \{M(H_I^{2m-1}\chi_I, \chi_I)_{L^2(I)}\}^{1/m} \leq (M|I|\,\|H_I\|^{2m-1})^{1/m},$$

where $\chi_I(r)=1$ on I and $|I|$ is the length of I. Letting m go to ∞ in (3.2), we have the assertion. q. e. d.

Remark 3.1. By Petti's theorem the weak measurability of $A(r)$ and the separability of $\mathfrak{H}$ show that $A(r)f$, $f\in\mathfrak{H}$, is strongly measurable on I. Moreover, $A(r)f$ is Bochner integrable on I since $\|A(r)\|$ is bounded in I (see Yosida [16], pp. 130—134).

Let $e_0=(A_{1/2}+\varepsilon, A_{1/2}+N)$, where N is chosen so large that $e=(\lambda_1, \lambda_2)\subset e_0$. Let $\zeta(\lambda)\in C_0^\infty(e_0)$ be a real function such that $\zeta(\lambda)=1$ on e, and let $\chi(r)\in C^\infty(\boldsymbol{R})$ satisfy the following: $\chi(r)=1$ for $r<1$, $=0$ for $r>2$ and $0<\chi(r)<1$ for $1<r<2$. We put for $\mu, \lambda\in e_0$, $r>R_2+1$ and $\tilde{x}\in S^{n-1}$,

$$(3.3)\qquad S_\pm(\mu, \lambda, r, \tilde{x}) = \pm\int_{R_2}^r \{\sqrt{\lambda-\eta(\lambda)V_1(s\tilde{x})} - \sqrt{\mu-\eta(\mu)V_1(s\tilde{x})}\}ds,$$

$$(3.4)\qquad p_R(\mu, \lambda, r, \tilde{x}) = \frac{1}{4\pi}\psi(r)^2\chi(r/R)^2\zeta(\lambda)\zeta(\mu)$$
$$\times\{\lambda-\eta(\lambda)V_1(r\tilde{x})\}^{-1/4}\{\mu-\eta(\mu)V_1(r\tilde{x})\}^{-1/4},$$

where $\psi(r)$ is as given in (2.1) and $R\geq(R_2+1)/2$.

For any $r, r'>R_2+1$, we have

$$(3.5)\qquad S_\pm(\mu, \xi, r, \tilde{x}) + S_\pm(\xi, \lambda, r', \tilde{x}) = \pm\int_{r'}^r \sqrt{\xi-\eta(\xi)V_1(s\tilde{x})}ds$$
$$\pm\left\{\int_{R_2}^{r'} \sqrt{\lambda-\eta(\lambda)V_1(s\tilde{x})}ds - \int_{R_2}^r \sqrt{\mu-\eta(\mu)V_1(s\tilde{x})}ds\right\}$$

and

$$(3.6)\qquad \partial_\xi^3 \exp\left\{\pm i\int_{r'}^r \sqrt{\xi-\eta(\xi)V_1(s\tilde{x})}ds\right\}$$
$$= \sigma_\pm(\xi, r, r', \tilde{x})\exp\left\{\pm i\int_{r'}^r \sqrt{\xi-\eta(\xi)V_1(s\tilde{x})}ds\right\};$$

(3.7)
$$\begin{aligned}&\sigma_{\pm}(\xi, r, r', \tilde{x})\\ &=\mp i\Big(\int_{r'}^{r}\partial_{\xi}\sqrt{\xi-\eta(\xi)V_1(s\tilde{x})}ds\Big)^3 \pm i\int_{r'}^{r}\partial_{\xi}^{3}\sqrt{\xi-\eta(\xi)V_1(s\tilde{x})}ds\\ &\quad -3\int_{r'}^{r}\partial_{\xi}\sqrt{\xi-\eta(\xi)V_1(s\tilde{x})}ds\Big(\int_{r'}^{r}\partial_{\xi}^{2}\sqrt{\xi-\eta(\xi)V_1(s\tilde{x})}ds\Big).\end{aligned}$$

Then the following inequalities are consequences of Lemma 1.2. Namely, there exists a constant $C_8 \geq 1$ such that for any $r, r' \geq R_2+1$, $\xi, \mu, \lambda \in e_0$, $\tilde{x} \in S^{n-1}$ and $R \geq (R_2+1)/2$,

(3.8) $$|\sigma_{\pm}(\xi, r, r', \tilde{x})| \geq C_8^{-1}|r-r'|^3 - C_8|r-r'|,$$

(3.9) $$|\partial_{\xi}^{l}\sigma_{\pm}(\xi, r, r', \tilde{x})| \leq C_8(1+|r-r'|^3) \quad (l=1, 2, 3),$$

(3.10) $$|\partial_{\xi}^{l}[p_R(\mu, \xi, r, \tilde{x})p_R(\xi, \lambda, r', \tilde{x})]| \leq C_8 \quad (l=0, 1, 2, 3).$$

With these inequalities we can apply Lemma 3.1 to prove the following

Lemma 3.2. *The operator* $P_{R,\pm}$ *defined by*

(3.11) $$[P_{R,\pm}\phi](\mu, \tilde{x}) = \int_{R_2+1}^{\infty}\int_{e_0}\exp\{iS_{\pm}(\mu, \lambda, r, \tilde{x})\}p_R(\mu, \lambda, r, \tilde{x})\phi(\lambda, \tilde{x})drd\lambda$$

for $\phi(\lambda, \tilde{x}) \in \mathfrak{H} = L^2(e_0 \times S^{n-1})$ *is bounded in* $\mathfrak{H}$, *and there exists a constant* $C_9 > 0$ *such that*

(3.12) $$\|P_{R,\pm}\| \leq C_9 \text{ for any } R \geq (R_2+1)/2.$$

Proof. We define the family $A_{R,\pm}(r)$, $r \in I_R = (R_2+1, 2R)$, of operators in $\mathfrak{H}$ by

(3.13) $$[A_{R,\pm}(r)\phi](\mu, \tilde{x}) = \int_{e_0}\exp\{iS_{\pm}(\mu, \lambda, r, \tilde{x})\}p_R(\mu, \lambda, r, \tilde{x})\phi(\lambda, \tilde{x})d\lambda.$$

Obviously, each $A_{R,\pm}(r)$ is bounded and selfadjoint in $\mathfrak{H}$. Since we have

