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Invariant States for Strongly Positive Operators on C^*-Algebras

By

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Abstract

A linear operator σ on a C^*-algebra A induces a contraction $\hat{\sigma}_\phi$ on the Hilbert space $\mathscr{H}_\phi$ associated with a σ-invariant state ϕ provided σ satisfies the Schwarz inequality: $\sigma(a^*a) \geq \sigma(a)^*\sigma(a)$. If ϕ is invariant under a class $\mathscr{S}$ of such operators, the following four properties are closely connected:

(i) abelianness of the reduction of $\pi_\phi(A)$ to the $\hat{\mathscr{S}}_\phi$-invariant part of $\mathscr{H}_\phi$,
(ii) asymptotic abelianness of ϕ,
(iii) abelianness of $\pi_\phi(A)' \cap \hat{\mathscr{S}}'_\phi$.
(iv) uniqueness of decompositions of ϕ into extremal $\mathscr{S}$-invariant states.

If $\mathscr{S}$ consists of 2-positive operators, almost all the same relationships between these properties hold as for the case of automorphism groups which has already been thoroughly investigated.

§ 1. Introduction

Much attention has been given to those states of a C^*-algebra A which are invariant under a group of *-automorphisms. A detailed account of the resulting theory is given in [4, Chapter 4]. The subject is particularly relevant to the algebraic model of statistical mechanics, where the automorphisms represent symmetries of the system. For example, the time-evolution of a reversible system corresponds to a strongly continuous one-parameter group of automorphisms. However, in an irreversible system, the time-evolution is represented by a one-parameter semigroup of positive operators σ_t on A, which are not necessarily multiplicative. In all the familiar examples, σ_t is completely positive, and it can be argued that there are good physical reasons for this (see for example the comments in [8, p. 136]). Completely positive semigroups can also be constructed by defining

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$$\sigma_t(a)=\int_G \alpha_g(a)d\mu_t(g)$$

where α is an action of a locally compact group G on A by automorphisms, and $\{\mu_t\colon t\geq 0\}$ is a convolution semigroup of probability measures on G.

This paper is concerned with semigroups $\mathscr{S}$ of linear operators σ on A which are *strongly positive* in the sense that they satisfy the Schwarz inequality:

$$\sigma(a^*a)\geq\sigma(a)^*\sigma(a).$$

Any 2-positive operator is strongly positive, but the converse fails [5]. If A is commutative, any positive operator is strongly positive, so the strongly positive one-parameter semigroups on A are precisely the diffusion (or Markov) semigroups on the spectrum of A [10]. The class of strongly positive semigroups is convenient since there is an associated semigroup $\hat{\mathscr{S}}_\phi$ of contractions on the Hilbert space $\mathscr{H}_\phi$ obtained from an $\mathscr{S}$-invariant state ϕ by GNS-construction. If $\mathscr{S}$ is a group of automorphisms, then $\hat{\mathscr{S}}_\phi$ is the unitary group covariant with the action of $\mathscr{S}$ in the representation π_ϕ of A on $\mathscr{H}_\phi$. Thus it is natural to try to establish connections between:

(i) abelianness of the reduction of $\pi_\phi(A)$ to the $\hat{\mathscr{S}}_\phi$-invariant part $\mathscr{K}_\phi$ of $\mathscr{H}_\phi$,

(ii) asymptotic abelianness of ϕ,

(iii) abelianness of $\pi_\phi(A)'\cap\hat{\mathscr{S}}'_\phi$,

(iv) uniqueness of decompositions of ϕ into $\mathscr{S}$-ergodic (extremal $\mathscr{S}$-invariant) states.

These properties are known to be very closely related for automorphism groups [4], and a study of them for strictly positive one-parameter semigroups has been initiated by Majewski and Robinson [15; 16]. Assuming that ϕ is a faithful on $\pi_\phi(A)''$, the equivalence of the four properties was shown in [16, Theorem 2]. The argument depended on two technical facts:

(a) $\pi_\phi(A)''\cap\hat{\mathscr{S}}'_\phi=\pi_\phi(A)''\cap\{p_\phi\}'=(\pi_\phi(A)'')^{\mathscr{S}}$,

(b) $\hat{\sigma}_\phi\pi_\phi(a)p_\phi=\pi_\phi(\sigma(a))p_\phi$,

where p_ϕ is the orthogonal projection of $\mathscr{H}_\phi$ onto $\mathscr{K}_\phi$, and $(\pi_\phi(A)'')^{\mathscr{S}}$ is the fixed point subspace of $\pi_\phi(A)''$ under $\mathscr{S}$.

In the non-faithful case, (a) can be replaced by a similar property:

$$\text{(a)}'\quad \pi_\phi(A)'\cap\hat{\mathscr{S}}'_\phi=\pi_\phi(A)'\cap\{p_\phi\}'$$
$$=\{x\in\pi_\phi(A)'\colon a\longrightarrow\langle\pi_\phi(a)x\xi_\phi,\,\xi_\phi\rangle \text{ is } \mathscr{S}\text{-invariant}\}.$$

In the theory relating to automorphism groups, (a)′ plays an important role which is sometimes obscured by the ease with which it is derived. The properties (a)′ and (b) are established in Propositions 2.1 and 4.2 respectively below, in the first case for any strongly positive semigroup, in the second for semigroups of operators σ which are (strongly) 2-positive, in the sense that

$$\sigma_2 = \mathbf{1}_2 \otimes \sigma : (a_{ij}) \longrightarrow (\sigma(a_{ij}))$$

is (strongly) positive on the C^*-algebra $M_2(A) = M_2(\mathbf{C}) \otimes A$ of 2×2 matrices over A. These two technical results play crucial roles in the subsequent arguments. For example it is immediately possible to link properties (iii) and (iv) (Theorem 3.1), and properties (i) and (ii) (Proposition 5.1).

