Real Algebraic Realization of Characteristic Classes

By

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§1. Introduction

Throughout this paper X denotes a real non-singular affine algebraic variety of dimension n. We will give a realization of the characteristic classes (the Stiefel-Whitney classes, the Pontrjagin classes and the Euler classes) of real affine algebraic vector bundles over X by algebraic subvarieties (Theorems 1, 2). For the complex field, Grothendieck [4] showed that the Chern classes of an algebraic vector bundle over a complex non-singular quasi-projective variety are realized by algebraic cycles. Morimoto [6] considered the complex analytic case. We prove Theorems 1, 2 by the method used there. If we work over a real analytic vector bundle, Thom's transversality theorem shows easily a realization of the characteristic classes by analytic subsets (see Suzuki [10]).

Theorem 1 was partially proved in [2], [8], and two different applications of them were given in [2], [9].

In Section 4, Theorems 3, 4 will show that the smoothing of algebraic subvarieties of X of codimension 1 for homological equivalence is always possible. The proof uses an idea in [8].

Given two cohomology classes of X which are realized by algebraic subvarieties, it seems likely that their cup product is realized by an algebraic subvariety. We prove this under some assumptions, applying Theorems 1, 2 (Theorem 5). We must remark that a realization of the cup product by an analytic subset is always possible according to the transversality theorem.

Section 6 considers an affine algebraic structure of a topological vector bundle over X. If the rank is 1, and if the Stiefel-Whitney class is realized by an algebraic subvariety, then the bundle has an affine algebraic structure.

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§2. Preliminaries

Let V be an algebraic subvariety of X of dimension k. We say that an element $\alpha \in H^{n-k}(X; \mathbf{Z}_2)$ or $H^{n-k}(X; \mathbf{Z})$ is *realized* by V if the Poincaré dual of α is the fundamental class $[V] \in H_k(X; \mathbf{Z}_2)$ or $H_k(X; \mathbf{Z})$ respectively where we use infinite chains if X is not compact. Here, if $\alpha \in H^{n-k}(X; \mathbf{Z})$ we require X, V to be orientable. See Appendix in [5] and [3] for the definitions of the Poincaré dual and the fundamental class.

Let $V \subset \mathbf{R}^m$, $V' \subset \mathbf{R}^{m'}$ be non-singular algebraic varieties. A C^∞ map f from V to V' is called *smooth rational* if it is the restriction of a rational map from $\mathbf{R}^m$ to $\mathbf{R}^{m'}$. We call a vector bundle $V \xrightarrow{f} V'$ an *affine algebraic vector bundle* if the coordinate functions, the inverses and the coordinate transformations are smooth rational. Here the general linear group is provided with the natural algebraic structure, and the coordinate neighborhoods are Zariski open sets.

Let $G_{m,m'}$ be the Grassmann manifold of m-linear subspaces in $\mathbf{R}^{m+m'}$ and $V_{m,m'}$ be the Stiefel manifold of orthogonal m-frames in $\mathbf{R}^{m+m'}$. Then $V_{m,m'}$ is clearly a locally closed non-singular algebraic subvariety of $\underbrace{\mathbf{R}^{m+m'} \times \cdots \times \mathbf{R}^{m+m'}}_{m}$. We give $G_{m,m'}$ an algebraic structure as follows [7]. Given $T \in G_{m,m'}$, let P_T denote the orthogonal projection of $\mathbf{R}^{m+m'}$ onto T. Thus $T \to P_T$ is an injection of $G_{m,m'}$ into the space of linear endomorphisms of $\mathbf{R}^{m+m'}$. We see easily that the image is a non-singular algebraic variety. We identify $G_{m,m'}$ with the image. Let p be the canonical map from $V_{m,m'}$ to $G_{m,m'}$ and $\xi_{m,m'}: E_{m,m'} \xrightarrow{q} G_{m,m'}$ be the vector bundle defined by

$$E_{m,m'} = \{(T, x) \in G_{m,m'} \times \mathbf{R}^{m+m'} \,|\, x \in T\}.$$

Then p is obviously smooth rational and $\xi_{m,m'}$ is an affine algebraic vector bundle. Let $\varphi_{m,m'}: E_{m,m'} \to \mathbf{R}^{m+m'}$ denote the projection onto the second factor.

$G_{m,m'}$ has a cellular subdivision by the Schubert varieties (see [15]). We will be concerned with some of the varieties. Take an affine coordinate system $(x_1, \ldots, x_{m+m'})$ of $\mathbf{R}^{m+m'}$, and denote by $\mathbf{R}^k$ the linear subspace defined by the equations:

$$x_{k+1} = \cdots = x_{m+m'} = 0 \quad (0 \leq k \leq m+m').$$

For each integers $1 \leq j \leq i \leq m$, let $F_j^i \subset G_{m,m'}$ denote the set of all linear subspaces T such that

$$\dim (T \cap \pmb{R}^{m+m'-i}) \geqq m+j-i \,.$$

Then F_j^i is an algebraic subvariety with the set of singularities F_{j+1}^i. It is well-known [15] that F_1^i realizes the $(m-i+1)$-th Stiefel-Whitney class of $\xi_{m,m'}$.