(3.14) $$\|A_{R,\pm}(r)\| \leq \{\sup_{\tilde{x}\in S^{n-1}}\int_{e_0}\int_{e_0}|p_R(\mu, \lambda, r, \tilde{x})|^2 d\lambda d\mu\}^{1/2},$$

it follows from (3.10) that

(3.15) $$\|A_{R,\pm}(r)\| \leq C_{10} \quad \text{for any} \quad R \geq (R_2+1)/2 \quad \text{and} \quad r \in I_R.$$

Further, by the Lebesgue theorem, $A_{R,\pm}(r)$ is strongly continuous in I_R. Thus, to complete the proof we have only to show the existence of a kernel $h_R(r, r')$ which satisfies the following inequalities:

(3.16) $$\|A_{R,\pm}(r)A_{R,\pm}(r')\| \leq h_R^2(r, r'),$$

(3.17) $$\int_{I_R}\Big|\int_{I_R}h_R(r, r')f(r')dr'\Big|^2 dr \leq C_{11}\int_{I_R}|f(r')|^2 dr'$$

for any $R \geq (R_2+1)/2$, $r, r' \in I_R$ and $f(r) \in L^2(I_R)$, where $C_{11}>0$ is independent of R.

We can choose $C_{12}>0$ and $C_{13}>0$ to satisfy

$$C_{12}+C_8^{-1}\tau^3-C_8\tau \geq C_{13}(1+\tau^3) \qquad \text{for any} \quad \tau \geq 0.$$

It then follows from (3.8) that

$$|\mp iC_{12}\operatorname{sgn}(r-r')+\sigma_{\pm}(\xi, r, r', \tilde{x})| \geq C_{13}(1+|r-r'|^3), \tag{3.18}$$

where $\operatorname{sgn} t=1$ if $t \geq 0$ and $=-1$ if $t<0$. So by (3.6),

$$\begin{aligned}\exp\left\{\pm i\int_{r'}^{r}\sqrt{\xi-\eta(\xi)V_1(s\tilde{x})}ds\right\} &= \{\mp iC_{12}\operatorname{sgn}(r-r')+\sigma_{\pm}(\xi, r, r', \tilde{x})\}^{-1} \\ &\times\{\mp iC_{12}\operatorname{sgn}(r-r')+\partial_\xi^3\}\exp\left\{\pm i\int_{r'}^{r}\sqrt{\xi-\eta(\xi)V_1(s\tilde{x})}ds\right\}.\end{aligned} \tag{3.19}$$

Note that the support in ξ of $p_R(\mu, \xi, r, \tilde{x})p_R(\xi, \lambda, r', \tilde{x})$ is contained in e_0. Then (3.5), (3.19) and integrations by parts give

$$\begin{aligned}&\int_{e_0}\exp\{iS_{\pm}(\mu, \xi, r, \tilde{x})+iS_{\pm}(\xi, \lambda, r', \tilde{x})\}p_R(\mu, \xi, r, \tilde{x})p_R(\xi, \lambda, r', \tilde{x})d\xi \\ &=\int_{e_0}\exp\{iS_{\pm}(\mu, \xi, r, \tilde{x})+iS_{\pm}(\xi, \lambda, r', \tilde{x})\}\{\mp iC_{12}\operatorname{sgn}(r-r')-\partial_\xi^3\} \\ &\quad\times[\{\mp iC_{12}\operatorname{sgn}(r-r')+\sigma_{\pm}(\xi, r, r', \tilde{x})\}^{-1}p_R(\mu, \xi, r, \tilde{x})p_R(\xi, \lambda, r', \tilde{x})]d\xi.\end{aligned}$$

Applying (3.9), (3.10) and (3.18) in this equality, we obtain

$$\begin{aligned}&\left|\int_{e_0}\exp\{iS_{\pm}(\mu, \xi, r, \tilde{x})+iS_{\pm}(\xi, \lambda, r', \tilde{x})\}p_R(\mu, \xi, r, \tilde{x})p_R(\xi, \lambda, r', \tilde{x})d\xi\right| \\ &\leq C_{14}(1+|r-r'|^3)^{-1},\end{aligned} \tag{3.20}$$

where $C_{14}>0$ is independent of $R \geq (R_2+1)/2$, $r, r' \in I_R$, $\mu, \lambda \in e_0$ and $\tilde{x} \in S^{n-1}$.

Now for any $\phi(\lambda, \tilde{x}) \in \mathfrak{H}$,

$$\begin{aligned}&[A_{R,\pm}(r)A_{R,\pm}(r')\phi](\mu, \tilde{x}) \\ &=\int_{e_0}\phi(\lambda, \tilde{x})d\lambda\int_{e_0}\exp\{iS_{\pm}(\mu, \xi, r, \tilde{x})+iS_{\pm}(\xi, \lambda, r', \tilde{x})\} \\ &\quad\times p_R(\mu, \xi, r, \tilde{x})p_R(\xi, \lambda, r', \tilde{x})d\xi.\end{aligned}$$

So (3.20) has shown the inequality

$$\|A_{R,\pm}(r)A_{R,\pm}(r')\| \leq C'_{14}(1+|r-r'|^3)^{-1} \quad \text{with} \quad C'_{14}=C_{14}|e_0|. \tag{3.21}$$

Hence, choosing $h_R(r, r')=\sqrt{C'_{14}}(1+|r-r'|^3)^{-1/2}$ for any $R \geq (R_2+1)/2$, we have (3.16) and (3.17) with $C_{11}=C'_{14}\left\{\int_{\boldsymbol{R}}(1+r^3)^{-1/2}dr\right\}^2<\infty$. q.e.d.

Remark 3.2. The method of the above proof, which apparently seems to be much different, however, follows the idea employed in Calderón-Vaillancourt [6]. If $V_1(x)$ is sufficiently smooth, e.g., $\in C^{19}(B(R_2))$, a general theory of Asada-Fujiwara [3] can be applied to obtain the above result.

As a corollary of Lemma 3.2 we can now prove the following

Proposition 3.1. *For any* $\phi(\lambda, \tilde{x}) \in C_0^\infty(e \times S^{n-1})$, *where* $e \Subset e_0$, *let* $K_\pm(e)\phi$ *be defined by* (2.13). *Then we have* $K_\pm(e)\phi \in L^2(\Omega)$ *and*

$$\|K_\pm(e)\phi\|^2 \leq C_9 \|\phi\|^2_{L^2(e\times S^{n-1})}. \tag{3.22}$$

Thus, $K_\pm(e)$ *can be extended to a bounded operator from* $L^2(e \times S^{n-1})$ *to* $L^2(\Omega)$.

