Publ. RIMS, Kyoto Univ.
18 (1982), 987–993

On the Oka-Cartan-Kawai Theorem B for the Sheaf ${}^E\tilde{\mathcal{O}}$

Dedicated to Professor Hisaaki Yoshizawa on his 60th birthday

By

Yoshifumi ITO*

The purpose of this note is to present a proof of the Oka-Cartan-Kawai Theorem B for the sheaf ${}^E\tilde{\mathcal{O}}$ of germs of slowly increasing vector valued holomorphic functions over $\tilde{\boldsymbol{C}}^n = \boldsymbol{D}^n \times \sqrt{-1}\boldsymbol{R}^n$. It was in June 1979 that this theorem was proved. Slightly afterward in the same year Junker proved it independently by another method [5]. Since our method is simpler than the Junker's, we report it here.

§1. The Sheaf ${}^E\tilde{\mathcal{O}}$

We denote by $\boldsymbol{D}^n$ the radial compactification of $\boldsymbol{R}^n$ in the sense of Kawai [6], [7], and by $\tilde{\boldsymbol{C}}^n$ the space $\boldsymbol{D}^n \times \sqrt{-1}\boldsymbol{R}^n$. We denote by E a quasi-complete locally convex topological vector space (LCTVS) (always assumed to be Hausdorff) unless the contrary is explicitly mentioned, and by $\mathscr{P} = \mathscr{P}_E$ the family of continuous seminorms of E defining a locally convex topology on E.

We now denote by ${}^E\tilde{\mathcal{O}}$ the sheafification of the presheaf $\{\tilde{\mathcal{O}}(\Omega; E)\}$, where, for an open set Ω in $\tilde{\boldsymbol{C}}^n$, the module $\tilde{\mathcal{O}}(\Omega; E)$ is defined as follows:

$$\tilde{\mathcal{O}}(\Omega; E) = \{f \in \mathcal{O}(\Omega \cap \boldsymbol{C}^n; E); \textit{for any positive } \varepsilon \textit{ and any compact set } K \textit{ in } \Omega \textit{ and any } q \in \mathscr{P}, \sup_{z \in K \cap \boldsymbol{C}^n} q(f(z)) e^{-\varepsilon |z|} < \infty \textit{ holds}\}.$$

Here we denote by $\mathcal{O}(\Omega \cap \boldsymbol{C}^n; E)$ the module of all E-valued holomorphic functions on the open set $\Omega \cap \boldsymbol{C}^n$ in $\boldsymbol{C}^n$.

We call this sheaf ${}^E\tilde{\mathcal{O}}$ the sheaf of germs of slowly increasing E-valued holomorphic functions.

It is easy to see that ${}^E\tilde{\mathcal{O}}|\boldsymbol{C}^n = {}^E\mathcal{O}$ holds, where ${}^E\mathcal{O}$ denotes the sheaf of germs

Communicated by M. Sato, June 10, 1981.
* Department of Mathematics, College of General Education, Tokushima University, Tokushima 770, Japan.

of E-valued holomorphic functions over $\boldsymbol{C}^n$, and that, for $E=\boldsymbol{C}$, ${}^{\boldsymbol{C}}\tilde{\mathcal{O}}=\tilde{\mathcal{O}}$ holds, where $\tilde{\mathcal{O}}$ is the sheaf of germs of slowly increasing holomorphic functions which was defined by Kawai [6], [7].

§2. The Sheaf ${}^{E}\tilde{\mathscr{E}}$

We recall the definition of the sheaf ${}^{E}\tilde{\mathscr{E}}$ of germs of slowly increasing E-valued C^∞-functions following Junker [4].

We define ${}^{E}\tilde{\mathscr{E}}$ to be the sheafification of the presheaf $\{\tilde{\mathscr{E}}(\Omega; E)\}$, where, for an open set Ω in $\tilde{\boldsymbol{C}}^n$, the module $\tilde{\mathscr{E}}(\Omega; E)$ is defined as follows:

$$\tilde{\mathscr{E}}(\Omega; E)=\{f\in\mathscr{E}(\Omega\cap\boldsymbol{C}^n; E);\ \textit{for any positive } \varepsilon \textit{ and any compact set } K \textit{ in } \Omega \textit{ and any } \alpha\in\bar{\boldsymbol{N}}^{2n} \textit{ and any } q\in\mathscr{P},\ \sup_{z\in K\cap\boldsymbol{C}^n} q(f^{(\alpha)}(z))e^{-\varepsilon|z|}<\infty \textit{ holds}\}.$$

Here $\bar{\boldsymbol{N}}=\boldsymbol{N}\cup\{0\}$ and $\mathscr{E}(\Omega\cap\boldsymbol{C}^n; E)$ is the module of E-valued C^∞-functions on the open set $\Omega\cap\boldsymbol{C}^n$ in $\boldsymbol{C}^n$. For $E=\boldsymbol{C}$, we put $\tilde{\mathscr{E}}={}^{\boldsymbol{C}}\tilde{\mathscr{E}}$.