When considering the global versions of some of the four properties, a new question is raised — is it sufficient that the property holds for the $\mathscr{S}$-ergodic states? For an automorphism group $\mathscr{S}$, suitable global versions of the four properties are equivalent [7; 4, Corollary 4.3.11], and it is sufficient that (i), (ii) or (iv) holds for $\mathscr{S}$-ergodic states [6; 3]. The corresponding result for (strongly) 2-positive semigroups is given in Theorem 6.1. Versions of the Kovacs-Szücs ergodic theorem and "quasi-largeness" of $\mathscr{S}$ are also obtained (Theorem 6.2).

Broadly speaking, this paper shows that results about invariant states for automorphism groups extend to strongly 2-positive semigroups, in particular to completely positive semigroups.

I am very grateful to D. W. Robinson for explaining the extent of his own work on this subject.

§2. The Induced Contraction Semigroups

Throughout the paper, A will be a C^*-algebra with identity $\mathbb{1}$, and $\mathscr{S}$ will be a semigroup of strongly positive linear operators σ on A satisfying $\sigma(\mathbb{1}) = \mathbb{1}$. If ϕ is an $\mathscr{S}$-invariant state of A with associated cyclic representation $(\mathscr{H}_\phi, \pi_\phi, \xi_\phi)$, then

$$\|\pi_\phi(\sigma(a))\xi_\phi\|^2 = \phi(\sigma(a^*)\sigma(a)) \leq \phi(\sigma(a^*a)) = \phi(a^*a) = \|\pi_\phi(a)\xi_\phi\|^2 .$$

Hence there is an induced semigroup $\hat{\mathscr{S}}_\phi = \{\hat{\sigma}_\phi : \sigma \in \mathscr{S}\}$ of linear contractions on $\mathscr{H}_\phi$ given by

$$\hat{\sigma}_\phi \pi_\phi(a)\xi_\phi = \pi_\phi(\sigma(a))\xi_\phi .$$

Let $\mathscr{K}_\phi$ be the set of all $\hat{\mathscr{S}}_\phi$-invariant vectors in $\mathscr{H}_\phi$, and $\tilde{\mathscr{K}}_\phi$ be the set of all vectors η in $\mathscr{H}_\phi$ satisfying

$$\hat{\sigma}_\phi \pi_\phi(a)\eta = \pi_\phi(\sigma(a))\eta \qquad (a \in A,\ \sigma \in \mathscr{S}).$$

Let p_ϕ and $\tilde{p}_\phi$ be the orthogonal projections of $\mathscr{H}_\phi$ onto the closed linear subspaces $\mathscr{K}_\phi$ and $\tilde{\mathscr{K}}_\phi$ respectively. The Alaoglu-Birkhoff mean ergodic theorem [4, Proposition 4.3.4] shows that p_ϕ belongs to the strongly closed convex hull of $\hat{\mathscr{S}}_\phi$. It will be seen later that $\mathscr{K}_\phi$ and $\tilde{\mathscr{K}}_\phi$ usually coincide.

Let Θ_ϕ be the order-isomorphism of $\pi_\phi(A)'$ into A^* defined by

$$\Theta_\phi(x)(a) = \langle \pi_\phi(a)x\xi_\phi, \xi_\phi\rangle .$$

The range of Θ_ϕ is the linear span of the face of the state space $S(A)$ of A generated by ϕ.

Proposition 2.1. *For x in $\pi_\phi(A)'$, the following are equivalent:*

(i) $x\hat{\sigma}_\phi = \hat{\sigma}_\phi x \qquad (\sigma \in \mathscr{S})$,

(ii) $xp_\phi = p_\phi x$,

(iii) *$\Theta_\phi(x)$ is $\mathscr{S}$-invariant,*

(iv) $x\tilde{p}_\phi = \tilde{p}_\phi x$.

Proof. The implication (i)⇒(ii) is immediate from the Alaoglu-Birkhoff mean ergodic theorem.

If x commutes with any projection $p \le p_\phi$ with $p\xi_\phi = \xi_\phi$, then

$$\begin{aligned}\Theta_\phi(x)(\sigma(a)) &= \langle px\pi_\phi(\sigma(a))\xi_\phi, \xi_\phi\rangle = \langle xp\hat{\sigma}_\phi\pi_\phi(a)\xi_\phi, \xi_\phi\rangle \\ &= \langle xp\pi_\phi(a)\xi_\phi, \xi_\phi\rangle = \Theta_\phi(x)(a).\end{aligned}$$

Thus $\Theta_\phi(x)$ is $\mathscr{S}$-invariant. This proves that (ii)⇒(iii) and (iv)⇒(iii).

To prove (iii)⇒(ii), it suffices to assume that $x \ge 0$ and $\langle x\xi_\phi, \xi_\phi\rangle = 1$, so that $\psi = \Theta_\phi(x)$ is an $\mathscr{S}$-invariant state. The implication then follows in a similar manner to the argument used to prove 1⇒2 in [16, Proposition 1], since the Alaoglu-Birkhoff theorem ensures that operators σ in the convex hull of $\mathscr{S}$ may be found so that $\hat{\sigma}_\phi$ and $\hat{\sigma}_\psi$ approximate p_ϕ and p_ψ respectively in the strong* topology.

Now let x be a projection in the von Neumann algebra $\pi_\phi(A)' \cap \{p_\phi\}'$. Then for σ in $\mathscr{S}$,

$$\begin{aligned}\|x\hat{\sigma}_\phi\pi_\phi(a)\xi_\phi\|^2 &= \|\pi_\phi(\sigma(a))x\xi_\phi\|^2 = \langle \pi_\phi(\sigma(a^*)\sigma(a))x\xi_\phi, x\xi_\phi\rangle \\ &\le \langle \pi_\phi(\sigma(a^*a))x\xi_\phi, x\xi_\phi\rangle = \langle x\hat{\sigma}_\phi\pi_\phi(a^*a)\xi_\phi, xp_\phi\xi_\phi\rangle \\ &= \langle x\pi_\phi(a^*a)\xi_\phi, x\xi_\phi\rangle = \|x\pi_\phi(a)\xi_\phi\|^2 .\end{aligned}$$

Hence

$$\|x\hat{\sigma}_\phi(\mathbb{1}_\phi - x)\eta\| \leq \|x(\mathbb{1}_\phi - x)\eta\| = 0 \qquad (\eta \in \mathscr{H}_\phi)$$

($\mathbb{1}_\phi$ is the identity operator on $\mathscr{H}_\phi$), so $x\hat{\sigma}_\phi(\mathbb{1}_\phi - x) = 0$. Replacing x by $\mathbb{1}_\phi - x$, it follows that $x\hat{\sigma}_\phi = x\hat{\sigma}_\phi x = \hat{\sigma}_\phi x$. Since $\pi_\phi(A)' \cap \{p_\phi\}'$ is generated by its projections, this completes the proof that (ii)⇒(i).