Proof. By integration by parts we have

$$\begin{aligned}
[K_\pm(e)\phi](x) &= \frac{\pm 1}{2\sqrt{\pi i}} r^{-(n-1)/2}\psi(r) \\
&\quad \times \int_e \exp\left\{\pm i \int_{R_2}^r \sqrt{\lambda - \eta(\lambda)V_1(s\tilde{x})}\,ds\right\}\{\lambda - \eta(\lambda)V_1(x)\}^{-1/4}\phi(\lambda, \tilde{x})\,d\lambda \\
&= \frac{1}{2\sqrt{\pi}} r^{-(n-1)/2}\psi(r)\int_e \exp\left\{\pm i\int_{R_2}^r \sqrt{\lambda - \eta(\lambda)V_1(s\tilde{x})}\,ds\right. \\
&\quad \times \partial_\lambda\left[\left\{\int_{R_2}^r \partial_\lambda\sqrt{\lambda - \eta(\lambda)V_1(s\tilde{x})}\,ds\right\}^{-1}\{\lambda - \eta(\lambda)V_1(x)\}^{-1/4}\phi(\lambda, \tilde{x})\right]d\lambda.
\end{aligned}$$

This with Lemma 1.2 shows that

$$|[K_\pm(e)\phi](x)| \leq C_{15} r^{-(n-1)/2} r^{-1} \quad \text{in} \quad B(R_2+1), \tag{3.23}$$

i.e., $K_\pm(e)\phi \in L^2(\Omega)$. Thus, we can apply the Lebesgue theorem and the Fubini theorem to obtain

$$\begin{aligned}
\|K_\pm(e)\phi\|^2 &= \lim_{R\to\infty}\int_\Omega |\chi(r/R)[K_\pm(e)\phi](x)|^2 dx \\
&= \lim_{R\to\infty}\int_e\int_{S^{n-1}} \overline{\phi(\mu, \tilde{x})}\,d\mu dS \int_{R_2}^\infty \int_e \exp\{iS_\pm(\mu, \lambda, r, \tilde{x})\} \\
&\quad \times p_R(\mu, \lambda, r, \tilde{x})\phi(\lambda, \tilde{x})\,dr d\lambda = \lim_{R\to\infty}(P_{R,\pm}\phi, \phi)_{L^2(e\times S^{n-1})} \\
&= \lim_{R\to\infty}(P_{R,\pm}\phi, \phi)_{L^2(e_0\times S^{n-1})}.
\end{aligned} \tag{3.24}$$

(3.12) and (3.24) imply (3.22). q.e.d.

§4. Theorem; Time Dependent Representations of $U_\pm(e)$

First we note the following lemma which can easily be proved by Lemma 1.2.

Lemma 4.1. *Let* ε *and* N *be any constants satisfying* $0<\varepsilon<1$ *and* $N>\Lambda_{\delta}-\Lambda_{1/2}+2$. *Then there exist some constants* $C_{16}\geq 1$ *and* $R_2\geq R_1'$ *such that for any* $(x, \lambda)\in B(R_2)\times[\Lambda_{\delta}+2\varepsilon, \Lambda_{1/2}+N]$,

$$C_{16}^{-1}\leq -\partial_{\lambda}^{2}\sqrt{\lambda-\eta(\lambda)V_1(x)}\leq C_{16}\,. \tag{4.1}$$

In this and next section we choose $R_2\geq R_1'$ defining $\rho_{\pm}(x, \lambda)$ as in the above lemma, and prove the following theorem which gives time dependent representations of the stationary wave operators $U_{\pm}(e)$ with $e\subset[\Lambda_{\delta}+2\varepsilon, \Lambda_{1/2}+N]$. Note that the operators $U_{\pm}(e)$ and $J_{\pm}(e)$ and functions $v_{\phi,\pm}(x, \lambda)$ and $g_{\phi,\pm}(x, \lambda)$ are now determined depending on the above R_2.

Theorem. *Let* ε, N *and* R_2 *be as in the above lemma. For any interval* $e=(\lambda_1, \lambda_2)\subset[\Lambda_{\delta}+2\varepsilon, \Lambda_{1/2}+N]$ *let* $J_{\pm}(e)$: $\mathscr{E}_0(e)L^2(\mathbb{R}^n)\to L^2(\Omega)$ *be defined by* (2.14). *Then the strong limits*

$$W_J^{\pm}(e)=\underset{t\to\pm\infty}{s\text{-}\lim}\ \exp\{iLt\}J_{\pm}(e)\exp\{-iL_0t\}\mathscr{E}_0(e) \tag{4.2}$$

exist in $L^2(\Omega)$ *and coincide with the stationary wave operators* $U_{\pm}(e)$ *defined by* (1.33). *Thus,* $W_J^{\pm}(e)$ *are unitary operators from* $\mathscr{E}_0(e)L^2(\mathbb{R}^n)$ *onto* $\mathscr{E}(e)L^2(\Omega)$ *satisfying*

$$\mathscr{E}(e)LW_J^{\pm}(e)f=W_J^{\pm}(e)\mathscr{E}_0(e)L_0f \quad \text{for any} \quad f\in\mathscr{D}(L_0)\,. \tag{4.3}$$

Remark 4.1. $K_{\pm}(e)$ and $J_{\pm}(e)$ depend on the function $\psi(r)$ given in (2.1). However, $W_J^{\pm}(e)$ does not depend on the choice of $\psi(r)$.

The following proposition will be proved in the next section by use of Lemma 4.1 and the stationary phase method.

Proposition 4.1. *For* $\phi(\lambda, \tilde{x})\in C_0^{\infty}(e\times S^{n-1})$ *let*

$$\hat{g}_{\phi,\pm}(x, t)=\int_e \exp\{-i\lambda t\}g_{\phi,\pm}(x, \lambda)d\lambda \quad (\pm t>0)\,, \tag{4.4}$$

where $g_{\phi,\pm}(x, \lambda)$ *is defined by* (2.2). *Then we have*

$$\pm\int_0^{\pm\infty}\|\hat{g}_{\phi,\pm}(\cdot, t)\|dt<\infty\,. \tag{4.5}$$

Based on Propositions 2.1, 3.1 and 4.1, we can now follow the idea employed in Kitada [13], Ikebe-Isozaki [10] and Kako [11], where is treated the case of "non-oscillating" long-range potentials, to prove the above theorem.