Proposition 2.1. *Let Ω be an open set in $\tilde{\boldsymbol{C}}^n$. Then $\tilde{\mathscr{E}}(\Omega)$ is a nuclear Fréchet space.*

Proof. See Junker [4], chapter III, Theorem 1.4. Q.E.D.

Proposition 2.2. *Let Ω be an open set in $\tilde{\boldsymbol{C}}^n$. Assume that E is a Fréchet space. Then we have the isomorphism $\tilde{\mathscr{E}}(\Omega; E)\cong\tilde{\mathscr{E}}(\Omega)\hat{\otimes}E$.*

Proof. See Junker [4], Chapter III, Theorem 1.6. Q.E.D.

Proposition 2.3. The sheaf ${}^{E}\tilde{\mathscr{E}}$ is soft.

Proof. Since ${}^{E}\tilde{\mathscr{E}}$ is obviously an $\tilde{\mathscr{E}}$-module, we have only to prove that $\tilde{\mathscr{E}}$ is soft by virtue of Theorem 9.12 of Bredon [1], Chapter II, p. 50. Since any open set in $\tilde{\boldsymbol{C}}^n$ is paracompact, any closed subset K has a fundamental system of paracompact neighborhoods. Now, let $f\in\tilde{\mathscr{E}}(K)$. Then by Theorem 9.4 of Bredon [1], Chapter II, p. 48, there are a closed neighborhood K' of K and $f'\in\tilde{\mathscr{E}}(K')$ extending f. Since $\tilde{\boldsymbol{C}}^n$ is paracompact and normal, there is a bounded C^∞ function g with bounded derivatives of any degree which is zero on the boundary $\partial K'$ of K' and 1 on K. The section gf': $x\to g(x)f'(x)$ in $\tilde{\mathscr{E}}(K')$ is zero on $\partial K'$ and coincides with f on K. Thus gf', and hence f, can be extended to $\tilde{\boldsymbol{C}}^n$. Thus we have proved that the sheaf $\tilde{\mathscr{E}}$ is soft, which completes the proof.
Q.E.D.

§3. The Dolbeault-Grothendieck Resolution of ${}^{E}\tilde{\mathcal{O}}^{p}$

Here we construct a soft resolution of the sheaf ${}^{E}\tilde{\mathcal{O}}^{p}$. First we introduce some notations. Let $\mathscr{F}$ be a sheaf over $\tilde{\mathbb{C}}^{n}$. Let Ω be an open set in $\tilde{\mathbb{C}}^{n}$. A differential from f with coefficients in $\mathscr{F}(\Omega)$ is said to be of type (p, q) if it can be written in the from:

$$f=\sum_{|I|=p}\sum_{|J|=q} f_{I,J}dz_{I}\wedge d\bar{z}_{J}\,,$$

where $I=(i_1,\dots, i_p)$ and $J=(j_1,\dots, j_q)$ are p-tuple and q-tuple of natural numbers $\{1,\dots, n\}$, respectively and we put

$$dz_{I}=dz_{i_1}\wedge\cdots\wedge dz_{i_p}\,,$$
$$d\bar{z}_{J}=d\bar{z}_{j_1}\wedge\cdots\wedge d\bar{z}_{j_q}\,,$$

and take

$$f_{I.J}\in\mathscr{F}(\Omega)\,.$$

Then we denote by $\mathscr{F}^{p,q}$ the sheaf of germs of differential forms of type (p, q) with coefficients in $\mathscr{F}$. We define the morphisms ∂, $\bar{\partial}$ of sheaves $\mathscr{F}^{p,q}$:

$$\partial:\ \mathscr{F}^{p,q}\longrightarrow\mathscr{F}^{p+1,q}$$
$$\bar{\partial}:\ \mathscr{F}^{p,q}\longrightarrow\mathscr{F}^{p,q+1}$$

as follows:

$$\partial f=\sum_{i=1}^{n}\sum_{|I|=p}\sum_{|J|=q}(\partial/\partial z_i)f_{I,J}dz_i\wedge dz_{I}\wedge d\bar{z}_{J}\,.$$
$$\bar{\partial} f=\sum_{i=1}^{n}\sum_{|I|=p}\sum_{|J|=q}(\partial/\partial \bar{z}_i)f_{I,J}d\bar{z}_i\wedge dz_{I}\wedge d\bar{z}_{J}\,.$$