Let $\mathfrak{m}_k(m, i)$ denote the set of all linear maps from $\pmb{R}^m$ to $\pmb{R}^i$ of rank $\leqq i-k$ for $0 \leqq k \leqq i \leqq m$. Then $\mathfrak{m}_k(m, i)$ is an algebraic subvariety of $\mathfrak{m}_0(m, i)=\pmb{R}^{mi}$ with the set of singularities $\mathfrak{m}_{k+1}(m, i)$ for $1 \leqq k$. We remark that the stratification $\{\mathfrak{m}_k-\mathfrak{m}_{k+1}\}_{k=0,\dots i}$ of $\mathfrak{m}_0(m, i)$ satisfies the Whitney condition A [14], namely, given $k<k'$, $A \in \mathfrak{m}_{k'}-\mathfrak{m}_{k'+1}$ and $A_j \in \mathfrak{m}_k-\mathfrak{m}_{k+1}$, $j=1, 2,\dots$ such that A_j and the tangent vector space of $\mathfrak{m}_k-\mathfrak{m}_{k+1}$ at A_j tend to A and an element τ in $G_{l,l'}$ respectively for some l, l', then τ contains the tangent space of $\mathfrak{m}_{k'}-\mathfrak{m}_{k'+1}$ at A (see [6] for a more detailed property).

Let $\eta : E \xrightarrow{\pi} M$ be a C^∞ vector bundle of rank m over a C^∞ manifold M. Let $f: E \to \pmb{R}^i$, $\Phi : M \to G_{m,m'-i}$ be C^∞ maps such that the induced bundle $\Phi^*(\xi_{m,m'-i}) \cong \eta$. We write the lift of Φ as $\tilde{\Phi} : E \to E_{m,m'-i}$. Put

$$\Psi = \varphi_{m,m'-i} \circ \tilde{\Phi} : E \longrightarrow \pmb{R}^{m+m'-i} \,,$$
$$\Psi_f = (f, \Psi) : E \longrightarrow \pmb{R}^{m+m'} \,.$$

We regard M as a submanifold of E through the zero cross-section. Since the restriction of Ψ_f on each fibre of E is an imbedding, the image of the tangent space of the fibre at a point of M under the derivative $d\Psi_f$ is an m-linear subspace in $\pmb{R}^{m+m'}$. This defines a C^∞ map $\Phi_f : M \to G_{m,m'}$.

A C^∞ map $f: M \times \pmb{R}^m \to \pmb{R}^i$ is called *in general position* (at a point $a \in M$) if the map $\bar{f} : M \to \mathfrak{m}_0(m, i)$ defined by

$$\bar{f}(x) = d(f|_{x \times \pmb{R}^m})_{x \times 0}$$

is transversal (in a neighborhood of a) to all the strata $\mathfrak{m}_k(m, i)-\mathfrak{m}_{k+1}(m, i)$, $k=0,\dots, i$. We can generalize naturally this definition to $f: E \to \pmb{R}^i$ for a C^∞ vector bundle E (see [6]). The following was proved in [6].

Lemma 1. *With the same notation as above, $f: E \to \pmb{R}^i$ is in general position if and only if $\Phi_f : M \to G_{m,m'}$ is transversal to the strata $F_1^i-F_2^i$, $F_2^i-F_3^i,\dots, F_i^i$.*

Lemma 2. *If M is compact, and if we provide the set of all C^∞ maps from E to $\pmb{R}^i$ with the compact-open C^2 topology, then the subset of maps in general position is open and dense.*

Assume that X is contained in $\pmb{R}^{n+n'}$. Then the normal bundle of X is

equivalent to the induced bundle $h^* \xi_{n',n}$, where h is the canonical smooth rational map from X to $G_{n',n}$ defined by $h(x)=$ the space of normal vectors of X at $x \in X$. The reason why h is smooth rational is as follows. Let $f_1, \ldots, f_{n'}$ be polynomials on $\mathbf{R}^{n+n'}$ vanishing on X. We denote by U the subset of X of points at which $\operatorname{grad} f_1, \ldots, \operatorname{grad} f_{n'}$ are independent, and by $v_1, \ldots, v_{n'}$ the orthogonalization of Schmidt of $\operatorname{grad} f_1, \ldots, \operatorname{grad} f_{n'}$ on U, namely,

$$v_1 = \operatorname{grad} f_1, \ v_2 = |v_1|^2 \operatorname{grad} f_2 - (v_1, \operatorname{grad} f_2) v_1, \ldots.$$

Then the map $v(x) = (v_1, \ldots, v_{n'})$ from U to $V_{n',n}$ is smooth rational and satisfies the equation $p \circ v = h$ on U. As X is covered by such U for suitable $f_1, \ldots, f_{h'}$, h is smooth rational on X. This shows also that given $V \xrightarrow{\pi} X$ an affine algebraic vector bundle of rank m, there exists a smooth rational map $g: X \to G_{m,m''}$ for some m'' such that $g^* \xi_{m,m''}$ is rationally equivalent to $V \xrightarrow{\pi} X$. Moreover it follows that for any smooth rational map $f: V \to \mathbf{R}^i$, $g_f: X \to G_{m,m'}$ is smooth rational with $m' = m'' + i$, and $g_f^* \xi_{m,m'}$ is also rationally equivalent to $V \xrightarrow{\pi} X$.

Lastly we need the concept of stratified map. Let h be a smooth rational map from X to another non-singular affine algebraic variety X'. Then we can stratify h, that is, there exist non-singular semi-algebraic subvarieties $Y_1, \ldots, Y_k$ of X' such that X' is the disjoint union of $Y_1, \ldots, Y_k$, that $Z_j = h^{-1}(Y_j)$ is non-singular for each j, that the restriction of h on Z_j is a submersion to Y_j for each j, and that the stratification $\{Y_j\}_j$ satisfies the Whitney condition A. For the proof, see [14].