Finally it is clear from the definition that $\tilde{p}_\phi$ belongs to the von Neumann algebra generated by $\pi_\phi(A)$ and $\hat{\mathscr{S}}_\phi$. Since (i) and (ii) are equivalent, this algebra is generated by $\pi_\phi(A)$ and p_ϕ. This gives the implication (ii)⇒(iv).

In the following, the von Neumann algebra generated by $\pi_\phi(A)$ and p_ϕ will be denoted by $\mathfrak{A}_\phi$. Proposition 2.1 shows that $\mathfrak{A}'_\phi = \pi_\phi(A)' \cap \hat{\mathscr{S}}'_\phi$, so $\hat{\mathscr{S}}_\phi \subset \mathfrak{A}_\phi$. Since $\hat{\mathscr{S}}_\phi$ is not self-adjoint in general, Proposition 2.1 is needed even to show that $\pi_\phi(A)' \cap \hat{\mathscr{S}}'_\phi$ is a von Neumann algebra. A simple calculation now shows that $\tilde{\mathscr{K}}_\phi$ contains $\mathfrak{A}'_\phi \xi_\phi$.

An extreme point of the compact convex set $S_{\mathscr{S}}(A)$ of all $\mathscr{S}$-invariant states will be called an *$\mathscr{S}$-ergodic* state.

Corollary 2.2. *An $\mathscr{S}$-invariant state ϕ is $\mathscr{S}$-ergodic if and only if $\pi_\phi(A) \cup \{p_\phi\}$ is irreducible.*

Proof. This follows from Proposition 2.1 which shows that the positive $\mathscr{S}$-invariant functionals majorised by ϕ are those of the form $\Theta_\phi(x)$ where $x \in \mathfrak{A}'_\phi$ and $0 \leq x \leq \mathbb{1}_\phi$.

Corollary 2.2 was proved by Robinson [16, Theorem 3] in the case when A is a von Neumann algebra, and ϕ is faithful and normal, and it is implicitly there in general. If ξ_ϕ is separating for $\pi_\phi(A)''$, then $\mathscr{S}$ induces a semigroup on $\pi_\phi(A)''$, and it follows from [16, Theorem 1] that $\pi_\phi(A)'' \cap \{p_\phi\}' = \pi_\phi(A)'' \cap \hat{\mathscr{S}}'_\phi$. This is closely related to Proposition 2.1 since $\mathscr{K}_\phi$ is invariant under the modular conjugation J, so that

$$\pi_\phi(A)'' \cap \{p_\phi\}' = J(\pi_\phi(A)' \cap \{Jp_\phi J\}')J = J\mathfrak{A}'_\phi J$$
$$\pi_\phi(A)'' \cap \hat{\mathscr{S}}'_\phi = J(\pi_\phi(A)' \cap (J\hat{\mathscr{S}}_\phi J)')J\,.$$

Proposition 2.1 may also be compared with [11, Proposition 2.3] in this setting.

A typical state ψ in the face F_ϕ of $S_{\mathscr{S}}(A)$ generated by some $\mathscr{S}$-invariant state ϕ is of the form $\psi = \Theta_\phi(x)$ for some x in $\mathfrak{A}'_\phi$ with $x \geq 0$ and $\langle x\xi_\phi, \xi_\phi\rangle = 1$. Now π_ψ may be identified with the restriction of π_ϕ to the subspace $\mathscr{H}_\psi = [\pi_\phi(A)x\xi_\phi]$, and $\xi_\psi = x^{\frac{1}{2}}\xi_\phi$. Since $\mathscr{H}_\psi$ is $\hat{\mathscr{S}}_\phi$-invariant, the orthogonal projection e of $\mathscr{H}_\phi$

onto $\mathscr{H}_\psi$ satisfies $\hat{\sigma}_\phi e = e\hat{\sigma}_\phi e$, and therefore $p_\phi e = ep_\phi e$ by the Alaoglu-Birkhoff theorem. Thus by Proposition 2.1, $\mathscr{H}_\psi$ is $\hat{\mathscr{S}}_\phi$-reducing, and calculation shows that $\hat{\sigma}_\psi = \hat{\sigma}_\phi e$ and $p_\psi = p_\phi e$.

Corollary 2.3. *Let E be any subset of $S_{\mathscr{S}}(A)$, and $(\pi, \hat{\mathscr{S}}, p) = \bigoplus_E (\pi_\phi, \hat{\mathscr{S}}_\phi, p_\phi)$. Then $\hat{\mathscr{S}}$ is contained in $(\pi(A) \cup \{p\})''$.*

Proof. It suffices to assume that E has finite cardinality n. Let $\psi = n^{-1} \sum_E \phi$, and $(\mathscr{H}_0, \pi_0, \hat{\mathscr{S}}_0, p_0)$ be the direct sum of n copies of $(\mathscr{H}_\psi, \pi_\psi, \hat{\mathscr{S}}_\psi, p_\psi)$. By Proposition 2.1, $\hat{\mathscr{S}}_0 \subset (\pi_0(A) \cup \{p_0\})''$, and by the above discussion, $(\pi, \hat{\mathscr{S}}, p)$ may be identified with the restriction of $(\pi_0, \hat{\mathscr{S}}_0, p_0)$ to some reducing subspace of $\mathscr{H}_0$.