We put $\mathscr{F}^{p}=\mathscr{F}^{p,0}$. Then we have the following

Theorem 3.1 (Dolbeault-Grothendieck resolution of ${}^{E}\tilde{\mathcal{O}}^{p}$). *Let E be a quasi-complete LCTVS. Then the sequence of sheaves over* $\tilde{\mathbb{C}}^{n}$

$$0\longrightarrow{}^{E}\tilde{\mathcal{O}}^{p}\longrightarrow{}^{E}\tilde{\mathscr{E}}^{p,0}\xrightarrow{\ \bar{\partial}\ }{}^{E}\tilde{\mathscr{E}}^{p,1}\xrightarrow{\ \bar{\partial}\ }\cdots\xrightarrow{\ \bar{\partial}\ }{}^{E}\tilde{\mathscr{E}}^{p,n}\longrightarrow 0$$

is exact.

Proof. The exactness of the sequence

$$0\longrightarrow{}^{E}\tilde{\mathcal{O}}^{p}\longrightarrow{}^{E}\tilde{\mathscr{E}}^{p,0}\xrightarrow{\ \bar{\partial}\ }{}^{E}\tilde{\mathscr{E}}^{p,1}$$

is evident. In fact, let Ω be a relatively compact open set in $\tilde{\mathbb{C}}^{n}$. Let $u\in$

$\tilde{\mathscr{E}}^{p,0}(\Omega; E)$ such that $\bar{\partial}u=0$. Then, if we write u in the from

$$u=\sum_{|I|=p} u_I dz_I,$$

we have

$$\partial u_I/\partial \bar{z}_j=0, \quad j=1, 2,\dots, n.$$

But this is the Cauchy-Riemann equation. Thus u_I is holomorphic in $\Omega \cap \boldsymbol{C}^n$. The condition that u_I is slowly increasing is already satisfied as the element of $\tilde{\mathscr{E}}(\Omega; E)$. Thus the exactness of the above sequence was proved.

Next we have to prove the exactness of the sequence

$$_E\tilde{\mathscr{E}}^{p,0} \xrightarrow{\bar{\partial}} {}_E\tilde{\mathscr{E}}^{p,1} \xrightarrow{\bar{\partial}} \cdots \xrightarrow{\bar{\partial}} {}_E\tilde{\mathscr{E}}^{p,n} \longrightarrow 0.$$

We will reason as in Hörmander [2], p. 32. Thus it follows from the following

Lemma 1. *Let Ω be a relatively compact open set in $\tilde{\boldsymbol{C}}^n$. Let $f \in \tilde{\mathscr{E}}^{p,q+1}(\Omega; E)$ $(p, q \geqq 0)$ satisfy the condition $\bar{\partial}f=0$. If Ω' is a relatively compact open set in Ω, we can find $u \in \tilde{\mathscr{E}}^{p,q}(\Omega'; E)$ with $\bar{\partial}u=f$ in Ω'.*

Proof of Lemma 1. We shall prove inductively that the Lemma is true if f does not involve $d\bar{z}_{k+1},\dots, d\bar{z}_n$. This is trivial if $k=0$, for f must then be zero since every term in f is of degree $q+1>0$ with respect to $d\bar{z}$. For $k=n$, the statement is identical to the Lemma. Assuming that it has already been proved when k is replaced by $k-1$, we write

$$f=d\bar{z}_k \wedge g+h,$$

where $g \in \tilde{\mathscr{E}}^{p,q}(\Omega; E)$, $h \in \tilde{\mathscr{E}}^{p,q+1}(\Omega; E)$ and g and h are independent of $d\bar{z}_k$, $\dots$, $d\bar{z}_n$. Write

$$g=\sum_{|I|=p}{}' \sum_{|J|=q}{}' g_{I,J} dz_I \wedge d\bar{z}_J,$$

where $\sum'$ means that we sum only over increasing multi-indices. Since $\bar{\partial}f=0$, we obtain

$$\partial g_{I,J}/\partial \bar{z}_j=0, \qquad j>k.$$

Thus $g_{I,J}$ is holomorphic in these variables.