A C^∞ function is called *regular* at a point if one of the first partial derivatives takes a non-zero value at the point.

§3. Algebraic Realization of Characteristic Classes

Theorem 1. *The Stiefel-Whitney classes of any affine algebraic vector bundle over X are realized by algebraic subvarieties. We can choose a non-singular algebraic subvariety of realization of the k-th class for $k=1$, $k \geq (n-1)/2$ or $k=$ rank of the bundle.*

Theorem 2. *Assume that X is orientable. Then the Pontrijagin classes of an affine algebraic vector bundle over X are realized by orientable algebraic subvarieties. If moreover the bundle is orientable, this statement holds for the Euler class. The subvarieties of realizations of the Eular class and the k-th Pontrjagin class can be non-singular, for $k \geq (n-2)/6$.*

Proof of Theorem 1.   Let m be the rank of the bundle, and $1 \leq i \leq m$ be an integer. We want to realize the $(m-i+1)$-th Stiefel-Whitney class by an algebraic subvariety. Through the argument in Section 2, we regard the bundle as the induced bundle $h^*\xi_{m,m'-i}$ for a smooth rational map $h: X \to G_{m,m'-i}$. Assume that there exists a smooth rational map $f: h^*E_{m,m'-i} \to \mathbb{R}^i$ in general position. Then, by Lemma 1 $h_f: X \to G_{m,m'}$ is transversal to the strata $F_1^i - F_2^i, F_2^i - F_3^i, \ldots$. Now F_1^i realizes the $(m-i+1)$-th Stiefel-Whitney class of $\xi_{m,m'}$. Hence $h_f^{-1}(F_1^i)$ is an algebraic subvariety of realization. Thus it is sufficient to show that at least one smooth rational map $f: h^*E_{m,m'-i} \to \mathbb{R}^i$ is in general position. This is immediately proved by Lemma 2 if X is compact.

By the remark at the end of Section 2, there exist non-singular semi-algebraic subvarieties $Y_1, \ldots, Y_l$ of $G_{m,m''}$ with $m'' = m' - i$ such that $G_{m,m''}$ is the disjoint union of $Y_1, \ldots, Y_l$, that $Z_j = h^{-1}(Y_j)$ is non-singular for each j, that $\{Y_j\}$ satisfies the Whitney condition A, that $h|_{Z_j}$ is a submersion to Y_j for each j, and that $\dim Y_j \leq \dim Y_{j'}$ for $j \leq j'$. Let $\xi_j: E_j \xrightarrow{q} Y_j$ denote the restriction of $\xi_{m,m''}$ on Y_j for $j = 1, \ldots, l$.

Now we want to find a C^∞ map $g: E_{m,m''} \to \mathbb{R}^i$ such that the restriction of g on E_j is in general position for $j = 1, \ldots, l$. Since Y_1 is compact, by Lemma 2 there exists a C^∞ map $g_1: E_1 \to \mathbb{R}^i$ in general position. Let g_1' be a C^∞ extension of g_1 on $E_{m,m''}$.

The restriction $g_1'|_{E_2}$ is in general position on an open neighborhood U_1 of Y_1 in $G_{m,m''}$. The reason is the following. As the problem is local, we can assume $E_{m,m''} = G_{m,m''} \times \mathbb{R}^m$. Assume that there does not exist such U_1. Then we have a sequence of points $a_1 a_2, \ldots,$ in Y_2 such that

$$\overline{g_1'|_{E_2}} = \overline{g_1'}|_{Y_2}: Y_2 \longrightarrow \mathfrak{m}_0(m, i)$$

is not in general position at $a_1, a_2, \ldots,$ and that $a_1, a_2, \ldots$ tend to a point b of Y_1. We write briefly $\mathfrak{m}_j(m, i) = \mathfrak{m}_j$. Choosing a subsequence, we can assume that $c_j = \overline{g_1'}(a_j)$, $j = 1, 2, \ldots$ are all contained in one $\mathfrak{m}_k - \mathfrak{m}_{k+1}$ for some k, and that the tangent vector space α_j of Y_2 at a_j and γ_j of $\mathfrak{m}_k - \mathfrak{m}_{k+1}$ at c_j, $j = 1, 2, \ldots$ tend to $\alpha \in G_{l,l'}$ and $\gamma \in G_{l'',m_1 - l''}$ respectively for some l, l', l''. Then, by the Whitney condition A on $\{Y_j\}$ and $\{\mathfrak{m}_j - \mathfrak{m}_{j+1}\}$, α and γ contain the tangent vector spaces β of Y_1 at b and δ of $\mathfrak{m}_{k'} - \mathfrak{m}_{k'+1}$ at $d = \overline{g_1'}(b)$ respectively where $d \in \mathfrak{m}_{k'} - \mathfrak{m}_{k'+1}$. Since $\overline{g_1'}|_{Y_1}$ is transversal to $\mathfrak{m}_{k'} - \mathfrak{m}_{k'+1}$ at b, we have

$$d(\overline{g_1'})_b(\beta) + \delta = T_d \text{ the tangent vector space of } \mathfrak{m}_0 \text{ at } d.$$

Hence

$$d(\bar{g}'_1)_b(\alpha) + \gamma = T_d.$$

It follows that $d(\bar{g}'_1)_{a_j}(\alpha_j) + \gamma_j = T_{a_j}$ for large j. This means that $\bar{g}'_1 |_{Y_2}$ is transversal to $\mathfrak{m}_k - \mathfrak{m}_{k+1}$ at a_j. This is a contradiction. Thus $g'_1 |_{E_2}$ is in general position on some U_1.