Corollary 2.4. *Let $\phi = \sum_{i=1}^{\infty} \lambda_i \phi_i$ be a σ-convex combination of $\mathscr{S}$-invariant states, and $(\mathscr{H}, \pi, \hat{\mathscr{S}}, \xi) = \bigoplus_{i=1}^{\infty} (\mathscr{H}_{\phi_i}, \pi_{\phi_i}, \hat{\mathscr{S}}_{\phi_i}, \lambda_i^{\frac{1}{2}} \xi_{\phi_i})$. Then $(\mathscr{H}_\phi, \pi_\phi, \hat{\mathscr{S}}_\phi, \xi_\phi)$ may be identified with the restriction of $(\mathscr{H}, \pi, \hat{\mathscr{S}}, \xi)$ to the $(\pi(A) \cup \hat{\mathscr{S}})$-reducing subspace $[\pi(A)\xi]$.*

Proof. It is clear that $(\mathscr{H}_\phi, \pi_\phi, \hat{\mathscr{S}}_\phi, \xi_\phi)$ may be identified with the restriction of $(\mathscr{H}, \pi, \hat{\mathscr{S}}, \xi)$ to the $\pi(A)$-reducing, $\hat{\mathscr{S}}$-invariant subspace $[\pi(A)\xi]$. It follows from Corollary 2.3 that this space is $\hat{\mathscr{S}}$-reducing.

Two $\mathscr{S}$-invariant states ϕ and ψ will be said to be *$\mathscr{S}$-equivalent* if there is a unitary mapping U of $\mathscr{H}_\phi$ onto $\mathscr{H}_\psi$, taking $\mathscr{K}_\phi$ onto $\mathscr{K}_\psi$ and intertwining π_ϕ with π_ψ, so that

$$U\pi_\phi(a)U^* = \pi_\psi(a) \qquad (a \in A).$$

Suppose ϕ is $\mathscr{S}$-ergodic, and η is a unit vector in $\tilde{\mathscr{K}}_\phi$ linearly independent of ξ_ϕ. Clearly the vector state $\psi = \omega_\phi^\eta$ $(\psi(a) = \langle \pi_\phi(a)\eta, \eta \rangle)$ is $\mathscr{S}$-invariant. Furthermore, for $a_1, \ldots, a_{n+1}$ in A and $\sigma_1, \ldots, \sigma_n$ in $\mathscr{S}$, let

$$b = a_1\sigma_1(a_2\sigma_2(a_3 \cdots \sigma_n(a_{n+1})) \cdots).$$

Then

$$\pi_\phi(b)\tilde{p}_\phi = \pi_\phi(a_1)(\sigma_1)\hat{}_\phi \pi_\phi(a_2) \cdots (\sigma_n)\hat{}_\phi \pi_\phi(a_{n+1})\tilde{p}_\phi.$$

It follows from the irreducibility of $\pi_\phi(A) \cup \{p_\phi\}$ (Corollary 2.2) and the Kadison transitivity theorem that $\xi_\phi \oplus \eta$ is cyclic for $(\pi_\phi \oplus \pi_\phi)(A)$ on $\mathscr{H}_\phi \oplus \mathscr{H}_\phi$. In particular, $(\mathscr{H}_\psi, \pi_\psi, \xi_\psi)$ may be identified with $(\mathscr{H}_\phi, \pi_\phi, \eta)$, and then

$$\hat{\sigma}_\psi \pi_\phi(a)\eta = \pi_\phi(\sigma(a))\eta = \hat{\sigma}_\phi \pi_\phi(a)\eta$$

so $\hat{\sigma}_\psi = \hat{\sigma}_\phi$. Thus $\mathfrak{A}'_\psi = \mathfrak{A}'_\phi$, so ψ is an $\mathscr{S}$-ergodic state distinct from, but $\mathscr{S}$-equivalent to, ϕ. The following result gives more precise information, and is an extension of [2, Corollary 4.3].

Proposition 2.5. *Let ϕ and ψ be distinct $\mathscr{S}$-ergodic states, and $\rho = \frac{1}{2}(\phi + \psi)$. The following are equivalent:*

(i) *ϕ and ψ are $\mathscr{S}$-equivalent,*

(ii) *$\psi = \omega_\phi^\eta$ for some η in $\tilde{\mathscr{K}}_\phi$,*

(iii) *$\mathfrak{A}'_\rho$ is a factor,*

(iv) *the line-segment between ϕ and ψ is not a face of $S_{\mathscr{S}}(A)$.*

In this case, $\mathfrak{A}'_\rho$ is a type I_2 factor, and the smallest face $F_{\phi\psi}$ of $S_{\mathscr{S}}(A)$ containing ϕ and ψ is affinely homeomorphic to a 3-dimensional Euclidean ball.

Proof. Let $(\mathscr{H}, \pi, \hat{\mathscr{S}}, p, \xi) = (\mathscr{H}_\phi, \pi_\phi, \hat{\mathscr{S}}_\phi, p_\phi, 2^{-\frac{1}{2}}\xi_\phi) \oplus (\mathscr{H}_\psi, \pi_\psi, \hat{\mathscr{S}}_\psi, p_\psi, 2^{-\frac{1}{2}}\xi_\psi)$, so that $(\mathscr{H}_\rho, \pi_\rho, \hat{\mathscr{S}}_\rho, p_\rho)$ identifies with the restriction of $(\mathscr{H}, \pi, \hat{\mathscr{S}}, p)$ to $[\pi(A)\xi]$ (Corollary 2.4). Note that $F_{\phi\psi} = F_\rho$.

(i)⇒(ii). There is a unitary U of $\mathscr{H}_\phi$ onto $\mathscr{H}_\psi$ intertwining π_ϕ with π_ψ and p_ϕ with p_ψ. It follows from Corollary 2.3 that U intertwines $\hat{\mathscr{S}}_\phi$ with $\hat{\mathscr{S}}_\psi$. Hence the vector $\eta = U^*\xi_\psi$ lies in $\tilde{\mathscr{K}}_\phi$ and satisfies $\psi = \omega_\phi^\eta$.

(ii)⇒(iii). By the above remarks, $\mathscr{H}_\rho = \mathscr{H}$ and $\mathfrak{A}'_\rho$ consists of those operators on $\mathscr{H} = \mathscr{H}_\phi \oplus \mathscr{H}_\psi$ whose matrix representations are of the form $(x_{ij})_{i,j=1,2}$ where $x_{ij} \in \mathfrak{A}'_\phi = \boldsymbol{C}\mathbf{1}_\phi$.