We now choose a solution $G_{I,J}$ of the equation

$$\partial G_{I,J}/\partial \bar{z}_k=g_{I,J}.$$

To do so, we choose a bounded function $\psi \in C_0^\infty(\Omega)$ with bounded derivatives of any degree so that $\psi(z)=1$ in a neighborhood $\Omega'' \Subset \Omega$ of $\bar{\Omega}'$, and set

$$G_{I,J}=(2i\pi)^{-1}\iint(\tau-z_k)^{-1}e^{-(\tau-z_k)^2}\psi(z_1,\dots,z_{k-1},\tau,z_{k+1},\dots,z_n)g_{I,J}(z_1,\dots,z_{k-1},\tau,z_{k+1},\dots,z_n)d\tau\wedge d\bar{\tau}$$
$$=-(2i\pi)^{-1}\iint\tau^{-1}e^{-\tau^2}\psi(z-\tau\eta_k)g_{I,J}(z-\tau\eta_k)d\tau\wedge d\bar{\tau}\,.$$

Here η_k denotes the vector $\eta_k=(\delta_{jk};\, j=1,2,\dots,n)$, δ_{jk} denoting Kronecker's δ. The last expression shows that $G_{I,J}\in\tilde{\mathscr{E}}(\Omega;E)$ by a simple estimate.

Here we prepare the following two lemmas.

Lemma 2. *Let ω be a relatively compact open set in $\tilde{\mathbf{C}}$ whose boundary $\partial\omega$ consists of a finite number of C^1 Jordan curves. Let u be a slowly increasing C^1-function in a neighborhood of $\bar{\omega}$. Then we have*

$$u(\zeta)=(2i\pi)^{-1}\left\{\int_{\partial\omega}(z-\zeta)^{-1}\exp(-(z-\zeta)^2)u(z)dz+\iint_{\omega}(z-\zeta)^{-1}\exp(-(z-\zeta)^2)\partial u/\partial\bar{z}\;dz\wedge d\bar{z}\right\},\qquad \zeta\in\omega\,.$$

Here $\partial\omega$ is oriented so that ω lies to the left of $\partial\omega$.

Lemma 3. *If μ is a measure with compact support in $\mathbf{C}$ such that, for any positive δ,*

$$\int_{\mathbf{C}}e^{-\delta|z|}d\mu(z)<\infty$$

holds, the integral

$$u(\zeta)=\int_{\mathbf{C}}(z-\zeta)^{-1}\exp(-(z-\zeta)^2)d\mu(z)$$

defines a slowly increasing holomorphic function outside the support of μ. In any open set ω where $d\mu=(2i\pi)^{-1}\phi dz\wedge d\bar{z}$ for some slowly increasing C^k-function with compact support in ω, we know that u is a slowly increasing C^k-function in ω and satisfies the equation $\partial u/\partial\bar{z}=\phi$ if $k\geqq 1$.

Then it follows from Lemma 3 that

$$\partial G_{I,J}/\partial\bar{z}_k=g_{I,J}$$

holds in Ω'', and we obtain by differentiating under the sign of integration

$$\partial G_{I,J}/\partial\bar{z}_j=0,\quad j>k\,.$$

If we set

$$G=\sum_{I,J}{}'\,G_{I,J}dz_I\wedge d\bar{z}_J\,,$$

it follows that in Ω''

$$\bar{\partial}G = d\bar{z}_k \wedge g + h_1 ,$$

where h_1 is independent of $d\bar{z}_k, \dots, d\bar{z}_n$. Hence $h - h_1 = f - \bar{\partial}G$ does not involve $d\bar{z}_k, \dots, d\bar{z}_n$, so by the inductive hypothesis we can find $v \in \tilde{\mathscr{E}}^{p,q}(\Omega'; E)$ so that $\bar{\partial}v = f - \bar{\partial}G$ there. But then $u = v + G$ satisfies the equation $\bar{\partial}u = f$, which completes the proof. Q. E. D.

In case of $E = \boldsymbol{C}$, we have the following

Corollary. *The sequence of sheaves over* $\boldsymbol{C}^n$.

$$0 \longrightarrow \tilde{\mathcal{O}}^p \longrightarrow \tilde{\mathscr{E}}^{p,0} \xrightarrow{\bar{\partial}} \tilde{\mathscr{E}}^{p,1} \xrightarrow{\bar{\partial}} \cdots \xrightarrow{\bar{\partial}} \tilde{\mathscr{E}}^{p,n} \longrightarrow 0$$

is exact.