Let W_1 be an open neighborhood of Y_1 in $G_{m,m''}$ whose closure is contained in U_1, E'_2 denote the restriction of E_2 on $Y_2 - W_1$. By Lemma 2 we can approximate $g'_1 |_{E'_2}$ by a C^∞ map g_2 in general position on E'_2. Let ψ be a C^∞ function defined on $G_{m,m''}$ such that

$$\psi = \begin{cases} 1 & \text{on} \quad W_1 \\ 0 & \text{on} \quad Y_2 - U_1. \end{cases}$$

Write

$$g'_2(x) = \psi \circ q(x) g'_1(x) - (1 - \psi) \circ q(x) \times g_2(x), \quad x \in E_1 \cup E_2.$$

Then, taking a closer approximation g_2, we can assume that $g'_2 |_{E_1} = g_1$ and $g'_2 |_{E_2}$ are in general position. Let g''_2 be a C^∞ extension of g'_2 on $E_{m,m''}$. We repeat this argument on $g''_2, E_3, \ldots$. Then we have a C^∞ map $g: E_{m,m''} \to \mathbf{R}^i$ such that $g |_{E_j}$ is in general position for $j = 1, \ldots, l$.

Moreover the argument above implies that the set of such maps g is open and dense in the set of C^∞ maps from $E_{m,m''}$ to $\mathbf{R}^i$. Therefore there exists a smooth rational map $g: E_{m,m''} \to \mathbf{R}^i$ such that $g |_{E_j}$ is in general position for $j = 1, \ldots, l$.

For each $j = 1, \ldots, l$, consider f_j the restriction of $f = g \circ \tilde{h}$ on $\tilde{Z}_j = \tilde{h}^{-1}(E_j)$ where $\tilde{h}: h^* E_{m,m''} \to E_{m,m''}$ is the lift of h. Then we see that f_j is in a general position as follows. As the problem is local, we can assume

$$E_j = Y_j \times \mathbf{R}^m, \quad \tilde{Z}_j = Z_j \times \mathbf{R}^m, \quad \tilde{h}(z, v) = (h(z), v) \quad \text{for} \quad (z, v) \in Z_j \times \mathbf{R}^m.$$

Then we have

$$\bar{f}_j = \bar{g} |_{Y_j} \circ h |_{Z_j}: Z_j \longrightarrow \mathfrak{m}_0.$$

We saw already that $h |_{Z_j}$ is a submersion to Y_j and that $\bar{g} |_{Y_j}: Y_j \to \mathfrak{m}_0$ is transversal to all strata $\mathfrak{m}_1 - \mathfrak{m}_2, \mathfrak{m}_2 - \mathfrak{m}_3, \ldots$. Hence $\bar{f}_j$ is transversal to $\mathfrak{m}_1 - \mathfrak{m}_2$, $\ldots$, namely f_j is in general position. Furthermore it follows that f is in general position at any point of Z_j, hence globally. Thus we have constructed an algebraic realization of the Stiefel-Whitney classes.

As the codimension of F_2^i (the set of singular points of F_1^i) in $G_{m,m'}$ is $2(m - i + 2)$, $h_f^{-1}(F_2^i)$ is of dimension at most $n - 2(m - i + 2)$. Hence the above

algebraic subvariety of realization of the k-th Stiefel-Whitney class is non-singular for $k \geq (n-1)/2$. Since F_1^1 is non-singular, the realization of m-th Stiefel-Whitney class is non-singular.

A smoothing of the algebraic realization of the first Stiefel-Whitney class follows from Theorem 3 below, because the subvariety is of codimension 1. Thus the theorem is proved.

Proof of Theorem 2. We prove this in the same way as above. For the Pontrjagin classes, we only have to choose m'' so that $m + m'$ is even. Then it is well-known [15] that $G_{m,m'}$ and F_2^{m-2i+2} are orientable, and that F_2^{m-2i+2} realizes the i-th Pontrjagin class of $\xi_{m,m'}$. Hence $h_f^{-1}(F_2^{m-2i+2})$ is an algebraic realization of the i-th Pontrjagin class of $h_f^* \xi_{m,m'}$. Furthermore $h_f^{-1}(F_2^{m-2i+2})$ is non-singular for $i \geq (n-2)/6$, because the set of singular points of F_2^{m-2i+2} is F_3^{m-2i+2}, and the codimension of F_3^{m-2i+2} is $6i+3$.

For the Euler class, choose m'' so that $m + m'$ is odd. Let $\tilde{G}_{m,m'}$ denote the oriented Grassmann manifold, and $p: \tilde{G}_{m,m'} \to G_{m,m'}$ denote the natural projection. Then p is a 2-fold covering. Since the bundle in problem is orientable, there exists a C^∞ map $\tilde{h}_f: X \to \tilde{G}_{m,m'}$ such that $p \circ \tilde{h}_f = h_f$. We know (see [15]) that the Euler class of $p^* \xi_{m,m'}$ is realized by $p^{-1}(F_1^1)$. Here we must remark that $\tilde{G}_{m,m'}$ and F_1^1 are orientable. Hence $h_f^{-1}(F_1^1)$ realizes the Euler class of $\xi_{m,m'}$. This subvariety is non-singular, because F_1^1 is non-singular. Thus we have proved the theorem.