Lemma 4.2. *We have for any* $\phi(\lambda, \tilde{x})\in C_0^{\infty}(e\times S^{n-1})$,

$$\left\|\int_e R_{\mp}(\lambda)g_{\phi,\pm}(\cdot,\lambda)d\lambda\right\| \le \pm\int_0^{\pm\infty} \|\hat{g}_{\phi,\pm}(\cdot,t)\|dt. \tag{4.6}$$

Proof. Noting that $g_{\phi,\pm}(\cdot,\lambda)\in L^2_{(1+\tilde{\beta})/2}(\Omega)$, where $\tilde{\beta}$ is as given in Proposition 1.3, we put

$$G_{\pm}(x)=\int_e R_{\mp}(\lambda)g_{\phi,\pm}(x,\lambda)d\lambda,$$

$$G_{\tau,\pm}(x)=\int_e R(\lambda\mp i\tau)g_{\phi,\pm}(x,\lambda)d\lambda \quad (\tau>0),$$

where the measurability of the integrands is guaranteed by Proposition 1.1 (b) and the continuity of $g_{\phi,\pm}(\cdot,\lambda)$ in $\lambda\in e$. In virtue of (2.4) we have

$$\begin{aligned} G_{\tau,\pm}(x) &= -i\int_e\left[\int_0^{\pm\infty}\exp\{i(L-\lambda\pm i\tau)t\}dt\right]g_{\phi,\pm}(x,\lambda)d\lambda \\ &= -i\int_0^{\pm\infty}\exp\{i(L\pm i\tau)t\}\hat{g}_{\phi,\pm}(x,t)dt. \end{aligned}$$

Thus,

$$\|G_{\tau,\pm}\|\le\pm\int_0^{\pm\infty}\|\hat{g}_{\phi,\pm}(\cdot,t)\|dt<\infty \qquad \text{for any} \quad \tau>0. \tag{4.7}$$

Further, since we have for any $f\in L^2_{(1+\tilde{\beta})/2}(\Omega)$ and $\tau>0$,

$$(G_{\tau,\pm},f)=\int_e (g_{\phi,\pm}(\cdot,\lambda), R(\lambda\pm i\tau)f)d\lambda,$$

it follows from Proposition 1.1 (a), (b), (c) and the Lebesgue theorem that

$$\begin{aligned} \lim_{\tau\downarrow 0}(G_{\tau,\pm},f) &= \int_e (g_{\phi,\pm}(\cdot,\lambda), R_{\pm}(\lambda)f)d\lambda \\ &= \int_e (R_{\mp}(\lambda)g_{\phi,\pm}(\cdot,\lambda), f)d\lambda=(G_{\pm},f). \end{aligned}$$

$L^2_{(1+\tilde{\beta})/2}(\Omega)$ being dense in $L^2(\Omega)$, this and (4.7) imply that $G_{\pm}$ is the weak limit as $\tau\downarrow 0$ of $G_{\tau,\pm}$ in $L^2(\Omega)$. Hence, $G_{\pm}\in L^2(\Omega)$ and

$$\|G_{\pm}\|\le\liminf_{\tau\downarrow 0}\|G_{\tau,\pm}\|\le\pm\int_0^{\pm\infty}\|\hat{g}_{\phi,\pm}(\cdot,t)\|dt,$$

which is to be proved. q. e. d.

Proof of Theorem. Let $f\in\mathscr{E}_0(e)L^2(\boldsymbol{R}^n)$ satisfy $[\mathscr{F}_0 f](\lambda,\tilde{x})\in C_0^\infty(e\times S^{n-1})$, and put $u(t)=\exp\{-iL_0 t\}f$. Since $\mathscr{F}_0 u(t)=\exp\{-i\lambda t\}\mathscr{F}_0 f$ by Proposition 1.3 (c), we see that $\mathscr{F}_0 u(t)$ also belongs to $C_0^\infty(e\times S^{n-1})$. By Propositions 1.3 (d) and 2.1 we then have

$$(4.8)\qquad \exp\{-iLt\}U_{\pm}(e)f=U_{\pm}(e)u(t)$$
$$=\frac{\pm 1}{2i}\int_{e}\{v_{\mathscr{F}_{0}u(t),\pm}(\cdot,\lambda)-R_{\mp}(\lambda)g_{\mathscr{F}_{0}u(t),\pm}(\cdot,\lambda)\}d\lambda.$$

Here by definition

$$(4.9)\qquad \frac{\pm 1}{2i}\int_{e}v_{\mathscr{F}_{0}u(t),\pm}(\cdot,\lambda)d\lambda=J_{\pm}(e)u(t)=J_{\pm}(e)\exp\{-iL_{0}t\}f.$$

On the other hand, the equality

$$g_{\mathscr{F}_{0}u(t),\pm}(\cdot,\lambda)=\exp\{-i\lambda t\}g_{\mathscr{F}_{0}f,\pm}(\cdot,\lambda)$$

and (4.4) show that

$$\hat{g}_{\mathscr{F}_{0}u(t),\pm}(\cdot,s)=\hat{g}_{\mathscr{F}_{0}f,\pm}(\cdot,s+t)\qquad\text{for any}\quad \pm s>0,$$

and hence, we have from Lemma 4.2 and Proposition 4.1,

$$(4.10)\qquad \left\|\int_{e}R_{\mp}(\lambda)g_{\mathscr{F}_{0}u(t),\pm}(\cdot,\lambda)d\lambda\right\|\leq\pm\int_{0}^{\pm\infty}\|\hat{g}_{\mathscr{F}_{0}f,\pm}(\cdot,s+t)\|ds$$
$$=\pm\int_{t}^{\pm\infty}\|\hat{g}_{\mathscr{F}_{0}f,\pm}(\cdot,s)\|ds\longrightarrow 0\quad\text{as}\quad t\longrightarrow\pm\infty.$$

(4.8), (4.9) and (4.10) prove the following:

$$(4.11)\qquad \lim_{t\to\pm\infty}\|\exp\{iLt\}J_{\pm}(e)\exp\{-iL_{0}t\}f-U_{\pm}(e)f\|=0.$$

Since $C_0^\infty(e\times S^{n-1})$ is dense in $L^2(e\times S^{n-1})$ and $\mathscr{F}_0$ is a unitary operator from $\mathscr{E}_0(e)L^2(\mathbb{R}^n)$ onto $L^2(e\times S^{n-1})$, (4.11) holds for any $f\in\mathscr{E}_0(e)L^2(\mathbb{R}^n)$.

The proof is thus completed. q.e.d.