(iii)⇒(iv). Since $\mathfrak{A}'_\rho$ is not two-dimensional, $F_{\phi\psi} = \Theta_\rho\{x \in \mathfrak{A}'_\rho : x \geq 0, \langle x\xi_\rho, \xi_\rho \rangle = 1\}$ is not one-dimensional.

(iv)⇒(i). For an operator $x = (x_{ij})$ in $\pi(A)' \cap \{p\}'$, x_{21} intertwines π_ϕ with π_ψ and p_ϕ with p_ψ, so $x_{21}^* x_{21} \in \mathfrak{A}'_\phi = \boldsymbol{C}\mathbf{1}_\phi$, $x_{21}x_{21}^* \in \mathfrak{A}'_\psi = \boldsymbol{C}\mathbf{1}_\psi$. If ϕ and ψ are not $\mathscr{S}$-equivalent, then there is no unitary intertwining operator, so $x_{21} = 0$. Thus

$$\pi(A)' \cap \{p\}' = \mathfrak{A}'_\phi \oplus \mathfrak{A}'_\psi = \boldsymbol{C}\mathbf{1}_\phi \oplus \boldsymbol{C}\mathbf{1}_\psi .$$

Hence $\mathscr{H}_\rho = \mathscr{H}$, and

$$\begin{aligned} F_{\phi\psi} &= \Theta_\rho\{\lambda\mathbf{1}_\phi \oplus \mu\mathbf{1}_\psi : \lambda \geq 0, \mu \geq 0, \lambda + \mu = 1\} \\ &= \{\lambda\phi + \mu\psi : \lambda \geq 0, \mu \geq 0, \lambda + \mu = 1\} . \end{aligned}$$

If ϕ and ψ are $\mathscr{S}$-equivalent, the proof of (ii)⇒(iii) shows that $\mathfrak{A}'_\rho$ is a type I_2 factor, and $F_{\phi\psi}$ is affinely homeomorphic to the set of 2×2 density matrices. These form a 3-dimensional Euclidean ball (see [1, p. 103]).

Part of Proposition 2.5 may be extended to show that the convex and σ-

convex hulls of any set of mutually $\mathscr{S}$-inequivalent $\mathscr{S}$-ergodic states are faces of $S_{\mathscr{S}}(A)$ (see [2, Corollary 4.3]).

§ 3. Local Decomposition Theory

Proposition 2.1 permits several aspects of the decomposition theory of individual states under a group of automorphisms (see [4, Section 4.3; 12]) to be extended to the case of strongly positive semigroups. Parts (i) and (iii) of Theorem 3.1 are proved in [16, Proposition 1].

Theorem 3.1. *Let ϕ be an $\mathscr{S}$-invariant state of A, $\mathfrak{B}$ be an abelian von Neumann subalgebra of $\mathfrak{A}'_\phi$, and $\mu_{\mathfrak{B}}$ be the orthogonal measure on $S(A)$ associated with $\mathfrak{B}$.*

(i) *$\mu_{\mathfrak{B}}$ is supported by $S_{\mathscr{S}}(A)$.*

(ii) *$\mu_{\mathfrak{B}}$ is a maximal measure on $S_{\mathscr{S}}(A)$ if and only if $\mathfrak{B}$ is a maximal abelian subalgebra of $\mathfrak{A}'_\phi$.*

(iii) *There is a unique maximal measure μ on $S_{\mathscr{S}}(A)$ representing ϕ if and only if $\mathfrak{A}'_\phi$ is abelian. In this case, μ is the $\mathfrak{A}'_\phi$-measure.*

Proof. In view of Proposition 2.1, the proofs of similar results in [4, Section 4.3] and [12, p. 106] can all be extended without significant modification.

There are however possible difficulties concerning the global decomposition theory. If $\mathscr{S}$ is a group of automorphisms, it is known [3; 4; 6; 7] that $S_{\mathscr{S}}(A)$ is a Choquet simplex if and only if $p_\phi\pi_\phi(A)p_\phi$ is abelian for each ϕ in $S_{\mathscr{S}}(A)$, or, equivalently, $\mathscr{K}_\phi$ is one-dimensional for each $\mathscr{S}$-ergodic ϕ. Furthermore abelianness of $p_\phi\pi_\phi(A)p_\phi$ is characterised by $\mathscr{S}$-abelianness:

$$\inf\{|\omega_\phi^\eta(a'b-ba')| : a'\in C_{\mathscr{S}}(a)\}=0 \qquad (a,\, b\in A;\, \eta\in\mathscr{K}_\phi)$$

where $C_{\mathscr{S}}(a)$ is the convex hull of $\mathscr{S}(a)=\{\sigma(a)\colon \sigma\in\mathscr{S}\}$. The proofs of these facts depended heavily on the fact that $\mathscr{K}_\phi=\tilde{\mathscr{K}}_\phi$, which is very easily established for groups of automorphisms. In the next section of this paper it will be shown that $\mathscr{K}_\phi=\tilde{\mathscr{K}}_\phi$ provided that $\mathscr{S}$ is strongly 2-positive, and the results mentioned above will be extended to such semigroups in the subsequent sections.

The method used in [3] involved passing to a projection in the weak closure of the crossed product of the C^*-dynamical system. This device is no longer available, but it is at least possible to use some of the earlier techniques of decomposition theory in separable cases [2; 6].

The semigroup $\mathscr{S}$ of operators on A will be said to be *separable* if each of its orbits $\mathscr{S}(a)$ is separable. Clearly $\mathscr{S}$ is separable if A is separable or if $\mathscr{S}$ is a strongly continuous one-parameter semigroup.