Remark 1. Consider the case of tangent bundle in Theorem 1. Let $1 \leq k \leq n$ be an odd integer or equal to n. The k-th Stiefel-Whitney class of the bundle is an element of $H^k(X; \mathbf{Z})$ where we use twisted coefficients. The Poincaré dual of this class is a homology class W_k of X with coefficients in $\mathbf{Z}$. Here we use infinite chain if X is not compact. The algebraic subvariety of realization of the k-th Stiefel-Whitney class which was constructed in the proof of Theorem 1 is orientable and has the oriented fundamental class W_k.

Remark 2. Even if a cohomology class of X is realized by a C^∞ manifold, it is not necessarily realized by an algebraic subvariety. See an example in [1].

§4. Smoothing of Algebraic Subvarieties

Given algebraic subvarieties V, V' of X, we call them $\mathbf{Z}_2$-*homologically*

equivalent if their fundamental classes are identical.   If the fundamental classes have the coefficients in Z, then V and V' are called Z-*homologically equivalent*. Moreover, "Z-(or Z_2-) *smoothing* V" means that we can find a non-singular V'.

Theorem 3.   *The Z_2-smoothing of any algebraic subvariety V of X of codimension 1 is possible.   Assume moreover that X is compact.   Let M be a C^∞ submanifold of X Z_2-homologically equivalent to V.   Then we can Z_2-smooth V so that the smoothed subvariety is the image of M under a C^∞ diffeomorphism of X arbitrarily close to the identity.*

Theorem 4.   *Assume that X is compact.   Let V, M be an orientable algebraic subvariety and an orientable C^∞ submanifold respectively which are Z-homologically equivalent.   Then we can Z-smooth V so that the smoothed subvariety is the image of M under a C^∞ diffeomorphism of X arbitrarily close to the identity.   Especially, if X is compact and orientable, the Z-smoothing of any orientable algebraic subvariety of codimension 1 is possible.*

Proofs.   The first half of Theorem 4 follows trivially from the latter half of Theorem 3.   The latter half of Theorem 4 is an immediate consequence of the first half of Theorem 4 and the fact that any first Z-cohomology class of a compact orientable manifold is realized by a C^∞ submanifold (see [11]).   Hence we need only to prove Theorem 3.

Let X' be the desingularization of Hironaka of the algebraic closure of X in $P^{n'}(R)$ where $X \subset R^{n'}$.   Let V' be the algebraic closure of V in X'.   If we can smooth V', the intersection of the smoothed subvariety and X is non-singular and Z_2-homologically equivalent to V.   Hence, from the beginning we can assume that X is compact.   Then the first half of Theorem 3 follows from the latter half and the fact that any first Z_2-cohomology class of a compact manifold is realized by a C^∞ submanifold [11].   Thus it is sufficient to prove the latter half of Theorem 3.   Obviously we can assume that M is analytic.

If an irreducible component of V is of codimension > 1, we can remove it, because the removing does not affect the fundamental class of V.   Hence we assume that all irreducible components of V are of codimension 1.   Moving M if necessary, we can assume also that the germs of M and of V at each point of $M \cup V$ are not identical.   Let $\Gamma, \mathcal{O}$ denote the rings of smooth rational functions, analytic functions on X respectively.   Let $\mathfrak{a} \subset \Gamma, \mathfrak{b} \subset \mathcal{O}$ be the ideals of functions vanishing on V, M respectively.

Now we will prove in the same way as Lemma 1 in [8] that $\mathfrak{c} = \mathfrak{a}\mathcal{O} \cap \mathfrak{b}$ is a

principal ideal. Let K be a triangulation of X, K' be a subcomplex such that $|K'| = V \cup M$ and that each $(n-2)$-simplex of K' is faces of even-numbered $(n-1)$-simplexes of K. Put

$$\Gamma_x = \{f/g \mid f, g \in \Gamma, g(x) \neq 0\} \quad \text{for} \quad x \in X,$$
$$\mathcal{O}_x = \text{the ring of analytic function germs at } x \in X.$$

We write

$$\mathfrak{a}_x = \mathfrak{a}\Gamma_x, \ \mathfrak{b}_x = \mathfrak{b}\mathcal{O}_x \quad \text{and} \quad \mathfrak{c}_x = \mathfrak{c}\mathcal{O}_x.$$

By the assumption on the irreducible components of V, $\mathfrak{a}$ is the intersection of prime ideals of height 1, hence so is $\mathfrak{a}_x$ for any $x \in X$. Clearly $\mathfrak{b}$ and $\mathfrak{b}_x$, $x \in X$, have the same property. Since X is non-singular, Γ_x and $\mathcal{O}_x$ are unique factorization rings for any $x \in X$. Therefore $\mathfrak{a}_x$, $\mathfrak{b}_x$ and hence $\mathfrak{c}_x$, $x \in X$, are principal ideals, and we have

$$(*) \qquad\qquad\qquad \mathfrak{c}_x = \mathfrak{a}_x \mathfrak{b}_x.$$

Here we remark that $\mathfrak{b}_x$ is the set of analytic function germs vanishing on M (Cartan Theorem A on Stein manifold).