§5. Proof of Proposition 4.1; The Stationary Phase Method

We put for the sake of simplicity

$$(5.1)\qquad \xi(x,\lambda)=\int_{R_2}^{r}\sqrt{\lambda-\eta(\lambda)V_1(s\tilde{x})}ds,$$

$$(5.2)\qquad \zeta_{\phi,\pm}(x,\lambda)=\sqrt{\pi}\exp\{\rho_{\pm}(x,\lambda)\}g_{\phi,\pm}(x,\lambda),$$

where $\phi(\lambda,\tilde{x})\in C_0^\infty(e\times S^{n-1})$ with $e=(\lambda_1,\lambda_2)\subset[\Lambda_\delta+2\varepsilon,\Lambda_{1/2}+N]$. We can find a concrete form of $\zeta_{\phi,\pm}(x,\lambda)$ in (2.3).

The following lemma is easily proved by a straight calculation (cf. Lemmas 1.2, 1.3 and 4.1).

Lemma 5.1. *There exists a constant* $C_{17}\geq 1$ *such that for any* $(x,\lambda)\in B(R_2+1)\times e$, (5.3) $$C_{17}^{-1}r \le \partial_\lambda \xi(x, \lambda) \le C_{17}r,$$

(5.4) $$C_{17}^{-1}r \le -\partial_\lambda^2 \xi(x, \lambda) \le C_{17}r,$$

(5.5) $$|\partial_\lambda^l \xi(x, \lambda)| \le C_{17}r \qquad (l=3, 4, 5),$$

(5.6) $$|\partial_\lambda^l \zeta_{\phi,\pm}(x, \lambda)| \le C_{17}r^{-1-\delta} \qquad (l=0, 1, 2).$$

Let $e_1=[\lambda_3, \lambda_4] \subset e$ be a closed interval which contains the support in λ of $\zeta_{\phi,\pm}(x, \lambda)$ for any $x \in B(R_2+1)$. We put

(5.7) $$t_j(x)=(\partial_\lambda \xi)(x, \lambda_j).$$

Then we have

(5.8) $$C_{17}^{-1}r \le t_2(x) < t_4(x) < t_3(x) < t_1(x) \le C_{17}r$$

since $(\partial_\lambda \xi)(x, \lambda)$ is by (5.4) a monotone decreasing (in a strong sense) function of $\lambda \in e$ for any $x \in B(R_2+1)$. Moreover, we have the

Lemma 5.2. *There exists a constant $C_{18} \ge 1$ such that for any $x \in B(R_2+1)$,*

(5.9) $$C_{18}^{-1}r \le t_1(x)-t_3(x) \le C_{18}r,$$

(5.10) $$C_{18}^{-1}r \le t_4(x)-t_2(x) \le C_{18}r.$$

Proof. Since we have

$$t_1(x)-t_3(x)=(\lambda_1-\lambda_3)\partial_\lambda^2 \xi(x, \lambda_1+(\lambda_3-\lambda_1)\theta)$$

for a suitable θ $(0<\theta=\theta(x)<1)$, (5.9) is a consequence of (5.4). (5.10) can similarly be proved. q.e.d.

With the above lemmas we shall estimate the function

(5.11) $$\hat{g}_{\phi,\pm}(x, t)=\int_e \exp\{-i\lambda t\} g_{\phi,\pm}(x, \lambda)d\lambda$$
$$=\frac{1}{\sqrt{\pi}} r^{-(n-1)/2} \int_{e_1} \exp\{-i\lambda t \pm i\xi(x, \lambda)\}\{\lambda-\eta(\lambda)V_1(x)\}^{-1/4}\zeta_{\phi,\pm}(x, \lambda)d\lambda.$$

Our estimation will be done in the each case $\pm t > t_1(x)$, $0 \le \pm t < t_2(x)$ or $t_2(x) \le \pm t \le t_1(x)$.

In the case $\pm t > t_1(x)$ or $0 \le \pm t < t_2(x)$, it holds that

(5.12) $$|\partial_\lambda\{\lambda t \mp \xi(x, \lambda)\}|=|t-\partial_\lambda \xi(x, \lambda)| \ge |t|-t_3(x) \quad \text{or} \quad t_4(x)-|t|$$

for any $(x, \lambda) \in B(R_2+1) \times e_1$. So we can prove the

Lemma 5.3. *There exists a $C_{19}>0$ such that*

(5.13) $$|\hat{g}_{\phi,\pm}(x, t)| \le C_{19}r^{-(n-1)/2}r^{-1-\delta}\{|t|-t_3(x)\}^{-2}[1+r^2\{|t|-t_3(x)\}^{-2}]$$

for any $x \in B(R_2+1)$ *and* $\pm t > t_1(x)$, *and*

(5.14) $$|\hat{g}_{\phi,\pm}(x, t)| \le C_{19} r^{-(n-1)/2} r^{-1-\delta} \{t_4(x) - |t|\}^{-2} [1 + r^2 \{t_4(x) - |t|\}^{-2}]$$

for any $x \in B(R_2+1)$ *and* $0 \le \pm t < t_2(x)$.

Proof. Integrating by parts gives

$$\begin{aligned}\hat{g}_{\phi,\pm}(x, t) &= \frac{1}{\sqrt{\pi}} r^{-(n-1)/2} \int_{e_1} \left[\left\{ \frac{1}{\partial_\lambda(-i\lambda t \pm i\xi)} \partial_\lambda \right\}^2 \exp\{-i\lambda t \pm i\xi(x, \lambda)\} \right] \\ &\qquad \times \{\lambda - \eta(\lambda) V_1(x)\}^{-1/4} \zeta_{\phi,\pm}(x, \lambda) d\lambda \\ &= -\frac{1}{\sqrt{\pi}} r^{-(n-1)/2} \int_{e_1} \exp\{-i\lambda t \pm i\xi(x, \lambda)\} (t \mp \partial_\lambda \xi)^{-2} \\ &\quad \times [\partial_\lambda^2 \{(\lambda - \eta V_1)^{-1/4} \zeta_{\phi,\pm}\} \pm 3(\partial_\lambda^2 \xi)(t \mp \partial_\lambda \xi)^{-1} \partial_\lambda \{(\lambda - \eta V_1)^{-1/4} \zeta_{\phi,\pm}\} \\ &\quad + \{3(\partial_\lambda^2 \xi)^2 (t \mp \partial_\lambda \xi)^{-2} \pm (\partial_\lambda^3 \xi)(t \mp \partial_\lambda \xi)^{-1}\} (\lambda - \eta V_1)^{-1/4} \zeta_{\phi,\pm}] d\lambda .\end{aligned}$$

Thus, noting (5.12), (5.4), (5.5) and the inequality

(5.15) $$|\partial_\lambda^l \{(\lambda - \eta V_1)^{-1/4} \zeta_{\phi,\pm}\}| \le C_{20} r^{-1-\delta} \qquad (l = 0, 1, 2)$$

which follows from (5.6) and Lemma 1.2, we obtain (5,13) and (5.14). q.e.d.