The next two lemmas present some of the technical measure-theoretic details needed. Here μ is a Baire measure on $S_{\mathscr{S}}(A)$ representing ϕ, and η is an arbitrary vector in $\mathscr{H}_\phi$. If $(a_n)_{n\geq 1}$ is a sequence in A chosen so that $\sum \|\pi_\phi(a_n)\xi_\phi-\eta\|<\infty$, it was shown in [2, Lemma 2.4] that $\eta_\psi=\lim \pi_\psi(a_n)\xi_\psi$ exists $\mu-$a.e.(ψ), and η_ψ is a.e. independent of the choice of (a_n). For η' in $\mathscr{H}_\phi$ and η'_ψ in $\mathscr{H}_\psi$ obtained from η' in this way,

$$\langle\eta, \eta'\rangle=\int\langle\eta_\psi, \eta'_\psi\rangle d\mu(\psi).$$

Lemma 3.2. *Suppose $\mathscr{S}$ is separable, and $\eta\in\mathscr{K}_\phi$. Then $\eta_\psi\in\mathscr{K}_\psi$ $\mu-$a.e. If A is separable and $\eta\in\tilde{\mathscr{K}}_\phi$, then $\eta_\psi\in\tilde{\mathscr{K}}_\psi$ $\mu-$a.e.*

Proof. For fixed σ in $\mathscr{S}$,

$$\sum \|\pi_\phi(\sigma(a_n))\xi_\phi-\eta\|=\sum \|\hat{\sigma}_\phi(\pi_\phi(a_n)\xi_\phi-\eta)\|<\infty.$$

Hence, for μ-almost all ψ,

$$\eta_\psi=\lim \pi_\psi(\sigma(a_n))\xi_\psi=\lim \hat{\sigma}_\psi\pi_\psi(a_n)\xi_\psi=\hat{\sigma}_\psi\eta_\psi.$$

By considering a countable subset $\mathscr{S}_0$ of $\mathscr{S}$ such that $\mathscr{S}_0(a_n)$ is dense in $\mathscr{S}(a_n)$ for each n, it follows that $\eta_\psi=\hat{\sigma}_\psi\eta_\psi$ for all σ in $\mathscr{S}$, so $\eta_\psi\in\mathscr{K}_\psi$ $\mu-$a.e.

A similar argument establishes the second statement.

Lemma 3.3. *For any η, η' in $\mathscr{H}_\phi$, and σ in $\mathscr{S}$,*

$$\langle\hat{\sigma}_\phi\eta, \eta'\rangle=\int\langle\hat{\sigma}_\psi\eta_\psi, \eta'_\psi\rangle d\mu(\psi).$$

Proof. With $\eta''=\hat{\sigma}_\phi\eta$,

$$\sum \|\pi_\phi(\sigma(a_n))\xi_\phi-\eta''\|=\sum \|\hat{\sigma}_\phi(\pi_\phi(a_n)\xi_\phi-\eta)\|<\infty.$$

Hence $\eta''_\psi=\lim \pi_\psi(\sigma(a_n))\xi_\psi=\hat{\sigma}_\psi\eta_\psi$ $\mu-$a.e. Thus

$$\langle\hat{\sigma}_\phi\eta, \eta'\rangle=\int\langle\eta''_\psi, \eta'_\psi\rangle d\mu(\psi)=\int\langle\hat{\sigma}_\psi\eta_\psi, \eta'_\psi\rangle d\mu(\psi).$$

§4. The Invariant Hilbert Space

If $\mathscr{S}$ is a group of *-automorphisms, then $\hat{\mathscr{S}}_\phi$ is a group of unitaries on $\mathscr{H}_\phi$ satisfying the covariance relation

$$\hat{\sigma}_\phi \pi_\phi(a) \hat{\sigma}_\phi^* = \pi_\phi(\sigma(a)).$$

It is then immediate that $\hat{\sigma}_\phi \pi_\phi(a)\eta = \pi_\phi(\sigma(a))\eta$ $(\eta \in \mathscr{K}_\phi)$, so the spaces $\mathscr{K}_\phi$ and $\tilde{\mathscr{K}}_\phi$ coincide. Although it is not clear whether these spaces always coincide if $\mathscr{S}$ is a strongly positive semigroup, it is possible to show that they do if $\mathscr{S}$ is 2-positive. A simple argument follows for the case when $\mathscr{S}$ is strongly 2-positive (including the case of 4-positivity).

Lemma 4.1. *Let σ be a strongly n-positive linear operator on A, and ϕ be a σ-invariant state of A. Then for a and b_i in A and η_i in $\mathscr{H}_\phi$ $(1 \le i \le n)$,*

$$\| \sum_{i=1}^{n} \pi_\phi(\sigma(ab_i))\eta_i \|^2 \le \|a\|^2 \sum_{i,j=1}^{n} \langle \pi_\phi(\sigma(b_j^* b_i))\eta_i, \eta_j \rangle .$$

Proof. Define $x = (x_{ij})$ and y in $M_n(A)$ by:

$$x_{1j} = b_j,\ x_{ij} = 0 \qquad (1 < i \le n;\ 1 \le j \le n)$$
$$y = I_n \otimes a .$$

The strong positivity of $\sigma_n = \mathbf{1}_n \otimes \sigma$ gives:

$$\sigma_n(x^* y^*)\sigma_n(yx) \le \sigma_n(x^* y^* y x) \le \|a\|^2 \sigma_n(x^* x).$$

Applying this inequality to the vector functional defined by $\bigoplus_{i=1}^{n} \eta_i$ in the representation $\bigoplus_{i=1}^{n} \pi_\phi$ gives the result.

Proposition 4.2. *Suppose $\mathscr{S}$ is a strongly 2-positive semigroup, and ϕ is an $\mathscr{S}$-invariant state. Then $\mathscr{K}_\phi = \tilde{\mathscr{K}}_\phi$.*