Let $\Delta_1, \ldots, \Delta_l$ be all the n-simplexes of K. We can suppose the existence of $f_1, \ldots, f_l$ in $\mathfrak{c}$ such that for each i, f_i is a generator of $\mathfrak{c}_x$ if x is a point of the star of Δ_i. Let $s_1, \ldots, s_r$ be all the $(n-1)$-simplexes of K'. Then the chain $s_1 + \cdots + s_r$ is a cycle. By the hypotheses, this is the boundary of a chain, say, $\Delta_1 + \cdots + \Delta_{l'}$, $l' < l$. We choose f_i so that f_i is positive in the interior of Δ_i if $i \leq l'$, and negative if $i > l'$. Then, for each Δ_i and each Δ_j in $\mathrm{St}(\Delta_i)$, f_i and f_j take the same sign in Δ_j, and f_j/f_i is positive and smooth there, because, if s_k is the common face of Δ_i and Δ_j, s_k contains non-singular points of V, hence the sign of values of f_i changes through s_k.

Let ρ_i, $i = 1, \ldots, l$ be elements of Γ such that $0 \leq \rho_i$, and $\rho_i \geq 1$ on Δ_i and $\leq \varepsilon$ outside of $\mathrm{St}(\Delta_i)$ where ε is a small positive number. We choose ε so small that for each Δ_i and each $\Delta_j \not\subset \mathrm{St}(\Delta_i)$, $\rho_j f_j/f_i$ is close to the zero function on Δ_i. Then the sum f of $\rho_i g_i$, $i = 1, \ldots, l$, generates $\mathfrak{c}_x$ for all x of X. Hence f is a generator of $\mathfrak{c}$, namely, $\mathfrak{c}$ is a principal ideal.

Let f be a generator of $\mathfrak{c}$, and $g_1, \ldots, g_k$ be a system of generators of $\mathfrak{a}$. Then we have analytic functions $\varphi_1, \ldots, \varphi_k$ defined on X such that

$$f = \varphi_1 g_1 + \cdots + \varphi_k h_k.$$

Let $\psi_1, \ldots, \psi_k$ be approximations of $\varphi_1, \ldots, \varphi_k$ by smooth rational functions. Put

$$f' = \psi_1 g_1 + \cdots + \psi_k g_k,$$
$$M' = \text{the closure of } f'^{-1}(0) - V.$$

Then M' is a non-singular algebraic subvariety and can be translated to M by a C^∞ diffeomorphism of X close to the identity. The reason is the following. Since $\mathfrak{a}_x$ is a principal ideal for each $x \in X$, there exists a closed neighborhood U of x where one of $g_1, \ldots, g_k$, say g_1, is a generator. Hence we have rational functions $h_2, \ldots, h_k$ on X smooth on U such that $g_i = h_i g_1$, $i = 2, \ldots, k$. Put

$$\Phi_U = \varphi_1 + \varphi_2 h_2 + \cdots + \varphi_k h_k \qquad \text{on} \quad U.$$

It follows that $f = \Phi_U g_1$ on U. By (*), Φ_U is a generator of $\mathfrak{b}_{x'}$ for any $x' \in U$. This means that the set of zero points of Φ_U is $U \cap M$ and that Φ_U is regular at $U \cap M$. Clearly

$$\Phi_U = \psi_1 + \psi_2 h_2 + \cdots + \psi_k h_k \qquad \text{on} \quad U$$

is an approximation of Φ_U by a smooth rational function, and we have

$$\Phi_U^{-1}(0) = U \cap M'.$$

Choose a close approximation, and cover X by such U's. Then we can assume that

$$U \cap M = \phi \quad \text{or} \quad \tau \colon (U, U \cap M) \overset{\sim}{\longrightarrow} ([0, 1]^n, 0 \times [0, 1]^{n-1}).$$

If $U \cap M = \phi$, we have $U \cap M' = \phi$. In the second case, we can assume moreover that the union W of U's such that $U \cap M \neq \phi$ is a tubular neighborhood of M, that the projection π of W onto M corresponds to

$$\pi' \colon [0, 1]^n \ni (z_1, \ldots, z_n) \longrightarrow (0, z_2, \ldots, z_n) \in 0 \times [0, 1]^{n-1},$$

and that $\Phi_U \circ \tau^{-1} = z_1$. Then $\Psi_U \circ \tau^{-1}$ is an approximation of z_1. Hence $\Psi_U \circ \tau^{-1}$ is a regular function, and $\tau(U \cap M')$ (= the set of zero points) is translated to $0 \times [0, 1]^{n-1}$ by π'. Thus $\pi|_{M'}$ is a diffeomorphism from M' to M. Obviously it is possible to extend $\pi|_{M'}$ onto X. It is easy to see that the closer the approximation of $\varphi_1, \ldots, \varphi_k$ is, the more the extension of $\pi|_{M'}$ is chosen close to the identity. The argument above shows also that M' is a non-singular algebraic subvariety; hence the theorem is proved.

Let V, V' be non-singular algebraic subvarieties of X, and α, α' denote the cohomology classes realized by V, V'. If V and V' intersect transversally, the cup product of α and α' is realized by $V \cap V'$. Hence we can generalize Theorems 3, 4 as follows.

Theorem 3′. *Let a cohomology class $\alpha \in H^k(X; \mathbf{Z}_2)$, $k = 1, \ldots, n$, be the cup k-product of the first cohomology classes $\beta_1, \ldots, \beta_k$. If $\beta_1, \ldots, \beta_k$ are realized by algebraic subvarieties, then α is realized by a non-singular algebraic subvariety.*

Theorem 4′. *Assume that X is orientable and compact. The above statement holds for $\mathbf{Z}$-coefficients.*

Example. From Theorems 1 and 3′, it follows that if X is homeomorphic to $P^n(\mathbf{R})$, n even, any $\mathbf{Z}_2$-cohomology class of X is realized by a non-singular algebraic subvariety.