Next we consider the case $t_2(x) \le \pm t \le t_1(x)$.

Lemma 5.4. *There exists a (unique) function* $\lambda_c(x, t)$ *such that for any* $x \in B(R_2+1)$ *and* $t_2(x) \le \pm t \le t_1(x)$,

(5.16) $$|t| = (\partial_\lambda \xi)(x, \lambda_c(x, t)),$$

(5.17) $$\lambda_1 \le \lambda_c(x, t) \le \lambda_2 ,$$

(5.18) $$\lambda_c(x, -t) = \lambda_c(x, t) .$$

Proof. We have only to solve in λ the equation $|t| = (\partial_\lambda \xi)(x, \lambda)$, which is possible by the monotonicity of $(\partial_\lambda \xi)(x, \lambda)$. q.e.d.

$\lambda_c(x, t)$ is the so-called critical point of $\lambda|t| - \xi(x, \lambda)$.

Let $\omega(\lambda)$ be a C^∞-function of $\lambda \in \boldsymbol{R}$ such that $0 \le \omega(\lambda) \le 1$, $\omega(\lambda) = 1$ for $|\lambda| \le 1/2$ and $= 0$ for $|\lambda| \ge 1$. By use of this function we divide $\hat{g}_{\phi,\pm}(x, t)$ into two parts:

(5.19) $$\begin{aligned}\hat{g}_{\phi,\pm}(x, t) &= \frac{1}{\sqrt{\pi}} r^{-(n-1)/2} \int_{e_1} \exp\{-i\lambda t \pm i\xi(x, \lambda)\} \\ &\quad \times \omega(\nu(x, t)\{\lambda - \lambda_c(x, t)\}) \{\lambda - \eta V_1(x)\}^{-1/4} \zeta_{\phi,\pm}(x, \lambda) d\lambda \\ &\quad + \frac{1}{\sqrt{\pi}} r^{-(n-1)/2} \int_{e_1} \exp\{-i\lambda t \pm i\xi(x, \lambda)\} \\ &\quad \times \{1 - \omega(\nu(x, t)\{\lambda - \lambda_c(x, t)\})\} \{\lambda - \eta V_1(x)\}^{-1/4} \zeta_{\phi,\pm}(x, \lambda) d\lambda \\ &= \hat{g}_{\phi,\pm}^{(1)}(x, t) + \hat{g}_{\phi,\pm}^{(2)}(x, t) ,\end{aligned}$$

where $v(x, t)\geq 1$ is given later. Note that

$$\lambda t\mp\xi(x, \lambda)=\lambda_c t\mp\xi(x, \lambda_c)\mp\frac{1}{2}(\lambda-\lambda_c)^2(\partial_\lambda^2\xi)(x, \lambda_c)$$
$$\mp\frac{1}{2}(\lambda-\lambda_c)^3\int_0^1(1-\tau)^2(\partial_\lambda^3\xi)(x, \lambda_c+(\lambda-\lambda_c)\tau)d\tau .$$

Then we have

$$\hat{g}_{\phi,\pm}^{(1)}(x, t)=\frac{1}{\sqrt{\pi}}r^{-(n-1)/2}\exp\{-i\lambda_c(x, t)t\pm i\xi(x, \lambda_c(x, t))\}$$
$$\times\int_{e_1}\exp\left\{\pm\frac{i}{2}(\lambda-\lambda_c)^2(\partial_\lambda^2\xi)(x, \lambda_c)\right\}a_\pm(x, t, \lambda)\exp\{\pm ib(x, t, \lambda)\}d\lambda, \tag{5.20}$$

where

$$a_\pm(x, t, \lambda)=\omega(v(x, t)\{\lambda-\lambda_c(x, t)\})\{\lambda-\eta(\lambda)V_1(x)\}^{-1/4}\zeta_{\phi,\pm}(x, \lambda), \tag{5.21}$$

$$b(x, t, \lambda)=\frac{1}{2}\{\lambda-\lambda_c(x, t)\}^3\int_0^1(1-\tau)^2(\partial_\lambda^3\xi)(x, \lambda_c+(\lambda-\lambda_c)\tau)d\tau . \tag{5.22}$$

By (5.15) and (5.5) we have noting $v(x, t)\geq 1$,

$$|\partial_\lambda^l a_\pm(x, t, \lambda)|\leq C_{21}v^l(x, t)r^{-1-\delta} \qquad (l=0, 1, 2) \tag{5.23}$$

$$|\partial_\lambda^l b(x, t, \lambda)|\leq C_{21}|\lambda-\lambda_c(x, t)|^{3-l}r \qquad (l=0, 1, 2). \tag{5.24}$$

We put

$$h_\pm(x, t, \lambda)=a_\pm(x, t, \lambda)\exp\{\pm ib(x, t, \lambda)\}. \tag{5.25}$$

Then obviously,

$$\partial_\lambda h_\pm=\{\partial_\lambda a_\pm\pm ia_\pm\partial_\lambda b\}\exp\{\pm ib\}, \tag{5.26}$$

$$\partial_\lambda^2 h_\pm=\{\partial_\lambda^2 a_\pm\pm 2i\partial_\lambda a_\pm\partial_\lambda b\pm ia_\pm\partial_\lambda^2 b-a_\pm(\partial_\lambda b)^2\}\exp\{\pm ib\}. \tag{5.27}$$

Lemma 5.5. *Let* $v(x, t)=r^{1/3}$ *in* (5.19). *Then there exists a* $C_{22}>0$ *such that for any* $x\in B(R_2+1)$ *and* $t_2(x)\leq\pm t\leq t_1(x)$,

$$|\hat{g}_{\phi,\pm}^{(1)}(x, t)|\leq C_{22}r^{-(n-1)/2}r^{-3/2-\delta}. \tag{5.28}$$