Proof. Take a in A, σ in $\mathscr{S}$, η in $\mathscr{K}_\phi$, and let $(b_r)_{r \ge 1}$ be a sequence in A such that $\|\pi_\phi(b_r)\xi_\phi - \eta\| \to 0$. Apply Lemma 4.1 with $n = 2$, $b_1 = b_r$, $b_2 = \mathbf{1}$, $\eta_1 = \xi_\phi$, $\eta_2 = -\eta$. This gives

$$\begin{aligned}
\|\hat{\sigma}_\phi \pi_\phi(ab_r)\xi_\phi - \pi_\phi(\sigma(a))\eta\|^2 &= \|\pi_\phi(\sigma(ab_r))\xi_\phi - \pi_\phi(\sigma(a))\eta\|^2 \\
&\le \|a\|^2 \{\langle \pi_\phi(\sigma(b_r^* b_r))\xi_\phi, \xi_\phi \rangle - \langle \pi_\phi(\sigma(b_r))\xi_\phi, \eta \rangle - \langle \pi_\phi(\sigma(b_r^*))\eta, \xi_\phi \rangle + \langle \eta, \eta \rangle\} \\
&= \|a\|^2 \{\phi(\sigma(b_r^* b_r)) - 2\mathrm{Re}\, \langle \hat{\sigma}_\phi \pi_\phi(b_r)\xi_\phi, \eta \rangle + \langle \eta, \eta \rangle\} \\
&= \|a\|^2 \{\phi(b_r^* b_r) - 2\,\mathrm{Re}\, \langle \pi_\phi(b_r)\xi_\phi, \eta \rangle + \langle \eta, \eta \rangle\} \\
&= \|a\|^2 \|\pi_\phi(b_r)\xi_\phi - \eta\|^2 \\
&\to 0 \text{ as } r \to \infty .
\end{aligned}$$

Thus $\hat{\sigma}_\phi \pi_\phi(a)\eta = \pi_\phi(\sigma(a))\eta$, as required.

If A is abelian, all positive operators are completely positive, so Proposition 4.2 covers diffusion processes [10].

There are various other circumstances in which one can establish the equality of $\mathscr{K}_\phi$ and $\tilde{\mathscr{K}}_\phi$ without assuming strong 2-positivity. These include the following cases:

(a) ξ_ϕ is separating for $\pi_\phi(A)''$ (see [16]).

(b) The class of all normal states ψ of $\pi_\phi(A)''$ which are $\mathscr{S}$-subinvariant ($\psi\circ\pi_\phi\circ\sigma\leq\psi\circ\pi_\phi$) is faithful for $\pi_\phi(A)''$ (see [13]).

(c) ϕ is $\mathscr{S}$-ergodic, and ω_ϕ^η is $\mathscr{S}$-invariant for each η in $\mathscr{K}_\phi$.

(d) $\mathscr{S}$ is separable, and $\mathscr{K}_\psi=\tilde{\mathscr{K}}_\psi$ for every $\mathscr{S}$-ergodic state ψ of A (see Lemma 3.2).

A short calculation similar to those in Lemma 4.1 and Proposition 4.2 shows that if $\mathscr{S}$ is a 2-positive semigroup, then ω_ϕ^η is $\mathscr{S}$-invariant for each η in $\mathscr{K}_\phi$. Combining this with (c), (d), and a reduction to a separable case, it follows that Proposition 4.2 is true for all 2-positive semigroups. However this proof is long and unnatural, so it has been omitted.

§5. Asymptotic Abelianness and Cluster Properties

Among the earliest results in the study of states invariant under an automorphism group were the equivalence of abelianness of $p_\phi\pi_\phi(A)p_\phi$ with weak asymptotic abelianness of the vector states ω_ϕ^η ($\eta\in\mathscr{K}_\phi$) [14], and the equivalence of one-dimensionality of $\mathscr{K}_\phi$ with weak clustering of ϕ [9]. For strongly positive semigroups, the first of these equivalences is related to the possible equality of $\mathscr{K}_\phi$ and $\tilde{\mathscr{K}}_\phi$, but the second is always valid.

A state ϕ of A is *weakly $\mathscr{S}$-abelian* if

$$\inf\{|\phi(a'b-ba')|: a'\in C_{\mathscr{S}}(a)\}=0 \qquad (a,\ b\in A);$$

ϕ is *weakly $\mathscr{S}$-clustering* if

$$\inf\{|\phi(a'b)-\phi(a)\phi(b)|: a'\in C_{\mathscr{S}}(a)\}=0 \qquad (a,\ b\in A).$$

Proposition 5.1. *Let ϕ be an $\mathscr{S}$-invariant state.*

(i) *The following are equivalent:*

(a) *$p_\phi\pi_\phi(A)p_\phi$ is abelian,*

(b) *$\mathscr{K}_\phi=\tilde{\mathscr{K}}_\phi$, and ω_ϕ^η is weakly $\mathscr{S}$-abelian, for each η in $\mathscr{K}_\phi$.*

(ii) *The following are equivalent:*

(a) *ϕ is $\mathscr{S}$-ergodic and $p_\phi\pi_\phi(A)p_\phi$ is abelian,*

(b) *$\mathscr{K}_\phi$ is one-dimensional,*

(c) *ϕ is weakly $\mathscr{S}$-clustering.*

Proof. If $p_\phi \pi_\phi(A)p_\phi$ is abelian, then it is immediate from [4, Theorem 4.1.25] that

$$\mathscr{K}_\phi = [\mathfrak{A}'_\phi \xi_\phi] \subset \tilde{\mathscr{K}}_\phi \subset \mathscr{K}_\phi.$$

Once this is established, both parts of the proposition may be proved exactly as for automorphism groups [4, Proposition 4.3.7, Theorem 4.3.22].

§6. Global Decomposition Theory

The local properties established above are sufficient to make it straightforward to extend [2, Corollary 4.4] to strongly 2-positive semigroups. Recall that a convex set K has the 1-*ball property* if the line segment joining any two extreme points is a face of K.