§5. Cup Product of Realizable Cohomology Classes

Let V, V' be algebraic subvarieties of X of codimension k, k'. Let α, α' be the cohomology classes realized by V, V' respectively, and $\{Y_i\}$, $\{Y'_j\}$ be respective stratifications of V, V'. If each Y_i and each Y'_j intersect transversally, then the cup product $\alpha \cup \alpha'$ is realized by the intersection $V \cap V'$. In this section we will weaken these conditions. The coefficient ring may be $\mathbf{Z}_2$ or $\mathbf{Z}$. For $\mathbf{Z}$, we must remember the arrangement at the beginning of Section 2.

Put $V'' = V \cap V'$. Let V_s, V'_s and V''_s denote the respective sets of singular points of V, V' and V''. Let $T\cdot$ denote the tangent vector space.

Theorem 5. *Let an algebraic subvariety S satisfy the following properties*
(i) $S \supset V'' \cap (V_s \cup V'_s \cup V''_s)$,
(ii) $\operatorname{codim} S > k + k'$,
(iii) $TR = T(V - V_s)|_R \cap T(V' - V'_s)|_R$ *with* $R = V'' - S$.
Then $\alpha \cup \alpha'$ is realized by an algebraic subvariety, and the set of singular points of the subvariety can be contained in S.

Proof. At first we suppose $S = \phi$. Then any point of V'' is a non-singular point of V and V'. Since V'' is non-singular, any irreducible component is a union of connected components. Hence, by the argument below we can assume that V'' is a manifold of dimension n'. Put $m = n + n' - k - k'$. Let $\xi: Y \xrightarrow{\tau} V''$ be the subbundle of $TX|_R$ of vectors orthogonal to $TV|_{V''} \oplus TV'|_{V''}$. We saw in Section 2 that the normal bundle of X is an affine algebraic bundle. By the same method it follows that ξ is an affine algebraic bundle. Let f be a C^∞ cross-section of ξ such that $f(V'')$ and V'' intersect transversally. Here we identify V'' with its image under the zero cross-section of ξ. We write the intersection as Z.

Using f, we can construct a C^∞ diffeomorphism θ of X arbitrarily close to the identity in the C^∞ Whitney topology such that $\theta(x)=x$ for any singular point x of V and V', that V and $\theta(V')$ intersect transversally and that $V \cap \theta(V')=Z$. Hence the C^∞ manifold Z realizes $\alpha \cup \alpha'$. On the other hand, by Lemma 3 below, Z realizes the m-th Stiefel-Whitney class of ξ. Apply Theorem 1 to ξ. Then it follows that $\alpha \cup \alpha'$ is realized by a non-singular algebraic subvariety. If the coefficient ring is $\mathbf{Z}$, it is sufficient to treat the Euler class of ξ instead of the Stiefel-Whitney class.

The case $S \ne \phi$. Apply the result above to $X-S$, $V-S$ and $V'-S$. We remark that the algebraic varieties are affine. Let $Z \subset X-S$ be an algebraic subvariety of realization of $(\alpha \cup \alpha')|_{X-S}$, Z' be the algebraic closure of Z in X. Then Z' is an algebraic subvariety of realization of $\alpha \cup \alpha'$ because of $\dim(Z' \cap S)$ $< \dim Z'$. Hence we proved the theorem.

Example. Assume that V, V' are non-singular. If $V \supset V'$, the conditions in Theorem 5 are satisfied.

Let $\eta: E \xrightarrow{\pi} M$ be a C^∞ vector bundle of rank m over a C^∞ manifold M. Let f be a C^∞ cross-section of η. Then we have

Lemma 3. *Assume that the image $f(M)$ and M intersect transversally. Then the intersection realizes the m-th Stiefel-Whitney class of η. Furthermore it realizes the Euler class if M and η are orientable.*

Proof. We use the notation in Section 2. First, consider the case $\eta = \xi_{m,m'}$. Then F_1^1 realizes the m-th Stiefel-Whitney class of $\xi_{m,m'}$. Hence it is sufficient to find a C^∞ cross-section f of $\xi_{m,m'}$ such that $f(G_{m,m})$ intersects transversally with $G_{m,m}$ at F_1^1. Put

$$f(T)=P_T((0,\dots,0,1)) \qquad \text{for} \quad T \in G_{m,m'}.$$

Then, obviously $f(G_{m,m'}) \cap G_{m,m} = F_1^1$. We see easily also that $f(G_{m,m'})$ and $G_{m,m'}$ intersect transversally, according to the formulas of local coordinate system of $G_{m,m'}$ [15].

The general case for the Stiefel-Whitney class. Let $h: M \to G_{m,m'}$ be a C^∞ classifying of map η. Modifying h if necessary, we can assume that h is transversal to F_1^1. Then $h^{-1}(F_1^1)$ realizes the m-th Stiefel-Whitney class of η. Put $f' = \tilde{h}^{-1} \circ f \circ h$, where $\tilde{h}$ is the lift of h. Then it follows immediately that $f'(M)$ intersects transversally with M at $h^{-1}(F_1^1)$.

For the Euler class, we need only to use $\tilde{G}_{m,m'}$ the oriented Grassmann

manifold like we did in Proof of Theorem 2. We omit the details.

We can prove the following in the same way as Theorems 3', 4'.

Theorem 6. *If* codim $V' = 1$, *then* $\alpha \cup \alpha'$ *is realized by an algebraic sub-variety, and the set of singular points of the subvariety is contained in that of* V. *Here, if the coefficient ring is* $\mathbb{Z}$, *we assume moreover the compactness of* X.