Proof. Note that the support in λ of $h_\pm(x, t, \lambda)$ is contained in e_1. Then by use of the equality

$$h_\pm(x, t, \lambda)=h_\pm(x, t, \lambda_c)+(\lambda-\lambda_c)\int_0^1(\partial_\lambda h_\pm)(x, t, \lambda_c+(\lambda-\lambda_c)\tau)d\tau$$

we have for any sufficiently large N,

$$\int_{e_1} \exp\{\pm 2^{-1} i(\lambda-\lambda_c)^2(\partial_\lambda^2\xi)(x,\lambda_c)\} h_\pm(x,t,\lambda)d\lambda \tag{5.29}$$
$$= h_\pm(x,t,\lambda_c)\int_{-N}^{N} \exp\{\pm 2^{-1} i(\lambda-\lambda_c)^2(\partial_\lambda^2\xi)(x,\lambda_c)\}d\lambda$$
$$+\int_{-N}^{N} \exp\{\pm 2^{-1} i(\lambda-\lambda_c)^2(\partial_\lambda^2\xi)(x,\lambda_c)\}(\lambda-\lambda_c)d\lambda$$
$$\times\int_0^1 (\partial_\lambda h_\pm)(x,t,\lambda_c+(\lambda-\lambda_c)\tau)d\tau\,.$$

Here applying the Fresnel integral formula, we have

$$\lim_{N\to\infty}\int_{-N}^{N} \exp\{\pm 2^{-1} i(\lambda-\lambda_c)^2(\partial_\lambda^2\xi)(x,\lambda_c)\}d\lambda \tag{5.30}$$
$$=\sqrt{2\pi}|(\partial_\lambda^2\xi)(x,\lambda_c)|^{-1/2}\exp(\mp\pi i/4)\,.$$

On the other hand, since the Lebesgue theorem shows that

$$\lim_{N\to\infty}\int_0^1 (\partial_\lambda h_\pm)(x,t,\lambda_c+(\pm N-\lambda_c)\tau)d\tau=0\,,$$

integrating by parts and changing the order of integration, we have

$$\lim_{N\to\infty}\int_{-N}^{N} \exp\{\pm 2^{-1} i(\lambda-\lambda_c)^2(\partial_\lambda^2\xi)(x,\lambda_c)\}(\lambda-\lambda_c)d\lambda \tag{5.31}$$
$$\times\int_0^1 (\partial_\lambda h_\pm)(x,t,\lambda_c+(\lambda-\lambda_c)\tau)d\tau$$
$$=\pm i\{(\partial_\lambda^2\xi)(x,\lambda_c)\}^{-1}\int_0^1 \tau d\tau\int_\Sigma \exp\{\pm 2^{-1} i(\lambda-\lambda_c)^2$$
$$\times(\partial_\lambda^2\xi)(x,\lambda_c)\}(\partial_\lambda^2 h_\pm)(x,t,\lambda_c+(\lambda-\lambda_c)\tau)d\lambda\,,$$

where

$$\Sigma=\{\lambda;\ \lambda_3-\lambda_c\le(\lambda-\lambda_c)\tau\le\lambda_4-\lambda_c \quad\text{and}\quad |(\lambda-\lambda_c)\tau|\le\nu^{-1}\}\,,$$

if we note that $\omega(\lambda)=0$ for $|\lambda|\ge 1$ and $h_\pm(x,t,\lambda)=0$ for $\lambda\notin e_1$. Taking account of (5.5), (5.23) and (5.24), we now have from (5.20), (5.27) and (5.29)—(5.31) the following inequalities which prove (5.28):

$$|\hat{g}^{(1)}_{\phi,\pm}(x,t)|\le C_{23}r^{-(n-1)/2}\Big[r^{-1-\delta}r^{-1/2}+r^{-1}\int_0^1\tau d\tau$$
$$\times\int_{|\lambda-\lambda_c|<(\nu\tau)^{-1}}\{\nu^2+\nu r|\lambda-\lambda_c|^2\tau^2+r|\lambda-\lambda_c|\tau+r^2|\lambda-\lambda_c|^4\tau^4\}r^{-1-\tilde\delta}d\lambda\Big]$$
$$=C_{23}r^{-(n-1)/2}r^{-1-\tilde\delta}\Big[r^{-1/2}+r^{-1}\int_0^1 2\tau\Big\{\nu\tau^{-1}+\frac{1}{3}r\nu^{-2}\tau^{-1}$$
$$+\frac{1}{2}r\nu^{-2}\tau^{-1}+\frac{1}{5}r^2\nu^{-5}\tau^{-1}\Big\}d\tau\Big]$$

$$=C_{23}r^{-(n-1)/2}r^{-1-\tilde{\delta}}\left[r^{-1/2}+r^{-1}\left\{2\nu+\frac{5}{3}r\nu^{-2}+\frac{2}{5}r^{2}\nu^{-5}\right\}\right]$$
$$=C_{23}r^{-(n-1)/2}r^{-1-\tilde{\delta}}\left(r^{-1/2}+\frac{61}{15}r^{-2/3}\right)$$

where in the last equality we have used $\nu=r^{1/3}$. q.e.d.

Lemma 5.6. *Let $\nu(x, t)$ be as in the above lemma. Then there exists a $C_{24}>0$ such that for any $x\in B(R_2+1)$ and $t_2(x)\leq \pm t\leq t_1(x)$,*

$$|\hat{g}^{(2)}_{\phi,\pm}(x, t)|\leq C_{24}r^{-(n-1)/2}r^{-5/3-\tilde{\delta}}. \tag{5.32}$$

Proof. We put

$$d_{\pm}(x, t, \lambda)=\{1-\omega(\nu(x, t)\{\lambda-\lambda_c(x, t)\})\}\{\lambda-\eta(\lambda)V_1(x)\}^{-1/4}\zeta_{\phi\,\pm}(x, \lambda). \tag{5.33}$$

Then it follows from (5.15) that

$$|\partial^l_\lambda d_{\pm}(x, t, \lambda)|\leq C_{25}\nu^l(x, t)r^{-1-\tilde{\delta}} \qquad (l=0, 1, 2) \tag{5.34}$$

in the whole e_1. Note that $d_{\pm}(x, t, \lambda)=0$ in $\{\lambda\in e_1;\ |\lambda-\lambda_c|<(2\nu)^{-1}\}$. On the other hand, it follows from (5.16) and (5.4) that for any $\lambda\in e_1$ satisfying $|\lambda-\lambda_c(x, t)|\geq(2\nu)^{-1}$,

$$\begin{aligned}|t\mp\partial_\lambda\xi(x, \lambda)|&=|\partial_\lambda\xi(x, \lambda_c)-\partial_\lambda\xi(x, \lambda)|\\&=|\lambda-\lambda_c|\,|(\partial^2_\lambda\xi)(x, \lambda_c+(\lambda-\lambda_c)\theta)| \qquad (0<\theta<1)\\&\geq(2\nu)^{-1}C_{17}^{-1}r=(2C_{17})^{-1}r^{2/3}.\end{aligned} \tag{5.35}$$