Theorem 6.1. *Let $\mathscr{S}$ be a separable (strongly) 2-positive semigroup on A. The following are equivalent*:

(i) *$p_\phi \pi_\phi(A)p_\phi$ is abelian for each ϕ in $S_{\mathscr{S}}(A)$,*
(ii) *Each $\mathscr{S}$-invariant state is weakly $\mathscr{S}$-abelian,*
(iii) *$\mathfrak{A}'_\phi$ is abelian for each ϕ in $S_{\mathscr{S}}(A)$,*
(iv) *$S_{\mathscr{S}}(A)$ is a Choquet simplex,*
(v) *Any $\mathscr{S}$-invariant state ϕ for which $\mathfrak{A}'_\phi$ is a factor is $\mathscr{S}$-ergodic,*
(vi) *No two distinct $\mathscr{S}$-ergodic states are $\mathscr{S}$-equivalent,*
(vii) *$S_{\mathscr{S}}(A)$ has the 1-ball property,*
(viii) *$\mathscr{K}_\phi$ is one-dimensional for each $\mathscr{S}$-ergodic state ϕ,*
(ix) *Each $\mathscr{S}$-ergodic state is weakly $\mathscr{S}$-clustering.*

Proof. Note first that $\mathscr{K}_\phi = \tilde{\mathscr{K}}_\phi$ for each $\mathscr{S}$-invariant state ϕ (Proposition 4.2). (This is the only way in which strong 2-positivity is used in the proof.)
(i)$\Leftrightarrow$(ii), (viii)$\Leftrightarrow$(ix). Proposition 5.1.
(i)$\Rightarrow$(iii). [4, Theorem 4.1.25].
(iii)$\Leftrightarrow$(iv). Proposition 3.1 (iii).
(iv)$\Rightarrow$(v). Corollary 2.2.
(v)$\Rightarrow$(vi)$\Leftrightarrow$(vii)$\Leftrightarrow$(viii). Proposition 2.5 and the remarks preceding it.
(viii)$\Rightarrow$(i). The argument is similar to that in [2, Theorem 2.5].
Thus for η in $\mathscr{K}_\phi$ and self-adjoint a and b in A, let $\zeta = p_\phi \pi_\phi(a)\eta$, $\zeta' = p_\phi \pi_\phi(b)\eta$. Let μ be any maximal measure on $S_{\mathscr{S}}(A)$ representing ϕ. In the notation of Section 3, Lemma 3.2 shows that η_ψ, ζ_ψ and ζ'_ψ belong to $\mathscr{K}_\psi$ for μ-almost all ψ. Hence if $\mathscr{K}_\psi$ is one-dimensional, all three vectors are scalar multiples of ξ_ψ.

Using the Alaoglu-Birkhoff theorem, explicit approximations can be found to show that

$$\zeta_\psi=\psi(a)\eta_\psi,\quad \zeta'_\psi=\psi(b)\eta_\psi.$$

Hence, according to (viii), there is a Baire set E carrying μ such that any $\mathscr{S}$-ergodic state in E satisfies the identity $\langle\zeta_\psi, \zeta'_\psi\rangle=\langle\zeta'_\psi, \zeta_\psi\rangle$. Since μ is pseudo-carried by the $\mathscr{S}$-ergodic states, this identity is valid $\mu-$a.e., so by Lemma 3.3,

$$\langle\zeta, \zeta'\rangle=\int\langle\zeta_\psi, \zeta'_\psi\rangle d\mu(\psi)=\int\langle\zeta'_\psi, \zeta_\psi\rangle d\mu(\psi)=\langle\zeta', \zeta\rangle.$$

This establishes that $p_\phi\pi_\phi(A)p_\phi$ is abelian.

Without assuming that $\mathscr{K}_\phi=\tilde{\mathscr{K}}_\phi$, but instead assuming that A is separable, one can still establish the equivalence of properties (iii)–(vii) of Theorem 6.1 with each of the following properties:

(i)$^\sim$ $\tilde{p}_\phi\pi_\phi(A)\tilde{p}_\phi$ is abelian for each $\mathscr{S}$-invariant state ϕ,

(viii)$^\sim$ $\tilde{\mathscr{K}}_\phi$ is one-dimensional for each $\mathscr{S}$-ergodic state ϕ.

Now let ϕ be an $\mathscr{S}$-invariant state for which $\mathscr{K}_\phi=\tilde{\mathscr{K}}_\phi$. Then there is a version of the Kovacs-Szücs ergodic theorem [4, Proposition 4.3.8]. Thus if $q_\phi=[\pi_\phi(A)'p_\phi]$ and $\mathfrak{M}_\phi=q_\phi\pi_\phi(A)''q_\phi$ and $\mathfrak{N}_\phi=\mathfrak{M}_\phi\cap\{p_\phi\}'$, there is a faithful normal projection M of norm one of $\mathfrak{M}_\phi$ onto $\mathfrak{N}_\phi$ such that

$$p_\phi x p_\phi=M(x)p_\phi \qquad (x\in\mathfrak{M}_\phi).$$

Furthermore there is an affine bijection between

(a) normal states ρ of $\mathfrak{N}_\phi$,

(b) $\mathscr{S}$-invariant states ψ of A which induce normal states $\tilde{\psi}$ of $\pi_\phi(A)''$ with $\tilde{\psi}\circ\pi_\phi=\psi$, $\tilde{\psi}(q_\phi)=1$.

This correspondence is given by

$$\psi(a)=\rho(M(q_\phi\pi_\phi(a)q_\phi)) \qquad \rho(x)=\tilde{\psi}(x).$$

A consequence of this is that if $S_{\mathscr{S}}(A)$ is a simplex, then $p_\phi\pi_\phi(A)p_\phi$ is abelian (see [4, Theorem 4.3.9]), so conditions (i)–(iv) of Theorem 6.1 are equivalent even for non-separable (strongly) 2-positive semigroups.

Still assuming that $\mathscr{K}_\phi=\tilde{\mathscr{K}}_\phi$, it is also possible to establish the equivalence of the following (see [4, 4.3.12, 4.3.14; 16, Section 5]):

(i) ω_ϕ^η is weakly $\mathscr{S}$-abelian for each η in $q_\phi\mathscr{H}_\phi$,

(ii) $\mathfrak{N}_\phi\subset\pi_\phi(A)'q_\phi$,

(iii) Each state ψ in the class (b) is represented by a unique maximal

measure μ_ψ on $S_{\mathscr{S}}(A)$, and μ_ψ is a subcentral measure on $S(A)$.
From this the following is obtained immediately:

Theorem 6.2. *Let $\mathscr{S}$ be a (strongly) 2-positive semigroup on A. The following are equivalent:*

(i) *Each state of A dominated by an $\mathscr{S}$-invariant state is weakly $\mathscr{S}$-abelian,*

(ii) *$S_{\mathscr{S}}(A)$ is a Choquet simplex, the maximal measures on which are subcentral measures on $S(A)$.*

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