§4. Oka-Cartan-Kawai Theorem B

We can now prove the Oka-Cartan-Kawai Theorem B for the sheaf ${}^E\tilde{\mathcal{O}}$.

Theorem 4.1 (Oka-Cartan-Kawai Theorem B). *Let E be a Fréchet space. For any $\tilde{\mathcal{O}}$-pseudoconvex domain Ω in $\tilde{\boldsymbol{C}}^n$, we have $H^p(\Omega, {}^E\tilde{\mathcal{O}}) = 0$ ($p \geqq 1$).*

Proof. Let Ω be an $\tilde{\mathcal{O}}$-pseudoconvex domain in the sense of Definition 2.1.3 of Kawai [7], p. 471 (see also Kawai [6]). Then the Oka-Cartan Kawai Theorem B for $\tilde{\mathcal{O}}$ shows that

$$H^p(\Omega, \tilde{\mathcal{O}}) = 0 \qquad (p \geqq 1),$$

(see Theorem 2.1.4 of Kawai [7], p. 471 and see also Kawai [6]). Thus the complex obtained from the Corollary to Theorem 3.1:

$$\tilde{\mathscr{E}}^{0,0}(\Omega) \xrightarrow{\bar{\partial}} \tilde{\mathscr{E}}^{0,1}(\Omega) \xrightarrow{\bar{\partial}} \cdots \xrightarrow{\bar{\partial}} \tilde{\mathscr{E}}^{0,n}(\Omega) \longrightarrow 0$$

is exact. Since $\tilde{\mathscr{E}}^{0,q}(\Omega)$'s are nuclear Fréchet spaces and E is a Fréchet space, the complex

$$\tilde{\mathscr{E}}^{0,0}(\Omega; E) \xrightarrow{\bar{\partial}} \tilde{\mathscr{E}}^{0,1}(\Omega; E) \xrightarrow{\bar{\partial}} \cdots \xrightarrow{\bar{\partial}} \tilde{\mathscr{E}}^{0,n}(\Omega; E) \longrightarrow 0$$

is also exact by virtue of the isomorphism

$$\tilde{\mathscr{E}}^{0,q}(\Omega; E) \cong \tilde{\mathscr{E}}^{0,q}(\Omega) \hat{\otimes} E$$

and the Theorem 1.10 of Ion and Kawai [3], p. 9. Hence we obtain

$$H^p(\Omega, {}^E\tilde{\mathcal{O}}) = 0 \qquad (p \geqq 1).$$

This completes the proof. Q. E. D.

Acknowledgment

The author wishes to express his sincere gratitude to Professors H. Yoshizawa and T. Hirai for their constant encouragements during the preparation of this work and to Professor M. Morimoto for his invaluable suggestions and to Dr. Junker for kindly sending the author his preprints of Diplomarbeit and Inaugural-Dissertation.

References

[1] Bredon, G. E., *Sheaf Theory*, McGraw-Hill, New York/St. Louis/San Francisco/Toronto/London/Sydney, 1967.

[2] Hörmander, L., *An Introduction to Complex Analysis in Several Variables*, North-Holland/American Elsevier, Amsterdam/London/New York, 1973.

[3] Ion, P. D. F. and T. Kawai, Theory of Vector-Valued Hyperfunctions. *Publ. RIMS, Kyoto Univ.*, **11** (1975), 1–19.

[4] Junker, K., Vektorwertige Fourierhyperfunktionen, Diplomarbeit, Düsseldorf, 1978.

[5] ———, Vektorwertige Fourierhyperfunktionen und Ein Satz von Bochner-Schwartz-Typ, Inaugural-Dissertation zur Erlangung des Doktorgrades der Mathematisch-Naturwissenschaftlichen Fakultät der Universität Düsseldorf, Düsseldorf, 1979.

[6] Kawai, T., The theory of Fourier transformations in the theory of hyperfunctions and its applications, *Surikaiseki Kenkyusho Kokyuroku* **108**, *RIMS, Kyoto Univ.*, (1969) (in Japanese).

[7] Kawai, T., On the theory of Fourier hyperfunctions and its applications to partial differential equations with constant coefficients. *J. Fac. Sci. Univ. Tokyo, Sect. IA*, **17** (1970), 467–517.