§6. Affine Algebraic Structure of Vector Bundles

Let $\eta : E \xrightarrow{\pi} X$ be a topological vector bundle of rank m. We say that η has an affine algebraic structure if η is equivalent to an affine algebraic vector bundle. It is proved in [1] that if $X \approx S^n$, η has an affine algebraic structure.

Conjecture. *Assume that all Stiefel-Whitney classes of* η *are realized by algebraic subvarieties. Then* η *has an affine algebraic structure.*

Theorem 7. *If* $m = 1$, *Conjecture is correct.*

Proof. Let $V \subset X$ be an algebraic subvariety which realizes the Stiefel-Whitney class of η. By Theorem 3, we can assume the smoothness of V. Let X' be the desingularization of Hironaka of the algebraic closure of X in $P^{n'}(\mathbf{R})$ where $X \subset \mathbf{R}^{n'}$, and V' be the algebraic closure of V in X'. Then there exists a line bundle η' over X' whose Stiefel-Whitney class is realized by V'. Since line bundles over X are determined by their Stiefel-Whitney classes, we can regard η as the restriction of η' on X. Hence it is sufficient to find an affine algebraic structure of η'. Therefore, from the beginning we assume that X is compact.

Let $V_0 = V, V_1, \ldots, V_n$ be distinct C^∞ submanifolds of X such that each V_i is the image of V under a C^∞ diffeomorphism close to the identity. Using the transversality theorem, we reduce V_i, $i = 1, \ldots, n$ to satisfy $\bigcap_{i=0}^{n} V_i = \phi$. Apply Theorem 3 to $V_1, \ldots, V_n$. Then we can assume also that V_i are non-singular algebraic subvarieties. Let V_{ij} denote the domain sandwiched in between V_i and V_j for each i, j. Put $U_i = X - V_i$, $i = 0, \ldots, n$. Then $\{U_i\}$ is an open algebraic covering of X. If we have smooth rational functions ψ_{ij} on $U_{ij} = U_i \cap U_j$ for each i, j such that

$$\psi_{ii} = 1, \; \psi_{ij}\psi_{jk} = \psi_{ik}, \quad \text{and}$$

$$\psi_{ij} \begin{cases} > 0 & \text{on} \quad U_{ij} - V_{ij} \\ < 0 & \text{on} \quad V_{ij}, \end{cases}$$

then there exists an algebraic line bundle over X which is equivalent to η. Moreover, if the bundle has a complexification, it is affine (see, e.g., [13]).

For each $i=1,\ldots, n$, we can construct a smooth rational function f_i on X in the same way as in Proofs of Theorems 3, 4 such that

(i) $f_i^{-1}(0) = V_0 \cup V_i$,

(ii) $f_i < 0$ in V_{0_i},

(iii) f_i is regular at $(V_0 - V_i) \cup (V_i - V_0)$, and

(iv) for any $x \in V_0 \cap V_i$, the germ of f_i at x is the product of two regular function germs.

Put $\psi_{ij} = f_i/f_j$. Then ψ_{ij} satisfy the conditions above and have complexifications. Hence Theorem is proved.

References

[1] Benedetti, R. and Tognoli, A., On real algebraic vector bundles, *Bull. Sci. Math.*, **104** (1980), 89–112.

[2] Bochnak, J., Kucharz, W. and Shiota, M., The divisor class groups of some rings of global real analytic, Nash or rational regular functions, to appear.

[3] Borel, A. and Haefliger, A., La classe d'homologie fondamental d'un espace analytique, *Bull. Soc. Math. France,* **89** (1961), 461–513.

[4] Grothendieck, A., La théorie des classes de Chern, *Bull. Soc. Math. France,* **86** (1958), 137–154.

[5] Milnor, J. W. and Stasheff, J. D., *Characteristic classes,* Ann. of Math. Studies, **76,** Princeton Univ. Press, 1974.

[6] Morimoto, H., Realization of Chern classes by subvarieties with certain singularities, *Nagoya Math. J.,* **80** (1980), 49–74.

[7] Palais, R., *Equivariant, real algebraic differential topology, Part I smoothness categories and Nash manifolds,* Notes Brandeis Univ., 1972.

[8] Shiota, M., Sur la factorialité de l'anneau des fonctions lisses rationnelles, *C.R.Acad. Sc. Paris,* **292** (1981), 67–70.

[9] Shiota, M., Equivalence of differentiable functions, rational functions and polynomials, *Ann. Inst. Fourier,* **32** (1982).

[10] Suzuki, H., On the realization of the Stiefel-Whitney characteristic classes by submanifolds, *Tôhoku Math. J.,* **10** (1958), 91–115.

[11] Thom, R., Quelques propriétés globales des variétés différentiables, *Comm. Math. Helv.* **28** (1954), 17–86.

[12] Tom, R., Un lemme sur less applications différentiables, *Bol. Soc. Mat. Mexicana,* **1** (1956), 59–71.

[13] Tognoli, A., *Algebraic geometry and Nash functions,* Academic Press, 1978.

[14] Whitney, H., Tangents to an analytic variety, *Ann. of Math.,* **81** (1965), 496–549.

[15] Wu, W. T., *Sur les classes caractéristiques des structures fibrées sphériques,* Act. Sci. Ind., **1183,** Hermann, 1952.

Added in proof: The author was informed that some of his results were obtained also by R. Benedetti and A. Tognoli (Remarks and counterexamples in real algebraic vector bundles and cycles, to appear).