Now, integrating by parts gives

$$\begin{aligned}\sqrt{\pi}r^{(n-1)/2}\hat{g}^{(2)}_{\phi,\pm}(x, t)&=\int_{e_1}\exp\{-i\lambda t\pm i\xi(x, \lambda)\}d_{\pm}(x, t, \lambda)d\lambda\\&=-\int_{e_1}\exp\{-i\lambda t\pm i\xi(x, \lambda)\}(t\mp\partial_\lambda\xi)^{-2}[\partial^2_\lambda d_{\pm}\pm3\partial^2_\lambda\xi(t\mp\partial_\lambda\xi)^{-1}\partial_\lambda d_{\pm}\\&\quad+\{3(\partial^2_\lambda\xi)^2(t\mp\partial_\lambda\xi)^{-2}\pm\partial^3_\lambda\xi(t\mp\partial_\lambda\xi)^{-1}\}d_{\pm}]d\lambda.\end{aligned} \tag{5.36}$$

So, applying (5.34) and (5.35) in the right side of (5.36), we obtain

$$|\hat{g}^{(2)}_{\phi,\pm}(x, t)|\leq C_{26}r^{-(n-1)/2}r^{-4/3}r^{-1-\tilde{\delta}}\{\nu^2+rr^{-2/3}\nu+(r^2r^{-4/3}+rr^{-2/3})\},$$

which proves (5.32) since $\nu=r^{1/3}$. q.e.d.

Proof of Proposition 4.1. Let μ be a constant satisfying $0<\mu<2\tilde{\delta}$. Then we have

$$\begin{aligned}&\left\{\pm\int_0^{\pm\infty}\|\hat{g}_{\phi,\pm}(\cdot, t)\|dt\right\}^2\\&\leq\int_0^{\pm\infty}(1+|t|)^{-1-\mu}dt\left\{\int_0^{\pm\infty}(1+|t|)^{1+\mu}dt\int_{B(R_2+1)}|\hat{g}_{\phi,\pm}(x, t)|^2dx\right\}\end{aligned} \tag{5.37}$$

$$= \pm \mu^{-1} \int_{B(R_2+1)} dx \int_0^{\pm\infty} |\hat{g}_{\phi,\pm}(x, t)|^2 (1+|t|)^{1+\mu} dt .$$

We divide the integrand of the right side as follows:

$$\begin{aligned}(5.38) \qquad & \pm \int_0^{\pm\infty} |\hat{g}_{\phi,\pm}(x, t)|^2 (1+|t|)^{1+\mu} dt \\ & = \pm \left[\int_0^{\pm t_2(x)} + \int_{\pm t_2(x)}^{\pm t_1(x)} + \int_{\pm t_1(x)}^{\pm\infty} \right] |\hat{g}_{\phi,\pm}(x, t)|^2 (1+|t|)^{1+\mu} dt \\ & = I_1 + I_2 + I_3 .\end{aligned}$$

By (5.14) of Lemma 5.3 we have

$$\begin{aligned} I_1 \leq & \pm 2 C_{19}^2 r^{-(n-1)} r^{-2-2\delta} \int_0^{\pm t_2(x)} (1+|t|)^{1+\mu} (t_4(x)-|t|)^{-4} \\ & \times [1 + r^4 (t_4(x)-|t|)^{-4}] \, dt .\end{aligned}$$

Thus, it follows from (5.10) and (5.8) that

$$\begin{aligned} I_1 & \leq 2(2+\mu)^{-1} C_{19}^2 C_{18}^4 r^{-(n-1)} r^{-2-2\delta} r^{-4} (1 + C_{17} r)^{2+\mu} (1 + C_{18}^4) \\ & \leq C_{27} r^{-(n-1)} r^{-4-2\delta+\mu} .\end{aligned}$$

By (5.13) of Lemma 5.3 we have

$$\begin{aligned} I_3 \leq & \pm 2 C_{19}^2 r^{-(n-1)} r^{-2-2\delta} \int_{\pm t_1(x)}^{\pm\infty} (1+|t|)^{1+\mu} (|t| - t_3(x))^{-4} \\ & \times [1 + r^4 (|t| - t_3(x))^{-4}] dt \\ \leq & \pm 2 C_{19}^2 r^{-(n-1)} r^{-2-2\delta} \int_{\pm t_1(x)}^{\pm\infty} 2^{\mu} \{ (|t| - t_3(x))^{1+\mu} + (t_3(x)+1)^{1+\mu} \} \\ & \times (|t| - t_3(x))^{-4} [1 + r^4 (|t| - t_3(x))^{-4}] dt .\end{aligned}$$

Thus, it follows from (5.9) and (5.8) that

$$\begin{aligned} I_3 \leq & 2^{1+\mu} C_{19}^2 r^{-(n-1)} r^{-2-2\delta} \{ (2-\mu)^{-1} (C_{18} r)^{-2+\mu} + 3^{-1} (1 + C_{17} r)^{1+\mu} \\ & \times (C_{18} r)^{-3} \} (1 + C_{18}^4) \\ \leq & C_{28} r^{-(n-1)} r^{-4-2\delta+\mu} .\end{aligned}$$

Further, by Lemmas 5.5 and 5.6,

$$\begin{aligned} I_2 \leq & \pm 2 \int_{\pm t_2(x)}^{\pm t_1(x)} (1+|t|)^{1+\mu} \{ |\hat{g}_{\phi,\pm}^{(1)}(x, t)|^2 + |\hat{g}_{\phi,\pm}^{(2)}(x, t)|^2 \} dt \\ \leq & 2(2+\mu)^{-1} r^{-(n-1)} (C_{22}^2 r^{-3-2\delta} + C_{24}^2 r^{-10/3-2\delta}) (1 + t_1(x))^{2+\mu} .\end{aligned}$$

Thus, it follows from (5.8) that

$$I_2 \leq C_{29} r^{-(n-1)} r^{-1-2\delta+\mu} .$$

Summarizing these inequalities, we have from (5.37) and (5.38),

$$\left\{\pm\int_0^{\pm\infty}\|\hat{g}_{\phi,\pm}(\cdot,t)\|dt\right\}^2$$

$$\leq\mu^{-1}\int_{B(R_2+1)}\{(C_{27}+C_{28})r^{-4-2\delta+\mu}+C_{29}r^{-1-2\delta+\mu}\}r^{-(n-1)}dx<\infty\,.$$

Thus, (4.5) holds and the proof of Proposition 4.1 is complete. q.e.d.

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